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## Dérivée de forme pour des problèmes non-cylindriques

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*Tube derivative for non-cylindrical problems  
Application to Shape-Newton Method*

Raja Dziri — Jean Paul Zolésio

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## Tube derivative for non-cylindrical problems Application to Shape-Newton Method

Raja Dziri , Jean Paul Zolésio

Thème 4 — Simulation et optimisation  
de systèmes complexes  
Projet Opale

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**Résumé :** The purpose of this preport is to give the expression of the shape derivative of functionals related to moving domains evolution problems. The main step is to define pertubations of a given non-cylindrical domain [also called tube] using transverse vector fields. Under a smoothness assumption, a tube can be generated by a suitable vector field starting from its initial domain. This relationship tube-vector field allows us to define the field derivative and to recognize the term which constitutes the shape derivative for a class of non-cylindrical functionals. We conclude the paper, by applying the obtained result to compute the second order optimality condition for a shape minimization problem.

**Mots-clés :** Non-cylindrical shape functionals, Eulerian derivatives, transverse vector fields, shape gradient and hessian

## Dérivée de forme pour des problèmes non-cylindriques

**Abstract:** Le but de ce rapport est de donner l'expression de la dérivée de forme de fonctionnelles associées à des problèmes d'évolution non-cylindrique. Le point clé de ce travail réside dans la définition des perturbations d'un domaine non-cylindrique [appelé aussi tube] en utilisant des champs de vecteurs transverses. Sous des hypothèses de régularités, un tube peut être généré par un champ de vecteur convenable connaissant son domaine initial. Cette relation tube-champ de vecteur nous permet de définir une dérivée par rapport aux champs de vitesse et d'en déduire l'expression de la dérivée par rapport à la forme. On analyse la dérivation de fonctionnelle de tube associées à des problèmes dynamiques non cylindriques. On considère le cas particulier de l'évolution d'une fonctionnelle de domaine et l'application aux méthode d'ordre 2 de type Newton; conclut ce rapport, en donnant la condition nécessaire s d'optimalité du second ordre( faisant intervenir le hessien de forme) pour un tel problème de minimisation.

**Key-words:** Fonctionnelles de forme non-cylindrique, dérivation par rapport aux champs de vitesse, champ transverse, dérivation par rapport à la forme

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## 1 Introduction

Dynamical shape control problems for systems modelized with dynamical non-cylindrical Partial Differential Equations (PDE) are encountered in fluid-structure issues, free boundary problems, etc...

The cost functional involved in such problems is expressed in terms of integrals over the non-cylindrical evolution domain and/or its lateral boundary. Following [] this non-cylindrical evolution domain will be called a tube and is of the following form :

$$Q = \bigcup_{0 < t < \tau} (\{t\} \times \Omega_t); \quad \text{At } t = 0, \Omega_0 = \Omega.$$

The initial geometry  $\Omega$  is called the base of the tube. Non smooth tubes are described in [11] with help of the time convection of the base by non lipschitzian vector field  $V$ . If the lateral boundary  $\Sigma$  of the tube  $Q$  is smooth enough to guarantee the existence of the outward normal field  $\vec{\nu}$  to  $\Sigma$ , there exists smooth non-autonomous vector fields  $V$  such that

$$T_t(V)\Omega = \Omega_t \subset R^N \quad \forall t \in [0, \tau] \quad (1)$$

where  $T(V)$  is the flow associated to  $V$ . More precisely, the *intrinsic* outward normal field  $\vec{\nu} \in R^{N+1}$  defined on the lateral boundary  $\Sigma$  of  $Q$ , can be written as (cf.[9])

$$\vec{\nu}(t) = \frac{1}{\sqrt{1 + v_\nu^2}} (-v_\nu(t), \vec{n}_{\Omega_t})$$

where  $\vec{n}_{\Omega_t} \in R^N$  is the ‘‘horizontal’’ outward normal field to  $\Omega_t$ . Then any sufficiently smooth non-autonomous vector field  $V$  such that

$$V(t) \cdot \vec{n}_{\Omega_t} = v_\nu(t) \quad \text{on } \Sigma \quad (2)$$

builds the tube  $Q$  (or equivalently : satisfies (1)).

Conversely, to any non-autonomous vector field  $V$ , one can associate, in the time interval  $[0, \tau]$ , a tube  $Q(V)$  (also denoted  $Q_V$ ) by setting  $\Omega_t = T_t(V)(\Omega)$ .

Obviously (2) is then satisfied on the lateral boundary  $\Sigma(V)$  of  $Q(V)$ .

The functionals considered may depend not only on the tube containing the evolution takes place but also on a field that builds this tube which verifies some extra physical condition (cf. [6]). In that specific situation, the functional is no more a tube functional as it depends on the tube but also on a specific vector field  $V$ . For that reason we consider now as independant variable in the problem, not the tube itself but the vector field  $V$  whose flow mapping builds the tube starting from a given base  $\Omega$ . We designate by  $Q_V$  the tube built by an admissible vector field  $V$  and we consider functionals  $\mathbf{j}(\mathbf{V})$  in the form

$$\mathbf{j}(\mathbf{V}) = \mathbf{J}(\mathbf{V}, \mathbf{Q}_V).$$

When  $J$  depends exclusively on  $Q_V$  we get :

$$\mathbf{j}(\mathbf{V} + \mathbf{W}) = \mathbf{j}(\mathbf{V}), \quad \forall \mathbf{W} \text{ s.t. } \langle \mathbf{W}, \mathbf{n}_{\Omega_t(\mathbf{V})} \rangle = \mathbf{0} \quad \text{on } \Sigma_{\mathbf{V}},$$

this means that  $\mathbf{j}$  depends only on the shape of the tube (it is, thus, called a tube functional). The dependence of  $J$  on a field  $V$  comes, generally, from boundary conditions associated to the state equation.

For a bounded tube  $Q$ , there exists a bounded open set  $D$  (called a hold-all) such that  $Q$  and its perturbations remain in  $(0, \tau) \times D$ . That cylindrical tube will be invariant under all the transformations we shall consider so that the vector field  $V$  as well as the perturbations fields  $W$  will satisfy the condition :

$$\langle W(t), \vec{n}_D \rangle = 0 \quad \text{on } (0, \tau) \times \partial D \quad (3)$$

where  $\vec{n}_D$  is the outward normal field to  $D$ , cf. for example [8] or [9].

The aim is to characterize the derivative with respect to the field of the functional  $\mathbf{j}'(\mathbf{V}, \mathbf{W})$ . In a first step we obtain a general expression in terms of the *transverse* field  $Z$ . We study the problem whose  $Z$  is solution and we explicit the gradient with use of the transposed equation solution  $\Lambda$ .

We give sufficient conditions under which  $\mathbf{j}$  is Gâteaux differentiable at a field  $V$ . For an optimization problem, we prove that if the state depends on the field only by the boundary conditions, the field gradient  $G(V)$  is supported by  $\Sigma_V$ . But if it depends only on the shape of the tube,  $G(V)$  has the following form

$$G(V)(t) = \gamma_{\Gamma_t(V)}^*(g(V)(t)n_{\Omega_t(V)})$$

where  $\Gamma_t(V) = \partial(\Omega_t(V))$  and  $\gamma_{\Gamma_t(V)}^*$  is the adjoint of the trace operator on  $\Gamma_t(V)$ ,  $g(V)(t) \in [\mathcal{D}^{k-1}(\Gamma_t(V))]'$ . For these results one has to assume  $\Omega_t$  to be  $\mathcal{C}^k$ ,  $\forall t \in [0, \tau]$ .



## 2 Basic tools

### 2.1 Transformations-Velocity method

First, let us recall some basic results dealing with velocity flows. Let  $\tau > 0$ ,  $D$  a convex, smooth and bounded open set in  $R^N$ . Consider a vector field

$$V : [0, \tau] \times \overline{D} \longrightarrow R^N, (t, x) \longmapsto V(t)(x) \stackrel{def}{=} V(t, x) \text{ such that}$$

$$V(t).n_D = 0 \text{ on } \partial D, \forall t \in [0, \tau]. \quad (4)$$

Moreover, we assume

$$(V) \left\{ \begin{array}{l} \forall x \in \overline{D}, V(\cdot, x) \in C([0, \tau]; R^N) \\ \exists c > 0, \forall x, y \in D, \|V(\cdot, x) - V(\cdot, y)\|_{C([0, \tau]; R^N)} \leq c|x - y| \end{array} \right.$$

where  $V(\cdot, x)$  is the function  $t \longmapsto V(t, x)$ . For any  $X \in \overline{D}$ , associate the solution  $x_V(\cdot, X)$  of the ordinary differential equation

$$\left\{ \begin{array}{l} \frac{dx}{dt}(t) = V(t, x(t)), t \in (0, \tau) \\ x(0) = X \end{array} \right.$$

