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Design and comparison of lattices of topological relations for spatial representation and reasoning

Florence Le Ber — Ludmila Mangelinck — Amedeo Napoli

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THÈME 3



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Design and comparison of lattices of topological relations for spatial representation and reasoning

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Abstract: This report presents an original approach to spatial representation and reasoning with topological relations based on the use of lattices of relations. This approach has been developed for spatial reasoning in the domain of agricultural landscape analysis. The report describes first the motivation of the present research work and the general framework of topological relations. Four different lattices of topological relations, including Galois lattices, are introduced. The choices made for spatial representation and reasoning with these lattices are discussed, together with a thorough study and comparison of the four lattices. The implementation of one of the lattices within an object-based representation system, as well as spatial lattice-based reasoning with topological relations in this system are then detailed.

Key-words: Topological relations, lattice-based classification, Galois connection, relation computation, raster images, object-based representation systems

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Définition et comparaison de treillis de relations topologiques pour la représentation et le raisonnement sur l'espace

Résumé : Ce rapport présente une approche originale pour la représentation de l'espace et le raisonnement spatial qualitatif. Cette approche s'appuie sur l'utilisation de treillis de relations topologiques et a été développée à des fins de reconnaissance d'organisations spatiales de paysages dans le domaine agronomique. Le rapport décrit d'abord la motivation de ce travail et le cadre théorique des relations topologiques dans lequel nous nous plaçons. Nous présentons quatre treillis de relations topologiques, dont deux sont des treillis de Galois. Nous discutons et comparons les capacités de ces treillis en matière de représentation et de raisonnement. Finalement, nous décrivons l'implantation et l'utilisation d'un de ces treillis pour la classification de structures spatiales dans le cadre d'un système de représentation de connaissances par objets.

Mots-clés : Relations topologiques, classification à partir de treillis, Correspondance de Galois, calcul de relations, images rasteur, représentation de connaissances par objets

Introduction

The study of land use is at present of first importance in agronomics, in order to understand the reorganization of farmland, the transformations of landscape and the alterations of the environment, depending on changes of the agriculture, e.g. enlargement of farm enterprises, modification of agricultural practices (cropping plans, etc.). This study leads to a number of qualitative and quantitative questions regarding the interpretation of present and past land use, the prediction of land use in the near future, the influences of the agricultural practices on the structure of land use, the conditions that have to be fulfilled to reach a desired landscape, etc. Moreover, it could be possible to understand the present state and the near future of land use and agriculture on the basis of the present observations.

Actually, these questions are linked together in the agronomics researches about agricultural landscape analysis, that is carried on the basis of field inquiries and spatial observations on maps, aerial photographs and satellite images. This is one of the fundamental reasons why landscape analysis is of main importance in the agronomics researches nowadays [Deffontaines 1986]. For that purpose, the agronomics researchers are particularly interested in the relationship between landscape and farming systems and their consequences on the environment. They try to discover and formalize landscape patterns that are particular structures of land use, which are linked to specific combinations of factors (soil, relief, settlement, farming system) [Deffontaines 1990, Benoît 1990]. Agronomics researchers assume that the structure of land use reveals the working of the farming system. Landscape patterns exist at several scales and are very useful for global diagnoses and forecasts on agriculture with regard to the environment.

Land-use maps based on satellite images are very useful for landscape analysis. However because of the quantity and frequency of these data, an automatic processing is needed. To be more efficient such processing must include the expertise of the researchers about farming systems and land use. Thus, it is worthwhile to use AI methods to build a knowledge-based system that could help agronomists to exploit land-use maps for landscape analysis. Therefore, the landscape patterns defined by the agronomics researchers have been adapted for classifying *zones* extracted on the maps [Le Ber and Mangelinck 1998]. Each zone corresponds to a “village territory” which is supposed to be exploited by the farmers of this village. A zone is made of raster regions, i.e. connected sets of pixels with the same label denoting the land-use category, e.g. crops, meadows, forest, village, etc. The adapted landscape patterns, called *landscape models*, are described with a list of agricultural spatial entities and a set of qualitative spatial relations between these entities.

From an implementation point of view, an object-based knowledge representation system, or OKR system, equipped with classification methods has been used [Napoli 1992, Napoli *et al.* 1994]. Indeed, exploiting land-use maps for landscape analysis may be considered as an instance classification problem, where landscape models correspond to classes, and zones correspond to instances that have to be classified according to landscape models. For solving such a classification problem, the landscape models have to be represented, implying that knowledge on agricultural entities and on spatial relations must be encoded. Besides, methods for extracting regions from the land-use maps and for checking spatial re-

lations on these regions have been defined [Mangelinck *et al.* 1996]. These methods are used as a basis for a classification process, whose goal is to compare the regions and the relations extracted from a land-use map under study to the entities and the relations associated with the landscape models.

In our application, the classification process is mainly based on the spatial relations, since they are the most characteristic elements of the landscape models. We have focused on the representation of topological relations because they can be described and manipulated on the basis of a well defined theoretical framework. Relying on this framework, we need a representation system of the topological relations that:

- defines primitives to compute relations on raster images: to compute –or to check– a relation R between two regions x and y means to test if $R(x, y)$ holds.
- integrates all the relations used in the landscape models,
- allows to store new information,
- allows reasoning on relations and composition of relations: to reason on relations means to deduce new relations from already computed ones.
- can be managed by an OKR system, and thus, the application must be low-sized in terms of memory space.

According to these needs, we have designed a representation of the topological relations based on a lattice structure. Such a representation has been already used for example in [Randell and Cohn 1992] to draw inferences on relations. We have built four lattices, including two lattices based on the Galois connection, that is used to create *concepts*, by associating sets of objects (the *extents* of the concepts) to sets of attributes (the *intents*) [Barbut and Monjardet 1970, Davey and Priestley 1990, Wille 1992]. In this way, relations that have to be recognized on an image can be associated to properties that can be computed on the image. The different lattices have been evaluated and implemented in the framework of our application.

The work presented here shows several new and original aspects in the management of spatial relations, regarding representation, reasoning and implementation aspects. For the representation and reasoning purposes, the theory of Galois lattices has been employed with success and has allowed to build different lattices providing different points of view on the topological relations, depending on the way the boundary of a raster region is considered. Moreover, it is one of the unique works relying on the Galois lattice theory, that is usually associated with formal concept analysis [Wille 1992], or for knowledge discovery purposes [Godin and Missaoui 1994, Simon and Napoli 1999]. Finally, classification-based reasoning is used for inferring spatial relations and spatial structures of agricultural entities, involving both classification of entities and relations. Another noticeable aspect is that this work allows a qualitative as well as a quantitative approach to the analysis of landscape on land-use maps, because the chosen implementation framework, i.e. OKR systems, provides both reasoning and computation facilities. This contrasts with existing works in the domain of

qualitative spatial reasoning, which mainly focus on reasoning about qualitative relations only. Indeed, our system combines symbolic and numerical techniques for reasoning about spatial entities and their qualitative relations and provides results that are easy to read and understand for agronomists. This research work can be naturally extended to the design of geographic information systems.

This article describes our work on the definition and the comparison of the different lattices of topological relations. It is structured as follows: the first part introduces the context of our work and describes the spatial theories on which it is based; the second part presents our proposition for computing topological relations on raster images; the third and fourth parts describe the various possibilities that have been explored for building lattices and a comparison of these lattices. The fifth part presents some elements about the implementation of the relations and their use for the classification of spatial structures. The sixth part gives an evaluation and some perspectives of the present research work. Finally we conclude.

1 Related work

In the section we present research works that are related to our work and that, for some of them, have inspired our work.

1.1 Topological relations: theories

Mathematical topology is based on the “open set” notion and defines the notions of “closed set” and “neighborhood”. Topology can also be defined with the notion of continuity: two forms are topologically equivalent if one form can be continuously transformed (i.e. with no hole) into the other: for instance, an egg is equivalent to a ball but is not equivalent to a torus. Topology can then be used to characterize forms.

In qualitative representations of space, topological relations are opposed to distance and orientation relations. Topological relations are binary and independent of the orientation of the entities. The basic ontological entity is generally a region. Regions represent portions of space determined by concrete objects, and are not considered as sets of spatial points [Vieu 1993].

Mereology is a theory of the *part-whole* relation. It was introduced by Lesniewski at the beginning of 20th century as an alternative to the set theory [Lesniewski 1927 1931]. It is an axiomatic base for topology and geometry of regions rather than of points. Mereology defines the relation $P(x, y)$, for “*x is a part of y*”, which is a partial order since it is reflexive, anti-symmetric and transitive:

$$\begin{aligned} \forall x \in \mathcal{D} & : P(x, x) \\ \forall x, y \in \mathcal{D} & : (P(x, y) \wedge P(y, x)) \rightarrow x = y \\ \forall x, y, z \in \mathcal{D} & : (P(x, y) \wedge P(y, z)) \rightarrow P(x, z) \end{aligned}$$

Seven relations can be defined from this primitive relation (see Table 1). The $PP(x, y)$ relation means “ x is a proper part of y ”. The $O(x, y)$ relation means “ x overlaps y ”. The $U(x, y)$ relation means “ x and y are part of the same region”. The remaining relations represent various cases of the “overlap” and “inclusion” notions.

$$\left| \begin{array}{lcl} PP(x, y) & \equiv_{def} & P(x, y) \wedge \neg P(y, x) \\ O(x, y) & \equiv_{def} & \exists z : P(z, x) \wedge P(z, y) \\ U(x, y) & \equiv_{def} & \exists z : P(x, z) \wedge P(y, z) \\ OX(x, y) & \equiv_{def} & O(x, y) \wedge \neg P(x, y) \\ UX(x, y) & \equiv_{def} & U(x, y) \wedge \neg P(y, x) \\ PO(x, y) & \equiv_{def} & OX(x, y) \wedge OX(y, x) \\ PU(x, y) & \equiv_{def} & UX(x, y) \wedge UX(y, x) \end{array} \right|$$

Table 1: Mereological relations (see [Vieu 1993, Varzi 1996]). The symbols \wedge and \neg represent the usual logical connectives.

Clarke has later proposed to formalize the part-whole relation using the “connection” notion [Clarke 1981, Clarke 1985]. Two regions x and y are said to be connected $-C(x, y)$ – if they share a point. A point is a zero-dimensional spatial element, according to its mathematical meaning. The Clarke’s theory is defined by the following axioms:

$$\begin{array}{lcl} \forall x \in \mathcal{D} & : & C(x, x) \\ \forall x, y \in \mathcal{D} & : & C(x, y) \rightarrow C(y, x) \\ \forall x, y \in \mathcal{D} & : & (\forall z \in \mathcal{D} : C(z, x) \leftrightarrow C(z, y)) \rightarrow x = y \end{array}$$

Relying on the connection relation, Clarke has defined five relations: $DC(x, y)$ “ x is disconnected from y ”, $P(x, y)$ “ x is a part of y ”, $PP(x, y)$ “ x is a proper part of y ”, $O(x, y)$ “ x overlaps y ”, $DR(x, y)$ “ x is discrete from y ”. The theory based on these relations is interesting because it makes a difference between a point and a region, thanks to the relations $C(x, y)$ – x and y share a point– and $O(x, y)$ – x and y share a region. This differentiation allows the definition of three additional relations: $EC(x, y)$ “ x is externally connected with y ”, $TP(x, y)$ “ x is a tangential part of y ”, $NTP(x, y)$ “ x is a non tangential part of y ” (see Table 2). Finally this differentiation is used to introduce topological notions (interior, boundary) into the purely set-based notions of the mereology: the Clarke’s theory is therefore called *mereotopology*.

Several theories have then been proposed for topological relations [Randell *et al.* 1992b, Vieu 1993]. These theories are generally formalized in first order logic. The differences between the theories are about the number of relations, and the interpretation of the relations. For instance, [Randell *et al.* 1992a] defines a theory with 10 relations including the “internal part” notion whereas [Cohn *et al.* 1993] defines a theory with 16 relations including relations based on the “convex hull” notion: *OUTSIDE*, *INSIDE*, etc. Each theory is defined by a set of axioms, depending on the desired properties of the spatial regions (open

$DC(x, y)$	\equiv_{def}	$\neg C(x, y)$
$P(x, y)$	\equiv_{def}	$\forall z : C(z, x) \rightarrow C(z, y)$
$PP(x, y)$	\equiv_{def}	$P(x, y) \wedge \neg P(y, x)$
$O(x, y)$	\equiv_{def}	$\exists z : P(z, x) \wedge P(z, y)$
$DR(x, y)$	\equiv_{def}	$\neg O(x, y)$
$EC(x, y)$	\equiv_{def}	$C(x, y) \wedge \neg O(x, y)$
$TP(x, y)$	\equiv_{def}	$P(x, y) \wedge \exists z : EC(z, x) \wedge EC(z, y)$
$NTP(x, y)$	\equiv_{def}	$P(x, y) \wedge \neg \exists z : EC(z, x) \wedge EC(z, y)$

Table 2: Mereotopological relations [Clarke 1981]. The relations $P(x, y)$, $PP(x, y)$, and $O(x, y)$ are identical to the relations $P(x, y)$, $PP(x, y)$, and $O(x, y)$ of mereology (Table 1).

and closed regions, atomicity, non-atomicity, etc., see [Varzi 1996, Vieu 1997, Varzi 1998, Masolo and Vieu 1999, Masolo 2000] for surveys and discussions on these points of view).

The most used and well-known topological relations are those of the so-called *RCC-8* theory [Randell *et al.* 1992b, Cohn *et al.* 1997] which is also based on the connection relation. This theory defines eight base relations which are exhaustive and mutually exclusive. Their names and iconic representations are given in Table 3 and their definitions in Table 4.

relation	notation	icons
“ <i>x is identical with y</i> ”	$EQ(x, y)$	
“ <i>x is a non tangential proper part of y</i> ”	$NTPP(x, y)$	
“ <i>x is a tangential proper part of y</i> ”	$TPP(x, y)$	
“ <i>x non tangentially contains as a proper part y</i> ”	$NTPP^{-1}(x, y)$	
“ <i>x tangentially contains as a proper part y</i> ”	$TPP^{-1}(x, y)$	
“ <i>x partially overlaps y</i> ”	$PO(x, y)$	
“ <i>x is externally connected with y</i> ”	$EC(x, y)$	
“ <i>x is disconnected from y</i> ”	$DC(x, y)$	

Table 3: Names and icons associated to the eight base relations of the *RCC-8* theory.

1.2 Computation of topological relations

In order to deal with applications, such as geographic databases or image interpretation, one needs to compute relations, i.e. one needs operations for testing if a relation holds between two regions. Such computation operations have been proposed in [Egenhofer 1989,

$EQ(x, y)$	\equiv_{def}	$P(x, y) \wedge P(y, x)$
$NTPP(x, y)$	\equiv_{def}	$PP(x, y) \wedge \neg \exists z : EC(z, x) \wedge EC(z, y)$
$TPP(x, y)$	\equiv_{def}	$PP(x, y) \wedge \exists z : EC(z, x) \wedge EC(z, y)$
$NTPP^{-1}(x, y)$	\equiv_{def}	$NTPP(y, x)$
$TPP^{-1}(x, y)$	\equiv_{def}	$TPP(y, x)$
$PO(x, y)$	\equiv_{def}	$O(x, y) \wedge \neg P(x, y) \wedge \neg P(y, x)$
$EC(x, y)$	\equiv_{def}	$C(x, y) \wedge \neg O(x, y)$
$DC(x, y)$	\equiv_{def}	$\neg C(x, y)$

Table 4: The base relations of the *RCC-8* theory [Randell *et al.*1992b]. Their definitions rely on the mereotopology.

Egenhofer and Sharma 1993, Clementini *et al.*1993]: they are based on the notions of *interior* and *boundary* of a region. They are linked to formal models of the topological relations.

In [Egenhofer 1989], computation operations are defined for the vector representation of geographic information systems: a n -dimensional region x is characterized by two sets, the interior (of dimension n), denoted by x° , and the boundary (of dimension $n - 1$), denoted by ∂x . Two regions are thus characterized by four sets whose intersections define four operations: $\partial x \cap \partial y$, $x^\circ \cap y^\circ$, $\partial x \cap y^\circ$ et $x^\circ \cap \partial y$. Figure 1 and Table 5 show the correspondence between the results of these operations and the eight relations of the *RCC-8* theory. For instance if the relation “ x partially overlaps y ”, $PO(x, y)$, holds between the two regions x and y , then the four operations applied to x and y give the following results: $\partial x \cap \partial y \neq \emptyset$, $x^\circ \cap y^\circ \neq \emptyset$, $\partial x \cap y^\circ \neq \emptyset$ and $x^\circ \cap \partial y \neq \emptyset$. Conversely, if the four operations give the previous results when applied to the two regions x and y , then the relation $PO(x, y)$ holds (Figure 1).

