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*Reachability problems for products of matrices
in semirings*

Stéphane Gaubert — Ricardo Katz

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Reachability problems for products of matrices in semirings

Stéphane Gaubert , Ricardo Katz

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Abstract: We consider the following matrix reachability problem: given r square matrices with entries in a semiring, is there a product of these matrices which attains a prescribed matrix? We define similarly the vector (resp. scalar) reachability problem, by requiring that the matrix product, acting by right multiplication on a prescribed row vector, gives another prescribed row vector (resp. when multiplied at left and right by prescribed row and column vectors, gives a prescribed scalar). We show that over any semiring, scalar reachability reduces to vector reachability which is equivalent to matrix reachability, and that for any of these problems, the specialization to any $r \geq 2$ is equivalent to the specialization to $r = 2$. As an application of this result and of a theorem of Krob, we show that when $r = 2$, the vector and matrix reachability problems are undecidable over the max-plus semiring $(\mathbb{Z} \cup \{-\infty\}, \max, +)$. We also show that the matrix, vector, and scalar reachability problems are decidable over semirings whose elements are “positive”, like the tropical semiring $(\mathbb{N} \cup \{+\infty\}, \min, +)$.

Key-words: Semigroup membership problem, orbit problem, matrix semigroups, projective linear semigroups, mortality, reachability, reduction, undecidability, max-plus algebra, tropical semiring

Problèmes d'accessibilité pour les produits de matrices dans les semi-anneaux

Résumé : Nous considérons le problème d'accessibilité matricielle suivant: étant données r matrices carrées à coefficients dans un semi-anneau, existe-t-il un produit de ces matrices qui atteint une matrice prescrite? On définit de même les problèmes d'accessibilité vectorielle (resp. scalaire), en exigeant que le produit de matrices, multiplié à gauche par un vecteur ligne prescrit, atteigne un autre vecteur ligne prescrit (resp. multiplié à gauche et à droite par des vecteurs ligne et colonne prescrits, atteigne un scalaire prescrit). Nous montrons que dans un semi-anneau arbitraire, le problème de l'accessibilité scalaire se réduit au problème de l'accessibilité vectorielle, qui est équivalent au problème de l'accessibilité matricielle, et que pour chacun de ces problèmes, le cas particulier obtenu en fixant une valeur $r \geq 2$ quelconque est équivalent au cas particulier $r = 2$. Nous déduisons de ce résultat ainsi que d'un théorème de Krob que pour $r \geq 2$, les problèmes d'accessibilité matricielle et vectorielle sont indécidable dans le semi-anneau max-plus $(\mathbb{Z} \cup \{-\infty\}, \max, +)$. Nous montrons aussi que ces problèmes deviennent décidables dans des semi-anneaux dont les éléments sont "positifs", comme le semi-anneau tropical $(\mathbb{N} \cup \{+\infty\}, \min, +)$.

Mots-clés : Problèmes d'accessibilité, semigroupe de matrices, orbites, semigroupes linéaires projectifs, mortalité, réduction, indécidabilité, algèbre max-plus, semi-anneau tropical

1. INTRODUCTION AND STATEMENT OF RESULTS

We consider the following problem:

Problem 1 (Matrix reachability). *Given $n \times n$ matrices A_1, \dots, A_r and M with entries in a semiring \mathcal{S} , is there a finite sequence $1 \leq i_1, \dots, i_k \leq r$ such that $A_{i_1} \cdots A_{i_k} = M$?*

(Let us recall that a semiring is a set \mathcal{S} equipped with an addition and a multiplication, such that: \mathcal{S} is a commutative monoid for addition, \mathcal{S} is a monoid for multiplication, multiplication left and right distributes over addition, and the zero element for addition is left and right absorbing for multiplication.) The matrix reachability problem, which asks whether M belongs to the semigroup generated by A_1, \dots, A_r , may be called more classically the *semigroup membership* problem. We chose our terminology to show the interplay with the two following problems:

Problem 2 (Vector reachability). *Given $n \times n$ matrices A_1, \dots, A_r and two $1 \times n$ matrices α, η , all with entries in a semiring \mathcal{S} , is there a finite sequence $1 \leq i_1, \dots, i_k \leq r$ such that $\alpha A_{i_1} \cdots A_{i_k} = \eta$?*

Problem 3 (Scalar reachability). *Given $n \times n$ matrices A_1, \dots, A_r , a $1 \times n$ matrix α , a $n \times 1$ matrix β , all with entries in a semiring \mathcal{S} , and a scalar $\gamma \in \mathcal{S}$, is there a finite sequence $1 \leq i_1, \dots, i_k \leq r$ such that $\alpha A_{i_1} \cdots A_{i_k} \beta = \gamma$?*

When M is the zero matrix, the matrix reachability problem is the well studied *mortality problem*. Paterson [Pat70] proved that when $\mathcal{S} = (\mathbb{Z}, +, \times)$ is the ring of integers, the mortality problem is undecidable, even when $n = 3$ and $r = 2n_{\text{P}} + 2$, where n_{P} is the minimal number of pairs of words for which Post's correspondence problem is undecidable (Matiyasevich and Sénizergues [MS96] proved that $n_{\text{P}} \leq 7$). Bournez and Branicky [BB02, Prop. 1] proved that the mortality problem remains undecidable when $n = 3$ and $r = n_{\text{P}} + 2$, and Halava and Harju [HH01] proved that the mortality problem remains undecidable even when $n = 3$ and $r = n_{\text{P}} + 1$. See Harju and Karhumäki [HK97], Blondel and Tsitsiklis [BT00], and Halava and Harju [HH01] for overviews. Blondel and Tsitsiklis [BT97], and independently, Cassaigne and Karhumäki [CK98], proved that the mortality problem for r matrices of dimension n reduces to the mortality problem for 2 matrices of dimension nr , which implies that there is an integer n_{mo} such that the mortality problem for two matrices of dimension n_{mo} is undecidable (it follows from [HH01] that one can take $n_{\text{mo}} = 3(n_{\text{P}} + 1)$).