For any  $t \in [0, \tau]$ , we have a transformation

$$T_t(V) : \overline{D} \longrightarrow \overline{D}; X \longmapsto T_t(V)(X) \stackrel{def}{=} x_V(t, X).$$

The mapping  $(t, X) \longmapsto T_t(V)(X)$  is denoted  $T(V)$  or  $T$  if no confusion is possible. Under assumption (V), the map  $T$  has the following properties :

$$\left\{ \begin{array}{l} (T_1) \quad \forall X \in \overline{D}, \quad T(\cdot, X) \in C^1([0, \tau]; R^N) \text{ and } \exists c > 0, \\ \quad \forall X, Y \in \overline{D}, \quad \|T(\cdot, X) - T(\cdot, Y)\|_{C^1([0, \tau]; R^N)} \leq c|X - Y| \\ (T_2) \quad \forall t \in [0, \tau], \quad X \longmapsto T_t(X) : \overline{D} \longrightarrow \overline{D} \text{ is bijective} \\ (T_3) \quad \forall x \in \overline{D}, \quad T^{-1}(\cdot, x) \in C([0, \tau]; R^N) \text{ and } \exists c > 0, \\ \quad \forall x, y \in \overline{D}, \quad \|T^{-1}(\cdot, x) - T^{-1}(\cdot, y)\|_{C([0, \tau]; R^N)} \leq c|x - y| \end{array} \right.$$

Introduce the Banach space,  $k \in N$

$$\mathcal{V}_o^k(D) = \{v \in C^k(\overline{D}, R^N) \mid v.n_D = 0 \text{ on } \partial D\}$$

and the following notations :

$$c(F) \stackrel{def}{=} \sup_{y \neq x} \frac{|F(y) - F(x)|}{|y - x|}; \quad c_k(F) \stackrel{def}{=} \sum_{|\alpha|=k} c(\partial^\alpha F) \text{ for } k \geq 1.$$

The regularity result stated below for the flow associated to a given vector field  $V$  is developed in [3].

**Proposition 1**

For all  $V \in C([0, \tau], \mathcal{V}_o^k(D))$ ,  $k \in \mathbb{N}$ , such that

$$c_k(V(t)) \leq c \text{ ( for a constant } c > 0 \text{ independent of } t), \tag{5}$$

it is associated a unique map

$$T(V) \in C^1([0, \tau], C^k(\overline{D}, R^N)) \cap C([0, \tau], W^{k+1, \infty}(D, R^N)).$$

Moreover the mapping  $t \rightarrow V(t) \circ T_t$  is in  $L^\infty(0, \tau; W^{k+1, \infty}(D, R^N))$ .

The transformation  $T_t(V)$  is one-to-one and maps  $\overline{D}$  into  $\overline{D}$ . For each  $t \in [0, \tau]$ , one can consider the transformation  $T_t^{-1}$  and notice that it is the flow, at  $s = t$ , of the vector field  $V_t$  defined by  $V_t(s) = -V(t - s)$  (cf. [8], [3]). Using this argument we can prove that  $T^{-1} \in C([0, \tau], C^k(\overline{D}, R^N))$  and with the implicit function theorem one can prove the following regularity result.

**Proposition 2**

The mapping  $[0, \tau] \rightarrow C^k(\overline{D}, R^N)$ ,  $t \mapsto T_t^{-1}$  is continuously differentiable.

**Proof.** Let  $t \in (0, \tau)$ . The map  $T_t \in C^k(\overline{D}, R^N)$ . Consider the mapping

$$\Phi : [0, \tau] \times C^{k-1}(\overline{D}, R^N) \rightarrow C^{k-1}(\overline{D}, R^N)$$

defined by

$$\Phi(t, S) = T(V)(t, S) - Id$$

It is clear that  $\Phi(t, T_t(V)^{-1}) = 0$ . Moreover  $\Phi$  is continuously differentiable. The partial derivatives are :

$$\partial_t \Phi(t, S) = V(t, T_t(V) \circ S); \quad \partial_S \Phi(t, S) \xi = (DT_t) \circ S \xi$$

It is clear that  $\partial_S \Phi(t, S)$  is in  $\text{aut}(C^{k-1}(\overline{D}, R^N))$ . We deduce from the implicit function theorem that  $T^{-1} \in C^1([0, \tau]; C^{k-1}(\overline{D}, R^N))$ . Then the mapping  $t \rightarrow DT_t^{-1} = (DT_t)^{-1} \circ T_t^{-1}$  is also in  $C^1([0, \tau]; C^{k-1}(\overline{D}, R^N))$ . Therefore the announced regularity is proved.  $\square$

**2.2 Tube perturbation**

From now on we assume  $k \geq 1$ . A perturbation of a tube  $Q_V$  in a direction  $W$  can be obtained by considering transverse transformations (which are transformations acting on domains built at the same time).

Let  $\Omega \subset\subset D$  of class  $C^k$  be given, for any sufficiently small positive parameter  $s$ , we consider the moving domain  $Q_{(V+sW)}$  as the perturbation of the tube  $Q_V$  in the direction of the field  $W$ . It is composed of the sets

$$\Omega_t(V + sW) = T_t(V + sW)(\Omega), \forall t \in [0, \tau].$$

A transverse transformation is a function which maps  $\Omega_t(V)$  onto  $\Omega_t(V + sW)$  for any  $t \in [0, \tau]$  (and  $\overline{D}$  onto  $\overline{D}$ ). A quite natural one is

$$\mathcal{T}_s^t = T_t(V + sW) \circ T_t(V)^{-1}.$$

If the regularity assumptions  $(T_1) - (T_3)$  are satisfied by the mapping  $(s, x) \mapsto \mathcal{T}_s^t(x)$  for any  $t \in [0, \tau]$ , this transformation can be considered as the flow of the vector field [ see for instance [1]]

$$\mathcal{Z}^t(s, \cdot) = \left( \frac{\partial}{\partial s} \mathcal{T}_s^t \right) \circ \mathcal{T}_s^t(\cdot)^{-1} = [\partial_s T_t(V + sW)] \circ T_t(V + sW)^{-1}. \quad (6)$$

**Lemme 1**

Let  $I_0$  be a neighborhood of zero;  $V$  and  $W$  being in  $\mathcal{C}([0, \tau]; \mathcal{V}_o^k(D))$  satisfying condition (5). The mapping

$$\begin{aligned} I_0 &\longrightarrow \mathcal{C}([0, \tau]; C^{k-1}(\overline{D}, R^N)) \\ s &\longrightarrow T(V + sW) \end{aligned}$$

is continuously differentiable and  $\partial_s(T_t(V + sW))$  satisfies for any  $t \in [0, \tau]$ ,

$$\begin{aligned} \partial_s[T_t(V + sW)] &= \int_0^t D(V + sW)(\mu, T_\mu(V + sW)) \partial_s[T_\mu(V + sW)] d\mu \\ &+ \int_0^t W(\mu, T_\mu(V + sW)) d\mu. \end{aligned} \quad (7)$$

**Proof.**

$$\begin{aligned} &T_t(V + sW) - T_t(V + s_o W) \\ &= \int_0^t (V + sW)(\mu, T_\mu(V + sW)) - (V + s_o W)(\mu, T_\mu(V + s_o W)) d\mu. \\ &\|T_t(V + sW) - T_t(V + s_o W)\|_{C^{k-1}(\overline{D})} \leq |s - s_o| \int_0^t \|W(\mu)\|_{C^{k-1}(\overline{D})} d\mu \\ &+ \max_{t \in [0, \tau]} \|DV(t) + s_o DW(t)\|_{C^{k-1}(\overline{D})} \int_0^t \|T_\mu(V + sW) - T_\mu(V + s_o W)\|_{C^{k-1}(\overline{D})} d\mu \end{aligned}$$

Applying the Gronwall inequality, it comes

$$\begin{aligned} & \|T_t(V + sW) - T_t(V + s_oW)\|_{C^{k-1}(\overline{D})} \leq \tau|s - s_o| \|W\|_{C([0,\tau];C^{k-1}(\overline{D}))} \\ & + \tau|s - s_o| \|W\|_{C([0,\tau];C^{k-1}(\overline{D}))} \int_0^t \exp(t - \mu) d\mu \end{aligned}$$

Thus for any  $t \in [0, \tau]$  :

$$\|T_t(V + sW) - T_t(V + s_oW)\|_{C^{k-1}(\overline{D})} \leq \tau|s - s_o| \|W\|_{C([0,\tau];C^{k-1}(\overline{D}))} e^t.$$

This proves that the considered map is in  $W^{1,\infty}(I_0; C([0, \tau]; C^{k-1}(\overline{D}); R^N))$  and also gives the following uniform boundedness (with respect to  $s$ ) :

$$\begin{aligned} & \frac{1}{|s - s_o|} \| (V + sW)(\mu, T_\mu(V + sW)) - (V + s_oW)(\mu, T_\mu(V + s_oW)) \|_{C^{k-1}(\overline{D})} \\ & \leq \|W(\mu)\|_{C^{k-1}(\overline{D})} + \tau \exp \tau \max_{t \in [0,\tau]} \|DV(t) + s_oDW(t)\|_{C^{k-1}(\overline{D})} \|W(\mu)\|_{C^{k-1}(\overline{D})} \end{aligned}$$

Then according to the Lebesgue theorem, the derivative exists everywhere in  $I_0$  and satisfies (7). It has the following expression :

$$\begin{aligned} \partial_s [T_t(V + sW)] &= \int_0^t \exp\left\{ \int_\xi^t D(V + sW)(\mu, T_\mu(V + sW)) d\mu \right\} W(\xi, T_\xi(V + sW)) d\xi. \\ &= \int_0^t DT_t(V + sW) \cdot [DT_\xi(V + sW)]^{-1} W(\xi, T_\xi(V + sW)) d\xi. \end{aligned}$$

It is clear that this expression is continuous in  $I_0$ .