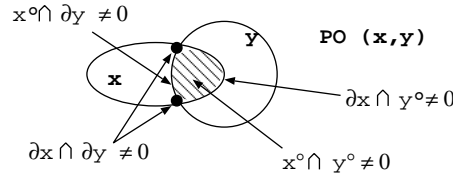


Figure 1: Correspondence between the operations defined in [Egenhofer 1989] and the relations of the *RCC-8* theory: the relation $PO(x, y)$.

The computation method based on these four operations is interesting since it associates the base relations of the *RCC-8* theory to a set of necessary and sufficient conditions to be checked on the regions. But the regions have to be characterized with an interior and a boundary, which can be difficult to define on raster images, as it is shown below (Section 2.1).

x,y	$\partial x \cap \partial y$	$x^\circ \cap y^\circ$	$\partial x \cap y^\circ$	$x^\circ \cap \partial y$
$EQ(x,y)$	$\neq \emptyset$	$\neq \emptyset$	\emptyset	\emptyset
$NTPP(x,y)$	\emptyset	$\neq \emptyset$	$\neq \emptyset$	\emptyset
$TPP(x,y)$	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$	\emptyset
$NTPP^{-1}(x,y)$	\emptyset	$\neq \emptyset$	\emptyset	$\neq \emptyset$
$TPP^{-1}(x,y)$	$\neq \emptyset$	$\neq \emptyset$	\emptyset	$\neq \emptyset$
$PO(x,y)$	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$
$EC(x,y)$	$\neq \emptyset$	\emptyset	\emptyset	\emptyset
$DC(x,y)$	\emptyset	\emptyset	\emptyset	\emptyset

Table 5: Correspondence between the relations of the *RCC-8* theory and the operations on interior and boundary sets proposed in [Egenhofer 1989].

1.3 Mathematical properties of topological relations

It is often useful to deduce new information from a partial knowledge about a situation or to verify the consistency of the available information in a given situation. This can be done thanks to the following mathematical properties of topological relations, where \mathcal{D} is a set of regions:

- *Conversion*: the converse relation of a relation R is the relation R^{-1} such that:
 $\forall (x,y) \in \mathcal{D}^2, R(x,y) \leftrightarrow R^{-1}(y,x)$
- *Disjoint*: two relations R_1 et R_2 are said to be disjoint if:
 $\forall (x,y) \in \mathcal{D}^2, R_1(x,y) \leftrightarrow \neg R_2(x,y)$
- *Complement*: the complement of a relation R is the relation R^c such that :
 R and R^c are disjoint and $\forall (x,y) \in \mathcal{D}^2, R(x,y) \vee R^c(x,y)$ holds.
- *Composition*: given three regions x, y, z and a pair of relations R_1, R_2 such that $R_1(x,y)$ and $R_2(y,z)$, the composition of R_1 and R_2 is the disjunction R of all the possible relations holding between x and z :
 $\forall (x,y,z) \in \mathcal{D}^3, R_1(x,y) \wedge R_2(y,z) \rightarrow R(x,z)$ and,
 $\forall R_3, (R_1(x,y) \wedge R_2(y,z) \wedge R_3(x,z)) \rightarrow (\forall (u,v) \in \mathcal{D}^2, R_3(u,v) \rightarrow R(u,v)).$

The last property is especially interesting: as proposed in [Randell and Cohn 1992], rules of composition are used to deduce the relation that holds between two regions x and z from the knowledge of the relations holding between those two regions and a third region y . For instance, Figure 2 depicts two configurations where three regions x, y and z are drawn with respect to the relations $TPP(x,y)$ and $EC(y,z)$. Thus $DC(x,z)$ or $EC(x,z)$ can be deduced. The two schemas of the figure correspond to the two possible relative locations of x and z .

The rules of composition on the topological relations are represented in *composition tables*. Such a table has been used first by Allen [Allen 1983] to deduce temporal relations

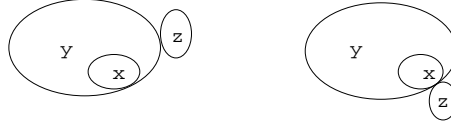


Figure 2: Composition of topological relations. The two configurations are described by $TPP(x, y)$ and $EC(y, z)$. In the left case, $DC(x, z)$ is true, and in the right case, $EC(x, z)$ is true.

between events. Composition tables have also been used in the connection-based theories: such a table is defined for 9 relations in [Randell *et al.* 1992a], and for 22 relations in [Cohn *et al.* 1993].

A composition table is defined for a subset of disjoint relations of a theory. Table 6 presents the composition table of the eight base relations of the *RCC-8* theory. The $EQ(x, y)$ “*x is identical with y*” relation is not represented because it is a neutral element: for any relation R , $R(x, y)$ and $EQ(y, z)$ implies $R(x, z)$; $EQ(x, y)$ and $R(y, z)$ implies $R(x, z)$.

1.4 Some applications in connection with topological relations

Mereotopological formalisms have been used in various applications.

In [Kawashima *et al.* 1994], topological relations have been used to describe the behavior of groups of soccer players: groups of players are extracted from image sequences, their position is described thanks to the relations of the *RCC-8* theory. Behaviors are defined according to the possible transitions between the relations. It has been recently suggested to continue this research, in order to analyze and comment robots playing soccer [Nebel 2000].

Clarke’s mereotopology has been extended to represent spatial expressions of natural language [Vieu 1993]. The part-whole relation, in particular, is used in several ways in natural language [Winston *et al.* 1987]: for instance, one should distinguish the expressions “the egg is in the bowl” and “the yolk is in the egg”. Various notions of *inside* are defined in [Cohn *et al.* 1997]. Furthermore, relying on the topological notions of open and closed sets, different notions of contact can also be defined: *external connection* ($EC(x, y)$) between two closed regions, *contact* between a closed region and an open one, *weak contact* between two open regions [Asher and Vieu 1995].

Computer-aided design systems and geographic information systems also use topological relations. Actually they use both numerical representations (based on the Euclidean space) and qualitative representations (distance, orientation, topology) to meet the users’ requirements [Cui *et al.* 1993, Frank 1997]. Furthermore, studies are done for applying spatio-terminological reasoning in the context of visual-query systems [Möller and Wessel 1999].

$R_2(y,z)$ $R_1(x,y)$	NTPP	TPP	NTPP ⁻¹	TPP ⁻¹	PO	EC	DC
NTPP	NTTP	NTPP	⊤	DC, EC, PO, TPP, NTPP	DC, EC, PO, TPP, NTPP	DC	DC
TPP	NTPP	TPP, NTPP	DC, EC, PO, TPP ⁻¹ , NTPP ⁻¹	DC, EC, PO, TPP, TPP ⁻¹ , EQ	DC, EC, PO, TPP, NTPP	DC, EC	DC
NTPP ⁻¹	PO, TPP, NTPP, TPP ⁻¹ , NTPP ⁻¹ , EQ	PO, TPP ⁻¹ , NTPP ⁻¹	NTPP ⁻¹	NTPP ⁻¹	PO, TPP ⁻¹ , NTPP ⁻¹	PO, TPP ⁻¹ , NTPP ⁻¹	DC, EC, PO, TPP ⁻¹ , NTPP ⁻¹
TPP ⁻¹	PO, TPP, NTPP	PO, TPP, TPP ⁻¹ , EQ	NTPP ⁻¹	TPP ⁻¹ , NTPP ⁻¹	PO, TPP ⁻¹ , NTPP ⁻¹	EC, PO, TPP ⁻¹ , NTPP ⁻¹	DC, EC, PO, TPP ⁻¹ , NTPP ⁻¹
PO	PO, TPP, NTPP	PO, TPP, NTPP	DC, EC, PO, TPP ⁻¹ , NTPP ⁻¹	DC, EC, PO, TPP ⁻¹ , NTPP ⁻¹	⊤	DC, EC, PO, TPP ⁻¹ , NTPP ⁻¹	DC, EC, PO, TPP ⁻¹ , NTPP ⁻¹
EC	PO, TPP, NTPP	EC, PO, TPP, NTPP	DC	DC, EC	DC, EC, PO, TPP, NTPP	DC, EC, PO, TPP, TPP ⁻¹ , EQ	DC, EC, PO, TPP ⁻¹ , NTPP ⁻¹
DC	DC, EC, PO, TPP, NTPP	DC, EC, PO, TPP, NTPP	DC	DC	DC, EC, PO, TPP, NTPP	DC, EC, PO, TPP, NTPP	⊤

Table 6: The composition table of the eight base relations of the *RCC-8* theory: the row gives the relation holding between x and y ; the column gives the relation holding between y and z . The case contains the composition of $R_1(x, y)$ and $R_2(y, z)$, i.e. the set of the relations possibly holding between x and z ; the \top mark corresponds to the undefined case, where all the relations of the *RCC-8* theory may hold between x et z .

2 Computing topological relations on raster images

In order to classify spatial structures from maps or images, methods are needed to check spatial relations on the images. In this section, we propose computational operations on raster images: raster images are, in opposition to vector images, made of points, called *pixels*, characterized by their position (line, column) and their label (color or gray level). The computational operations are based on the interior and boundary sets of raster regions, leading to a fundamental problem of digital topology: defining the boundary of a raster region actually requires raster elements, that are two-dimensional and thus contradict the topology [Kong and Rosenfeld 1989].

2.1 The boundary problem

When the raster representation is considered on its own, the boundary of a region can be defined by the pixels that are externally connected to the region: if the region is made of only one pixel, then its boundary is made of its eight neighbors. The boundary can also be internal, and composed of the pixels that are externally connected to the complementary set of the region (its exterior): then, if the region is a square made of nine pixels, its boundary contains eight pixels. These two definitions lead to topological paradoxes [Kong and Rosenfeld 1989]. Furthermore, they do not meet the Clarke's definition of connection, *x and y share a point* : if the boundary is internal, two externally connected regions do not share a boundary point; conversely, if the boundary is external, two disconnected regions may share a boundary point.

Nevertheless, computational operations have been defined with such raster boundaries [Egenhofer and Sharma 1993]. The regions considered are regular raster regions that are 4-connected, non empty, and with no hole. Due to these restrictions, the boundary of a region respects the topology. Besides, operations based on the intersection of the internal and boundary sets allow to distinguish the relations $EC(x, y)$ and $DC(x, y)$. For instance, in the case of an external boundary, $\partial x \cap y^\circ$ and $x^\circ \cap \partial y$ are empty when the two regions are disconnected, and non-empty when the two regions are connected.

Another way is to complete the raster representation with one-dimensional boundaries. In [Kovalevsky 1989], the raster representation is completed with edges and nodes representing the boundary of the pixels, to become a full topological model. The same representation is used in [Winter and Frank 1999], and the topological relations are defined with functions that apply to pixels, edges and nodes.

Our proposition is to solve the boundary problem by introducing an *abstract boundary* whose elements are *abstract pixels*. An abstract pixel stands across four real pixel, as shown in Figure 3. An abstract pixel f is a boundary pixel of a region x if there is at least one pixel belonging to x and one pixel belonging to the complement of x among the four real pixels intersected by f . The boundary intersection of two regions is then easily obtained, as shown in Figure 4.

This representation is well adapted for the computation of topological relations since we deal with two raster representations. It can be applied either to 8-connected or to

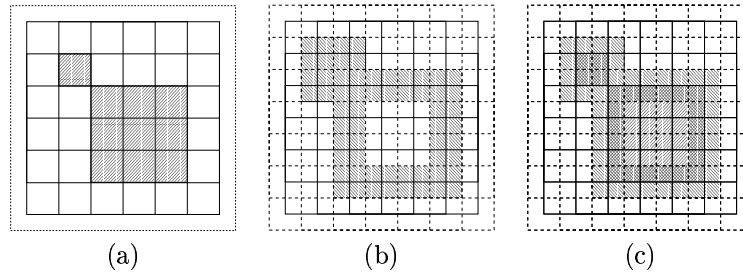


Figure 3: Defining the boundary of a region: (a) the interior of a region is made of real pixels (hatched); (b) the boundary of the region is made of abstract pixels standing across the real pixels (inverse hatched); (c) the combination of the interior and the boundary [Mangelinck 1998].

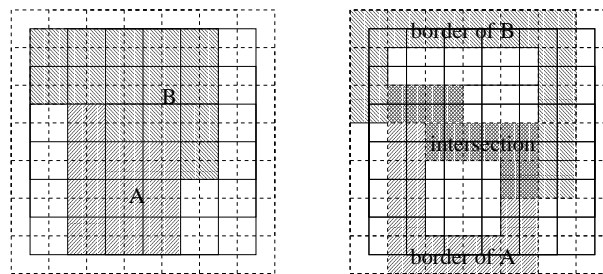


Figure 4: The boundary intersection of two regions on a raster image: the plain grid corresponds to the real pixels, the dotted grid corresponds to the abstract pixels.

4-connected regions. In both cases, the boundary is made of 4-connected abstract pixels. An abstract pixel of a boundary may have more than 2 neighbors (see figure 3), thus the boundary is not a Jordan curve as it is classically defined for digital topology [Latecki *et al.* 1995]. A possible solution is to eliminate some configurations, as it is done in [Egenhofer and Sharma 1993], but this leads to eliminate 8-connected regions, which are frequent on land-use maps. Another solution is to define an oriented tracing of the boundary (e.g. first right and up), as it is proposed in [Kovalevsky 1989]. This problem should be analysed further, but this is not the purpose of the present paper. Actually, the fact that the boundary of a region is generally not a Jordan curve, is not a real problem for the intersection of boundaries, that is one of the basic operation used below.

2.2 Computational operations for defining topological relations

In [Egenhofer 1989], computational operations are defined for topological relations: the computational operations are the intersection of the interiors of two regions, $x^\circ \cap y^\circ$; the intersection of the boundaries of two regions, $\partial x \cap \partial y$; the intersections of the boundary of a region and the interior of the other region, $\partial x \cap y^\circ$ and $x^\circ \cap \partial y$. We call *CE-8* the set of the eight conditions derived from these operations, according to their result (empty or not empty, see Table 5).

These operations are unfortunately not convenient for our problem because of the definition of the region boundary. The interior and boundary sets are actually of different nature (real pixels *versus* abstract pixels) and it is thus incorrect to intersect those two sets. We propose a method for checking relations similar to the method of [Egenhofer 1989] but we use the set-difference of the interior sets rather than the intersections of boundary set and interior set. The regions of concern are closed, non empty and not necessarily internally connected. We propose to use the following four computational operations on these regions: the intersection of the interior sets, $x^\circ \cap y^\circ$; the intersection of the boundary sets, $\partial x \cap \partial y$; the two differences of the interior sets, $x^\circ - y^\circ$ and $y^\circ - x^\circ$. From these four operations we have derived the eight following conditions:

- $(x^\circ - y^\circ = \emptyset)$: “*x is a part of y*”, denoted by $P(x, y)$ ¹
- $(x^\circ - y^\circ \neq \emptyset)$: “*x is not a part of y*”, denoted by $Dx(x, y)$
- $(y^\circ - x^\circ = \emptyset)$: “*x contains y*” (“*y is a part of x*”), denoted by $P^{-1}(x, y)$
- $(y^\circ - x^\circ \neq \emptyset)$: “*x does not contain y*” (“*y is not a part of x*”), denoted by $Dy(x, y)$
- $(x^\circ \cap y^\circ = \emptyset)$: “*x is discrete from y*”, denoted by $DR(x, y)$
- $(x^\circ \cap y^\circ \neq \emptyset)$: “*x overlaps y*”, denoted by $O(x, y)$
- $(\partial x \cap \partial y = \emptyset)$: “*x does not share a boundary with y*”, denoted by $NA(x, y)$
- $(\partial x \cap \partial y \neq \emptyset)$: “*x shares a boundary with y*”, denoted by $A(x, y)$

This set of conditions is called *CM-8*. It is important to notice that some conditions imply others because of the properties of the image regions. For instance, each of the conditions $x^\circ - y^\circ = \emptyset$ or $y^\circ - x^\circ = \emptyset$ implies the condition $x^\circ \cap y^\circ \neq \emptyset$ (the regions are non empty); the condition $\partial x \cap \partial y \neq \emptyset$ is implied by the conjunction of the same conditions (the regions are non empty and closed). Finally these four operations allow to check the *RCC-8* theory relations on the images. The correspondence between each relation and the conditions are described on Table 7: each relation is equivalent to a subset of conditions of *CM-8*. Note that, if $PO(x, y)$ holds, $A(x, y)$ or $NA(x, y)$ may hold when x or y are not internally connected. However, since the *RCC-8* theory does not distinguish different kind of partial overlaps, assuming $A(x, y)$ is not conflicting. Computing $Dx(x, y)$, $Dy(x, y)$ and $O(x, y)$ is actually enough to check $PO(x, y)$.