The scalar reachability problem previously appeared in the literature in the following form:

Problem 4 (Corner reachability). *Given $n \times n$ matrices A_1, \dots, A_r with entries in \mathcal{S} , and a scalar $\gamma \in \mathcal{S}$, is there a finite sequence $1 \leq i_1, \dots, i_k \leq r$ such that $(A_{i_1} \cdots A_{i_k})_{1n} = \gamma$?*

When γ is zero, this becomes the *zero corner problem* [Man74, HK97, CK98], which is undecidable over $(\mathbb{Z}, +, \times)$ when $n = 3$ and $r = n_{\text{P}}$ [Man74], and also when $r = 2$ and $n = 3n_{\text{P}} + 3$ [CK98, Theorem 2 and § 2.3]. An easy observation ((20) below) shows that the scalar and corner reachability problems are essentially equivalent.

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In this paper, we will show that over any semiring, matrix reachability is a problem equivalent to vector reachability, which is harder than scalar reachability, and we will also show that for $r \geq 2$, the r -generators version of each of these problems is equivalent to its 2-generators variant. To formalize what “harder” and “equivalent” means, we have to define the notion of reduction. We shall assume that the elements of the semiring \mathcal{S} are represented in some effective way, and that we have oracles taking the representations of two elements $a, b \in \mathcal{S}$ as input and returning representations of the sum of a and b , of the product of a and b , together with the truth value $a = b$, as output. Then, we say that a problem P' *reduces* to a problem P , and we write $P' \rightarrow P$, if there is an algorithm solving problem P' , using an oracle solving Problem P together with the oracles computing the sum, the product, and checking the equality in \mathcal{S} . The notion we just defined is a special case of *Turing reduction* [HU79] with respect to oracles. We shall also say that P and P' are *equivalent*, and write $P \leftrightarrow P'$, if P reduces to P' and P' reduces to P .

To state more precise results, we need to introduce restricted versions of the above problems. Thus, $\text{MREACH}(r, n)$ will denote the specialization of the matrix reachability problem to r generators of dimension n and $\text{MREACH}(r)$ will denote the specialization of the matrix reachability problem to r generators (of arbitrary dimension). We will use a similar notation for the vector, scalar, and corner reachability problems, whose r, n specializations will be denoted by $\text{VREACH}(r, n)$, $\text{SREACH}(r, n)$, $\text{CREACH}(r, n)$, etc. The following theorem is proved in §4.

Theorem 1. *In an arbitrary semiring,*

- (1) $\text{SREACH}(r, n)$ reduces to $\text{SREACH}(2, rn)$
- (2) $\text{VREACH}(r, n)$ reduces to $\text{VREACH}(2, rn)$
- (3) $\text{MREACH}(r, n)$ reduces to $\text{MREACH}(2, rn)$
- (4) $\text{VREACH}(r, n)$ reduces to $\text{MREACH}(r + 1, k)$
where $k = n + 1$ if $\eta \neq 0$, and $k = n + 3$ otherwise
- (5) $\text{SREACH}(r, n)$ reduces to $\text{VREACH}(r + 1, k)$
where $k = n + 1$ if $\gamma \neq 0$, and $k = n + 3$ otherwise
- (6) $\text{SREACH}(r, n)$ reduces to $\text{MREACH}(r + 1, k)$
where $k = n + 2$ if $\gamma \neq 0$, and $k = n + 5$ otherwise.

Moreover, the value of γ is preserved in Reduction (1), whereas in Reductions (2) and (3), the zero or non zero character of η or M is preserved.

Additionnaly, V. Blondel [Blo02] observed that in an arbitrary semiring,

- (7) $\text{MREACH}(r, n)$ reduces to $\text{VREACH}(r, rn)$.

For completeness, we reproduce the (simple) proof in §4.7. As an immediate corollary of Theorem 1 and Reduction (7), we get:

Corollary 1. *In an arbitrary semiring, for all $r, r', r'', r''' \geq 2$, the scalar reachability problem for r matrices is equivalent to the scalar reachability problem for r' matrices, which reduces to the vector reachability problem for r'' matrices, which is equivalent to the matrix reachability problem for r''' matrices.*

All the reductions in the proofs of the present paper take a polynomial time (but the problems should not be expected to be polynomial, except in very special cases).

It would be surprising if the reduction $\text{SREACH}(r) \rightarrow \text{MREACH}(r+1)$ stated in (6) could be improved to give $\text{SREACH}(r) \rightarrow \text{MREACH}(r)$. Indeed, when $\mathcal{S} = \mathbb{Z}$, $\text{SREACH}(1)$ is equivalent to the Pisot problem, a well known unsolved problem consisting in deciding the existence of a zero in an integer linear recurrent sequence, whereas the matrix reachability problem $\text{MREACH}(1)$ becomes:

$$\text{given } A, M \in \mathbb{Z}^{n \times n}, \text{ is there some } k \geq 1 \text{ such that } A^k = M ,$$

a much simpler problem, which is even solvable in polynomial time, see [KL86], and also [CLZ00, BBC⁺96]. The vector reachability problem for one matrix, $\text{VREACH}(1)$, which was called the *orbit problem* in [KL86], is also solvable in polynomial time [KL86], so that the existence of a reduction $\text{SREACH}(r) \rightarrow \text{VREACH}(r)$ seems unlikely.