Let  $\mathcal{S}^t(s) = \partial_s [T_t(V + sW)]$ . Then  $\mathcal{Z}^t(s, x) = \mathcal{S}^t(s) \circ T_t(V + sW)^{-1}$ . In the next section, it will be shown that the field derivatives of non-cylindrical functionals are expressed in terms of the transverse vector field  $\mathbf{Z}(t, x) = \mathcal{Z}^t(0, x)$ . Therefore, one has to know more about this vector field. In particular, it will be shown that  $\mathbf{Z}$  can be characterized as the unique solution of

$$\partial_t \mathbf{Z} + [\mathbf{Z}, V] = W \text{ in } (0, \tau) \times D \quad (8)$$

$$\mathbf{Z}(0, \cdot) = 0 \text{ in } D \quad (9)$$

where  $[ \cdot, \cdot ]$  denotes the Lie Brackets. For that purpose let us consider the vector field  $S$ ,  $S(t, \cdot) \stackrel{def}{=} \mathcal{S}^t(0, \cdot) = \mathbf{Z}(t) \circ T_t(V)$ .

**Lemme 2**

The function  $S$  is the unique vector field, in  $\mathcal{C}^1([0, \tau]; C^{k-1}(\overline{D}, R^N))$ , satisfying

$$S(t) = \int_0^t W(\mu, T_\mu(V)) d\mu + \int_0^t DV(\mu, T_\mu(V)) S(\mu) d\mu. \quad (10)$$

**Proof.** Let  $\mathcal{F}$  be the mapping defined by

$$\begin{aligned} \mathcal{F} : [0, \tau] \times C^{k-1}(\overline{D}, R^N) &\rightarrow C^{k-1}(\overline{D}, R^N) \\ (t, \varphi) &\rightarrow DV(t, T_t(V))\varphi + W(t, T_t(V)). \end{aligned}$$

For any  $t \in [0, \tau]$  and any  $\varphi \in C^{k-1}(\overline{D})$ ,  $\mathcal{F}(t)$  is affine and  $\mathcal{F}(\cdot, \varphi)$  is continuous. So the existence and uniqueness of (10) are given by the Cauchy-Lipschitz theorem. Moreover the solution has the following expression

$$\begin{aligned} S(t) &= \int_0^t \exp\left\{ \int_s^t DV(\mu, T_\mu(V)) d\mu \right\} W(s, T_s(V)) ds, \quad \forall t \in [0, \tau]. \\ &= \int_0^t DT_t(V) \cdot [DT_s(V)]^{-1} W(s, T_s(V)) ds \end{aligned}$$

□

**Lemme 3**

If  $\psi \in C^1([0, \tau], C^{k-1}(\overline{D}, R^N))$  is such that

$$\partial_t \psi + D\psi \cdot V \in C([0, \tau], C^{k-1}(\overline{D}, R^N))$$

and satisfies (8)-(9), then  $\psi \circ T(V)$  belongs to  $C^1([0, \tau], C^{k-1}(\overline{D}, R^N))$  and satisfies (10). Conversely, if  $\varphi \in C^1([0, \tau], C^{k-1}(\overline{D}, R^N))$  is solution of (10), then  $\varphi \circ T(V)^{-1} \in C^1([0, \tau], C^{k-1}(\overline{D}, R^N))$  such that

$$\partial_t [\varphi \circ T(V)^{-1}] + D[\varphi \circ T(V)^{-1}] \cdot V \text{ is in } C([0, \tau], C^{k-1}(\overline{D}, R^N))$$

and satisfies (8)-(9).

**Proof.** If  $\psi$  belongs to  $C^1([0, \tau], C^{k-1}(\overline{D}, R^N))$ , then it is the same for the mapping  $t \mapsto \psi(t) \circ T_t$  and we have :

$$\partial_t (\psi(t, T_t)) = [DV(t)] \circ T_t \cdot \psi(t, T_t) + W(t) \circ T_t.$$

Thus  $\psi \circ T(V)$  satisfies (10). Conversely, since  $\varphi \in C^1([0, \tau], C^{k-1}(\overline{D}, R^N))$  is solution of (10) and  $T^{-1}$  is in  $C^1([0, \tau], C^{k-1}(\overline{D}, R^N))$ , it comes that  $\varphi \circ T(V)^{-1}$  is in  $C^1([0, \tau], C^{k-1}(\overline{D}, R^N))$  and we have :

$$\begin{aligned} \partial_t (\varphi(t, T_t^{-1})) &= [\partial_t \varphi] \circ T_t^{-1} + [D\varphi] \circ T_t^{-1} \cdot \partial_t (T_t^{-1}) \\ &= \{ [DV] \circ T_t \cdot \varphi + W \circ T_t \} \circ T_t^{-1} - (D\varphi) \circ T_t^{-1} D(T_t^{-1}) \cdot V(t) \\ &= DV \cdot (\varphi \circ T_t^{-1}) + W - D(\varphi \circ T_t^{-1}) \cdot V(t) \end{aligned}$$

which concludes the proof. □

**Theorem 1**

The field  $\mathbf{Z}$  is the unique vector field in

$C^1([0, \tau], C^{k-1}(\overline{D}, R^N))$ , such that  $\partial_t \mathbf{Z} + D\mathbf{Z}.V \in C([0, \tau], C^{k-1}(\overline{D}, R^N))$ , solution of problem (8)-(9).

**Proof.** Consider the solution  $S$  of (10), compose by  $T(V)^{-1}$  and make use of the lemma 3. □

**Remark 1**

Taking into account the characterization of  $\mathbf{Z}$ , we deduce that it has the following expression :

$$\mathbf{Z}(t) = \left\{ \int_0^t DT_t(V) \cdot [DT_s(V)]^{-1} W(s, T_s(V)) ds \right\} \circ T_t(V)^{-1}. \quad (11)$$

By the way a useful property of  $\mathbf{Z}$  is that if  $V$  and  $W$  are of free divergence then  $\mathbf{Z}$  too. Indeed

**Proposition 3**

Let  $V$  and  $W$  in  $C([0, \tau], \mathcal{V}_o^k(D))$ ,  $k \geq 1$ , such that (5) holds. Assume

$$\operatorname{div} V = \operatorname{div} W = 0 \text{ in } D.$$

Then the field  $\mathbf{Z}$  is of divergence free :

$$\operatorname{div} \mathbf{Z} = 0 \text{ in } D.$$

**Proof.** Let  $f \in \mathcal{D}(D)$ . The transformations  $T_t(V + sW)$  and  $T_t(V)^{-1}$  maps  $D$  onto  $D$ . Then,

$$\int_D f \circ T_s^t dx = \int_D f dx.$$

Indeed since  $V$  and  $W$  are of free divergence, we have

$$\int_D f dx = \int_{T_t(V+sW)(T_t(V)^{-1}(D))} f dx = \int_{T_t(V)^{-1}(D)} f \circ T_t(V + sW) dx = \int_D f \circ T_s^t dx.$$

From this we deduce that

$$\frac{d}{ds} \left( \int_D f \circ T_s^t dx \right) = 0$$

The mapping  $s \mapsto T_s^t$  is in  $C^1(I_0; C^{k-1}(\overline{D}, R^N))$ , thus

$$\int_D \nabla f \cdot \mathbf{Z} dx = 0, \forall f \in \mathcal{D}(D).$$

or equivalently  $\operatorname{div} \mathbf{Z} = 0$  in  $\mathcal{D}'(D)$ . □

### 2.3 Explicit expression of $\mathbf{Z}(t).n_{\Omega_t(V)}$

In the sequel, we consider a set  $\Omega$  of class  $C^k$  ( $k$  is kept greater than 1). As we will see in the applications, the expression of the Eulerian derivative depends on  $\mathbf{Z}$ . Hence, it seems to be necessary to introduce two adjoint states. One associated to the state equation and the other to the field  $\mathbf{Z}$ . For a functional defined on a tube  $Q_V$  (with the initial domain  $\Omega$ ), this unusual situation might be avoided by considering the function  $\mathbf{z}$  defined by

$$\mathbf{z}(t) = (\mathbf{Z}(t).n_t) \circ T_t(V) \text{ on } (0, \tau) \times \Gamma \quad (\Gamma = \partial\Omega).$$