Four conditions of *CM-8* are identical to four conditions of *CE-8*: those concerning the intersection of the interior sets and the intersection of the boundary sets of two regions. We

¹The regions are closed and thus $x^\circ - y^\circ = \emptyset \leftrightarrow x - y = \emptyset$.

x, y	$x^\circ - y^\circ$	$y^\circ - x^\circ$	$x^\circ \cap y^\circ$	$\partial x \cap \partial y$
$EQ(x, y)$	\emptyset	\emptyset	$\neq \emptyset$	$\neq \emptyset$
$NTPP(x, y)$	\emptyset	$\neq \emptyset$	$\neq \emptyset$	\emptyset
$TPP(x, y)$	\emptyset	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$
$NTPP^{-1}(x, y)$	$\neq \emptyset$	\emptyset	$\neq \emptyset$	\emptyset
$TPP^{-1}(x, y)$	$\neq \emptyset$	\emptyset	$\neq \emptyset$	$\neq \emptyset$
$PO(x, y)$	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$	$\neq \emptyset$
$EC(x, y)$	$\neq \emptyset$	$\neq \emptyset$	\emptyset	$\neq \emptyset$
$DC(x, y)$	$\neq \emptyset$	$\neq \emptyset$	\emptyset	\emptyset

Table 7: The correspondence between our computational operations for checking relations on the image and the *RCC-8* theory relations [Mangelinck 1998]. See Table 5 for a comparison with the operations proposed in [Egenhofer 1989].

have eliminated the conditions concerning the intersection of the boundary set of a region and the interior set of the other region. We have in turn added conditions concerning the set-difference of the interior sets of two regions. As previously said, this choice has been done in order to have consistent operations involving sets of the same kind : interiors (real pixels) or boundaries (abstract pixels).

It can be noticed that Ligozat has recently proposed to use the regions themselves and their interiors to compute the relations of the *RCC-8* theory [Ligozat 1999]. The operations he uses are the intersection and the inclusion. Thus, the relation $TPP(x, y)$ is defined with $\{x \subset y, x \not\subseteq y^\circ\}$. This approach could also be used with raster images but the boundary problem should be reexamined.

3 Two lattices of topological relations

In this section, we introduce two lattices of topological relations that we describe according to their abilities for representing the relations, their reasoning properties and the associated computational operations. The first lattice is based on the power-set of the so-called set \mathcal{B} of the eight base relations of the *RCC-8* theory. The second lattice is a Galois lattice, that is based both on the set \mathcal{B} and on the set *CM-8* of conditions. In the following, we denote by \mathcal{D} a set of spatial regions that are non empty, closed, and non necessarily internally connected. The definitions and proofs of this section are detailed in the appendix.

3.1 A lattice based on the subsets of \mathcal{B}

The first lattice that we have built is a “reference” lattice, denoted by $\mathcal{T}_{\mathcal{P}}$ and based on the power-set of \mathcal{B} . Its cardinality is $2^8 = 256$. The ordering relation is the set-inclusion. This lattice is defined as follows:

Definition 1 The lattice $\mathcal{T}_{\mathcal{P}}$ is the structure $\langle RCC-8, \subseteq, \cap, \cup, \emptyset, \mathcal{B} \rangle$ where:

- the set of elements is the power-set of \mathcal{B} , denoted by $RCC-8$,
- the ordering is the set-inclusion \subseteq ,
- the greatest lower bound (glb) of two elements E_1 and E_2 is the intersection \cap of the sets E_1 and E_2 ,
- the least upper bound (lub) of two elements E_1 and E_2 is the union \cup of the sets E_1 and E_2 ,
- the bottom element is the empty-set \emptyset ,
- the top element is the set \mathcal{B} .

This lattice is actually associated to the boolean algebra $(RCC-8, \cap, \cup, ^c, \emptyset, \mathcal{B})$ (see [Davey and Priestley 1990]). It is complemented: every element E has a complement in $\mathcal{T}_{\mathcal{P}}$, which is its complement E^c in \mathcal{B} . It contains all the subsets of \mathcal{B} and thus represents all the compositions of \mathcal{B} relations (see Table 6). In order to exploit this lattice we have defined a function π_r that maps an element E of $\mathcal{T}_{\mathcal{P}}$ to the disjunction of all the relations of E (denoted by $\pi_r(E)$):

$$\forall (x, y) \in \mathcal{D}^2, \quad \pi_r(E)(x, y) =_{def} \bigvee_{r \in E} r(x, y)$$

It can be noticed that the relation $\pi_r(E)$ associated to the *lub* E of two elements E_1, E_2 , is the disjunction of the two relations $\pi_r(E_1), \pi_r(E_2)$:

$$\pi_r(E) = \pi_r(E_1 \cup E_2) = \pi_r(E_1) \vee \pi_r(E_2)$$

Furthermore, the ordering on the lattice elements is equivalent to the implication between relations. Firstly, the relation $\pi_r(E)$ associated to E is implied by the relation $\pi_r(E')$ associated to a subset E' of E . Thus:

$$\forall E, E' \in \mathcal{T}_{\mathcal{P}} : (E' \subseteq E) \rightarrow (\forall (x, y) \in \mathcal{D}^2, \pi_r(E')(x, y) \rightarrow \pi_r(E)(x, y))$$

Secondly the inverse implication is demonstrated thanks to the disjoint-property of the base relations of the $RCC-8$ theory: if a relation of \mathcal{B} is true for a pair of regions of \mathcal{D}^2 , the seven other relations are false. Then, if a disjunction of elements of \mathcal{B} implies another disjunction, the elements of the first subset are necessarily elements of the second subset. Finally we prove the following theorem:

$$\forall E, E' \in \mathcal{T}_{\mathcal{P}} : (E' \subseteq E) \leftrightarrow (\forall (x, y) \in \mathcal{D}^2, \pi_r(E')(x, y) \rightarrow \pi_r(E)(x, y))$$

For example, the element $(TPP, NTPP)$ is included in the element $(TPP, NTPP, EQ)$. The first one is associated to the relation $PP(x, y)$ while the second one is associated to the relation $P(x, y)$: $PP(x, y) \rightarrow P(x, y)$ (see Table 1).

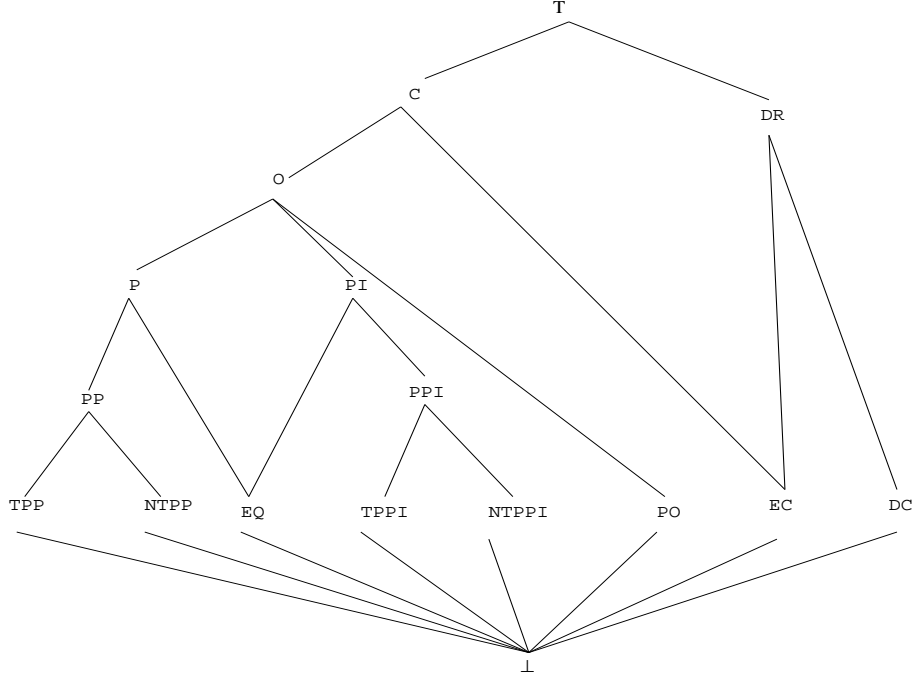


Figure 5: The lattice described in [Randell and Cohn 1992]. The labels denote relations that are disjunctions of \mathcal{B} elements. This lattice is automatically completed into a boolean (i.e. distributive and complemented) lattice to be exploited.

A boolean lattice of topological relations similar to the lattice $\mathcal{T}_{\mathcal{P}}$ has been defined in [Randell and Cohn 1992] (see Figure 5). This lattice is used to prove theorems and to infer rules of composition between relations. The composition of all relations can actually be deduced from this lattice and from Table 6. For instance, if it is known that, for the three regions x , y and z , $PP(x, y)$ and $EC(y, z)$, it can be inferred that $DR(x, z)$ (DR is the *lub* of the pair (EC, DC) , see Figure 5):

$$\begin{array}{ll}
 \text{from the lattice:} & PP(x, y) = TPP(x, y) \vee NTPP(x, y) \\
 \text{from the table:} & TPP(x, y) \circ EC(y, z) = DC(x, z) \vee EC(x, z) \\
 \text{and:} & NTPP(x, y) \circ EC(y, z) = DC(x, z) \\
 \text{thus:} & PP(x, y) \circ EC(y, z) = DC(x, z) \vee EC(x, z) = DR(x, z)
 \end{array}$$

3.2 A smooth introduction to the Galois theory and formal concept analysis

This introduction to the Galois lattice theory is based on [Barbut and Monjardet 1970, Guénoche and Van Mechelen 1993].

A number of classification problems can be formalized by means of a class of objects and a class of attributes, and a binary *connection* between the two classes, which indicates for each object-attribute pair whether the attribute applies to the object or not. The attributes may be features that are present or absent, or attribute values that have been dichotomized into boolean variables. From such a binary connection, it is possible to derive for each object o its *attribute pattern*, i.e. the set of all attributes that apply to o , denoted by $A(o)$. Similarly, it is possible to derive for each attribute a its *object pattern*, i.e. the set of all objects to which a applies, denoted by $O(a)$. One may further derive *rectangles*, i.e. pairs $O \times A$ of an object set O and an attribute set A , such that every attribute of A applies to every object of O . Moreover, *maximal rectangles* $O \times A$ have the additional property that the attribute set A consists of all common attributes of the objects of O , and that the object set O consists of all objects to which the attributes of A jointly apply.

Maximal rectangles are called *formal concepts* within the context of the binary connection under study [Davey and Priestley 1990, Ganter and Wille 1999]. They are called *concepts* because they actually represent a class of objects, the object set O being the *extension* of the class, denoted by $Ext(C)$ for a concept C , and the attribute set the *intension* of the class, denoted by $Int(C)$. They are called *formal concepts* because they are mathematical entities that do not necessarily refer to a psychological reality. On the set of all concepts derived from a given binary connection, a partial order may be defined based on the set inclusion between the intensions (or between the extensions in a dual way). For this partial order, each pair of formal concepts has a least upper bound (*lub*), and a greatest lower bound (*glb*): hence, formal concepts are organized within a *lattice*. More precisely, the partial order between two concepts C_1 and C_2 , denoted by $C_1 \sqsubseteq C_2$, verifies that the intension of C_2 (the *subsumer* concept) is included in the intension of C_1 (the *subsumee* concept), i.e. $Int(C_2) \subseteq Int(C_1)$, or, dually, that the extension of C_1 is included in the extension of C_2 , i.e. $Ext(C_1) \subseteq Ext(C_2)$. Thus, there exists an order-reversing one-to-one connection between the extensions and the intensions of formal concepts. This connection meets the criteria of a *Galois connection* (made precise below) and the lattice of formal concepts is called a *Galois lattice* [Barbut and Monjardet 1970].

From a mathematical point of view, let E and F be two finite sets and R a binary relation on $E \times F$. One may define a mapping $f : 2^E \rightarrow 2^F$ such that, if x is an element of E , $f(\{x\})$ consists of all elements of F related to x by R . Furthermore, if X is an arbitrary part of E , $f(X) = \{y \in F | \forall x \in X : xRy\}$. Dually, we may define a mapping $g : 2^F \rightarrow 2^E$ such that, if y is an element of F , $g(\{y\})$ consists of all elements of E that are related to y by R . Furthermore, if Y is an arbitrary part of F , $g(Y) = \{x \in E | \forall y \in Y : xRy\}$. The couple $\{f, g\}$ is said to be a *Galois connection* between the sets E and F .

In terms of objects and attributes, $f(X)$ is the set of all attributes shared by all objects in X , and $g(Y)$ is the set of all objects that have all attributes of Y . Moreover, $X \subseteq X' \Rightarrow$

$f(X') \subseteq f(X)$, and $Y \subseteq Y' \Rightarrow g(Y') \subseteq g(Y)$: the mappings f and g are decreasing. By applying first f and then g , one may construct a composed mapping $h = g[f] = g \circ f$, which maps every part of E onto a part of E . Similarly, by applying first g and then f , one may construct a composed mapping $h' = f[g] = f \circ g$, which maps every part of F onto a part of F . It is then possible to show that h (and respectively h') is: (i) monotonously increasing, that is if X and X' are subsets of E : $X \subseteq X' \Rightarrow h(X) \subseteq h(X')$, (ii) extensive, that is $X \subseteq h(X)$, and (iii) idempotent, that is $h(X) = h[h(X)]$. Furthermore, h and h' are *closure operators*: a subset X of E is said to be *closed* if and only if $X = h(X)$; $Y = h'(Y)$ for a subset Y of F . The closure operators h for E and h' for F are said to be *Galois closures*.

Let L_E and L_F the sets of all closed parts of E and F (respectively), partially ordered by set inclusion. Equipped with this partial order, L_E and L_F have lattice structures: the *glb* of two parts is their intersection, whereas the *lub* of two parts is the closure of their union. The Galois connection $\{f, g\}$ restricted to the closed parts of E and F is a one-to-one correspondence between the lattices L_E and L_F .

We may now consider the set L of all couples of corresponding parts of L_E and L_F , i.e. each element of L is the Cartesian product of closed parts of E and F , which may be denoted by $X \times f(X)$, or also, $g(Y) \times Y$, with X , $f(X)$, Y , and $g(Y)$ being closed. We may further define the partial order relation \sqsubseteq on L : $X \times Y \sqsubseteq X' \times Y'$ if and only if $X' \subseteq X$ and $Y \subseteq Y'$. The structure (L, \sqsubseteq) is the *Galois lattice* of the relation R on $E \times F$, and it can be demonstrated that the elements of L are the formal concepts derived from the relation R .

3.3 A Galois lattice based on topological relations

3.3.1 The design of the lattice

Regarding our needs, we must build a lattice that respects the following requirements: it should contain the \mathcal{B} and $CM-8$ sets, it should be ordered by the implication relation between the eight relations of \mathcal{B} and the eight conditions of $CM-8$. Such a lattice can be built thanks to the Galois theory as described above. We have defined a Galois connection $\{f, g\}$ between \mathcal{B} and $CM-8$ as follows:

Definition 2 f maps a relation r of \mathcal{B} to a subset of $CM-8$ where all conditions hold for a pair (x, y) whenever the relation r holds for (x, y) :

$$f: \begin{array}{l} 2^{\mathcal{B}} \quad \mapsto \quad 2^{CM-8} \\ \{r\} \quad \mapsto \quad f(\{r\}) = \{c \in CM-8 \mid \forall (x, y) \in \mathcal{D}^2 : r(x, y) \rightarrow c(x, y)\} \end{array}$$

Furthermore, f maps every subset R of \mathcal{B} to the subset C of $CM-8$ whose conditions are implied by *all* the relations of R : $f(R) = \bigcap_{r \in R} f(\{r\})$.

Definition 3 g maps a condition c of $CM-8$ to the subset of \mathcal{B} where all relations imply the condition c :

$$g: \begin{array}{l} 2^{CM-8} \quad \mapsto \quad 2^{\mathcal{B}} = RCC-8 \\ \{c\} \quad \mapsto \quad g(\{c\}) = \{r \in \mathcal{B} \mid \forall (x, y) \in \mathcal{D}^2 : r(x, y) \rightarrow c(x, y)\} \end{array}$$

Furthermore g maps every subset C of $CM-8$ to a subset R of \mathcal{B} whose relations imply *all* the conditions of C : $g(C) = \bigcap_{c \in C} g(\{c\})$. The Galois connection $\{f, g\}$ is expressed in Table 8.

	P	Dx	P^{-1}	Dy	DR	O	NA	A
EQ	1	0	1	0	0	1	0	1
$NTPP$	1	0	0	1	0	1	1	0
TPP	1	0	0	1	0	1	0	1
$NTPP^{-1}$	0	1	1	0	0	1	1	0
TPP^{-1}	0	1	1	0	0	1	0	1
PO	0	1	0	1	0	1	0	1
EC	0	1	0	1	1	0	0	1
DC	0	1	0	1	1	0	1	0

Table 8: The table represents elements (r, c) , with $r \in \mathcal{B}$ (line) and $c \in CM-8$ (column). An element $(r, c) = 1$ if $\forall(x, y) \in \mathcal{D}^2, r(x, y) \rightarrow c(x, y)$. Otherwise $(r, c) = 0$.