In the statement of Theorem 1, we needed to distinguish the cases where M , η , or γ , are zero. Indeed, the reductions depend critically on the zero or non-zero character of the instance. For instance, the proof of (3) when $M = 0$ follows merely from the argument of Blondel and Tsitsiklis [BT97] and of Cassaigne and Karhumaki [CK98], whereas the $M \neq 0$ case is proved using a very different method (compare §4.3.1 with §4.3.2).

We next derive some consequences of Theorem 1. Let us consider the case when \mathcal{S} is the max-plus semiring $\mathbb{Z}_{\max} = (\mathbb{Z} \cup \{-\infty\}, \max, +)$. In \mathbb{Z}_{\max} , the matrix product is given by

$$(AB)_{ij} = \max_k (A_{ik} + B_{kj}) .$$

In \mathbb{Z}_{\max} , the scalar reachability problem was solved negatively by Krob:

Theorem 2 (See [Kro93]). *For $r = 2$, the scalar reachability problem over the max-plus semiring \mathbb{Z}_{\max} is undecidable.*

In fact, Krob did not make explicit Theorem 2, but we shall see in § 2.5 that Theorem 2 is contained in his proof. Note also that Krob stated the results in the *equatorial semiring* $\mathbb{Z}_{\min} = (\mathbb{Z} \cup \{+\infty\}, \min, +)$, which is effectively isomorphic to \mathbb{Z}_{\max} (by a change of sign), so that decidability issues in \mathbb{Z}_{\max} and \mathbb{Z}_{\min} are equivalent. We get as a corollary of Theorem 2 and of the reductions (5), (6), (2), and (3) in Theorem 1:

Theorem 3. *For $r = 2$, the matrix and vector reachability problems over the max-plus semiring \mathbb{Z}_{\max} are undecidable.*

The $r \geq 2$ bound is optimal, since when $r = 1$, the matrix reachability problem in \mathbb{Z}_{\max} is known to be decidable (see §4.13 below). Moreover, a simple argument shows that for any r , the *mortality* problem in the max-plus semiring is decidable (use the third remark after Theorem 2 in [BT97]).

The proof of Theorem 3 and Krob's proof of Theorem 2 show that the restrictions of the scalar, vector, and matrix reachability problems to matrices of some fixed, sufficiently large, dimension n , remain undecidable. Indeed, Matiyasevich's theorem (see in particular the corollary in the introduction of [DMR76], [Mat93], and the references therein) shows that the Hilbert's tenth problem remains undecidable for a subclass of instances consisting of a family of polynomials of bounded degree, with a fixed number of variables, and one can check that Krob's proof, when applied to this family, yields linear representations of bounded dimension n .

A natural question would be to find an alternative proof which would allow a more precise control of the dimension. The reader should note, here, that the Post-correspondence based technique of Paterson [Pat70], which relies on the embedding of a free monoid with at least two letters into matrices over \mathbb{Z} , has no natural extension to \mathbb{Z}_{\max} . In fact, the possibility of such an extension was considered when the equality problem for max-plus rational series was still open, and it was remarked independently by Krob and by Simon [Sim88], that $\mathbb{Z}_{\max}^{n \times n}$ contains no free submonoid. (To see this, define, for all $A \in \mathbb{Z}_{\max}^{n \times n}$, $n(A) = \sup\{|A_{ij}| \mid 1 \leq i, j \leq n, A_{ij} \neq -\infty\}$, observe that $n(AB) \leq n(A) + n(B)$, and deduce that any finitely generated matrix submonoid of $\mathbb{Z}_{\max}^{n \times n}$ has a growth function $O(k^{n^2})$.)

As easy corollaries of Theorem 3, we will get in §4.10 undecidability results for *projective* variants of the reachability problem. Recall that the proportionality relation \sim on \mathbb{Z}_{\max}^n and $\mathbb{Z}_{\max}^{n \times n}$ is defined by $u \sim v$ if $u = \lambda v$, for some $\lambda \in \mathbb{R}$ (that is, $u_i = \lambda + v_i$ when $u, v \in \mathbb{Z}_{\max}^n$, or $u_{ij} = \lambda + v_{ij}$ when $u, v \in \mathbb{Z}_{\max}^{n \times n}$).

Corollary 2 (Projective matrix reachability over \mathbb{Z}_{\max} is undecidable). *The following problem is undecidable: given $A_1, A_2, M \in \mathbb{Z}_{\max}^{n \times n}$, is there a finite sequence $1 \leq i_1, \dots, i_k \leq 2$ such that $A_{i_1} \cdots A_{i_k} \sim M$?*

Corollary 3 (Projective vector reachability over \mathbb{Z}_{\max} is undecidable). *The following problem is undecidable: given $A_1, A_2 \in \mathbb{Z}_{\max}^{n \times n}$, and $\alpha, \eta \in \mathbb{Z}_{\max}^{1 \times n}$, is there a finite sequence $1 \leq i_1, \dots, i_k \leq 2$ such that $\alpha A_{i_1} \cdots A_{i_k} \sim \eta$?*