#### Lemme 4

The mapping  $t \mapsto T_t(V)$  is in  $C^1([0, \tau]; C^k(\overline{D}, R^N))$ . Therefore

$$t \mapsto n_t \circ T_t = \frac{*(DT_t)^{-1} n}{\|*(DT_t)^{-1} n\|} \text{ is in } C^1([0, \tau]; C^{k-1}(\Gamma))$$

$n$  and  $n_t$  are the outward normal fields respectively to  $\Omega$  and  $\Omega_t(V)$ , on  $\Gamma$  and  $\Gamma_t$ . Its derivative is given by :

$$\partial_t(n_t \circ T_t) = \langle DV.n_t, n_t \rangle \circ T_t n_t \circ T_t - *DV \circ T_t n_t \circ T_t.$$

#### Proposition 4

The function  $\mathbf{z} \in C^1([0, \tau]; C^{k-1}(\Gamma))$  is the unique solution of

$$\partial_t \mathbf{z}(t) - \alpha(t) \circ T_t(V) \mathbf{z}(t) = \beta(t) \circ T_t(V) \text{ on } (0, \tau) \times \Gamma \quad (12)$$

$$\mathbf{z}(0) = 0 \text{ on } \Gamma \quad (13)$$

where  $\alpha(t) = \langle DV.n_t, n_t \rangle$ ,  $\beta(t) = W(t).n_t$ .

**Proof.** The mapping  $t \mapsto \mathbf{Z}(t, T_t) = S(t)$  is differentiable. Thus

$$\begin{aligned} \partial_t [\mathbf{Z}(t, T_t).n_t \circ T_t] &= \partial_t(\mathbf{Z} \circ T_t).n_t \circ T_t + \mathbf{Z}(t, T_t).\partial_t(n_t \circ T_t) \\ &= \langle (W(t) + DV(t).Z(t)) \circ T_t, n_t \circ T_t \rangle + \langle DV.n_t, n_t \rangle \circ T_t (\mathbf{Z}(t).n_t) \circ T_t \\ &\quad - \langle (DV.\mathbf{Z}) \circ T_t, n_t \circ T_t \rangle \\ &= (W(t).n_t) \circ T_t + \langle DV.n_t, n_t \rangle \circ T_t (\mathbf{Z}(t).n_t) \circ T_t. \end{aligned}$$

Eventually the desired result is obtained. □

**Remark 2**

Expression of  $\mathbf{z}$  in terms of the data :

From (12) we deduce that

$$\begin{aligned} \partial_t[\mathbf{z} \exp(\int_0^t \alpha(s) \circ T_s ds)] &= [\alpha(t) \circ T_t \mathbf{z} + \partial_t \mathbf{z}] \exp(\int_0^t \alpha(s) \circ T_s ds) \\ &= \beta(t) \circ T_t \exp(\int_0^t \alpha(s) \circ T_s ds). \end{aligned}$$

Hence  $\mathbf{z}$  can be expressed as follows

$$\begin{aligned} \mathbf{z}(t) &= \int_0^t \beta(s) \circ T_s(V) \exp(-\int_s^t \alpha(r) \circ T_r(V) dr) ds. \quad (14) \\ &= \int_0^t [W(s).n_s] \circ T_s(V) \exp(-\int_s^t \langle DV(r)n_r, n_r \rangle \circ T_r(V) dr) ds. \end{aligned}$$

**2.4 Adjoint problem associated to  $\mathbf{Z}$** 

As shown in the proof of Lemma 3 the solution of (8)-(9) is obtained via a change of variable. Then if  $H(D)$  is a Banach space of functions defined on  $D$ , stable by multiplication by functions in  $C^{k-1}(\overline{D})$ , the same process generates the solution of the adjoint problem associated to  $\mathbf{Z}$ .

**Theorem 2**

Let  $F \in L^2((0, \tau); H(D))$ . There exists a unique  $\Lambda \in C([0, \tau]; H(D))$  such that  $\partial_t \Lambda + D\Lambda.V \in L^2((0, \tau); H(D))$  solution of

$$-\partial_t \Lambda - D\Lambda.V - {}^*DV.\Lambda - (\operatorname{div}V)\Lambda = F \quad (15)$$

$$\Lambda(\tau) = 0. \quad (16)$$

**Proof.** Consider  $\theta \in C^1([0, \tau]; H(D))$  the unique solution of the backward problem

$$\begin{aligned} -\partial_t \theta - [{}^*(DV) \circ T_t + (\operatorname{div}V) \circ T_t] \theta &= F \circ T_t \\ \theta(\tau) &= 0. \end{aligned}$$

Applying  $\exp \int_0^t [({}^*DV(s)) \circ T_s + (\operatorname{div}V(s)) \circ T_s \mathbf{I}] ds$ , we get

$$\begin{aligned} -\partial_t \left[ \exp \left\{ \int_0^t [{}^*(DV(s)) \circ T_s + (\operatorname{div}V(s)) \circ T_s \mathbf{I}] ds \right\} \theta(t) \right] &= \\ \exp \left\{ \int_0^t [{}^*(DV(s)) \circ T_s + (\operatorname{div}V(s)) \circ T_s \mathbf{I}] ds \right\} F \circ T_t. \end{aligned}$$



By integration we deduce an explicit expression of  $\theta$  :

$$\begin{aligned}\theta(t) &= \int_t^\tau \exp\left\{-\int_s^t [{}^*DV(\xi) \circ T_\xi + (\operatorname{div}V(\xi)) \circ T_\xi \mathbf{I}] d\xi\right\} F(s) \circ T_s ds \\ &= \int_t^\tau {}^*(DT_t)^{-1} (DT_s) F(s) \circ T_s \gamma(s) \gamma(t)^{-1} ds\end{aligned}$$

Then taking for  $\Lambda = \theta \circ T_t^{-1}$ , it is easy to see that  $\Lambda$  is the unique solution of (15)-(16) which, for a suitable right-hand term, will represent the adjoint problem associated to  $\mathbf{Z}$  associated to a given cost functional.

□

## 2.5 A right-hand term supported by $\Sigma(V)$

Let  $f \in L^2(\Sigma(V))$  and assume that the mapping  $t \mapsto \gamma_{\Gamma_t}^*(f(t)n_t)$  belongs to  $L^2((0, \tau), H(D))$ . We proved in Theorem 2 the existence of a unique  $\Lambda \in C([0, \tau], H(D))$  such that

$$\begin{aligned}-\partial_t \Lambda - D\Lambda.V - {}^*DV.\Lambda - \operatorname{div}V \Lambda &= \gamma_{\Gamma_t}^*(f(t)n_t) \\ \Lambda(\tau) &= 0.\end{aligned}\tag{17}$$

We shall prove that the solution  $\Lambda$  is, in fact, supported by the lateral boundary  $\Sigma(V)$  since the right-hand term in this problem is itself supported by  $\Sigma(V)$  and there is no diffusion term.

### Lemme 5

Let  $f \in L^2(0, \tau; L^2(\Gamma_t))$ . There exists a unique solution in  $C([0, \tau]; L^2(\Gamma_t))$  such that  $\partial_t \lambda + \nabla_{\Gamma_t} \lambda.V \in L^2(0, \tau; L^2(\Gamma_t))$  of the following problem

$$\begin{cases} \partial_t \lambda(t) + \nabla_{\Gamma_t} \lambda.V + \lambda \operatorname{div}V &= f(t) \text{ on } \cup_t (\{t\} \times \Gamma_t) \\ \lambda(\tau) &= 0 \text{ on } \Gamma_\tau \end{cases}$$

**Proof.** Notice that

$$[\partial_t \lambda(t) + \nabla_{\Gamma_t} \lambda.V]_{|\Gamma_t} \circ T_t = \partial_t (\lambda \circ T_t)_{|\Gamma}$$

and consider  $\mu \in C([0, \tau]; L^2(\Gamma))$  the unique solution of

$$\begin{aligned}\partial_t \mu + (\operatorname{div}V) \circ T_t \mu &= f(t) \circ T_t \text{ on } (0, \tau) \times \Gamma \\ \mu(\tau) &= 0 \text{ on } \Gamma.\end{aligned}$$

which can be expressed as :

$$\begin{aligned}\mu(t) &= -\int_t^\tau \exp\left\{\int_t^s (\operatorname{div}V) \circ T_r dr\right\} f(s) \circ T_s ds \\ &= -\int_t^\tau \det DT_s \cdot (\det DT_t)^{-1} f(s) \circ T_s ds\end{aligned}$$

Then  $\lambda$ , defined by  $\lambda(t) = \mu(t) \circ T_t^{-1}$ , is solution of the considered problem. Uniqueness is obvious. □

**Theorem 3**

Let  $f \in L^2(0, \tau; L^2(\Gamma_t))$ . The solution  $\Lambda$  of (17) is supported by  $\Sigma(V)$ . Precisely

$$\Lambda(t) = -\gamma_{\Gamma_t}^*(\lambda(t)n_t), \quad t \in (0, \tau). \quad (18)$$

where  $\lambda$  is defined in Lemma 5.

