



**HAL**  
open science

## The clique number of unit quasi-disk graphs

Stephan Ceroi

► **To cite this version:**

| Stephan Ceroi. The clique number of unit quasi-disk graphs. RR-4419, INRIA. 2002. inria-00072169

**HAL Id: inria-00072169**

**<https://inria.hal.science/inria-00072169>**

Submitted on 23 May 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

***The clique number of unit quasi-disk graphs***

Stéphan CEROI

**N° 4419**

Mars 2002

THÈME 1

 ***rapport  
de recherche***



## The clique number of unit quasi-disk graphs

Stéphan CEROI \*

Thème 1 — Réseaux et systèmes  
Projet Mascotte

Rapport de recherche n° 4419 — Mars 2002 — 9 pages

**Abstract:** For  $\epsilon \in [0, 1]$ , a *unit  $\epsilon$ -quasi-disk* is a connected compact set  $Q$  of the plane such that there exists a point  $P$  such that  $D(P, 1 - \epsilon) \subseteq Q \subseteq D(P, 1)$ , where  $D(C, r)$  denotes the disk of centre  $C$  and radius  $r$ . We prove that for any fixed  $\epsilon > 0$ , the clique number problem on the class of intersection graphs of unit  $\epsilon$ -quasi-disks is NP-complete.

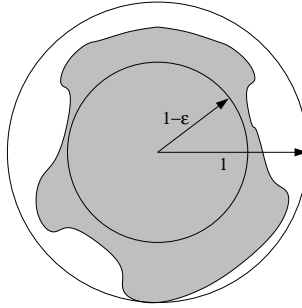
**Key-words:** Clique number, intersection graph, quasi-disk.

\* Supported by a FET-CRESCCO grant (CNRS). E.mail : [sceroi@sophia.inria.fr](mailto:sceroi@sophia.inria.fr)

## Clique maximum des graphes d'intersection de quasi-disques unités

**Résumé :** Pour  $\epsilon \in [0, 1]$ , un  $\epsilon$ -*quasi-disque unité* est un compact connexe  $Q$  du plan tel qu'il existe un point  $P$  tel que  $D(P, 1 - \epsilon) \subseteq Q \subseteq D(P, 1)$ , où  $D(C, r)$  désigne le disque de centre  $C$  et de rayon  $r$ . Nous montrons que pour tout  $\epsilon > 0$  fixé, le problème de la clique maximum sur la classe des graphes d'intersection de  $\epsilon$ -quasi-disques unité est NP-complet.

**Mots-clés :** Clique maximum, graphe d'intersection, quasi-disque.

Figure 1: A unit  $\epsilon$ -quasi-disk.

## 1 Introduction

Unit disk graphs (also known as *proximity graphs*) have received a lot of attention recently because of their applications to radio telecommunication and mobile phone networks ([5, 2, 10, 9, 11]). The set  $\mathcal{V}$  of centres of the disks represents the set of transmitters, and to avoid interferences, two transmitters whose corresponding disks intersect must have distinct channel. Then the minimum number of channels turns out to be the chromatic number  $\chi(G)$  of the corresponding unit disk graph  $G$  ([12, 4, 8]). Of course,  $\chi(G)$  is at least the clique number  $\omega(G)$ . In [5], it is proved that  $\chi(G) \leq 3\omega(G) - 2$ , and a  $O(n^{4.5})$  algorithm to compute  $\omega(G)$  is given (the input being  $\mathcal{V}$ ). Very recently, Raghavan and Spinrad [13] gave an polynomial time algorithm whose input is the abstract graph, and the author [3] extended this result to intersection graph of translates of a fixed convex set.

But the modeling of the covering area of a transmitter as a disk is unrealistic, and a modeling using *unit  $\epsilon$ -quasi-disks* has been introduced [1]. A unit  $\epsilon$ -quasi-disk (or *quasi-disk* when the context is clear) is a connected compact set  $Q$  of the plane such that there exists a point  $O$  such that  $D(O, 1 - \epsilon) \subseteq Q \subseteq D(O, 1)$ , where  $D(C, r)$  denotes the disk of centre  $C$  and radius  $r$  (see Fig. 1).