Projective reachability problems arise in relation with the problem of determining whether a max-plus rational series is subsequential (i.e. has a deterministic linear representation). A general result which was first understood by Choffrut, see [Cho78, Ch. 3] and [Cho03, Th. 1] for a recent overview, see also [Gau95, Th. 4] and [Moh97, Th. 10], yields a partial-decision algorithm to determine whether a max-plus rational series is subsequential (by "partial decision", we mean that the algorithm need not terminate, even when the series is subsequential). In the case of \mathbb{Z}_{\max} , this algorithm consists in computing the set $\{\alpha A_{i_1} \cdots A_{i_k} \mid k \geq 1, 1 \leq i_1, \dots, i_k \leq r\}$ modulo the equivalence relation \sim . Thus, Corollary 3 shows that it is undecidable whether this algorithm will produce the equivalence class of η . Corollary 3 also has an interesting discrete event systems interpretation. In this context [Gau95, Gau96, GM98, BJJG98, GM99b], the vector $\alpha' := \alpha A_{i_1} \cdots A_{i_k}$ gives the completion time of different events, after the execution of a schedule represented by a sequence i_1, \dots, i_k , and the equivalence class of α' modulo \sim represents inter-event delays ($\alpha'_i - \alpha'_j$ represents typically the time a part stays in a storage resource).

Even, in the classical case $\mathcal{S} = \mathbb{Z}$, Theorem 1 suggests some results. For instance, the result of [Man74, HK97, CK98] showing that the zero corner problem is undecidable over \mathbb{Z} implies that the vector reachability problem is undecidable, and this does not seem to have been stated previously. In fact, we can prove a more precise result by a small modification of the proof of Paterson [Pat70] (see §4.9):

Proposition 1. *The vector mortality problem over \mathbb{Z} , for $n_P + 1$ matrices of dimension 3, is undecidable.*

Theorem 1 also improves by one unit the dimension obtained in Theorem 2 of [CK98] (see §4.8):

Corollary 4. *The zero corner problem for 2 matrices of dimension $3n_P + 2$ over \mathbb{Z} is undecidable.*

It is natural to ask whether the reachability problems become decidable in other semirings. Many variants of \mathbb{Z}_{\max} can be found in the literature. In particular, the following semirings are listed in [Pin98]:

$$\begin{array}{ll} \mathbb{N}_{\min} = (\mathbb{N} \cup \{+\infty\}, \min, +) & \text{Tropical semiring [Sim90]} \\ \mathbb{N}_{\max} = (\mathbb{N} \cup \{-\infty\}, \max, +) & \text{Boreal semiring [Kro93]} \\ \bar{\mathbb{N}}_{\max} = (\mathbb{N} \cup \{\pm\infty\}, \max, +) & \text{Masclé's semiring [Mas86]} \\ \mathcal{L} = (\mathbb{N} \cup \{\omega, +\infty\}, \min, +) & \text{Leung's semiring [Leu91]} \end{array}$$

In $\bar{\mathbb{N}}_{\max}$, $(+\infty) + (-\infty) = (-\infty) + (+\infty) = -\infty$. Leung's semiring \mathcal{L} is the one point compactification of the semiring \mathbb{N}_{\min} equipped with its discrete topology: the minimum is defined with respect to the order $0 < 1 < 2 < \dots < \omega < +\infty$, and the addition of \mathbb{N}_{\min} is completed by $\omega + a = a + \omega = \max(a, \omega)$. The semiring \mathbb{N}_{\max} is a subsemiring of \mathbb{Z}_{\max} , and the map $x \mapsto -x$ is an isomorphism from \mathbb{N}_{\min} to a subsemiring of \mathbb{Z}_{\max} .

To show that the reachability problems are decidable over these semirings we will need the following definitions. We say that a semiring \mathcal{S} is *separated by morphisms of finite image* if for all $\gamma \in \mathcal{S}$, there is a finite semiring \mathcal{S}_γ and a semiring morphism π_γ from \mathcal{S} to \mathcal{S}_γ such that $\pi_\gamma^{-1}(\pi_\gamma(\gamma)) = \{\gamma\}$. We shall say that \mathcal{S} is *effectively separated by morphisms of finite image* if the maps $\gamma \mapsto \mathcal{S}_\gamma$ and $\gamma \mapsto \pi_\gamma$ are effective, in the sense that for any $\gamma \in \mathcal{S}$, we can compute the (finite) set of elements of the semiring \mathcal{S}_γ , together with the addition and multiplication tables of \mathcal{S}_γ , and that we can compute $\pi_\gamma(y)$ for any $y \in \mathcal{S}$. We prove in §4.11:

Theorem 4. *The matrix, vector, and scalar reachability problems are decidable over a semiring that is effectively separated by morphisms of finite image.*

We will show in fact a slightly more precise result (Theorem 5 in §4.11).

Our method applies not only to max-plus type semirings, but also to the semiring of natural numbers, $\mathbb{N} = (\mathbb{N}, +, \times)$, and to its completion, $\bar{\mathbb{N}} = (\mathbb{N} \cup \{+\infty\}, +, \times)$ (in $\bar{\mathbb{N}}$, we adopt the convention $0 \times (+\infty) = (+\infty) \times 0 = 0$). A simple argument, which is given in § 4.12, shows that:

Proposition 2. *The semirings \mathbb{N}_{\min} , \mathbb{N}_{\max} , $\bar{\mathbb{N}}_{\max}$, \mathcal{L} , \mathbb{N} , and $\bar{\mathbb{N}}$, all are effectively separated by morphisms of finite image.*

As a corollary of Theorem 4 and Proposition 2 we get:

Corollary 5. *The matrix, vector, and scalar reachability problems are decidable over the semirings \mathbb{N}_{\min} , \mathbb{N}_{\max} , $\bar{\mathbb{N}}_{\max}$, \mathcal{L} , \mathbb{N} , and $\bar{\mathbb{N}}$.*

When the semiring is \mathbb{N}_{\min} or \mathbb{N}_{\max} , the decidability of the scalar reachability problem was stated by Krob in [Kro94, Proposition 2.2].