For further properties of  $p(\cdot)$  see [2] or [5].

**Proof.** Set  $\mathbf{X}(t) = -\gamma_{\Gamma_t}^*(\lambda(t)n_t) (\in H^{-1}(D, R^N))$ . We should identify the distribution

$$-\partial_t \mathbf{X} - D\mathbf{X}.V - {}^*DV.\mathbf{X} - (\operatorname{div}V) \mathbf{X}.$$

For that let  $\varphi \in \mathcal{D}((0, \tau) \times D)$ , thus

$$\begin{aligned} & \langle -\partial_t \mathbf{X} - D\mathbf{X}.V - {}^*DV.\mathbf{X} - (\operatorname{div}V) \mathbf{X}, \varphi \rangle_{\mathcal{D}'((0, \tau) \times D), \mathcal{D}((0, \tau) \times D)} \\ &= - \int_0^\tau \int_{\Gamma_t} \lambda(t) \langle \partial_t \varphi, n_t \rangle d\Gamma_t dt + \int_0^\tau \int_{\Gamma_t} \lambda(t) \langle -D\varphi.V + DV.\varphi, n_t \rangle d\Gamma_t dt \end{aligned}$$

The first term  $E_1 = - \int_0^\tau \int_{\Gamma_t} \lambda (\partial_t \varphi).n_t d\Gamma_t dt$  is treated as follows : Using the transformation  $T_t(V)$

$$\begin{aligned} E_1 &= - \int_0^\tau \int_{\Gamma} \lambda \circ T_t [(\partial_t \varphi) \circ T_t].n_t \circ T_t \omega(t) d\Gamma dt, \quad \omega(t) = \det(DT_t) \| {}^*DT_t^{-1}.n \|_{R^N} \\ &= - \int_0^\tau \int_{\Gamma} \langle \partial_t(\varphi \circ T_t) - (D\varphi.V) \circ T_t, n_t \circ T_t \rangle \lambda \circ T_t \omega(t) d\Gamma dt \\ &= \int_0^\tau \int_{\Gamma} \langle \varphi \circ T_t, n_t \circ T_t \rangle \partial_t(\lambda \circ T_t) \omega(t) + \langle \varphi \circ T_t, \partial_t(\omega(t) n_t \circ T_t) \rangle \lambda \circ T_t d\Gamma dt \\ &+ \int_0^\tau \int_{\Gamma} \langle (D\varphi.V) \circ T_t, n_t \circ T_t \rangle \lambda \circ T_t \omega(t) d\Gamma dt. \end{aligned}$$

But  $\omega(t)n_t \circ T_t = \gamma(t) * DT_t^{-1}n$ ; where  $\gamma(t) = \det DT_t$  so

$$\begin{aligned}
E_1 &= \int_0^\tau \int_\Gamma \partial_t(\lambda \circ T_t) \langle \varphi \circ T_t, n_t \circ T_t \rangle \omega(t) d\Gamma dt \\
&+ \int_0^\tau \int_\Gamma \langle \varphi \circ T_t, \partial_t(\gamma(t) * DT_t^{-1}n) \rangle \lambda \circ T_t d\Gamma dt, \\
&+ \int_0^\tau \int_{\Gamma_t} \lambda \langle D\varphi.V, n_t \rangle d\Gamma_t dt, \\
&= \int_0^\tau \int_\Gamma [\partial_t \lambda + \nabla_{\Gamma_t} \lambda.V] \circ T_t \langle \omega(t) n_t \circ T_t, \varphi \circ T_t \rangle d\Gamma dt \\
&+ \int_0^\tau \int_\Gamma \lambda \circ T_t \langle \varphi \circ T_t, \partial_t(*DT_t^{-1}n) \rangle \gamma(t) d\Gamma dt \\
&+ \int_0^\tau \int_\Gamma \lambda \circ T_t \langle \varphi \circ T_t, \partial_t(\gamma(t)) * DT_t^{-1}n \rangle d\Gamma dt + \int_0^\tau \int_{\Gamma_t} \lambda \langle D\varphi.V, n_t \rangle d\Gamma_t dt \\
&= \int_0^\tau \int_{\Gamma_t} [\partial_t \lambda + \nabla_{\Gamma_t} \lambda.V] \varphi.n_t d\Gamma_t dt + \int_0^\tau \int_{\Gamma_t} \langle D\varphi.V, n_t \rangle \lambda d\Gamma_t dt \\
&+ \int_0^\tau \int_\Gamma \lambda \circ T_t \langle \varphi \circ T_t, n_t \circ T_t \rangle \omega(t) (\operatorname{div} V) \circ T_t d\Gamma dt \\
&- \int_0^\tau \int_\Gamma \lambda \circ T_t \langle \varphi \circ T_t, *(DV) \circ T_t n_t \circ T_t \rangle \omega(t) d\Gamma dt \\
&= \int_0^\tau \int_{\Gamma_t} (\partial_t \lambda + \nabla_{\Gamma_t} \lambda.V) \varphi.n_t d\Gamma_t dt + \int_0^\tau \int_{\Gamma_t} \lambda \langle D\varphi.V, n_t \rangle d\Gamma_t dt \\
&+ \int_0^\tau \int_{\Gamma_t} \lambda \varphi.n_t \operatorname{div} V d\Gamma_t dt - \int_0^\tau \int_{\Gamma_t} \lambda \langle DV.\varphi, n_t \rangle d\Gamma_t dt.
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
&\langle -\partial_t \mathbf{X} - D\mathbf{X}.V - *DV.\mathbf{X} - \operatorname{div} V \mathbf{X}, \varphi_{\mathcal{D}', \mathcal{D}} \rangle \\
&= \int_0^\tau \int_{\Gamma_t} (\partial_t \lambda + \nabla_{\Gamma_t} \lambda.V + \lambda \operatorname{div} V) \varphi.n_t d\Gamma_t dt = \int_0^\tau \int_{\Gamma_t} f(t) \varphi(t).n_t d\Gamma_t dt
\end{aligned}$$

which is equivalent, in a distribution sense, to

$$-\partial_t \mathbf{X} - D\mathbf{X}.V - *DV.\mathbf{X} - \operatorname{div} V \mathbf{X} = \gamma_{\Gamma_t}^*(f(t)n_t)$$

Moreover it is clear that  $\mathbf{X}(\tau) = 0$ .

From the uniqueness theorem 2, we deduce that  $\Lambda(t) = -\gamma_{\Gamma_t}^*(\lambda(t)n_t)$ .

□

### 3 Derivability with respect to the field

As mentioned before, we are interested in the structure of the Eulerian derivative of non-cylindrical functionals of the following type

$$\left\{ \begin{array}{l} j(V) = \int_{Q(V)} F(t, x, u(V)(t, x)) \, dx dt \\ \text{where } Q(V) = \bigcup_{0 < t < \tau} (\{t\} \times T_t(V)(\Omega)) \end{array} \right.$$

$\tau$  is a non-negative scalar.

$\Omega$  a domain in  $R^N$ .

$F : I(= [0, \tau]) \times D \times R^{N'} \rightarrow R \quad C^1$ .

The function  $u$  is solution of a well-posed non-cylindrical PDE, of order  $2m$ ,  $m \in N^*$ , in  $Q(V)$  :

$$\partial_t u + A(u) = f \quad \text{in } Q(V) \quad (19)$$

$$B_j(u) = g_j \quad \text{on } \Sigma(V), \quad 0 \leq j \leq m-1 \quad (20)$$

$$u(0) = u_0 \quad \text{in } \Omega \quad (21)$$

$A$  is a differential operator of order  $2m$ .

$B_j$  is a boundary differential operator of order  $m_j$  ( $0 \leq m_j \leq 2m-1$ ).

In the sequel, the following notations will be used.