The two functions f and g are used to define a closure operator h of $RCC-8$ and a closure operator h' of 2^{CM-8} :

$$h : \begin{array}{l} RCC-8 \mapsto RCC-8 \\ h(R) = g \circ f(R) \end{array} \quad h' : \begin{array}{l} 2^{CM-8} \mapsto 2^{CM-8} \\ h'(C) = f \circ g(C) \end{array}$$

The Galois lattice $\mathcal{T}_{\mathcal{G}}$ is built on the basis of the two closure operators h and h' . Its elements are pairs $(C, R) \in 2^{CM-8} \times RCC-8$, where $h(R) = R$ and $C = f(R) = h'(C)$. Such a pair is also a formal concept derived from the implication relation between \mathcal{B} and $CM-8$, C being the intension and R the extension of this concept. $\mathcal{T}_{\mathcal{G}}$ is defined as follows:

Definition 4 *The Galois lattice $\mathcal{T}_{\mathcal{G}}$ based on the implication between the relations of \mathcal{B} and the conditions of $CM-8$ is the structure $(\mathcal{E}_{\mathcal{G}}, \sqsubseteq, \frown, \smile, (CM-8, \emptyset), (\emptyset, \mathcal{B}))$ where:*

- $\mathcal{E}_{\mathcal{G}}$ is the set of all pairs (C, R) where R is a subset of \mathcal{B} closed for h , C is a subset of $CM-8$ closed for h' , $f(R) = C$ and $g(C) = R$.
- The ordering \sqsubseteq between two elements (C_1, R_1) and (C_2, R_2) is defined as follows:

$$((C_1, R_1) \sqsubseteq (C_2, R_2)) \leftrightarrow \begin{cases} C_2 \subseteq C_1 \\ R_1 \subseteq R_2 \end{cases}$$

where \subseteq is the set-inclusion.

- The greatest lower bound (denoted by \frown) of two elements is defined as follows:

$$(C_1, R_1) \frown (C_2, R_2) = (h'(C_1 \cup C_2), R_1 \cap R_2)$$

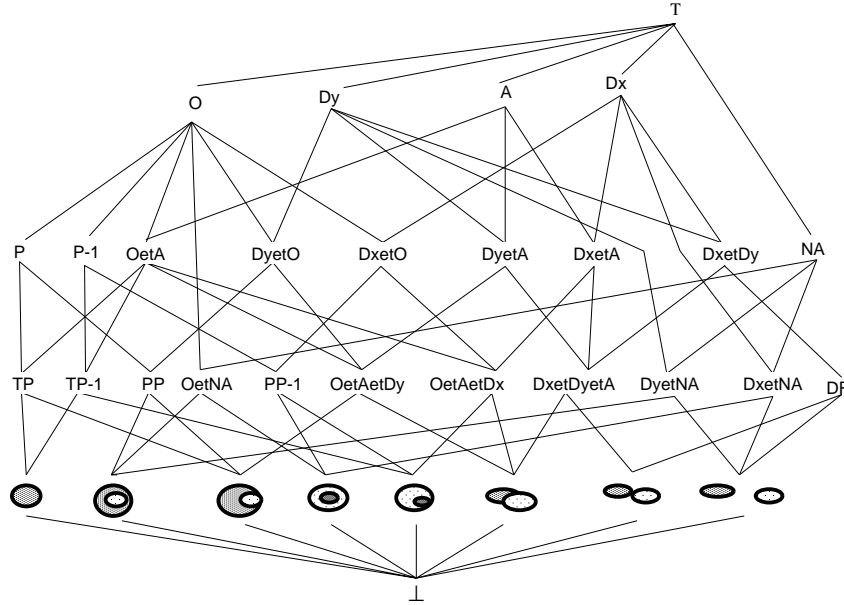


Figure 6: The Galois lattice built on the basis of Table 8: the pairs (C, R) are denoted by the relation, or by the condition, to which they are associated by the relations π_r , or π_c (see Section 3.3.2).

- The least upper bound (denoted by \smile) of two elements is defined as follows:

$$(C_1, R_1) \smile (C_2, R_2) = (C_1 \cap C_2, h(R_1 \cup R_2))$$

where \cup and \cap are the set-union and the set-intersection.

- The bottom element is $(CM-8, \emptyset)$,
- The top element is (\emptyset, B) .

In a more concrete way, the Galois lattice \mathcal{T}_G is built by searching for the maximal rectangles in Table 8 or by searching for the subsets of B that are closed with respect to h . The lattice \mathcal{T}_G is shown in Figure 6, the denotations of the nodes are explained below.

3.3.2 The lattice ordering and the implication between relations

As we have done before for the $\mathcal{T}_{\mathcal{P}}$ lattice (see paragraph 3.1) we now define the π_r function that maps an element (C, R) of $\mathcal{T}_{\mathcal{G}}$ to the disjunction of the relations of R (denoted $\pi_r(C, R)$):

$$\forall(x, y) \in \mathcal{D}^2, \quad \pi_r(C, R)(x, y) =_{def} \bigvee_{r \in R} r(x, y)$$

Similarly we introduce the π_c function that is concerned with the conditions of $CM-8$. The π_c function maps an element (C, R) of $\mathcal{T}_{\mathcal{G}}$ to the conjunction of the C elements:

$$\forall(x, y) \in \mathcal{D}^2, \quad \pi_c(C, R)(x, y) =_{def} \bigwedge_{c \in C} c(x, y)$$

We show below that the two formulas resulting from the application of π_c and π_r are equivalent. This equivalence is used to name the elements of $\mathcal{T}_{\mathcal{G}}$, either as a disjunction of relations or as a conjunction of conditions. On Figure 6, elements are denoted by a name or an icon: a name represents a conjunction of conditions (e.g. DetA denotes the element $(\{O, A\}, R)$, where et stands for “and”) or a disjunction of relations (e.g. PP denotes the element $(C, \{TPP, NTPP\})$, since $PP = TPP \vee NTPP$)². The icons at the bottom of the lattice represent the elements associated to the eight base relations of the $RCC-8$ theory (e.g. $\text{EQ} = (\{O, A, P, P^{-1}\}, \{EQ\})$).

The π_c and π_r functions are also used to make explicit the link between the lattice ordering and the implication relation on the topological relations. Actually the following property is proved for all pairs of elements of $\mathcal{T}_{\mathcal{G}}$:

$$(C_1, R_1) \sqsubseteq (C_2, R_2) \leftrightarrow \begin{cases} \forall(x, y) \in \mathcal{D}^2, \bigvee_{r \in R_1} r(x, y) \rightarrow \bigvee_{r \in R_2} r(x, y) \\ \forall(x, y) \in \mathcal{D}^2, \bigwedge_{c \in C_1} c(x, y) \rightarrow \bigwedge_{c \in C_2} c(x, y) \end{cases}$$

Thus, if an element $E_1 = (C_1, R_1)$ (e.g. EQ) is smaller than another element $E_2 = (C_2, R_2)$ (e.g. TP), the disjunction of the relations of R_1 implies the disjunction of the relations of R_2 and the conjunction of the conditions of C_1 implies the conjunction of the conditions of C_2 :

$$\text{EQ} \sqsubseteq \text{TP} \leftrightarrow \begin{array}{l} \forall(x, y) \quad (EQ(x, y) \rightarrow (EQ \vee TPP)(x, y)) \\ \text{and} \quad ((A \wedge P \wedge P^{-1} \wedge O)(x, y) \rightarrow (A \wedge P \wedge O)(x, y)) \end{array}$$

3.4 Properties of the $\mathcal{T}_{\mathcal{G}}$ lattice

3.4.1 The characteristics of the elements of $\mathcal{T}_{\mathcal{G}}$

The following properties characterize the elements of $\mathcal{T}_{\mathcal{G}}$:

²In the following A denotes the condition and \mathbf{A} denotes the lattice element where the condition is represented (respectively EC and \mathbf{EC} to denote the topological relation and the element of the lattice).

Property 1 *If a pair $\mathbf{E} = (C, R)$ is an element of the $\mathcal{T}_{\mathcal{G}}$ lattice, then the relation $\pi_r(\mathbf{E}) = \pi_r(C, R)$ is equivalent to the condition $\pi_c(\mathbf{E}) = \pi_c(C, R)$:*

$$(C, R) \in \mathcal{E}_{\mathcal{G}} \rightarrow \left(\forall (x, y) \in \mathcal{D}^2, \bigvee_{r \in R} r(x, y) \leftrightarrow \bigwedge_{c \in C} c(x, y) \right)$$

This property shows that the relation $\pi_r(\mathbf{E})$ associated with an element of $\mathcal{T}_{\mathcal{G}}$ is logically equivalent to the condition $\pi_c(\mathbf{E})$ associated with the same element. Thus, checking a relation on the image requires to look for the element of $\mathcal{T}_{\mathcal{G}}$ that represents this relation and to check the associated conditions.

The second property that characterizes the elements of the $\mathcal{T}_{\mathcal{G}}$ lattice is as follows:

Property 2 *Let R be a subset of \mathcal{B} and C a subset of $CM-8$. If the relation formed by the disjunction of the elements of R is equivalent to the condition formed by the conjunction of the elements of C , then the pair (\bar{C}, R) , where $\bar{C} = h'(C)$, is an element of $\mathcal{T}_{\mathcal{G}}$:*

$$\left(\forall (x, y) \in \mathcal{D}^2, \bigvee_{r \in R} r(x, y) \leftrightarrow \bigwedge_{c \in C} c(x, y) \right) \rightarrow (\bar{C}, R) \in \mathcal{T}_{\mathcal{G}}$$

The next property characterizes the elements of \mathcal{B} . It can be deduced from Property 2 or by examining Table 8:

Property 3 *For each element r of \mathcal{B} , $(f(\{r\}), \{r\})$ is an element of $\mathcal{T}_{\mathcal{G}}$.*

Property 3 shows that the base relations of \mathcal{B} are represented as individuals in the $\mathcal{T}_{\mathcal{G}}$ lattice, situated at the bottom of the lattice, as displayed in Figure 6. The next property characterizes the conditions of $CM-8$ and their representation in $\mathcal{T}_{\mathcal{G}}$:

Property 4 *Each condition c of $CM-8$ can be associated to a distinct element of $\mathcal{T}_{\mathcal{G}}$, $\mathbf{E}_c = (\bar{C}, R)$, where $R = g(\{c\})$, $\bar{C} = h'(\{c\})$, and:*

$$\forall (x, y) \in \mathcal{D}^2, c(x, y) \leftrightarrow \bigvee_{r \in R} r(x, y) \leftrightarrow \bigwedge_{c_i \in C} c_i(x, y)$$

This last property can directly be checked on the conditions. In particular:

$$\begin{array}{ll} g(\{P\}) = \{EQ, NTPP, TPP\} & \\ h'(\{P\}) = (P, O) & \text{and } P(x, y) \leftrightarrow P(x, y) \wedge O(x, y) \\ g(\{P^{-1}\}) = \{EQ, NTPP^{-1}, TPP^{-1}\} & \\ h'(\{P^{-1}\}) = (P^{-1}, O) & \text{and } P^{-1}(x, y) \leftrightarrow P^{-1}(x, y) \wedge O(x, y) \\ g(\{DR\}) = \{DC, EC\} & \\ h'(\{DR\}) = (DR, Dx, Dy) & \text{and } DR(x, y) \leftrightarrow DR(x, y) \wedge Dx(x, y) \wedge Dy(x, y) \end{array}$$

The other singletons of 2^{CM-8} are closed for $h' : h'(\{c\}) = \{c\}$. Thus, all conditions of $CM-8$ are represented as individuals in the $\mathcal{T}_{\mathcal{G}}$ lattice. They are represented respectively by the elements P, P⁻¹, DR, O, Dy, A, Dx and NA (see Figure 6).

3.4.2 Characterizing the *glb* (\frown)

The operator \frown in the lattice is equivalent to the operator \wedge (conjunction) on the relations.

Property 5 *An element E of $\mathcal{T}_{\mathcal{G}}$ is the *glb* of two elements E_1 and E_2 if and only if $\pi_r(E)$ is equivalent to the conjunction of the two relations $\pi_r(E_1)$ and $\pi_r(E_2)$:*

$$(E = E_1 \frown E_2) \leftrightarrow (\forall(x, y) \in \mathcal{D}^2, \pi_r(E)(x, y) \leftrightarrow \pi_r(E_1)(x, y) \wedge \pi_r(E_2)(x, y))$$

Furthermore, each element E of $\mathcal{T}_{\mathcal{G}}$ is characterized by the equivalence between the disjunction $\pi_r(E)$ and the conjunction $\pi_c(E)$. Thus:

Property 6 *Let E, E_1, E_2 be three elements of the $\mathcal{T}_{\mathcal{G}}$ lattice:*

$$(E = E_1 \frown E_2) \leftrightarrow (\forall(x, y) \in \mathcal{D}^2, \pi_c(E)(x, y) \leftrightarrow \pi_c(E_1)(x, y) \wedge \pi_c(E_2)(x, y))$$

3.4.3 The minimal set of conditions for checking a relation

The relations expressed in the $\mathcal{T}_{\mathcal{G}}$ lattice are associated to a set C of conditions which are sufficient to check this relation. But all conditions of C are not necessary, and some of these conditions should be eliminated to minimize the calculation time. Let us take an example: $EQ(x, y)$, “ x is identical with y ”, is associated to the set of conditions $C = \{A, O, P, P^{-1}\}$ (see Table 8). But, the checking of $EQ(x, y)$ only relies on the checking of the conditions $P(x, y)$ (“ x is a part of y ”) and $P^{-1}(x, y)$ (“ x contains y ”) since the conjunction of these two conditions imply the two other conditions $A(x, y)$ (“ x shares a boundary with y ”) and $O(x, y)$ (“ x overlaps y ”). This implication is due to the properties of the image regions (non-emptiness, closure) as previously mentioned. In $\mathcal{T}_{\mathcal{G}}$, EQ is represented by the element $EQ = (\{A, O, P, P^{-1}\}, \{EQ\})$. According to Property 1:

$$\forall(x, y) \in \mathcal{D}^2, EQ(x, y) \leftrightarrow (A \wedge O \wedge P \wedge P^{-1})(x, y)$$

Furthermore, P and P^{-1} are represented by the elements P and P^{-1} . The *glb* of P and P^{-1} is EQ (see Figure 6). According to Property 6:

$$\forall(x, y) \in \mathcal{D}^2, \pi_c(EQ)(x, y) \leftrightarrow \pi_c(P)(x, y) \wedge \pi_c(P^{-1})(x, y)$$

Then (Property 4): $\forall(x, y) \in \mathcal{D}^2, (A \wedge O \wedge P \wedge P^{-1})(x, y) \leftrightarrow (P \wedge P^{-1})(x, y)$

Then: $\forall(x, y) \in \mathcal{D}^2, EQ(x, y) \leftrightarrow (P \wedge P^{-1})(x, y)$

Finally, the properties of the operator \frown can be used to find out the minimal set of conditions that are necessary to check the EQ relation. This minimal set can directly be read on the lattice.

The minimal sets associated to the other relations of \mathcal{B} can be found accordingly:

$$\begin{aligned} \forall (x, y) \in \mathcal{D}^2 : \quad & NTPP(x, y) \leftrightarrow (P \wedge NA)(x, y) \\ & TPP(x, y) \leftrightarrow (P \wedge A \wedge Dy)(x, y) \\ & NTPP^{-1}(x, y) \leftrightarrow (P^{-1} \wedge NA)(x, y) \\ & TPP^{-1}(x, y) \leftrightarrow (P^{-1} \wedge A \wedge Dx)(x, y) \\ & PO(x, y) \leftrightarrow (Dx \wedge Dy \wedge O)(x, y) \\ & EC(x, y) \leftrightarrow (A \wedge DR)(x, y) \\ & DC(x, y) \leftrightarrow (NA \wedge DR)(x, y) \end{aligned}$$

3.4.4 Characterizing the *lub* (\smile)

Property 7 For all pairs of lattice elements (E_1, E_2) , the *lub* $E' = E_1 \smile E_2$ is such that the disjunction of $\pi_r(E_1)$ and $\pi_r(E_2)$ implies the relation $\pi_r(E')$:

$$(E' = E_1 \smile E_2) \rightarrow (\forall (x, y) \in \mathcal{D}^2, \pi_r(E')(x, y) \leftarrow \pi_r(E_1)(x, y) \vee \pi_r(E_2)(x, y))$$

This property is easy to prove since the relation set R' of the element $E' = E_1 \smile E_2$ is the closure of the union of the two relation sets R_1, R_2 of the elements E_1 and E_2 (cf. Definition 4). The reciprocal property of Property 7 is not true in the $\mathcal{T}_{\mathcal{G}}$ lattice because an element of $\mathcal{T}_{\mathcal{G}}$ can be the *lub* of several elements: E' can be both the *lub* of the pair E_1, E_2 and the *lub* of the pair E_3, E_4 . Then:

$$\begin{aligned} E' = E_1 \smile E_2 : \pi_r(E_1) \vee \pi_r(E_2) &\rightarrow \pi_r(E') \\ E' = E_3 \smile E_4 : \pi_r(E_3) \vee \pi_r(E_4) &\rightarrow \pi_r(E') \\ \text{and } \pi_r(E') &\rightarrow \pi_r(E_1) \vee \pi_r(E_2) \vee \pi_r(E_3) \vee \dots \end{aligned}$$

For instance, the *lub* of the elements TP and TP^{-1} of $\mathcal{T}_{\mathcal{G}}$ is the element $0etA$ which is also the *lub* of the elements $0etAetDx$ and $0etAetDy$ (see Figure 6). This property has to be linked to a property of closed sets: the *lub* of two closed sets is generally not a closed set, whereas the *glb* is a closed set [Barbut and Monjardet 1970, Davey and Priestley 1990].