In this paper, we prove the following theorem :

**Theorem 1** *The clique number on the class of intersection graphs of unit  $\epsilon$ -quasi-disk is NP-complete for any  $\epsilon > 0$ .*

## 2 Proof of the theorem

The Euclidean plane is given with an orthonormal set of axis  $(O, Ox, Oy)$ . The coordinates of a point  $P$  are denoted  $(x_p, y_p)$ . For  $P$  a point of the plane and  $r \geq 0$ ,  $D(P, r)$  (resp.  $C(P, r)$ ) denotes the closed disk (resp. circle) of centre  $P$  and radius  $r$ . We denote by  $\rho_\alpha$  the rotation of centre  $O$  and of angle  $\alpha$ . For  $S$  a subset of the plane and  $\vec{t}$  a vector,  $S + \vec{t}$  is the translated of  $S$  by vector  $\vec{t}$ .

To prove Theorem 1, we use a reduction of the following problem :

**Problem :** Maximum independent set (MIS) on cubic graphs.

**Instance :** A cubic graph  $G$ ; an integer  $K$ .

**Question :** Is there an independent set  $X$  of  $G$  such that  $|X| \geq m$  ?

This problem is known to be NP-complete, even if restricted to planar cubic graphs [7, 6].

Let  $\epsilon > 0$ . Throughout this proof, quasi-disk stands for unit  $\epsilon$ -quasi-disk. Clearly, the MIS problem is also NP-complete on the class of *connected* cubic graphs distinct from  $K_4$ . So let  $G = (V, E)$  be a connected cubic graph, different from  $K_4$ . Then by Brook's theorem,  $G$  is 3-colorable. Let  $C_1, C_2, C_3$  a partition of  $V$  in 3 independent sets. To prove Theorem 1, we are going to construct a set of quasi-disks  $\{Q_1, Q_2, \dots, Q_n\}$  whose intersection graph is the complement  $\overline{G}$  of  $G$ . Thus the clique number of  $\overline{G}$  is exactly the size of an MIS of  $G$ , and this proves that the clique number is NP-complete on the class of unit  $\epsilon$ -quasi-disk graphs

For  $S$  a subset of  $\{1, 2, \dots, n\}$ , we construct a quasi-disk  $D_S^n$  the following way : let  $D(O, 1)$  be the unit disk centered in the origin. Remove from it the open half plane  $\Pi : x > 1 - \epsilon$ , let  $D'$  be the resulting set. Let  $A$  and  $B$  be the two endpoints of the segment  $D(O, 1) \cap \Pi$ ,  $A$  being the one with positive ordinate. Let  $O', A', B'$  denote the middle of  $[A, B]$ ,  $[A, O']$ ,  $[B, O']$  respectively. Let  $A''$  (resp.  $B''$ ) denotes the point of  $C(O, 1)$  having the same ordinate as  $A'$  (resp.  $B'$ ) and positive abscissae. We denote by  $l$  this positive abscissae (an easy computation gives  $l = \frac{1}{2}\sqrt{4 - 2\epsilon + \epsilon^2}$  and  $y_{A'} = \frac{1}{2}\sqrt{2\epsilon - \epsilon^2}$ ). Note that the rectangle  $A'A''B''B'$  is between the disks  $D(O, 1)$  and  $D(O, 1 - \epsilon)$ . We consider the  $n + 2$  points  $P_0 = B', P_1, P_2, \dots, P_n, P_{n+1} = A'$  that subdivide regularly the segment  $[A'B']$ , and we denote by  $\mu$  half the distance between two consecutive points, i.e.  $\mu = y_{A'}/(n + 1)$ . We denote by  $R_i$  the rectangle with vertices  $(1 - \epsilon, y_{P_i} - \mu/2)$ ,  $(1 - \epsilon, y_{P_i} + \mu/2)$ ,  $(l, y_{P_i} + \mu/2)$  and  $(l, y_{P_i} - \mu/2)$ .

For  $S \subseteq \{1, 2, \dots, n\}$ , the set  $D_S^n$  is defined as  $D' \cup \bigcup_{i \in S} R_i$  (see Fig 2). For  $i \in \{1, 2, \dots, n\}$ , we denote by  $E_{S,i}^n$  the set  $D_S^n \cap \rho_{\pi/3}(D_{\{n-i+1\}}^n)$  (see Fig. 3). It is easy to