We already observed from Theorem 1 that matrix reachability, or equivalently, vector reachability, are harder problems than scalar reachability. However, for all the examples of semirings that we considered, either all problems were undecidable, or they were all decidable. This raises the question of the existence of a semiring with undecidable matrix reachability problem but decidable scalar reachability problem.

Let us finally mention some additional motivation and point out other references. Automata with multiplicities and semigroups of matrices over the tropical semiring have been much studied in connection with decision problems in language theory, see [Sim78, Sim94], [Has82, Has90], [Mas86], [Leu91], [Kro93], and [Pin98] for a survey. Automata with multiplicities over the max-plus semiring and max-plus linear semigroups appear in the modelling of discrete event dynamic systems, see [Gau95, GM98, BJG98, GM99b, GM99a]. General references about max-plus algebra are [BCOQ92, CG79, KM97, GM02]. Some of the present results have been announced in [GK03].

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2. PREPARATION

In this section, we collect preliminary results.

2.1. Reformulation in terms of linear representations. The proof of the results uses rational series and automata notions: we next recall basic definitions. See [BR88] or [Lal79] for more background.

Let $\Sigma_r = \{a_1, \dots, a_r\}$ denote an alphabet with r letters, and let Σ_r^* denote the *free monoid* on Σ_r , that is the set of finite (possibly empty) words with letters in Σ_r . A subset of Σ_r^* is called a *language*. We will also consider the *free semigroup* Σ_r^+ , which is the subsemigroup of Σ_r^* composed of nonempty words. We say that a map $s : \Sigma_r^* \rightarrow \mathcal{S}$ is *recognizable* or *rational* if there exists an integer n , $\alpha \in \mathcal{S}^{1 \times n}$, $\beta \in \mathcal{S}^{n \times 1}$, and a morphism $\mu : \Sigma_r^* \rightarrow \mathcal{S}^{n \times n}$ such that $s(w) = \alpha \mu(w) \beta$ for all $w \in \Sigma_r^*$. We say that (α, μ, β) is a *linear representation* of s , and that n is the *dimension* of the representation. We denote by $\mathcal{S}^{\text{rat}}(\langle \Sigma_r \rangle)$ the set of rational

maps, which are also called *rational series*. Problems 1–4, can be rewritten as:

- (8) $\text{MREACH}(r, n) :$
 μ morphism $\Sigma_r^* \rightarrow \mathcal{S}^{n \times n}, M \in \mathcal{S}^{n \times n}; \exists w \in \Sigma_r^+, \mu(w) = M ?$
- (9) $\text{VREACH}(r, n) :$
 μ morphism $\Sigma_r^* \rightarrow \mathcal{S}^{n \times n}, \alpha, \eta \in \mathcal{S}^{1 \times n}; \exists w \in \Sigma_r^+, \alpha \mu(w) = \eta ?$
- (10) $\text{SREACH}(r, n) :$
 $s \in \mathcal{S}^{\text{rat}} \langle \langle \Sigma_r \rangle \rangle$ with a linear representation of dimension $n, \gamma \in \mathcal{S};$
 $\exists w \in \Sigma_r^+, s(w) = \gamma ?$
- (11) $\text{CREACH}(r, n) :$
 μ morphism $\Sigma_r^* \rightarrow \mathcal{S}^{n \times n}, \gamma \in \mathcal{S}; \exists w \in \Sigma_r^+, \mu_{1,n}(w) = \gamma ?$

2.2. Variants allowing the empty word. We required that the word w belongs to Σ_r^+ and not to Σ_r^* in the formulations (8)–(11), because in the statements of Problems 1–4, we considered sequences i_1, \dots, i_k of length at least 1. However, some simple observations will show that putting $w \in \Sigma_r^*$ or $w \in \Sigma_r^+$ in (8)–(11) is essentially irrelevant.

Let us denote by $\text{MREACH}'(r, n)$, $\text{VREACH}'(r, n)$, $\text{SREACH}'(r, n)$, and $\text{CREACH}'(r, n)$ the variants of the above problems with Σ_r^* instead of Σ_r^+ in (8)–(11). We will denote by $\mathbb{0}$ and $\mathbb{1}$ the zero and unit elements of \mathcal{S} , respectively, by $\mathbb{0}_{pq} \in \mathcal{S}^{p \times q}$ or simply $\mathbb{0}$ the $p \times q$ zero matrix, and by $I_n \in \mathcal{S}^{n \times n}$ or simply I the $n \times n$ identity matrix. Since the cases where $M = I$ or $M = \mathbb{0}$, or $\eta = \mathbb{0}$, or $\gamma = \mathbb{0}$ will sometimes require a special treatment, we will incorporate restrictions about M , η , or γ in the notation, writing for instance $\text{MREACH}(r, n, M \neq I)$ for the restriction of the matrix reachability problem to r generators of dimension n and a matrix M different from the identity.