4 Elements for the comparison of the lattices

4.1 Comparing the lattices $\mathcal{T}_{\mathcal{P}}$ and $\mathcal{T}_{\mathcal{G}}$

From an implementation point of view, the $\mathcal{T}_{\mathcal{G}}$ lattice is more easy to manage than the $\mathcal{T}_{\mathcal{P}}$ lattice. An object-based knowledge representation system must be reasonably sized in terms of memory space. The $\mathcal{T}_{\mathcal{P}}$ lattice is complemented and contains $2^8 = 256$ elements. This means that the designer has to implement and to manage 256 classes of relations. The $\mathcal{T}_{\mathcal{G}}$ lattice by contrast contains only 34 elements (see Figure 6). Thus, it can be worth to choose the $\mathcal{T}_{\mathcal{G}}$ lattice rather than the $\mathcal{T}_{\mathcal{P}}$ lattice.

Another major advantage of the $\mathcal{T}_{\mathcal{G}}$ lattice is the representation of the conditions that are used to check the relations on images, and the representation of the implication relations

holding between the conditions and the relations. This is not done in the $\mathcal{T}_{\mathcal{P}}$ lattice: the conditions are only implicitly represented, as they are equivalent to disjunctions of \mathcal{B} relations. The implication relations between the topological relations and the conditions thus do not appear in the $\mathcal{T}_{\mathcal{P}}$ lattice, making this lattice not very useful for checking relations on the image. On the contrary, the links between relations and conditions are singled out in the $\mathcal{T}_{\mathcal{G}}$ lattice, which is built from the Galois connection between \mathcal{B} and $CM-8$.

The last point is about reasoning in the lattices: the reasoning is complete in $\mathcal{T}_{\mathcal{P}}$ whereas it is not in $\mathcal{T}_{\mathcal{G}}$. One of the reasons is that some compositions of the base relations are missing in $\mathcal{T}_{\mathcal{G}}$. In this case, the $\mathcal{T}_{\mathcal{P}}$ lattice has to be preferred: it includes all the disjunctions of \mathcal{B} elements and thus contains all the compositions of Table 6. Conversely the $\mathcal{T}_{\mathcal{G}}$ lattice contains the following compositions (see Figure 6):

$$\begin{aligned}
PO \vee TPP \vee NTPP \vee TPP^{-1} \vee NTPP^{-1} \vee EQ &= \pi_r(0) \\
DC \vee EC \vee PO \vee TPP \vee NTPP &= \pi_r(Dy) \\
DC \vee EC \vee PO \vee TPP^{-1} \vee NTPP^{-1} &= \pi_r(Dx) \\
PO \vee TPP \vee NTPP &= \pi_r(Dy \text{ et } 0) \\
PO \vee TPP^{-1} \vee NTPP^{-1} &= \pi_r(Dx \text{ et } 0) \\
PO \vee TPP \vee TPP^{-1} \vee EQ &= \pi_r(0 \text{ et } A) \\
NTPP \vee TPP &= \pi_r(PP) \\
NTPP^{-1} \vee TPP^{-1} &= \pi_r(PP^{-1}) \\
EC \vee DC &= \pi_r(DR)
\end{aligned}$$

and lacks three compositions:

$$\begin{aligned}
&EC \vee PO \vee NTPP \vee TPP \\
&EC \vee PO \vee NTPP^{-1} \vee TPP^{-1} \\
&DC \vee EC \vee PO \vee TPP \vee TPP^{-1} \vee EQ
\end{aligned}$$

In order to combine the advantages of $\mathcal{T}_{\mathcal{G}}$ (computation) and the advantages of $\mathcal{T}_{\mathcal{P}}$ (inference), we propose to extend the $\mathcal{T}_{\mathcal{G}}$ lattice with the lacking compositions. The extended lattice is described in the next paragraph and is compared to the $\mathcal{T}_{\mathcal{G}}$ lattice.

4.2 Extending the lattice $\mathcal{T}_{\mathcal{G}}$

4.2.1 The design of the extended lattice $\mathcal{T}_{\mathcal{G}\mathcal{E}}$

The extended lattice $\mathcal{T}_{\mathcal{G}\mathcal{E}}$ is built progressively: the lacking relations are added step by step as it is done in incremental algorithms for building lattices [Missikoff and Scholl 1989,

Carpineto and Romano 1993, Godin *et al.* 1995]. The addition of a relation in $\mathcal{T}_{\mathcal{G}\mathcal{E}}$ relies on two steps:

- Addition of the relation according to the order of the original lattice, i.e. searching for the most specific ascendants and the most general descendants of the new relation;
- Addition of a *glb* or of a *lub* for the pairs lacking one.

According to the first step, the addition of the two first relations relies on the following implications with the elements of the lattice $\mathcal{T}_{\mathcal{G}}$:

$$\begin{aligned}
EC \vee PO \vee NTPP \vee TPP &\rightarrow \pi_r(\text{Dy}) \\
EC \vee PO \vee NTPP \vee TPP &\leftarrow \pi_r(\text{Dy et 0}) \\
EC \vee PO \vee NTPP \vee TPP &\leftarrow \pi_r(\text{Dy et A}) \\
EC \vee PO \vee NTPP^{-1} \vee TPP^{-1} &\rightarrow \pi_r(\text{Dx}) \\
EC \vee PO \vee NTPP^{-1} \vee TPP^{-1} &\leftarrow \pi_r(\text{Dx et 0}) \\
EC \vee PO \vee NTPP^{-1} \vee TPP^{-1} &\leftarrow \pi_r(\text{Dx et A})
\end{aligned}$$

Following these implications, the relation $EC \vee PO \vee NTPP \vee TPP$ is expressed by the element Dy et -Aou 0- that is placed between the element Dy (up) and the elements Dy et A and Dy et 0 (down, see Figure 8). The relation $EC \vee PO \vee NTPP^{-1} \vee TPP^{-1}$, as for it, is expressed by the element Dx et -Aou 0- that is placed between the element Dx (up) and the elements Dx et A and Dx et 0 (down). The lattice structure is then preserved and no other element has to be added.

The insertion of the last relation in the extended lattice is more complicated; according to the first step, we find the following implications with the elements of the lattice $\mathcal{T}_{\mathcal{G}}$:

$$\begin{aligned}
DC \vee EC \vee PO \vee TPP \vee TPP^{-1} \vee EQ &\leftarrow \pi_r(\text{A}) \\
DC \vee EC \vee PO \vee TPP \vee TPP^{-1} \vee EQ &\leftarrow \pi_r(\text{Dx et Dy}) \\
DC \vee EC \vee PO \vee TPP \vee TPP^{-1} \vee EQ &\rightarrow \top
\end{aligned}$$

The relation is expressed by the element Aou -Dx et Dy- and placed in the lattice above the two elements A and Dx et Dy . The lattice structure is lost (see Figure 7, left):

- the pair $(\text{Dy et A}, \text{Dx et Dy})$ has two *lubs*, Aou -Dx et Dy- and Dy ,
- the pair $(\text{Dx et A}, \text{Dx et Dy})$ has also two *lubs*, Aou -Dx et Dy- and Dx .

In order to preserve the structure of the lattice $\mathcal{T}_{\mathcal{G}\mathcal{E}}$, two new elements have to be added, -Aou Dx -et Dy , being the new *lub* of $(\text{Dy et A}, \text{Dx et Dy})$, and -Aou Dy -et Dx , being the new *lub* of $(\text{Dx et A}, \text{Dx et Dy})$ (see Figure 7). Finally we have added five elements to $\mathcal{T}_{\mathcal{G}}$ to build $\mathcal{T}_{\mathcal{G}\mathcal{E}}$. These five elements corresponds to the five following subsets of \mathcal{B} :

1. Dy et -Aou 0- : $(EC, PO, NTPP, TPP)$

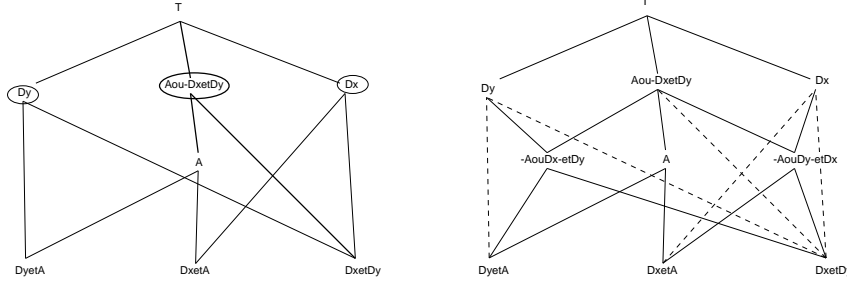


Figure 7: Left: adding the element $Aou-DxetDy$ - in the lattice \mathcal{T}_G (partial view). The two *lubs* of the pairs $(Dy \circ etA, Dx \circ etDy)$ and $(Dx \circ etA, Dx \circ etDy)$ are encircled. Right: adding the two elements $-AouDx-etDy$ and $-AouDy-etDx$ to preserve the structure of the lattice.

2. $Dx \circ etAou0$:- $(EC, PO, NTPP^{-1}, TPP^{-1})$
3. $Aou-Dx \circ etDy$:- $(DC, EC, PO, TPP, TPP^{-1}, EQ)$
4. $-AouDy-etDx$: (DC, EC, PO, TPP^{-1})
5. $-AouDx-etDy$: (DC, EC, PO, TPP)

No other element has to be added and the lattice structure is preserved. The lattice $\mathcal{T}_{G\mathcal{E}}$ is displayed in Figure 8.

Finally, the lattice $\mathcal{T}_{G\mathcal{E}}$ contains all the necessary relations for representing the compositions of \mathcal{B} relations. The lattice $\mathcal{T}_{G\mathcal{E}}$ is thus sufficient to infer the compositions of all disjunctions of \mathcal{B} relations from the distributivity of \circ with respect to \vee (cf. Section 3.1).

4.2.2 Comparing \mathcal{T}_G and $\mathcal{T}_{G\mathcal{E}}$

Since we want to represent the relations with computation and inference capabilities, we can work either with \mathcal{T}_G or $\mathcal{T}_{G\mathcal{E}}$. The last one seems preferable since it contains all the relations for representing the compositions of \mathcal{B} relations: thus all possible relations can be inferred from what is already computed on the image. Furthermore, the $\mathcal{T}_{G\mathcal{E}}$ lattice contains a reasonable number of relations and can be easily managed in an object-based knowledge representation system.

Besides, one should notice that the ordering in $\mathcal{T}_{G\mathcal{E}}$ is distinct from the one in \mathcal{T}_G that is defined by the inclusion relation both on $CM-8$ subsets and on \mathcal{B} subsets (cf. Definition 4). The lattice $\mathcal{T}_{G\mathcal{E}}$ is an extension of \mathcal{T}_G and is actually based on the implication between relations, being equivalent to the inclusion on \mathcal{B} subsets, as it has been shown before (Section 3.3.2). But it is not possible to associate subsets of $CM-8$ to the five relations added to \mathcal{T}_G (to obtain $\mathcal{T}_{G\mathcal{E}}$) in the same way as it is done for the original elements of \mathcal{T}_G . Actually the new elements of $\mathcal{T}_{G\mathcal{E}}$ are not formal concepts as introduced in Section 3.2. Thus the ordering of \mathcal{T}_G is not preserved in $\mathcal{T}_{G\mathcal{E}}$.

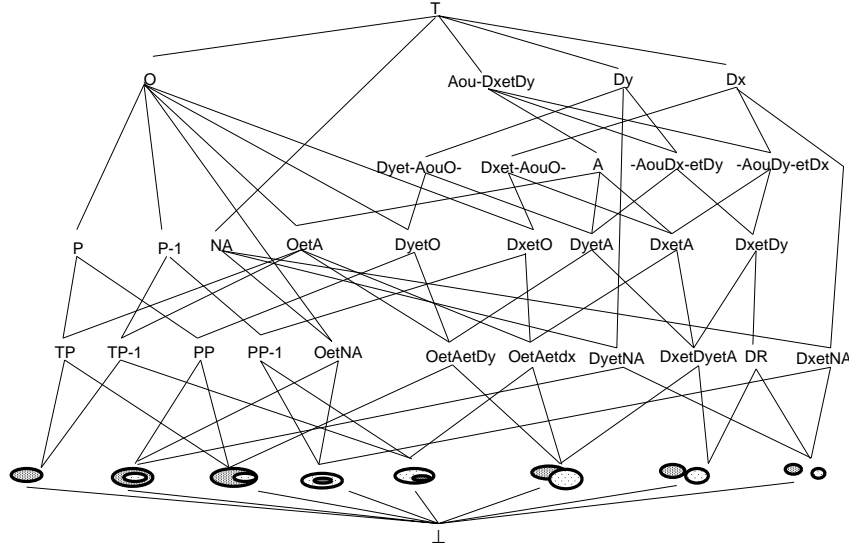


Figure 8: The lattice $\mathcal{T}_{G\epsilon}$, extending \mathcal{T}_G , for representing the compositions of \mathcal{B} relations.

Therefore, the $\mathcal{T}_{G\epsilon}$ lattice cannot be easily used for the computation of relations: the five added relations are not equivalent to a conjunction of conditions, and they cannot be associated to a minimal set of conditions. For example, the element $Dx et Aou O-$ is placed between the element $Dx et A$ and the element Dx in \mathcal{T}_G (see Figure 6 and 8). According to the \mathcal{T}_G ordering, the condition set C of $Dx et Aou O-$ would be such as: $\{Dx, A\} \subsetneq C \subsetneq \{Dx\}$, and that is impossible.

4.3 A second Galois lattice

In this section we describe a second Galois lattice that is based on the operations proposed in [Egenhofer 1989] to compute topological relations. In our present work, we have chosen to use other operations and we have built a Galois lattice to exploit the links between these operations and the \mathcal{B} relations. It is interesting to build a Galois lattice based on another set of operations and to compare this new lattice to the Galois lattice \mathcal{T}_G in order to make explicit the strengths and the limitations of the two approaches from this point of view.

4.3.1 The design of the lattice

The eight conditions based on the operations proposed in [Egenhofer 1989] have been described previously (see Section 2.2). Their set is named $CE-8$. Four of these conditions are identical to these of $CM-8$ (A , NA , DR , O). The four others are:

$$\begin{aligned}
\partial x \cap y^\circ = \emptyset & \text{ denoted } NFO \\
\partial x \cap y^\circ \neq \emptyset & \text{ denoted } FO \\
x^\circ \cap \partial y = \emptyset & \text{ denoted } NOF \\
x^\circ \cap \partial y \neq \emptyset & \text{ denoted } OF
\end{aligned}$$

The implication relation between these eight conditions and the relations of \mathcal{B} has been made explicit in Table 5. A Galois connection $\{f, g\}$ can then be defined between the sets $CE\text{-}8$ and \mathcal{B} as it is done in Section 3.3.1. The connection is described in Table 9.

	<i>DR</i>	<i>O</i>	<i>NA</i>	<i>A</i>	<i>NFO</i>	<i>FO</i>	<i>NOF</i>	<i>OF</i>
<i>DC</i>	1	0	1	0	1	0	1	0
<i>EC</i>	1	0	0	1	1	0	1	0
<i>PO</i>	0	1	0	1	0	1	0	1
<i>TPPI</i>	0	1	0	1	1	0	0	1
<i>TPP</i>	0	1	0	1	0	1	1	0
<i>NTPP</i>	0	1	1	0	0	1	1	0
<i>NTPPI</i>	0	1	1	0	1	0	0	1
<i>EQ</i>	0	1	0	1	1	0	1	0

Table 9: The table represents elements (r, c) , with $r \in \mathcal{B}$ (line) and $c \in CE\text{-}8$ (column). An element $(r, c) = 1$ if $\forall(x, y) \in \mathcal{D}^2, r(x, y) \rightarrow c(x, y)$. Otherwise $(r, c) = 0$.

The Galois lattice built from this connection is named \mathcal{T}_G' , its elements are pairs (C, R) , where C is a subset of $CE\text{-}8$ and R is a subset of \mathcal{B} . It is displayed in Figure 9. The notation of the elements is similar to the notation used for the \mathcal{T}_G lattice: an element is denoted with a name or a icon describing a relation, or with a name or a conjunction (“et” standing for “and”) of names of conditions.

4.3.2 Comparing the Galois lattices \mathcal{T}_G and \mathcal{T}_G'

As they are both Galois lattices, the \mathcal{T}_G and \mathcal{T}_G' lattices have basically the same properties. The main differences between the two lattices are about the conditions and the sets of relations that are represented.