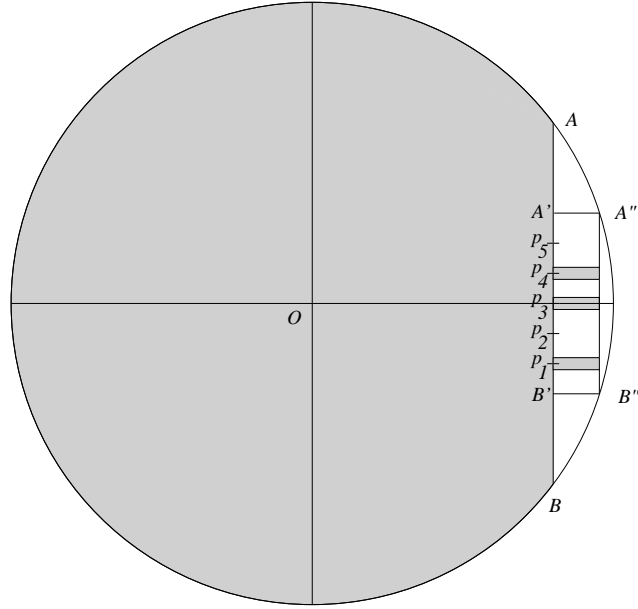


Figure 2: The set  $D_{\{1,3,4\}}^5$ , in grey.

see that  $E_{S,i}^n$  is a quasi-disk. The points of  $E_{S,i}^n \cap \Pi$  is a set of  $|S|$  rectangles called *the connector* of  $E_{S,i}^n$ . Similarly, the rectangle  $E_{S,i}^n \cap \rho_{\pi/3}(\Pi)$  is called *the identifier* of  $E_{S,i}^n$ .

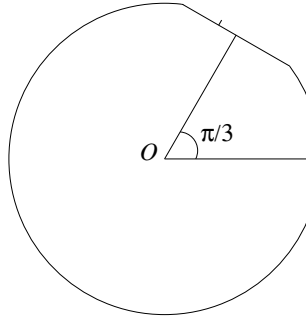
Recall that  $l$  is the abscissae  $A'$ , and let  $O_1 = O$ ,  $O_2 = (2l, 0)$  and  $O_3 = (l, \sqrt{3}l)$  be the vertices of an equilateral triangle of size  $2l$ . The quasi-disks  $E_{S,i}^n$  are defined to satisfy the following property :

**Lemma 1** *Let  $Q_1 = E_{S,i}^n$  and  $Q_2 = \rho_{2\pi/3}(E_{S',i'}^n) + \overrightarrow{O_1O_2}$  be two quasi-disks. They intersect if and only if  $i' \in S$ .*

**Proof.** Due to the value of  $l$  and the relative positions of  $Q_1$  and  $Q_2$ , these two quasi-disks intersect if and only if the identifier of  $Q_2$  intersects a rectangle of the connector of  $Q_1$ .

By construction, the rectangles of the connector of  $Q_1$  are vertically centered in the  $y_{P_j}$ 's,  $j \in S$ . It is easy to see that the identifier of  $Q_2$  is vertically centered in  $y_{P_{i'}}$ . Thus this rectangle intersects the connector of  $Q_1$  if and only if  $i' \in S$ . ■



Figure 3: A set  $E_{S_i}^n$ 

For any vertex  $v_i$  of  $G$ , we note  $S_i$  the set of indices of the non-neighbours of  $v_i$  :  $S_i = \{j \neq i : \{v_i, v_j\} \notin E\}$ , and we associate to  $v_i$  the quasi-disk  $Q_i$  defined as

- $E_{S_i}^n$  if  $v_i \in C_1$ ;
- $\rho_{2\pi/3}(E_{S_i}^n) + \overrightarrow{O_1O_2}$  if  $v_i \in C_2$ ;
- $\rho_{-2\pi/3}(E_{S_i}^n) + \overrightarrow{O_1O_3}$  if  $v_i \in C_3$ ;

Thus the quasi-disks associated to the elements of  $C_i$  are centered in  $O_i$ , see Fig 4.

Now we shall prove that the intersection graph of this set of quasi-disks is the complement of  $G$ .

**Lemma 2** *Two vertices  $v_i$  and  $v_j$  are adjacent in  $G$  if and only if  $Q_i$  and  $Q_j$  are disjoint.*

First suppose that  $v_i$  and  $v_j$  are adjacent, thus they do not belong to the same color class. Without loss of generality, we may suppose that  $v_i \in C_1$  and  $v_j \in C_2$ . By definition of  $S_i$ ,  $j \notin S_i$ , and thus by Lemma 1,  $Q_i$  and  $Q_j$  do not intersect.