Lemma 1. *The following reductions hold:*

- (12) $\text{VREACH}(r, n) \leftrightarrow \text{VREACH}'(r, n)$
- (13) $\text{SREACH}(r, n) \leftrightarrow \text{SREACH}'(r, n)$
- (14) $\text{MREACH}(r, n, M \neq I) \leftrightarrow \text{MREACH}'(r, n, M \neq I)$
- (15) $\text{MREACH}(r, n, M = I) \rightarrow \text{MREACH}'(r, n + 1, M \neq I)$
- (16) $\text{CREACH}(r, n \geq 2, \gamma \neq \mathbb{0}) \leftrightarrow \text{CREACH}'(r, n \geq 2, \gamma \neq \mathbb{0}) .$

Proof. Consider an instance of $\text{VREACH}(r, n)$, which consists of $\alpha, \eta \in \mathcal{S}^{1 \times n}$ and a morphism $\mu : \Sigma_r^* \rightarrow \mathcal{S}^{n \times n}$. Since

$$(17) \quad (\exists w \in \Sigma_r^+, \alpha \mu(w) = \eta) \iff (\exists a \in \Sigma_r, \exists z \in \Sigma_r^*, \alpha \mu(a) \mu(z) = \eta) ,$$

$\text{VREACH}(r, n)$ reduces to $\text{VREACH}'(r, n)$. Conversely,

$$(18) \quad (\exists w \in \Sigma_r^*, \alpha \mu(w) = \eta) \iff (\alpha = \eta \text{ or } \exists w \in \Sigma_r^+, \alpha \mu(w) = \eta) ,$$

shows that $\text{VREACH}'(r, n)$ reduces to $\text{VREACH}(r, n)$, which shows (12).

Similarly, consider an instance of $\text{SREACH}(r, n)$, consisting of $s \in \mathcal{S}^{\text{rat}}\langle\langle \Sigma_r \rangle\rangle$ with a linear representation of dimension n , (α, μ, β) , and $\gamma \in \mathcal{S}$. Let $_$ denote the empty word of Σ_r^* , and let $a^{-1}s$ denote the series defined by $a^{-1}s(w) := s(aw)$. Observe that $a^{-1}s$ is recognized by the linear representation $(\alpha\mu(a), \mu, \beta)$, which is still of dimension n . Since

$$(\exists w \in \Sigma_r^+, s(w) = \gamma) \iff (\exists a \in \Sigma_r, \exists w \in \Sigma_r^*, a^{-1}s(w) = \gamma) ,$$

$\text{SREACH}(r, n)$ reduces to $\text{SREACH}'(r, n)$. Conversely,

$$(19) \quad (\exists w \in \Sigma_r^*, s(w) = \gamma) \iff (s(_) = \gamma \text{ or } \exists w \in \Sigma_r^+, s(w) = \gamma) ,$$

shows that $\text{SREACH}'(r, n)$ reduces to $\text{SREACH}(r, n)$, which shows (13).

The problems $\text{MREACH}'(r, n, M \neq I)$ and $\text{MREACH}(r, n, M \neq I)$ are trivially equivalent, because μ sends the empty word to the identity matrix. This shows (14).

Before showing (15), we introduce a notation that we shall use repeatedly in the sequel. If U_1, \dots, U_k are square matrices with entries in a semiring \mathcal{S} , we denote by $\text{diag}(U_1, \dots, U_k)$ the block diagonal matrix whose diagonal blocks are U_1, \dots, U_k . If μ_1, \dots, μ_k are morphisms from a free monoid to matrix monoids, we denote by $\text{diag}(\mu_1, \dots, \mu_k)$ the morphism which sends a word w to $\text{diag}(\mu_1(w), \dots, \mu_k(w))$. For all $1 \leq p$, we denote by $\mathbf{0}_{pp}$ the zero morphism from a free monoid to $\mathcal{S}^{p \times p}$. Let us define now the morphism $\mu' : \Sigma_r^* \rightarrow \mathcal{S}^{(n+1) \times (n+1)}$, $\mu' = \text{diag}(\mu, \mathbf{0}_{11})$. Since $\mu'(_) = I$,

$$(\exists w \in \Sigma_r^+, \mu(w) = I) \iff (\exists w \in \Sigma_r^*, \mu'(w) = \text{diag}(I, \mathbf{0}_{11})) ,$$

which shows (15).

Finally, the problems $\text{CREACH}'(r, n \geq 2, \gamma \neq \mathbf{0})$ and $\text{CREACH}(r, n \geq 2, \gamma \neq \mathbf{0})$ are trivially equivalent, because $\mu(_)_{1n} = \mathbf{0} \neq \gamma$, as soon as $n \geq 2$ and $\gamma \neq \mathbf{0}$. This shows (16). \square

We did not consider the problems $\text{MREACH}'(r, n, M = I)$ and $\text{CREACH}'(r, n \geq 2, \gamma = \mathbf{0})$ in Lemma 1, since the answer to these problems is trivially “yes”.

2.3. Equivalence of corner and scalar reachability. The following elementary reductions show that up to an increase of the dimension of matrices, the corner and scalar reachability problems are equivalent:

$$(20) \quad \text{SREACH}(r, n) \rightarrow \text{CREACH}(r, n+2) ,$$

$$(21) \quad \text{CREACH}(r, n) \rightarrow \text{SREACH}(r, n) .$$

Indeed, consider an instance of $\text{SREACH}(r, n)$, which consists of a series $s \in \mathcal{S}^{\text{rat}}\langle\langle \Sigma_r \rangle\rangle$ with a linear representation (α, μ, β) of dimension n , and a scalar $\gamma \in \mathcal{S}$. We build the morphism $\mu' : \Sigma_r^* \rightarrow \mathcal{S}^{(n+2) \times (n+2)}$ such that

$$(22) \quad \mu'(a_i) = \begin{pmatrix} \mathbf{0}_{11} & \alpha\mu(a_i) & \alpha\mu(a_i)\beta \\ \mathbf{0}_{n1} & \mu(a_i) & \mu(a_i)\beta \\ \mathbf{0}_{11} & \mathbf{0}_{1n} & \mathbf{0}_{11} \end{pmatrix} \quad \forall 1 \leq i \leq r .$$

An immediate induction on the length of w shows that

$$(23) \quad \mu'(w) = \begin{pmatrix} \mathbb{0}_{11} & \alpha\mu(w) & s(w) \\ \mathbb{0}_{n1} & \mu(w) & \mu(w)\beta \\ \mathbb{0}_{11} & \mathbb{0}_{1n} & \mathbb{0}_{11} \end{pmatrix}, \quad \forall w \in \Sigma_r^+.$$

Thus,

$$(24) \quad \forall w \in \Sigma_r^+, \mu'_{1,n+2}(w) = s(w),$$

which shows (20).