Let us first examine the conditions of $CE\text{-}8$. Five of them are expressed within elements that are immediate descendants of \top , the top element of the lattice (see Figure 9). These elements are *A*, *O*, *NA*, *NFO*, *NOF*, whose set of conditions $C = \{c\}$ is closed for h' :

$$\forall c \in \{A, O, NA, NOF, NFO\}, h'(\{c\}) = \{c\}$$

The other conditions behave as follows:

$$\begin{aligned}
h'(\{OF\}) &= \{OF, O\} & \text{and } OF(x, y) &\leftrightarrow OF(x, y) \wedge O(x, y) \\
h'(\{FO\}) &= \{FO, O\} & \text{and } FO(x, y) &\leftrightarrow FO(x, y) \wedge O(x, y) \\
h'(\{DR\}) &= \{DR, NOF, NFO\} & \text{and } DR(x, y) &\leftrightarrow DR(x, y) \wedge NOF(x, y) \wedge NFO(x, y)
\end{aligned}$$

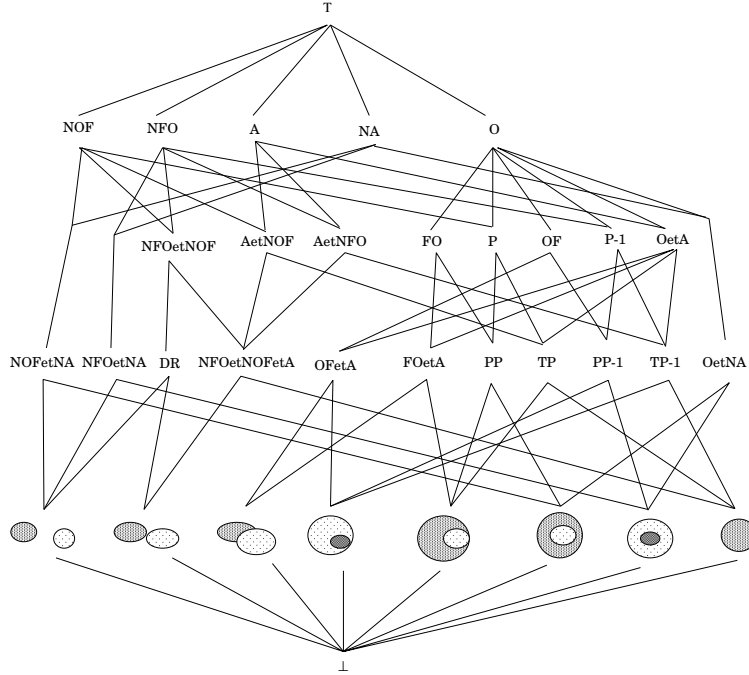


Figure 9: The Galois lattice \mathcal{T}_G' relying on the conditions proposed in [Egenhofer 1989].

We now compare the number of conditions used to check the relations in \mathcal{T}_G and \mathcal{T}_G' . Table 10 shows the number of conditions used for computing the base relations in each lattice. The first column gives the names of the corresponding elements in the two lattices (E' and E). The two following columns give the sets of conditions C' and C . The two last columns give the number (n' , n) of conditions used to check the relations (i.e. the cardinality of the minimal set of conditions, see Section 3.4.3). The number of conditions used to check all the relations expressed in the two lattices is described in the appendix.

Checking a relation does not necessarily require the same number of conditions in the two lattices (see Table 10): for instance, checking the relation EQ relies on two conditions in \mathcal{T}_G , $\{P, P^{-1}\}$, whereas it needs three conditions in \mathcal{T}_G' , $\{O, NFO, NOF\}$. The conditions P and P^{-1} respectively correspond to the pairs $\{O, NFO\}$ and $\{O, NOF\}$. Conversely, checking the relation PO relies on three conditions in \mathcal{T}_G , $\{Dx, Dy, A\}$, whereas it needs only the two conditions $\{FO, OF\}$ in \mathcal{T}_G' .

To summarize, the two lattices use the same mean number of conditions to check the relations, common or not common. Thus the choice between the two Galois lattices is not a matter of computation optimization (minimization of the number of tests) but rather a

E' / E	C'	C	n'	n
DC	$\{DR, NOF, NFO, NA\}$	$\{DR, Dx, Dy, NA\}$	2	2
EC	$\{DR, NOF, NFO, A\}$	$\{DR, Dx, Dy, A\}$	2	2
PO	$\{FO, OF, O, A\}$	$\{Dx, Dy, O, A\}$	2	3
TPP ⁻¹	$\{OF, O, A, NFO\}$	$\{P^{-1}, O, A, Dx\}$	3	3
TPP	$\{FO, O, A, NOF\}$	$\{P, O, A, Dy\}$	3	3
NTPP	$\{FO, O, NA, NOF\}$	$\{P, O, NA, Dy\}$	2	2
NTPP ⁻¹	$\{OF, O, NA, NFO\}$	$\{P^{-1}, O, NA, Dx\}$	2	2
EQ	$\{NFO, NOF, O, A\}$	$\{P, P^{-1}, O, A\}$	3	2

Table 10: Comparing the number of conditions used to check the base relations in the Galois lattices \mathcal{T}_G' and \mathcal{T}_G : E' / E are the names of the elements in the two lattices, n' and n the number of conditions used to check a relation (see the appendix for the complete comparison).

matter of adequation with the needs of the current application data: in one hand, it can be preferable to compute the intersection of a boundary set and an interior set ($\partial x \cap y^\circ$, $x^\circ \cap \partial y$), in the other hand, it can be preferable to compute the difference of two interior sets ($x^\circ - y^\circ$, $y^\circ - x^\circ$). According to our definition of the boundary of a region (Section 2.1), the second solution is better in our case.

4.4 Reasoning with lattices of topological relations

An important reasoning problem in the *RCC-8* theory, denoted by RSAT, is deciding consistency of a set of constraints of the form xRy , where R is a disjunction of the base relations, and x, y are spatial regions. This problem is NP-hard in general, but there exist subsets of *RCC-8* where it is tractable [Renz and Nebel 1999, Renz 1999]. Three maximal tractable subsets of *RCC-8* containing all base relations are defined in [Renz 1999]. According to these results, we conclude that RSAT (\mathcal{T}_G) and RSAT (\mathcal{T}_G') are NP-hard since the two lattices contain the relations $\{NTPP, NTPP^{-1}\}$ (element $0etNA$) and $\{DC, NTPP, NTPP^{-1}\}$ (NA , see Appendix). All other relations represented in \mathcal{T}_G and \mathcal{T}_G' are member of the maximal tractable subset $\hat{\mathcal{H}}_8$ defined in [Renz and Nebel 1999, Renz 1999].

The constraint satisfaction problem has been examined for internally connected planar regions in [Grigni *et al.* 1995]. The authors consider two notions of satisfiability, the relational consistency and the realizability. The realizability subproblem corresponds to planar constraints that must be satisfied besides the relational consistency ones, and it is shown to be almost everywhere NP-hard. According to the authors the realizability subproblem only appears for configurations involving more than 12 regions. Furthermore, in the case of regions that are not necessarily internally connected, relational consistency implies realizability [Renz 1998].

Lattices have been used for automatic theorem proving and relations inferring in the framework of the *RCC-8* theory [Randell and Cohn 1992]. Moreover, they are well adapted for classification-based reasoning. The Galois lattices in particular are well adapted for the classification and computation of topological relations. Their main drawback is that reasoning is not complete, even in the extended lattices like $\mathcal{T}_{\mathcal{G}\mathcal{E}}$, due to the building method of these lattices and their resulting properties. Actually, the lattices are not complemented and the *lub* of two elements is not equivalent to the disjunction of the corresponding relations (see Section 3.4.4). For instance, if the relation $(TP \vee TP^{-1})(x, y)$ holds, it is concluded in $\mathcal{T}_{\mathcal{G}}$ that $(TPP \vee TPP^{-1} \vee EQ \vee PO)(x, y)$ holds (0_{etA} is the *lub* of TP and TP^{-1}), whereas it is concluded in $\mathcal{T}_{\mathcal{P}}$ that $(TPP \vee TPP^{-1} \vee EQ)(x, y)$ holds. The last conclusion cannot be found in $\mathcal{T}_{\mathcal{G}}$ since the corresponding element does not exist.

This drawback is in balance with the low number of elements in the Galois lattices, making them practically more manageable than the boolean lattice $\mathcal{T}_{\mathcal{P}}$.

5 Lattice-based classification of spatial relations and structures

The lattices $\mathcal{T}_{\mathcal{G}}$ and $\mathcal{T}_{\mathcal{G}\mathcal{E}}$ have been implemented within the frame-based representation system Y3 [Ducournau 1991, Masini *et al.* 1991]. The elements of the lattices have been represented within relation classes organized according to the lattice ordering. In the following, the same notation is used for an element and the class representing this element, i.e. E stands for the element E and for the class representing E.

5.1 Inferences on the relations

Figure 10 shows the generic class SPATIAL-RELATION that represents the top element of the lattice, and three other classes. The class 0 is a subclass of the generic class, the class DR is a subclass of the class D \times etDy and the class EC is a subclass of the classes DR and D \times etDy \times etA.

The classes are described with attributes representing the various properties of the lattice elements, such as the attributes `complement`, `converse` and `condition`. The value of the `complement` attribute of class E is the class E^c such that $\pi_r(E^c)$ is the complement of the relation $\pi_r(E)$. For example the complement of the class 0 is the class DR since the two classes represent respectively the lattice elements ($\{O\}, \{EQ, PO, NTPP, TPP, NTPP^{-1}, TPP^{-1}\}$) and ($\{DR\}, \{DC, EC\}$). The value of the `converse` attribute of a class E is the element E^{-1} such that $\pi_r(E^{-1})$ is the converse relation of $\pi_r(E)$ (see Section 1.3). For example, the class EC fixes the value of the attribute `converse`: the converse relation of EC is EC itself. The value of the `condition` attribute of a class E is the set of conditions associated to this class. For example, the class 0 assigns the value of this attribute to C0 that represents the condition $O(x, y) = (x^\circ \cap y^\circ \neq \emptyset)$.

The composition of relations is represented within a method named `transitivity`. This method is only defined in the classes that represent the \mathcal{B} relations. The method is detailed

```

(defclass SPATIAL-RELATION
  (is-a . RELATION)
  (complement (a . SPATIAL-RELATION))
  (converse (a . SPATIAL-RELATION))
  (condition (a . CONDITION))
  (search-conditions (method) (...))
  (verify-relation (method) (O1 O2) (...))
  (transitivity (method) (...))
  .... )

(defclass O
  (is-a . SPATIAL-RELATION)
  (converse (value . O))
  (complement (value . DR))
  (condition (value . CO))
  .... )

(defclass DR
  (is-a . DxetDy)
  (converse (value . DR))
  (condition (value . CDR))
  .... )

(defclass EC
  (is-a . (DR DxetDyA))
  (converse (value . EC))
  .... )

```

Figure 10: Classes representing the elements of the lattices \mathcal{T}_G or $\mathcal{T}_{G\mathcal{E}}$. The properties of the relations are represented within attributes and methods.

for the class DC in Figure 11: for example the relation associated to DC, composed with the relation associated to EC gives the relation associated to Dy.

```

(defclass DC
  (is-a DR NAetDx NAetDy)
  (transitivity (DC . T)
               (EC . Dy)
               (PO . Dy)
               (TPP . Dy)
               (NTPP . Dy)
               (TPP-1 . DC)
               (NTPP-1 . DC)
               (EQ . DC))
  ...)

```

Figure 11: Representing the rules of composition within methods. T corresponds to the lattice element T.

The computation of the composition of two arbitrary relations is done as follows: to each element E_i of \mathcal{T}_G corresponds a disjunction of \mathcal{B} relations ($\pi_r(E_i)$). Thus, a set of descendants \mathcal{I}_{E_i} where the method `transitivity` is defined can be associated to each class E_i . For example, the class DR is associated with the set $\mathcal{I}_{DR} = \{DC, EC\}$. Considering the sets of descendants \mathcal{I}_{E_i} for a class E_i and \mathcal{I}_{E_j} for a class E_j , it is possible to apply the `transitivity` method of all the classes of \mathcal{I}_{E_i} to all the classes of \mathcal{I}_{E_j} and thus obtain a new set of classes. The *lub* of these classes is the class that represents the relation resulting from the composition of $\pi_r(E_i)$ and $\pi_r(E_j)$.

For example, the sets associated to DC and PP are $\mathcal{I}_{DC} = \{DC\}$ and $\mathcal{I}_{PP} = \{TPP, NTPP\}$. Composing DC and PP, the system will apply the `transitivity` method of the class DC to the classes TPP and NTPP. The resulting set is $\{Dy, Dy\}$ whose *lub* is Dy (see Figure 11).

5.2 The computation of the relations

As shown in Figure 10, each relation class E also includes an attribute `condition` recording the set of conditions of the lattice element E. The value of this attribute is inherited from the classes Dx, Dy, 0, NA, A, (see Figure 6) where its value is fixed. The inheritance mechanism is *cumulative* for all classes, but P, P^{-1} and DR. In those last three classes, the value of `condition` is overridden with a unique condition: for example this value for DR is equal to CDR that corresponds to the condition $x^\circ \cap y^\circ = \emptyset$ (Figure 10). Finally, the classes where the value of `condition` is a singleton correspond to the singletons of 2^{CM-8} .

We have previously shown that the conjunction (\wedge) and the glb (\frown) were equivalent in the \mathcal{T}_G lattice (Property 5). This property can be used to find out the minimal set of necessary conditions to check a relation $\pi_r(E)$, and, as well, to find out the classes representing this set among the ascendants of E.

For example, let us suppose that the EQ relation, expressed by the class EQ, has to be checked for two regions in the image; this class has a set of superclasses whose `condition` attribute value is reduced to a unique condition. This set is $\mathcal{S} = \{P, P^{-1}, A, 0\}$. The associated conditions are: P , P^{-1} , A and O . The common descendants of the elements of \mathcal{S} are then searched in the class hierarchy: P and P^{-1} have a common descendant (their *glb*, in fact) that is EQ. Using Property 5, we deduce that the conjunction of the two conditions P and P^{-1} is equivalent to the relation EQ (cf. Section 3.4.3). Thus, checking the relation EQ can be reduced to checking the two conditions P and P^{-1} .

Practically, the `verify-relation` methods of the classes P and P^{-1} are used on the image regions. These methods respectively compute the conditions P , or P^{-1} , associated to the classes; if they succeed, an instance of the *glb* of P and P^{-1} is created, i.e. an instance of EQ is created. If the `verify-relation` method of one of the classes fails then the `complement` attribute of this class can be used to find out which relation has been verified. For example, if the `verify-relation` method of P fails, the system can infer that the relation Dx holds (the value of the `complement` attribute of P is Dx, representing the condition Dx); the system finally creates an instance of the *glb* of the two classes: $P^{-1} \frown Dx = PP^{-1}$ (see Figure 6).

5.3 Classification of spatial structures

Let us return to our original motivation, i.e. to recognize spatial structures on land-use maps and to classify them according to landscape models. These models are represented within classes, and organized into several lattice structures. Model classes are linked through instances of the relation classes. For example (Figure 12), the class `TERRITORY` is specialized into the classes `TERRITORY-DC-FOREST` and `TERRITORY-EC-FOREST`, that are defined by the following attributes: `t-dcn-f` and `t-ecn-f`, that are respectively instance of the class `DC` and instance of the class `EC` [Mangelinck 1998].

```
(defclass TERRITORY-DC-FOREST
  (is-a TERRITORY)
  (t-dcn-f (is-a . DC)
           (a . FOREST)))

(defclass TERRITORY-EC-FOREST
  (is-a TERRITORY)
  (t-ecn-f (is-a . EC)
           (a . FOREST)))
```

Figure 12: Examples of classes representing models of spatial structures. The attributes represent the relations between spatial entities: for instance the attribute `t-dcn-f` is an instance of the class `DC`; its range is the class `FOREST`.

We have defined 6 terminal models that describe the main global structures of village territories in the Lorraine region (East of France) [Le Ber and Mangelinck 1998]. These models are built from more basic and general models (see Figure 13). For instance, the model `VALLEY` is a specification of the following models:

```
TERRITORY-EC-FOREST,
TERRITORY-BETWEEN-FORESTS,
TERRITORY-WITH-CROP-GROUP,
TERRITORY-WITH-CROP-EC-FOREST,
TERRITORY-WITH-MEADOW,
TERRITORY-WITH-COVERING-MEADOW, ...
```

These models are linked to the models `CROP`, `CROP-GROUP`, `CROP-EC-FOREST`, `MEADOW`, `COVERING-MEADOW`, ... that are elements of the lattice describing the models of crop fields and meadows. This lattice itself contains about 140 elements.