Now suppose conversely that  $v_i$  and  $v_j$  are not adjacent. If they belong to the same color class  $C_k$  then  $Q_i$  and  $Q_j$  intersect obviously. Otherwise, as previously we may suppose without loss of generality that  $v_i \in C_1$  and  $v_j \in C_2$ , and then again by Lemma 1 we prove that  $Q_i$  and  $Q_j$  intersect. ■

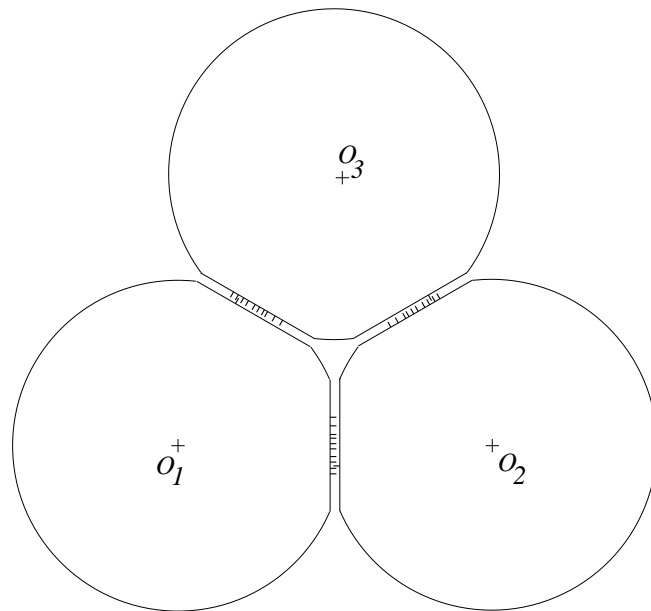


Figure 4: Configuration of the  $Q_i$ 's.

## References

- [1] Lali Barriere, Pierre Fraigniaud, Lata Narayanan, and Jaroslav Opatrny. Robust position-based routing in wireless ad hoc networks with unstable transmission ranges. In *5th ACM International Workshop on Discrete Algorithms and Methods for Mobile Computing and Communications (DIALM '01)*, Rome, July 2001.
- [2] Heinz Breu and David G. Kirkpatrick. Unit disk graph recognition is NP-hard. *Comput. Geom.*, 9(1-2):3–24, 1998.
- [3] Stéphan Ceroi. The clique number of intersection graphs of convex bodies. Technical report, LIRMM n° 02006, 2002. To appear in the *European Journal of Combinatorics*.
- [4] R. J. Cimikowski. Coloring certain proximity graphs. *Comput. Math. Appl.*, 20(3):69–82, 1990.
- [5] B.N. Clark, C.J. Colbourn, and D.S. Johnson. Unit disk graphs. *Discrete Math.*, 86(1):165–177, 1990.
- [6] M. R. Garey, D. S. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theoretical Computer Science*, 1:237–267, 1976.
- [7] Michael R. Garey and David S. Johnson. *Computers and intractability*. W.H. Freeman and company, 1979.
- [8] A. Gräf, M. Stumpf, and G. Weïßenfels. On coloring unit disk graphs. *Algorithmica*, 20:277–293, 1998.
- [9] M. V. Marathe, H. Breu, H. B. Hunt, III, S. S. Ravi, and D. J. Rosenkrantz. Simple heuristics for unit disk graphs. *Networks*, 25(2):59–68, 1995.
- [10] Madhav V. Marathe, Venkatesh Radhakrishnan, Harry B. Hunt, III, and S. S. Ravi. Hierarchically specified unit disk graphs. *Theoret. Comput. Sci.*, 174(1-2):23–65, 1997.
- [11] Tomomi Matsui. Approximation algorithms for maximum independent set problems and fractional coloring problems on unit disk graphs. In *Discrete and computational geometry (Tokyo, 1998)*, pages 194–200. Springer, Berlin, 2000.
- [12] Colin McDiarmid and Bruce Reed. Colouring proximity graphs in the plane. *Discrete Math.*, 199(1-3):123–137, 1999.

- [13] Vijay Raghavan and Jeremy Spinrad. Robust algorithms for restricted domains. In *Proceedings of the Twelfth Annual Symposium on Discrete Algorithms*, pages 460–467, Washington DC, USA, January 2001. ACM/SIAM.

## Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>Proof of the theorem</b>	<b>4</b>



---

Unité de recherche INRIA Sophia Antipolis

2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique  
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)

Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)

Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38330 Montbonnot-St-Martin (France)

Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

---

Éditeur

INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)

<http://www.inria.fr>

ISSN 0249-6399