Reduction (21) holds because $\text{CREACH}(r, n)$ is merely a special case of $\text{SREACH}(r, n)$. Indeed, consider an instance of the corner reachability problem, consisting of μ as above, and $\gamma \in \mathcal{S}$, and let $\alpha = (\mathbb{1}, \mathbb{0}, \dots, \mathbb{0}) \in \mathcal{S}^{1 \times n}$, and $\beta = (\mathbb{0}, \dots, \mathbb{0}, \mathbb{1})^T \in \mathcal{S}^{n \times 1}$. Then, for all $w \in \Sigma_r^*$, $\mu_{1n}(w) = \alpha\mu(w)\beta$, which shows (21).

2.4. Matrix representation of trim unambiguous automata. We shall use several times the following essentially classical constructions. To any automaton \mathcal{A} over Σ_r with set of states $\{1, \dots, p\}$, and set of initial (resp. final) states I (resp. F), we associate the morphism $\nu_{\mathcal{A}} : \Sigma_r^* \rightarrow \mathcal{S}^{p \times p}$,

$$(25) \quad \forall x \in \Sigma_r, \nu_{\mathcal{A}}(x)_{ij} = \begin{cases} \mathbb{1} & \text{if there is an arrow from } i \text{ to } j \text{ labeled } x \text{ in } \mathcal{A}, \\ \mathbb{0} & \text{otherwise,} \end{cases}$$

the vectors

$$\alpha_{\mathcal{A}} \in \mathcal{S}^{1 \times p}, (\alpha_{\mathcal{A}})_k = \begin{cases} \mathbb{1} & \text{if } k \in I \\ \mathbb{0} & \text{otherwise,} \end{cases} \quad \beta_{\mathcal{A}} \in \mathcal{S}^{p \times 1}, (\beta_{\mathcal{A}})_k = \begin{cases} \mathbb{1} & \text{if } k \in F \\ \mathbb{0} & \text{otherwise,} \end{cases}$$

together with

$$\begin{aligned} M_{\mathcal{A}} &= \{\nu_{\mathcal{A}}(v) \mid v \in \Sigma_r^* \text{ and } [\nu_{\mathcal{A}}(v)]_{i\phi} = \mathbb{1} \text{ for some } i \in I, \phi \in F\}, \\ F_{\mathcal{A}} &= \{\alpha_{\mathcal{A}}\nu_{\mathcal{A}}(v) \mid v \in \Sigma_r^* \text{ and } [\alpha_{\mathcal{A}}\nu_{\mathcal{A}}(v)]_{\phi} = \mathbb{1} \text{ for some } \phi \in F\}. \end{aligned}$$

Recall that \mathcal{A} is *unambiguous* if for all $w \in \Sigma_r^*$, there is at most one path with label w from an input state to an output state, and that \mathcal{A} is *trim* if for all state k , there is a path from some input state to k , and a path from k to some output state.

Lemma 2. *If \mathcal{A} is trim and unambiguous, then $M_{\mathcal{A}}$ and $F_{\mathcal{A}}$ can be effectively computed, and the language L recognized by \mathcal{A} is $\{w \in \Sigma_r^* \mid \nu_{\mathcal{A}}(w) \in M_{\mathcal{A}}\} = \{w \in \Sigma_r^* \mid \alpha_{\mathcal{A}}\nu_{\mathcal{A}}(w) \in F_{\mathcal{A}}\}$.*

Proof. If \mathcal{A} is trim and unambiguous, for all $1 \leq i, j \leq p$ and $w \in \Sigma_r^*$, there is at most one path from i to j with label w . Then, it follows from the well known graph interpretation of the matrix product (see e.g. [Sta98, § 4.7]), that all the matrices $\nu_{\mathcal{A}}(v)$ have $\mathbb{0}, \mathbb{1}$ entries (which implies that $M_{\mathcal{A}}$ is finite and can be effectively computed), and that $L = \nu_{\mathcal{A}}^{-1}(M_{\mathcal{A}})$. The analogous property for $F_{\mathcal{A}}$ is proved in a similar way. \square

2.5. Derivation of Theorem 2 from Krob's proof. Krob considered the following problems for series $s, t \in \mathcal{S}^{\text{rat}} \langle \langle \Sigma_r \rangle \rangle$ and $\mathcal{S} = \mathbb{N}_{\min}, \mathbb{N}_{\max}, \mathbb{Z}_{\min}$:

$$\begin{aligned}
(\text{Equality}) & \quad s, t \in \mathcal{S}^{\text{rat}} \langle \langle \Sigma_r \rangle \rangle; s = t ? \\
(\text{Inequality}) & \quad s, t \in \mathcal{S}^{\text{rat}} \langle \langle \Sigma_r \rangle \rangle; s \leq t ? \\
(\text{Local Inequality}) & \quad s, t \in \mathcal{S}^{\text{rat}} \langle \langle \Sigma_r \rangle \rangle; \exists w \in \Sigma_r^*, s(w) \leq t(w) ? \\
(\text{Local Equality}) & \quad s, t \in \mathcal{S}^{\text{rat}} \langle \langle \Sigma_r \rangle \rangle; \exists w \in \Sigma_r^*, s(w) = t(w) ?
\end{aligned}$$