The classification of a spatial structure requires the classification of a set of related spatial structures and the corresponding relations, as it is illustrated in the following simplified example. The system first recognizes a territory entity on the image thanks to its label (the image that must be analyzed has been previously labelled [Bachacou 1995]). The system represents this entity as an instance, say `t1`, of the model class `TERRITORY`. The system tries then to classify the instance `t1` into a more specific class, for example the

TERRITORY-EC-FOREST class³ (see Figure 12). It checks therefore whether the instance $t1$ matches the properties of TERRITORY-EC-FOREST, i.e. $t\text{-ecn-f} = (\text{is-a . EC})(a . \text{FOREST})$. It searches for forest entities in the neighborhood of $t1$ in the image, and computes the topological relation between each forest entity and the territory, until it finds one forest entity externally connected to $t1$. If it succeeds, then the $t\text{-ecn-f}$ property is verified and the instance $t1$ is classified into the TERRITORY-EC-FOREST class. If it fails, the system tries to classify $t1$ into another subclass of TERRITORY (e.g. TERRITORY-DC-FOREST). The classification process goes on down the hierarchy of spatial structures until all classes have been checked.

5.4 Related work

Our approach presents some similarities with the one used in the VEIL system for the recognition process of objects in an image [Price *et al.* 1994, Russ *et al.* 1996]: the domain model is described within hierarchies of concepts embedded in the LOOM system [MacGregor 1991, MacGregor and Brill 1992]. The LOOM classifier is used to class the image objects into concepts according to the available information. The VEIL system combines both declarative knowledge and computational methods as in our system.

The work presented in [Haarslev *et al.* 1998] and [Haarslev *et al.* 1999] has also objectives that are similar to ours: it holds on qualitative spatial representation for managing map databases and spatial query processing. Spatial objects and relations are represented within the RACE system, that is based on description logics, and is extended to the polygon concrete domain, for taking into account spatial objects. Special modeling constructs can be used to represent topological relations as defined roles. Spatial reasoning relies on two main operations in description logics reasoning, namely consistency checking and classification. From the application point of view, regions are represented by polygons (elements of the concrete domain), and reasoning is carried on relations between polygons, for recognizing regions with specific characteristics and answering queries to a map database.

The objectives of this work and those of our work can be considered as complementary. The former work holds on polygons and is carried out in the context of description logics, with respect to the *RCC-8* theory. In our work, we focus on the representation and reification of *RCC-8* relations –relations correspond to concepts and not to roles–, within an object-based representation system, and on a lattice-based classification of spatial relations. The regions considered in our work are raster regions –and not vector regions–, but this does not make a fundamental difference between the two approaches from the reasoning point of view. The underlying reasoning mechanism is based in both cases on classification, and especially relation classification. Moreover, in [Möller and Wessel 1999], spatiotemporal default reasoning is introduced: a specific query completion problem is studied and default knowledge is used for completing and making more precise queries. This is still a complementary view with respect to our work. In our approach, classification and default knowledge represented within the classes are used to allow classification-based reasoning on spatial relation

³Forests cover a large area in Lorraine and are related to agriculture in many ways.

and concepts. However, a more theoretical study of our approach has still to be carried out, as it is done in [Möller and Wessel 1999].

6 Evaluation and research perspectives

The system has been used on land-use maps representing a part of the Lorraine region (East of France) in 1990 [Bachacou 1995]. It has produced maps where the village territories are classified according to the landscape models of the agronomists. About 25% of the territories are misclassified or not classified: the first ones show that the landscape models must be made more precise. The second ones are territories that cannot be classified according to the models. Actually, the experts are very interested both in the village territories that are classified and in those that are not classified. The first ones confirm and generalize their knowledge which is acquired from field studies. The last ones are 'special' territories that can indicate changes of the land use and thus that have to be further investigated. Besides, the system is still a prototype under development. Only partial evaluations have been performed until now, and a real-sized evaluation of the system remains to be done on other regions and dates (partial results are reported in [Mangelinck 1998]). By now, the present results of the system have indicated in which ways researches must be lead, for example by adding other spatial relations to describe the models (distance, orientation, and extended topology) and by refining the indices that describe the spatial entities.

The system and the associated research work can be enhanced and continued in a number of directions:

- The lattices considered above rely on the Galois connection theory. They are well adapted to the computation of relations, considering vector or raster data (images or geographic information systems). The choice of a set of computational primitives (two have been studied here, but other could be taken into account, such as the one proposed in [Ligozat 1999]) and then of the corresponding lattice $\mathcal{T}_{\mathcal{G}}$ or $\mathcal{T}_{\mathcal{G}}$ ' depends on the characteristics of the problem to be solved, as well as on the concrete definition of the boundaries, which is different if raster or vector data are used. This choice should rely on precise guidelines.
- The spatial domain contains regions which are not necessarily internally connected, in order to solve the planar realizability problem [Grigni *et al.* 1995]. However, this lead to another problem about the partial overlap relation ($PO(x, y)$) that may hold whereas the regions have no common boundary point ($NA(x, y)$). In the general case, PO can be checked, using the conditions Dx , Dy , and O . But when $NA(x, y)$ has been already checked on two regions, then $PO(x, y)$ can not be inferred in the $\mathcal{T}_{\mathcal{G}}$ lattice: $(Dx \wedge Dy \wedge O \wedge NA)(x, y) \rightarrow \perp(x, y)$. In the future, it could thus be useful to extend the lattice by including two kinds of partial overlap, according to the boundary overlap.

- In order to reason with spatial relations, it is better to exploit a complemented lattice such as $\mathcal{T}_{\mathcal{P}}$, or, at least, a lattice including all the relations of the composition table. A convenient compromise between relation computation and reasoning is to extend the Galois lattice $\mathcal{T}_{\mathcal{G}}$ in order to include the lacking relations. This leads to the lattice $\mathcal{T}_{\mathcal{G}\varepsilon}$, where some elements are not computable but where all compositions of the base relations are present.
- Lattices such as $\mathcal{T}_{\mathcal{P}}$ are frequently used in systems based on the mereotopology for representing spatial knowledge and for spatial reasoning [Randell and Cohn 1992]. By contrast, lattices combining computation and reasoning purposes, such as the lattices $\mathcal{T}_{\mathcal{G}}$ and $\mathcal{T}_{\mathcal{G}'}$, have not yet been used in the field of spatial representation and reasoning (at least to our knowledge). In addition to this “double competence”, these two lattices have the main advantage of being minimal in terms of memory space (34 elements for $\mathcal{T}_{\mathcal{G}}$ versus 256 elements for $\mathcal{T}_{\mathcal{P}}$) and of being easily extensible: indeed, it is possible to modify, extend or reduce the set of computational primitives or the set of base relations without loosing the properties associated to the Galois connection theory.
- Lattices are well adapted to classification-based reasoning. For instance, the classification mechanism that we have developed for our application manages two related lattices: the first one is a lattice of relation classes based on $\mathcal{T}_{\mathcal{G}}$, and the second one is a lattice of classes representing the models of spatial structures. In this context, hierarchical case-based reasoning for interpretation and prediction purposes is a natural extension of classification-based reasoning (see for example [Lieber and Napoli 1996, Lieber and Napoli 1998]).
- A last word has to be said about the reification of relations in an object-based representation system. Actually, we have used an *ad hoc* implementation [Manginck 1998, Le Ber *et al.* 1999]. A more general reification model has still to be studied, allowing to compose different types of relations and take advantage of these relations: inheritance of properties, representation of context, quantification, etc.

Conclusion

In this paper we have presented an original study for qualitative spatial representation and reasoning with topological relations. We have presented three lattices that can be used to compute and infer topological relations on raster images or data in geographic information systems. We have used the base relations of the *RCC-8* theory and a set of computational primitives on raster images and integrated them in a same lattice thanks to the Galois connection theory. We have shown that this lattice, named $\mathcal{T}_{\mathcal{G}}$, provides interesting properties for both computing and inferring relations, whereas the lattice based on the \mathcal{B} subsets, named $\mathcal{T}_{\mathcal{P}}$, is well adapted for reasoning. We have extended $\mathcal{T}_{\mathcal{G}}$ into a lattice $\mathcal{T}_{\mathcal{G}\varepsilon}$ in order to improve its reasoning capabilities. Finally we have compared $\mathcal{T}_{\mathcal{G}}$ with

a fourth lattice, named \mathcal{T}_G' , built from the Galois connection between the base relations and the computational primitives proposed in [Egenhofer 1989].

This work has been done in the context of the design of a knowledge-based system for helping agricultural landscape analysis. The aim of this project is to automatically match spatial structures on land-use maps with landscape models defined by agronomists. These models are expressed as sets of spatial entities linked with qualitative spatial relations. The matching problem can be expressed in the framework of an object-based knowledge representation system, as a problem of instance classification, where the spatial structures appearing on the maps are instances and the landscape models are classes. Actually, the lattices \mathcal{T}_G and $\mathcal{T}_{G\mathcal{E}}$ have been implemented within the frame-based representation system Y3 and used for the classification of spatial structures.

For the future, other spatial relations such as qualitative distance relations and orientation relations are planned to be added to the system, leading to new research problems both in the theoretical field of space representation and in the field of object-based knowledge representation systems.

Appendix

Composition of the base relations of the *RCC-8* theory

There are 12 disjunctions of relations in the transitivity table of \mathcal{B} (Table 6):

$$\begin{aligned}
 & DC \vee EC \vee PO \vee TPP \vee NTPP \\
 & DC \vee EC \vee PO \vee TPP^{-1} \vee NTPP^{-1} \\
 & DC \vee EC \vee PO \vee TPP \vee TPP^{-1} \vee EQ \\
 & \quad DC \vee EC \\
 & \quad \quad TPP \vee NTPP \\
 & \quad \quad \quad TPP^{-1} \vee NTPP^{-1} \\
 & PO \vee TPP \vee NTPP \vee TPP^{-1} \vee NTPP^{-1} \vee EQ \\
 & \quad PO \vee TPP^{-1} \vee NTPP^{-1} \\
 & \quad \quad PO \vee TPP \vee NTPP \\
 & \quad \quad \quad PO \vee TPP \vee TPP^{-1} \vee EQ \\
 & EC \vee PO \vee TPP^{-1} \vee NTPP^{-1} \\
 & \quad EC \vee PO \vee TPP \vee NTPP
 \end{aligned}$$

The elements of the lattice \mathcal{T}_G

The 32 elements of the lattice (except the top and bottom elements) are described in the Table 11.

Comparing the Galois lattices \mathcal{T}_G and \mathcal{T}_G'

The table 12 shows the number of conditions used for computing each relation that is expressed in the lattices \mathcal{T}_G and \mathcal{T}_G' . The first column gives the sets of relations R expressed in the lattices. The two following columns give the names of the corresponding elements in the two lattices (\mathbf{E}' and \mathbf{E}). The two last columns give the number (n' , n) of conditions used to check the relations (i.e. the cardinality of the minimal set of conditions, see paragraph 3.4.3). The eight base relations are listed in the first frame. The other relations common to the two lattices are represented in the second frame. The third and fourth frames list respectively the relations that are only represented in \mathcal{T}_G' , or in \mathcal{T}_G .

Definitions and notations

The definitions are based on [Barbut and Monjardet 1970].

R	C	E
$\{DC\}$	$\{DR, NA, Dx, Dy\}$	DC
$\{EC\}$	$\{DR, A, Dx, Dy\}$	EC
$\{PO\}$	$\{O, Dx, Dy, A\}$	PO
$\{TPP^{-1}\}$	$\{O, P^{-1}, Dx, A\}$	TPP ⁻¹
$\{TPP\}$	$\{O, P, Dy, A\}$	TPP
$\{NTPP\}$	$\{O, P, Dy, NA\}$	NTPP
$\{NTPP^{-1}\}$	$\{O, P^{-1}, Dx, NA\}$	NTPP ⁻¹
$\{EQ\}$	$\{O, P, P^{-1}, A\}$	EQ
$\{DC, NTPP\}$	$\{NA, Dy\}$	Dy et NA
$\{DC, NTPP^{-1}\}$	$\{NA, Dx\}$	Dx et NA
$\{PO, TPP^{-1}\}$	$\{O, Dx, A\}$	O et A et Dx
$\{PO, TPP\}$	$\{O, Dy, A\}$	O et A et Dy
$\{TPP, NTPP\}$	$\{O, P, Dy\}$	PP
$\{TPP, EQ\}$	$\{O, P, A\}$	TP
$\{TPP^{-1}, NTPP^{-1}\}$	$\{O, P^{-1}, Dx\}$	PP ⁻¹
$\{TPP^{-1}, EQ\}$	$\{O, P^{-1}, A\}$	TP ⁻¹
$\{NTPP, NTPP^{-1}\}$	$\{O, NA\}$	O et NA
$\{PO, TPP, NTPP\}$	$\{O, Dy\}$	Dy et O
$\{PO, TPP^{-1}, NTPP^{-1}\}$	$\{O, Dx\}$	Dx et O
$\{TPP, TPP^{-1}, EQ, PO\}$	$\{O, A\}$	O et A
$\{EC, PO\}$	$\{A, Dx, Dy\}$	Dx et Dy et A
$\{EC, PO, TPP^{-1}\}$	$\{A, Dx\}$	Dx et A
$\{EC, PO, TPP\}$	$\{A, Dy\}$	Dy et A
$\{EC, PO, DC\}$	$\{Dx, Dy\}$	Dx et Dy
$\{TPP, NTPP, EQ\}$	$\{O, P\}$	P
$\{TPP^{-1}, NTPP^{-1}, EQ\}$	$\{O, P^{-1}\}$	P ⁻¹
$\{DC, EC\}$	$\{DR, Dx, Dy\}$	DR
$\{DC, NTPP, NTPP^{-1}\}$	$\{NA\}$	NA
$\{TPP, TPP^{-1}, EQ, PO, EC\}$	$\{A\}$	A
$\{TPP, NTPP, TPP^{-1}, NTPP^{-1}, EQ, PO\}$	$\{O\}$	O
$\{EC, PO, DC, TPP^{-1}, NTPP^{-1}\}$	$\{Dx\}$	Dx
$\{EC, PO, DC, TPP, NTPP\}$	$\{Dy\}$	Dy

Table 11: The elements of the lattice $\mathcal{T}_{\mathcal{G}}$: E is the name of the element, R the subset of \mathcal{B} , C the subset of $CM-8$.

R	E'	E	n'	n
$\{DC\}$	DC	DC	2	2
$\{EC\}$	EC	EC	2	2
$\{PO\}$	PO	PO	2	3
$\{TPP^{-1}\}$	TPP^{-1}	TPP^{-1}	3	3
$\{TPP\}$	TPP	TPP	3	3
$\{NTPP\}$	NTPP	NTPP	2	2
$\{NTPP^{-1}\}$	$NTPP^{-1}$	$NTPP^{-1}$	2	2
$\{EQ\}$	EQ	EQ	3	2
$\{DC, EC\}$	DR	DR	1	1
$\{DC, NTPP\}$	NOFetNA	DyetaNA	2	2
$\{DC, NTPP^{-1}\}$	NFOetNA	DxetaNA	2	2
$\{PO, TPP^{-1}\}$	OFetA	OetaAetDx	2	3
$\{PO, TPP\}$	FOetA	OetaAetDy	2	3
$\{TPP, NTPP\}$	PP	PP	2	2
$\{TPP, EQ\}$	TP	TP	3	2
$\{TPP^{-1}, NTPP^{-1}\}$	PP^{-1}	PP^{-1}	2	2
$\{TPP^{-1}, EQ\}$	TP^{-1}	TP^{-1}	3	2
$\{NTPP, NTPP^{-1}\}$	OetNA	OetNA	2	2
$\{DC, NTPP, NTPP^{-1}\}$	NA	NA	1	1
$\{PO, TPP, NTPP\}$	FO	Dyeta0	1	2
$\{PO, TPP^{-1}, NTPP^{-1}\}$	OF	Dxeta0	1	2
$\{TPP, NTPP, EQ\}$	P	P	2	1
$\{TPP^{-1}, NTPP^{-1}, EQ\}$	p^{-1}	p^{-1}	2	1
$\{TPP, TPP^{-1}, EQ, PO\}$	OetaA	OetaA	2	2
$\{TPP, TPP^{-1}, EQ, PO, EC\}$	A	A	1	1
$\{TPP, NTPP, TPP^{-1}, NTPP^{-1}, EQ, PO\}$	O	O	1	1
$\{EC, EQ\}$	NFOetNOFetA		3	
$\{EC, EQ, TPP\}$	AetNOF		2	
$\{EC, EQ, TPP^{-1}\}$	AetNFO		2	
$\{EC, EQ, DC\}$	NFOetNOF		2	
$\{EC, EQ, DC, TPP, NTPP\}$	NOF		1	
$\{EC, EQ, DC, TPP^{-1}, NTPP^{-1}\}$	NFO		1	
$\{EC, PO\}$		DxetaDyeta		3
$\{EC, PO, TPP^{-1}\}$		DxetaA		2
$\{EC, PO, TPP\}$		DyetaA		2
$\{EC, PO, DC\}$		DxetaDy		2
$\{EC, PO, DC, TPP^{-1}, NTPP^{-1}\}$		Dx		1
$\{EC, PO, DC, TPP, NTPP\}$		Dy		1

Table 12: Comparing the number of conditions used to check the relations in the Galois lattices \mathcal{T}_G and \mathcal{T}_G' : E' and E are the names of the elements in the two lattices, n' and n the number of conditions used to check a relation.