#### Notations 1

Assume the existence of the derivative, at  $s = 0$ , of the mapping  $s \rightarrow u^s(\cdot, \cdot) = u(V + sW)(\cdot, T_s(\mathcal{Z}(s)(\cdot)))$  in  $L^2(I, H^{2m}(\Omega_t(V)))$  for the weak or the strong topology. It is denoted

$$\dot{u}(V; W).$$

Under the same assumption, let

$$u'(V; W) = \dot{u}(V; W) - \partial_x u \cdot \mathbf{Z}.$$

#### Lemme 6

Assume that, for any direction  $W \in \mathcal{C}([0, \tau], \mathcal{D}(D, R^N))$ , the derivative

$\dot{u}(V; W)$  exists in  $L^2(I, H^{2m}(\Omega_t(V)))$  and that

$$u'(V; W) \quad \text{depends linearly on } W.$$

Then the functional  $j(\cdot)$  is Gâteaux differentiable at  $V$  and there exists a time-dependent distribution  $G(V) \in L^1(0, \tau; \mathcal{D}'(D, R^N))$  with  $\text{spt}[G(V)(t)] \subset \overline{\Omega}_t(V)$  such that

$$j'(V; W) = \int_0^\tau \langle G(V)(t), W(t) \rangle_{\mathcal{D}'(D, R^N), \mathcal{D}(D, R^N)} \, dt.$$

**Proof.** In the perturbed tube  $Q(V + sW)$ , the cost functional has the following expression :

$$\begin{aligned} j(V + sW) &= \int_0^\tau \int_{\Omega_t(V+sW)} F(t, x, u(V + sW)(t, x)) dx dt \\ &= \int_0^\tau \int_{T_s(\mathcal{Z}^t)(\Omega_t(V))} F(t, x, u(V + sW)(t, x)) dx dt \\ &= \int_0^\tau \int_{\Omega_t(V)} F(t, T_s(\mathcal{Z}^t)(x), u^s(t, x)) dx dt \end{aligned}$$

Hypothesis *iii*) ensures the existence of

$$\mathbf{j}'(\mathbf{V}; \mathbf{W}) = \frac{d}{ds} \mathbf{j}(\mathbf{V} + s\mathbf{W})|_{s=0}.$$

Precisely we have

$$\begin{aligned} j'(V; W) &= \int_0^\tau \int_{\Omega_t(V)} \partial_x F(t, x, u(t, x)) \cdot \mathbf{Z}(t, x) + \partial_y F(t, x, u(t, x)) \cdot \dot{u}(t, x) \\ &\quad + F(t, x, u(t, x)) \operatorname{div} \mathbf{Z}(t, x) dx dt \end{aligned}$$

which is equivalent to

$$\begin{aligned} j'(V; W) &= \int_0^\tau \langle \partial_y F(t, x, u(t, x)); \dot{u}(t, x) - \partial_x u(t, x) \cdot \mathbf{Z}(t, x) \rangle dt \\ &\quad + \int_0^\tau \int_{\Omega_t(V)} \operatorname{div}[F(t, x, u(t, x)) \mathbf{Z}(t)] dx dt \end{aligned}$$

which we rewrite, under smoothness assumptions, as follows

$$\begin{aligned} j'(V; W) &= \int_0^\tau \int_{\Omega_t(V)} \partial_y F(t, x, u(t, x)) \cdot u'(t, x) dx dt \\ &\quad + \int_0^\tau \int_{\Gamma_t(V)} F(t, x, u(t, x)) \mathbf{Z}(t) \cdot n_t d\Gamma_t dt. \end{aligned} \tag{22}$$

According to assumption *ii*) and the linear dependence of  $\mathbf{Z}$  on  $W$ , we obtain the linear dependence of  $\mathbf{j}'(\mathbf{V}; \mathbf{W})$  on  $W$ .

□

Considering  $\Lambda$  the solution of problem (15)-(16) we can express the boundary integral on  $\mathbf{Z}(t) \cdot n_t$  explicitly in terms of  $W(t) \cdot n_t$ . It is the object of the following lemma.

**Lemme 7**

Let  $F$  be a sufficiently smooth function defined on  $\Sigma(V)$ . Then

$$\int_0^\tau \int_{\Gamma_t(V)} F(t) \mathbf{Z}(t) \cdot n(t) d\Gamma_t dt =$$

$$\int_0^\tau \int_{\Gamma_t(V)} \left\{ \int_t^\tau F(s) \circ T_s(V) \circ T_t(V)^{-1} ds \right\} W(t).n(t) d\Gamma_t dt$$

**Proof.**

$$\begin{aligned} \int_0^\tau \int_{\Gamma_t} F(t) \mathbf{Z}.n_t d\Gamma_t dt &= \int_0^\tau \langle -\partial_t \Lambda - D\Lambda.V - *DV.\Lambda, \mathbf{Z} \rangle dt \\ &= \int_0^\tau \langle \partial_t \mathbf{Z} + D\mathbf{Z}.V - DV.\mathbf{Z}, \Lambda \rangle dt = - \int_0^\tau \int_{\Gamma_t(V)} \lambda(t) W.n_t d\Gamma_t dt \\ &= \int_0^\tau \int_{\Gamma_t} \int_t^\tau F(s) \circ T_s \circ T_t(V)^{-1} W(t).n(t) [\gamma(s)\gamma(t)] \circ T_t(V)^{-1} ds d\Gamma_t dt \end{aligned}$$

□

**Remark 3**

In fact, we can use the explicite expression of  $\mathbf{Z}$  given by (11) to obtain the expression of the integral in terms of  $W$  and by-pass the adjoint problem associated to  $\mathbf{Z}$ .

In the sequel, it will be shown that the eulerian derivative coincides with the shape derivative when the functional depends only on the shape of the tube. First let us define a *tube function* (resp. *tube functional*).

**Lemme 8**

If  $u(V + W) = u(V)$  (resp.  $\mathbf{j}(\mathbf{V} + \mathbf{W}) = \mathbf{j}(\mathbf{V})$ ), for any sufficiently smooth  $V$  and  $W$  s.t.  $W(t).n_{\Omega_t(V)} = 0$  on  $\Sigma(V)$ , then  $u$  (resp.  $\mathbf{j}$ ) depends only on the shape of the considered tube. It is called a *tube function* (resp. *tube functional*).

**Proposition 5**

The hypotheses of Lemma 6 are assumed to be satisfied.

1. If  $u$  depends only on the trace, on the lateral boundary  $\Sigma(V)$ , of the field  $V$ , then the gradient  $G(V)$  is supported on  $\Sigma(V)$  and there exists  $R(V) \in L^1(0, \tau; \mathcal{D}'(\Gamma_t(V)))$  s.t.

$$G(V)(t) = \gamma_{\Gamma_t(V)}^*(R(V)(t))$$

where  $\gamma_{\Gamma_t(V)}^*$  is the adjoint of the trace operator on  $\Gamma_t$ .

2. If  $u$  is a tube function ( so it is denoted  $u(Q_V)$ ), then

$$j'(V; W) = 0 \quad \text{for any } W \text{ s.t. } W(t).n_{\Omega_t(V)} = 0 \text{ in } \Gamma_t(V) \text{ for a.e. } t \in [0, \tau].$$

3. Assume  $\Omega$  of class  $\mathcal{C}^k$ ,  $k \geq 1$  and the linear mapping  $W \mapsto \mathbf{j}(\mathbf{V}; \mathbf{W})$  continuous in  $\mathcal{C}([0, \tau]; C^k(\overline{\mathbf{D}}, R^N))$ . Under the assumptions of Lemma 6 and if  $u$  is a tube function, there exists a time-dependent distribution  $g$ ,  $g(t) \in [\mathcal{D}^{k-1}(\Gamma_t(V))]'$ , such that

$$\mathbf{j}'(\mathbf{V}; \mathbf{W}) = \int_0^\tau \langle \gamma_{\Gamma_t(V)}^*(\mathbf{g}(\mathbf{V})(t)\mathbf{n}_t), \mathbf{W}(t) \rangle_{\mathcal{D}^{k-1}(\mathbf{D})', \mathcal{D}^{k-1}(\mathbf{D})} dt$$

**Proof.**

1. Let  $W$  be a field such that  $W(t) \in \mathcal{D}(D, R^N)$  and  $\text{spt}W(t) \cap \overline{\Omega_t(V)} = \emptyset$  for any  $t \in [0, \tau)$ . Thus  $T_t(V + sW)\Omega = T_t(V)\Omega (= \Omega_t(V))$ . Therefore  $Q(V + sW) = Q(V)$ . On the other hand,  $u(V + sW)(t)|_{\Gamma_t(V)} = u(V)(t)|_{\Gamma_t(V)}$  a.e. in  $[0, \tau]$ . The well-posedness of the PDE satisfied by  $u$  implies its uniqueness. So, in this case, we have  $u(V + sW) = u(V)$  a.e. in  $Q(V)$ . This proves that  $\mathbf{j}'(\mathbf{V}; \mathbf{W}) = \mathbf{0}$ . So  $\text{spt}G(V)(t) \subset \overline{\Omega_t(V)}$  a.e. in  $[0, \tau]$ . By similar arguments and considering vector fields  $W$  such that  $W(t) \in \mathcal{D}(\Omega_t(V), R^N)$  for any  $t \in [0, \tau)$ , we prove that  $\text{spt}G(V)(t) \subset \overline{\Omega_t(V)}^c$  for any  $t \in [0, \tau)$ . Hence, we can conclude that  $\text{spt}G(V)(\cdot) \subset \Sigma(V)$ .
2. Expressing the functional  $\mathbf{j}$  at the ‘‘point’’  $V + sW$ , we obtain

$$j(V + sW) = J(V + sW, Q(V + sW)) = J\left(V + sW, \bigcup_{0 < t < \tau} [\{t\} \times \mathcal{T}_s^t \Omega_t(V)]\right)$$

where  $\mathcal{T}_s^t = T_t(V + sW) \circ T_t(V)^{-1}$ .

The condition  $W(t).n_{\Omega_t(V)} = 0$  for any  $t \in [0, \tau)$  implies that  $T_t(V + sW)\Omega = T_t(V)\Omega (= \Omega_t(V))$ . It comes that  $Q(V + sW) = Q(V)$ . Moreover  $u(V + sW) = u(V)$ , then  $\mathbf{j}(V + sW) = \mathbf{j}(V)$ . Therefore  $\mathbf{j}'(\mathbf{V}; \mathbf{W}) = \mathbf{0}$ .

3. The continuity of the mapping  $W \mapsto \mathbf{j}(\mathbf{V}; \mathbf{W})$  in  $\mathcal{C}([0, \tau]; C^k(\overline{D}, R^N))$  and the fact that the gradient  $G(V)(t) = \gamma_{\Gamma_t(V)}^*(R(V)(t))$ , give that  $R(V) \in L^1(0, \tau; \mathcal{D}^{-k+1}(\Gamma_t(V)))$ . Moreover

$$\begin{aligned} j'(V; W) &= \int_0^\tau \langle R(V)(t), W(t) \rangle_{\mathcal{D}^{-k+1}, \mathcal{D}^{k-1}} dt \\ &= \int_0^\tau \langle R(V)(t), C(t) \rangle dt + \int_0^\tau \langle R(V)(t), (W(t).n_t(V))n_t(V) \rangle dt \\ &= \int_0^\tau \langle R(V)(t), (W(t).n_t(V))n_t(V) \rangle dt \end{aligned}$$

where  $C$  is an admissible vector field such that

$$C(t) = W(t) - (W(t).n_t(V))n_t(V) \text{ on } \Gamma_t(V).$$

We know that  $j'(V; C) = 0$ . So under the specified hypotheses

$$R(V)(t) = \gamma_{n_t}^*(g(V)(t)n_t(V)).$$

where  $\gamma_{n_t}$  is the normal trace.

□

**Remark 4**

1. More generally if  $u$  depends only on the trace, on the lateral boundary  $\Sigma(V)$ , of the field  $V$  and  $R(V) \in L^1(0, \tau; \mathcal{D}^{-k+1}(\Gamma_t(V)))$  then

$$j'(V; W) = \int_0^\tau \langle R(V)(t), (W(t).n_t(V)) n_t(V) \rangle dt + \int_0^\tau \langle R(V)(t), C(t) \rangle dt$$

where the first integral of the right-hand term is the shape derivative and the second one is purely dynamic and is due to the variation of the tangential component of  $W$ .

2. Assume the existence of a vector function  $R$  with  $R(t) \in L^p(\Gamma_t(V), \mathbb{R}^N)$ ,  $p \geq 1$ , such that  $G(V)(t) = \gamma_{\Gamma_t(V)}^* R(t)$ . Then, under the assumptions of Lemma 6 and if the density  $g$  satisfies  $g(t) = R(t).n_t \in L^p(\Gamma_t(V))$ , we have an integral representation for the derivative

$$j'(V; W) = \int_0^\tau \int_{\Gamma_t(V)} g(V)(t) W(t).n_t d\Gamma_t dt.$$

## 4 Newton-Shape Method

We apply the previous tube analysis to a usual shape functional  $J(\Omega)$ . For a given field  $V$ , the domain  $\Omega_t(V)$  is defined for any  $t$  then we get the classical expansion (25) that we treat following the previous tube functional approach. The “initial” domain  $\Omega$  being given as well as the time  $t_0$ , from (25) the functional  $J(\Omega_{t_0}(V))$  turns to be a functional  $j(V)$  of the speed vector field  $V$ ,  $j(V) = J(\Omega_{t_0}(V))$  being defined by the right hand side of (25). Then we apply our previous tube derivative calculus with the use of the transverse field  $Z$  associated to the perturbation field  $W$ . Concerning the shape analysis we adopt now the classical terminologie introduced in [8], [12] concerning the notions of shape derivative (resp. boundary shape derivative) of distributions defined on a moving domain (resp. on a moving boundary or surface). For example the boundary shape derivative of the normal vector field, denoted by  $n'_\Gamma(V)$  is given by  $n'_\Gamma(V) = -\nabla_\Gamma \langle V(0), n \rangle$  see [5].

Assume the family of domains to be smooth enough, say  $\Omega$  of class  $C^k$  with the integer  $k \geq 1$ . We know that for any

$$V \in \mathcal{E}_{0,k} \stackrel{def}{=} C^0([0, \tau], \mathcal{V}_o^k(D))$$

the associated flow and its inverse,  $T(V)$  and  $T(V)^{-1}$ , are elements of  $\mathcal{C}^1([0, \tau], \mathcal{C}^k(\overline{D}, \mathbb{R}^N))$ . Consider a shape functional  $J(\cdot)$  defined on a such family of domains  $\mathcal{A}$  containing the domains

$$\{\Omega_s(Y) = T_s(Y)\Omega; \forall s \in [0, \tau], \forall Y \in \mathcal{V}_o^k(D)\}.$$

Suppose  $J$  shape differentiable on any  $B \in \mathcal{A}$ . We denote by  $dJ(B; Y)$  the shape derivative of  $J$  on any  $B \in \mathcal{A}$  in the direction  $Y$ . If  $B$  is in  $\mathcal{C}^k(\overline{D}, \mathbb{R}^N)$ , there exists a Distribution



$G(B)$  of finite order,  $G(B) \in \mathcal{D}^{k-1}(D, \mathbb{R}^n)'$  such that

$$dJ(B; Y) = \langle G(B), Y \rangle_{\mathcal{D}^{k-1}(D, \mathbb{R}^n)', \mathcal{D}^{k-1}(D, \mathbb{R}^n)}.$$

When  $k = 1$  that gradient is a (vector) measure supported by the boundary.

#### 4.1 expansion

We assume that the gradient is smooth enough in time, ( $s \rightarrow G(\Omega_s(V)) \in L^1(0, \tau, \mathcal{D}'(D, \mathbb{R}^N))$ ), then we obtain the following expansion :

$$j(V) = J(\Omega_{t_0}(V)) = J(\Omega) + \int_0^{t_0} \langle G(\Omega_s(V)), V(s) \rangle ds \quad (23)$$

We say that the shape functional  $J$  is twice shape differentiable (see [10]) if

- The mapping  $\Omega \rightarrow G(\Omega)$  is shape differentiable, that is :
- $\forall V$ , the mapping

$$\begin{aligned} R &\longrightarrow \mathcal{D}^{k-1}(D, \mathbb{R}^N)' \\ s &\longmapsto G(\Omega_s(V)) \end{aligned}$$

is derivable. The shape derivative, which is the derivative at  $s = 0$  is denoted  $G'(\Omega; V)$ .

We assume that the mapping  $Z \rightarrow G'(\Omega_s(V), Z)$  is continuous and then ( see [7], [8] ) depends only on  $V(s)$  then we get  $G'(\Omega_s(V), Z) = G'(\Omega_s(V), Z(s))$ .

Moreover we assume that the mapping  $Z \rightarrow G'(\Omega_s(V), Z)$  is linear and continuous and since  $\partial\Omega_s(V)$  is smooth, from the generic argument developped in [10] we derive that  $G'(\Omega, \cdot)$  will depend on the autonomous field  $V$  only through its normal component on the boundary, that is  $G'(\Omega_s(V)) \cdot Z = \tilde{G}'(\Omega_s(V)) \cdot (z(s))$  where  $z(s) = \langle Z(s), n_s \rangle$ .

We define the associated *Shape Hessian* ( which can be seen as a continuous linear operator)  $\tilde{G}'(\Omega_s(V)) \in \mathcal{L}(\mathcal{D}(\Gamma, \mathbb{R}^N), \mathcal{D}(D, \mathbb{R}^N)')$ .

- We suppose the mapping  $s \mapsto G'(\Omega_s(V))$ , to be continuous.
- The derivative of  $j$  on  $V$  in the direction  $W$  is

$$j'(V; W) = \int_0^{t_0} \langle G'(\Omega_s(V)) \cdot \mathbf{Z}, V \rangle + \langle G(\Omega_s(V)), W \rangle ds$$

From [9] the structure theorem ( [9] [8], [4] ) for gradient, there existe a scalar distribution, called shape density gradient,  $g(B)$  in  $\mathcal{D}^{k-1}(\partial B)'$  such that

$$dJ(B; Y) = \langle g(B); \gamma_{\partial B}(Y) \cdot n \rangle_{\mathcal{D}^{k-1}(\Gamma)', \mathcal{D}^{k-1}(\Gamma)} \quad (24)$$

We introduce the class of Shape differentiable functionals as follows :

##### Definition 1

The shape functional  $J$  is in  $\mathcal{H}^1(\mathcal{A})$  if it is shape differentiable at any  $B \in \mathcal{A}$  and if the density gradient  $g(B)$  of  $J$  at  $B \in \mathcal{A}$  verifies :  $g(B) \in H^{\frac{1}{2}}(\partial B)$ .