Partial ordering. X is a non empty set, \leq is a binary relation defined on X , (X, \leq) is a *partial ordering* if-f:

1. \leq is reflexive: $\forall x \in X, x \leq x$,
2. \leq is antisymmetric: $\forall x, y \in X, (x \leq y \text{ et } y \leq x) \Rightarrow (x = y)$,
3. \leq is transitive: $\forall x, y, z \in X, (x \leq y \text{ et } y \leq z) \Rightarrow (x \leq z)$.

Infimum or glb. (X, \leq) is a partial ordering, A is a subset of X , m is an element of X , m is an *infimum* of A if-f:

- $\forall a \in A, (m \leq a)$,
- $\forall c \in X, (\forall a \in A, c \leq a) \Rightarrow (c \leq m)$.

Supremum or lub. (X, \leq) is a partial ordering, A is a subset of X , m is an element of X , m is a *supremum* of A if-f:

- $\forall a \in A, (a \leq M)$,
- $\forall c \in X, (\forall a \in A, a \leq c) \Rightarrow (M \leq c)$.

Lattice. A *lattice* $\langle X, \leq, \wedge, \vee \rangle$ is a partial ordering (X, \leq) where each pair $\{x, y\} \in X^2$ has a supremum \vee and an infimum \wedge ⁴.

Complemented lattice. $\langle X, \leq, \wedge, \vee \rangle$ is a lattice with a bottom element O and a top element I . X is said to be *complemented* if all the elements of X have a complement element in X that is if:

$$\forall x \in X, \exists y \in X : (x \vee y = I \text{ et } x \wedge y = O)$$

Distributive lattice. A lattice $\langle X, \leq, \wedge, \vee \rangle$ is said to be distributive if it obeys to the following expressions for all the triplets (x, y, z) :

$$(D_{\wedge}) \quad x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

$$(D_{\vee}) \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

$$(Md) \quad (x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee y)$$

It can be demonstrated that any of the three expressions implies the two others.

⁴The infimum and the supremum are generally denoted with \wedge and \vee but we rather use these symbols to denote the logical conjunction and disjunction.

Closure. h is an application in a ordered set (E, \leq) , h is a closure operator if it respects the three following properties:

- $\forall x, \forall y, x \leq y \Rightarrow h(x) \leq h(y)$ (h is monotone)
- $\forall x, x \leq h(x)$ (h is extensive)
- $\forall x, h \circ h(x) = h(x)$ (h is idempotent)

An element of E is closed with respect to h if it is equal to its closure by h ($x = h(x)$).

Proofs

We first admit the following hypotheses:

- (1) The relations of the *RCC-8* theory are exhaustive and mutually exclusive, i.e. for all pair $(x, y) \in \mathcal{D}^2$, there is one and only one relation of \mathcal{B} that holds: $\forall (x, y) \in \mathcal{D}^2, \exists r \in \mathcal{B} : r(x, y)$ and $\forall r' \in \mathcal{B}, r' \neq r \rightarrow \neg r'(x, y)$
- (2) A relation of \mathcal{B} does imply a condition of *CM-8* or its negation: $\forall r \in \mathcal{B}, \forall c \in \text{CM-8}, r \notin \mathcal{C}'(c) \leftrightarrow r \in \mathcal{C}'(\neg c)$ or $\mathcal{C}'(c) \cup \mathcal{C}'(\neg c) = \mathcal{B}$
- (3) For all relation of \mathcal{B} , there is no other relation that implies the same conditions of *CM-8*: $\forall r \in \mathcal{B}, \neg \exists r' \in \mathcal{B}, \mathcal{C}(r) \subseteq \mathcal{C}(r')$

Lemma 1 For all subset R of \mathcal{B} , $f(R) = f(\bar{R})$ where $\bar{R} = h(R)$ (that is, all subset $C = f(R)$ of *CM-8* is closed).

Proof: $f(R) = \bigcap_{r \in R} \mathcal{C}(r)$ and $f(\bar{R}) = \bigcap_{r \in \bar{R}} \mathcal{C}(r)$
 $R \subseteq \bar{R}$ (extensivity of h) then $f(\bar{R}) \subseteq f(R)$

Conversely: $f(\bar{R}) = f \circ h(R) = f \circ g \circ f(R) = h' \circ f(R)$
then $f(R) \subseteq f(\bar{R})$ (extensivity of h').

Lemma 2 For all subset C of *CM-8*, $g(C) = g(\bar{C})$ où $\bar{C} = h'(C)$ (that is, all subset $R = g(C)$ of \mathcal{B} is closed).

Proof: $g(C) = \bigcap_{c \in C} \mathcal{C}'(c)$ and $g(\bar{C}) = \bigcap_{c \in \bar{C}} \mathcal{C}'(c)$
 $C \subseteq \bar{C}$ (extensivity of h') then $g(\bar{C}) \subseteq g(C)$

Conversely: $g(\bar{C}) = g \circ h'(C) = g \circ f \circ g(C) = h \circ g(C)$
then $g(C) \subseteq g(\bar{C})$ (extensivity of h).

Lemma 3 For all subset R of \mathcal{B} , the disjunction of the relations of R implies the conjunction of the conditions of $f(R)$, i.e.:

$$\forall (x, y) \in \mathcal{D}^2, \bigvee_{r \in R} r(x, y) \rightarrow \bigwedge_{c \in f(R)} c(x, y)$$

Proof: For all $c \in f(R)$:
 $(c \in f(R)) \leftrightarrow (c \in \bigcap_{r \in R} \mathcal{C}(r)) \rightarrow (\forall r \in R, \forall (x, y) \in \mathcal{D}^2, r(x, y) \rightarrow c(x, y)).$
Then: $\forall r \in R, \forall (x, y) \in \mathcal{D}^2, r(x, y) \rightarrow \bigwedge_{c \in f(R)} c(x, y).$
Then: $\forall (x, y) \in \mathcal{D}^2, \bigvee_{r \in R} r(x, y) \rightarrow \bigwedge_{c \in f(R)} c(x, y).$

Lemma 4 For all subset C of CM -8, the conjunction of the conditions of C implies the disjunction of the relations of $g(C)$, i.e.:

$$\forall (x, y) \in \mathcal{D}^2, \bigwedge_{c \in C} c(x, y) \rightarrow \bigvee_{r \in g(C)} r(x, y)$$

Proof: we prove the contraposition: if no relation of $g(C)$ holds for a pair (x, y) then at least one condition of C does not hold for this pair.

Suppose $g(C) \neq \mathcal{B}$.

$(\exists (x, y) \in \mathcal{D}^2, \forall r \in g(C), \neg r(x, y)) \rightarrow \exists r' \in (\mathcal{B} - g(C)), r'(x, y)$ (cf. Hypothesis (1)).
 $r' \notin g(C) = \bigcap_{c \in C} \mathcal{C}'(c)$ then: $\exists c \in C, r' \notin \mathcal{C}'(c).$

According to the hypothesis (2), $r'(x, y)$ implies the negation of $c(x, y)$. Finally:

$$\forall (R, C) \in \mathcal{E}_g, \forall (x, y) \in \mathcal{D}^2, \neg \bigvee_{r \in R} r(x, y) \rightarrow \neg \bigwedge_{c \in C} c(x, y)$$

If $g(C) = \mathcal{B}$, then the disjunction of \mathcal{B} elements holds (cf. Hypothesis (1)).

Lemma 5 For all subset C of CM -8, the conjunction of the conditions of C is equivalent to the conjunction of the conditions of the h' -closure of C :

$$\forall (x, y) \in \mathcal{D}^2, \bigwedge_{c \in C} c(x, y) \leftrightarrow \bigwedge_{c \in h'(C)} c(x, y)$$

Proof: We denote $\bar{C} = h'(C)$.

h' is monotonously growing then $\forall C \in 2^{CM-8}, C \subseteq \bar{C}$.

Then $\forall (x, y) \in \mathcal{D}^2, \bigwedge_{c \in \bar{C}} c(x, y) \rightarrow \bigwedge_{c \in C} c(x, y).$

Conversely, according to the lemmas 3 and 4:

$$\forall (x, y) \in \mathcal{D}^2, \bigwedge_{c \in C} c(x, y) \rightarrow \bigvee_{r \in g(C)} r(x, y) \rightarrow \bigwedge_{c \in f \circ g(C)} c(x, y) = \bigwedge_{c \in \bar{C}} c(x, y).$$

Proof of the property 1: If a pair $E = (C, R)$ is an element of the lattice \mathcal{T}_g then the disjunction of the elements of R is equivalent to the conjunction of the elements of C .

According to the definition of the Galois lattice, $(C, R) \in \mathcal{E}_G \leftrightarrow C = f(R)$ and $R = f'(C)$.

Direct implication: it is directly based on the lemma 3 and on $C = f(R)$.
 $\forall (R, C) \in \mathcal{E}_G, \forall (x, y) \in \mathcal{D}^2, \bigvee_{r \in R} r(x, y) \rightarrow \bigwedge_{c \in C} c(x, y)$.

Converse implication: it is directly based on the lemma 4 and on $R = g(C)$.
 $\forall (R, C) \in \mathcal{E}_G, \forall (x, y) \in \mathcal{D}^2, \bigwedge_{c \in C} c(x, y) \rightarrow \bigvee_{r \in R} r(x, y)$.

Proof of the property 2: R is a subset of \mathcal{B} and C is a subset of $CM-8$. If the conjunction of the C elements is equivalent to the disjunction of the R elements, then the pair (\bar{C}, R) is an element of the lattice \mathcal{T}_G .

According to the lemma 5 and the hypothesis:

$$\forall (x, y) \in \mathcal{D}^2, \bigvee_{r \in R} r(x, y) \leftrightarrow \bigwedge_{c \in C} c(x, y) \leftrightarrow \bigwedge_{c \in \bar{C}} c(x, y).$$

$(\bar{C}, g(\bar{C}))$ is a element of the lattice (definition of the Galois lattice).

$g(\bar{C})$ is denoted with R' . The aim is to prove $R = R'$.

According to the property 1,

$$(\bar{C}, R') \in \mathcal{T}_G \rightarrow \left(\forall (x, y) \in \mathcal{D}^2, \bigwedge_{c \in \bar{C}} c(x, y) \leftrightarrow \bigvee_{r \in R'} r(x, y) \right)$$

$$\text{Hence } \forall (x, y) \in \mathcal{D}^2, \bigvee_{r \in R} r(x, y) \leftrightarrow \bigvee_{r \in R'} r(x, y)$$

For all relation $r \in R$, (x, y) is a pair of \mathcal{D}^2 for which r holds: then it exists a relation $r' \in R'$ that holds for (x, y) . According to the hypothesis (1), $r = r'$. Conversely, for all relation $r' \in R'$, it exists a relation $r \in R$ such that $r = r'$. The two sets are equal.

Proof of the property 3: For all element r of \mathcal{B} , $(f(\{r\}), \{r\})$ is an element of the lattice \mathcal{T}_G .

This property can be proved in different ways. The first one is to observe (cf. Table 7) that any relation of \mathcal{B} is equivalent to a conjunction of four conditions of $CM-8$. C_r denotes the subset of the four conditions. According to the property 2, $(\bar{C}_r, \{r\})$ is an element of the lattice. Hence $\{r\}$ is closed for h and $\bar{C}_r = f(\{r\})$.

The second way is to prove directly that $\{r\}$ is closed by h . Suppose it is false, i.e. it exists a relation $r' \neq r$ that belongs to $h(\{r\})$.

$$h(\{r\}) = g \circ f(\{r\}) = g(\mathcal{C}(r)) = \bigcap_{c \in \mathcal{C}(r)} \mathcal{C}'(c).$$

$$r' \in h(\{r\}) \rightarrow (\forall c \in \mathcal{C}(r), \forall (x, y), r'(x, y) \rightarrow c(x, y)).$$

$$r' \in h(\{r\}) \rightarrow \mathcal{C}(r) \subseteq \mathcal{C}(r') \text{ which is impossible according to the hypothesis (3).}$$

Proof of the property 4: Any condition c of $CM-8$ can be associated to an element $E_c = (C, R)$, where $C = h'(\{c\})$ and $R = g(C) = g(\{c\})$. This element is singular for each condition.

First, according to the lemma 5, any condition c of $CM-8$ is equivalent to the conjunction of the conditions of its h' -closure. Hence: $\forall(x, y) \in \mathcal{D}^2, c(x, y) \leftrightarrow \bigwedge_{c \in \overline{\{c\}}} c(x, y)$

$\overline{\{c\}} = h'(\{c\})$ being closed by h' , the element $E_c = (\overline{\{c\}}, g(\overline{\{c\}}))$ is an element of the lattice \mathcal{T}_g ; $g(\overline{\{c\}}) = g(\{c\})$ according to the lemma 2; Finally, according to the property 1:

$$\forall(x, y) \in \mathcal{D}^2, c(x, y) \leftrightarrow \bigwedge_{c \in \overline{\{c\}}} c(x, y) \leftrightarrow \bigvee_{r \in g(\{c\})} r(x, y)$$

The element E_c is singular for each condition since the conditions are pairwise not equivalent.

Proof of the property 5: An element $E = (C, R)$ of the lattice is the glb of two elements $E_1 = (C_1, R_1)$ and $E_2 = (C_2, R_2)$ if and only if the relation $\pi_r(E)$ is equal to the conjunction of the two relations ($\pi_r(E_1)$) and of R_2 ($\pi_r(E_2)$):

$$(E = E_1 \frown E_2) \leftrightarrow (\forall(x, y) \in \mathcal{D}^2, \pi_r(E)(x, y) \leftrightarrow \pi_r(E_1)(x, y) \wedge \pi_r(E_2)(x, y))$$

The glb of two elements of the lattice is defined as follows:

$$(C_1, R_1) \frown (C_2, R_2) = (h'(C_1 \cup C_2), R_1 \cap R_2).$$

Direct implication: E is the glb of the elements E_1 and E_2 .

Suppose $\bigvee_{r \in R} r(x, y)$ holds for a pair of objects $(x, y) \in \mathcal{D}^2$.

Hence: $\exists r \in R, r(x, y)$

$$(r \in R_1 \cap R_2 \text{ and } r(x, y)) \rightarrow \bigvee_{r \in R_1} r(x, y) \wedge \bigvee_{r \in R_2} r(x, y).$$

Conversely, suppose $\bigvee_{r \in R_1} r(x, y) \wedge \bigvee_{r \in R_2} r(x, y)$ holds for a pair of objects (x, y) .

Hence $\exists r_1 \in R_1, \exists r_2 \in R_2, r_1(x, y) \wedge r_2(x, y)$.

According to the hypothesis (2), $r_1 = r_2$, hence $r_1 \in R_1 \cap R_2$ and $\bigvee_{r \in R} r(x, y)$ holds.

Converse implication: E, E_1 and E_2 are three elements of the lattice such that:

$$\bigvee_{r \in R} r(x, y) \leftrightarrow \bigvee_{r \in R_1} r(x, y) \wedge \bigvee_{r \in R_2} r(x, y)$$

Proving $R = R_1 \cap R_2$ is enough; $C = h'(C_1 \cup C_2)$ can be deduced immediately by $C = f(R)$ and $g(C_1) \cap g(C_2) = g(C_1 \cup C_2)$.

Let be any relation $r \in R$ and a pair $(x, y) \in \mathcal{D}^2$ such that $r(x, y)$ holds.

Hence $\bigvee_{r \in R} r(x, y)$ holds and thus $\bigvee_{r \in R_1} r(x, y) \wedge \bigvee_{r \in R_2} r(x, y)$ holds.

Hence $\exists r_1 \in R_1, \exists r_2 \in R_2, r_1(x, y) \wedge r_2(x, y)$.

According to the hypothesis (1), $r_1 = r_2 = r$ hence $R \subseteq R_1 \cap R_2$.

Conversely, let be a relation $r \in R_1 \cap R_2$ and a pair of objects $(x, y) \in \mathcal{D}^2$ such that $r(x, y)$ holds. The same reasoning leads to $R_1 \cap R_2 \subseteq R$.

Proof of the property 7: For any pair of elements of the lattice (E_1, E_2) , the *lub* $E' = E_1 \smile E_2$ verifies that the disjunction of the relations $\pi_r(E_1)$ and $\pi_r(E_2)$ implies the relation $\pi_r(E')$.

The *lub* of two elements of the lattice is defined as follows:

$$(C_1, R_1) \smile (C_2, R_2) = (C_1 \cap C_2, h(R_1 \cup R_2)).$$

Let be $E' = (C', R')$ and $R' = h(R_1 \cup R_2) \supseteq R_1 \cup R_2$

Hence:

$$\left(\bigvee_{r \in R'} r(x, y) = \bigvee_{r \in h(R_1 \cup R_2)} r(x, y) \right) \leftarrow \left(\bigvee_{r \in R_1 \cup R_2} r(x, y) = \bigvee_{r \in R_1} r(x, y) \vee \bigvee_{r \in R_2} r(x, y) \right)$$

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