Assume  $J$  in  $\mathcal{H}^2(\mathcal{A})$  and consider the following expansion :

$$J(\Omega_{t_0}(V)) = J(\Omega) + \int_0^{t_0} \langle G(\Omega_s(V)), V(s) \rangle ds \quad (25)$$

We are interested in computing the first order optimality condition for the minimization problem

$$\min_V J(\Omega_{t_0}(V)), \quad t_0 \text{ being fixed.}$$

We suppose that,  $\forall V$ , the mapping  $s \mapsto \tilde{G}'(\Omega_s(V))$ , to be continuous. (That is equivalent to say that the mapping  $\Omega \rightarrow G'(\Omega)$  is continuous fom  $\mathcal{A}$  equipped with the *Courant Metric of domains* (see [4]) in  $\mathcal{D}(D, R^N)$ ).

## 4.2 Second derivative with density gradient formulation

The derivative of  $j$  on  $V$  in the direction  $W$  is

$$j'(V; W) = \int_0^{t_0} \langle G'(\Omega_s(V)) \cdot \mathbf{Z}(s), V \rangle + \langle G(\Omega_s(V)), W \rangle ds$$

If we rewrite  $j$  in terms of the density gradient  $g(\Gamma_s(V))$  associated to  $G(\Omega_s(V))$ , using the fact that

$$G(\Omega_s(V)) = \gamma_{\Gamma_s(V)}^*(g(\Gamma_s(V)) n(\Gamma_s(V))) \quad (26)$$

So that

$$j(V) = J(\Omega) + \int_0^{t_0} \int_{\Gamma_s(V)} g(\Gamma_s(V)) \langle V(s), n(\Gamma_s(V)) \rangle d\Gamma_s ds$$

thus the derivative of  $j$  at a point  $V$ , in the direction  $W$ , has a more explicit expression.

### Proposition 6

Assume

1. the data sufficiently smooth
2. the shape derivative  $g'(\Gamma_s(V); Y)$  exists for any admissible autonomous direction  $Y$  and any  $s \in [0, t_0]$ .

Then  $\Gamma_s(V)$ ,

$$\begin{aligned} j'(V; W) = & \int_0^{t_0} \int_{\Gamma_s(V)} ( g'(\Gamma_s(V); \mathbf{Z}(s)) \langle V(s), n(\Gamma_s(V)) \rangle + \\ & g(\Gamma_s(V)) \langle W(s), n(\Gamma_s(V)) \rangle + \operatorname{div}_{\Gamma_s(V)}(g(\Gamma_s(V)) V(s)) \langle \mathbf{Z}(s), n_s \rangle + \\ & \langle \nabla_{\Gamma_s}(g(\Gamma_s(V))), V_{\Gamma_s(V)}(s) \rangle \langle \mathbf{Z}(s), n_s \rangle \\ & + g(\Gamma_s(V)) \langle DV(s).n_s, n_s \rangle \langle Z(s), n_s \rangle ) d\Gamma_s ds \end{aligned}$$

and it depends on  $W(s)$  through  $Z(s)$ .

**Proof.** By direct calculation, following derivatives rules and notations ([8], [5]),  $\kappa$  being the mean curvature of the manifold, we get :

$$\begin{aligned} j'(V; W) &= \int_0^{t_0} \int_{\Gamma_s(V)} (g'(\Gamma_s(V); \mathbf{Z}(s)) \langle V(s), n(\Gamma_s(V)) \rangle + \\ &g(\Gamma_s(V)) \langle W(s), n(\Gamma_s(V)) \rangle + g(\Gamma_s(V)) \langle V(s), n'(\Gamma_s(v); Z(s)) \rangle + \\ &\kappa g(\Gamma_s(V)) \langle V(s), n(\Gamma_s(V)) \rangle \langle \mathbf{Z}(s), n(\Gamma_s(V)) \rangle \\ &+ \langle DV(s).n_s, n_s \rangle \langle Z(s), n_s \rangle) d\Gamma_s ds \end{aligned}$$

Now as  $n'(\Gamma_s(v); Z(s)) = -\nabla_{\Gamma_s(v)} \langle Z(s), n_s \rangle$ , using tangential by part integration we obtain (27).  $\square$

We assume the following linearity :  $g'(\Gamma_s(V); \mathbf{Z}(s)) = \tilde{g}'(\Gamma_s(V)). \langle \mathbf{Z}(s), n_s \rangle$ , which is true from the structure derivative theorem ([7], [8]), as soon as  $g'(\Gamma_s(V); \mathbf{Z}(s))$  is linear and continuous with respect to  $Z$ , together with the regularity of the boundary. The first term can be rewritten as :

$$\begin{aligned} &\int_0^{t_0} \int_{\Gamma_s(V)} (g'(\Gamma_s(V); \mathbf{Z}(s)) \langle V(s), n(\Gamma_s(V)) \rangle) d\Gamma_s ds \\ &= \int_0^{t_0} \int_{\Gamma_s(V)} (\tilde{g}'(\Gamma_s(V))^* \cdot \langle V(s), n(\Gamma_s(V)) \rangle \langle \mathbf{Z}(s), n(\Gamma_s(V)) \rangle) d\Gamma_s ds \end{aligned}$$

Let

$$F(s) = \tilde{g}'(\Gamma_s(V))^* \cdot \langle V(s), n(\Gamma_s(V)) \rangle + g(\Gamma_s(V)) \operatorname{div} V(s) + \langle \nabla_{\Gamma_s(V)} g(\Gamma_s(V)), V(s) \rangle$$

Then

$$j'(V; W) = \int_0^{t_0} \int_{\Gamma_s(V)} F(s) \mathbf{Z}(s) \cdot n(s) + g(\Gamma_s(V)) W(s) \cdot n(s) d\Gamma_s ds$$

Finally we derive the result concerning the necessary Optimality condition for  $j(V) = J(\Omega_{t_0}(V))$  :

### Proposition 7

The necessary optimality condition associated to considered minimization problem is :

$$a.e.s \text{ in } (0, t_0) \left\{ \int_s^{t_0} F(r) \circ T_r(V) \circ T_s(V)^{-1} dr \right\} + g(\Gamma_s(V)) = 0 \quad a.e. \text{ in } \Gamma_s(V)$$

The proof is based on Lemma 7. We can also use the adjoint problem solution  $\lambda$  and derive explicit expression for the gradient of the functional  $j$  in order to derive a second order descent method for the shape functional  $J$ . From 23 we get

$$\int_0^\tau \int_{\Gamma_t} F(t) \mathbf{Z} \cdot n_t d\Gamma_t dt = - \int_0^\tau \int_{\Gamma_t(V)} \lambda(t) W \cdot n_t d\Gamma_t dt$$

So that :

$$j'(V; W) = \int_0^{t_0} \int_{\Gamma_s(V)} ( \lambda(s) + g(\Gamma_s(V)) ) \langle W(s), n(s) \rangle d\Gamma ds$$

Where  $\lambda$  solves the backward problem :

$$\partial_t \lambda(t) + \nabla_{\Gamma_t} \lambda \cdot V + \lambda \operatorname{div} V = F(t) \text{ on } \cup_t (\{t\} \times \Gamma_t), \quad \lambda(t_0) = 0 \text{ on } \Gamma_{t_0}$$

### 4.3 Example

Finally, let us consider the following functional and the associated minimization over  $V$  :

$$j(V) = J(\Omega) - \frac{1}{2} \int_0^{t_0} \int_{\Gamma_t(V)} \left( \frac{\partial y_t}{\partial n_t} \right)^2 \langle V(t), n_t \rangle d\Gamma dt$$

The eulerian derivative of  $j$  in the direction  $W$  is given by :

$$\begin{aligned} j'(V; W) = & -\frac{1}{2} \int_0^{t_0} \int_{\Gamma_t(V)} \left( [2 \left( \frac{\partial y_t}{\partial n_t} \right)^2 \langle V(t), n_t \rangle + \operatorname{div} (|\nabla y_t(V)|^2 V)] \langle \mathbf{Z}, n_t \rangle \right. \\ & \left. + \left( \frac{\partial y_t}{\partial n_t} \right)^2 \langle W, n_t \rangle \right) d\Gamma dt \end{aligned}$$

So the necessary optimality condition would be the following :

$$\int_t^{t_0} [2 \left( \frac{\partial y_s}{\partial n_s} \right)^2 \langle V(s), n_s \rangle + \operatorname{div} (|\nabla y_s(V)|^2 V(s))] \circ T_s(V) \circ T_t^{-1}(V) ds + \left( \frac{\partial y_t}{\partial n_t} \right)^2 = 0$$

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