Corollary 4.3 of [Kro93] shows that all these problems are undecidable when $\mathcal{S} = \mathbb{N}_{\min}$ or $\mathcal{S} = \mathbb{N}_{\max}$, provided that the number of letters r is at least 2. The undecidability of the scalar reachability problem does not follow from this statement, but it does follow from the proof of [Kro93]. Indeed, in § 3 of [Kro93], Krob associates effectively to any instance (I) of Hilbert's tenth problem a rational series denoted by HD , with coefficients in \mathbb{Z}_{\min} , over an alphabet A , with the property that $HD(w) \leq 0$ for all $w \in A^*$ and that there is a word $z \in A^*$ such that $HD(z) = 0$ if, and only if, instance (I) has a solution. Since Hilbert's tenth problem is undecidable, this implies that the scalar reachability problem over the semiring \mathbb{Z}_{\min} is undecidable, when $\gamma = 0$. Moreover, the coding argument given at the beginning of the proof of Theorem 3.1 of [Kro93] associates effectively to HD a rational series $\sigma(HD)$ with coefficients in \mathbb{Z}_{\min} over a two letters alphabet, and this series takes the same finite values as HD . This shows Theorem 2.

3. EMBEDDING MATRIX SEMIGROUPS WITH r GENERATORS IN MATRIX SEMIGROUPS WITH 2 GENERATORS

The proof of Theorem 1 relies on two different embeddings of semigroups of $n \times n$ matrices with r -generators in semigroups of $nr \times nr$ matrices with 2-generators.

3.1. First embedding. Let b, c denote two letters. To any morphism $\mu : \Sigma_r^* \rightarrow \mathcal{S}^{n \times n}$, we associate the morphism $\bar{\mu} : \{b, c\}^* \rightarrow \mathcal{S}^{nr \times nr}$, defined by:

$$(26) \quad \bar{\mu}(b) = \begin{pmatrix} \mu(a_1) & \mathbb{0}_{n, (r-1)n} \\ \vdots & \vdots \\ \mu(a_r) & \mathbb{0}_{n, (r-1)n} \end{pmatrix}, \quad \text{and} \quad \bar{\mu}(c) = \begin{pmatrix} \mathbb{0}_{(r-1)n, n} & I_{(r-1)n} \\ \mathbb{0}_{n, n} & \mathbb{0}_{n, (r-1)n} \end{pmatrix}$$

(recall that I_n denotes the $n \times n$ identity matrix).

Following an usual device, we shall associate to any word of Σ_r^* a word of $\{b, c\}^*$ by way of the coding function $\delta : \Sigma_r^* \rightarrow \{b, c\}^*$,

$$(27) \quad \delta(a_{i_1} \dots a_{i_k}) = c^{i_1-1} b \dots c^{i_k-1} b ,$$

for all $1 \leq i_1, \dots, i_k \leq r$. The function δ is a bijection from Σ_r^* to the language $\delta(\Sigma_r^*) = \{b, cb, \dots, c^{r-1}b\}^*$. The following result can be proved by an immediate induction on k .

Proposition 3. For all $a_{i_1}, \dots, a_{i_k} \in \Sigma_r$,

$$(28) \quad \bar{\mu} \circ \delta(a_{i_1} \dots a_{i_k}) = \begin{pmatrix} \mu(a_{i_1} a_{i_2} \dots a_{i_k}) & \mathbb{0}_{n, (r-1)n} \\ \mu(a_{i_1+1} a_{i_2} \dots a_{i_k}) & \mathbb{0}_{n, (r-1)n} \\ \vdots & \vdots \\ \mu(a_r a_{i_2} \dots a_{i_k}) & \mathbb{0}_{n, (r-1)n} \\ \mathbb{0}_{(i_1-1)n, n} & \mathbb{0}_{(i_1-1)n, (r-1)n} \end{pmatrix}.$$

We shall use in particular the specialization of (28) to $i_1 = r$:

$$(29) \quad \forall z \in \Sigma_r^*, \bar{\mu} \circ \delta(a_r z) = \begin{pmatrix} \mu(a_r z) & \mathbb{0}_{n, (r-1)n} \\ \mathbb{0}_{(r-1)n, n} & \mathbb{0}_{(r-1)n, (r-1)n} \end{pmatrix}.$$

3.2. Second embedding. This embedding is borrowed from the proof of [BT97, Th. 1] and [CK98, Th. 1]. To any morphism $\mu : \Sigma_r^* \rightarrow \mathcal{S}^{n \times n}$, we associate the morphism $\hat{\mu} : \{b, c\}^* \rightarrow \mathcal{S}^{rn \times rn}$:

$$(30) \quad \hat{\mu}(b) = \begin{pmatrix} \mathbb{0}_{(r-1)n, n} & I_{(r-1)n} \\ I_{nn} & \mathbb{0}_{n, (r-1)n} \end{pmatrix}, \quad \hat{\mu}(c) = \text{diag}(\mu(a_1), \dots, \mu(a_r)).$$

To simplify notations, we will use a convention of cyclic indexing of the letters of Σ_r , so that $a_{r+1} = a_1$, $a_{r+2} = a_2$, etc. We shall use the trivial fact that any word $v \in \{b, c\}^*$ can be written (uniquely) as:

$$(31) \quad v = c^{i_1} b \dots c^{i_k} b c^{i_{k+1}}$$

where $k \geq 0$ and $0 \leq i_1, \dots, i_{k+1}$, with the convention that $v = c^{i_{k+1}} = c^{i_1}$ when $k = 0$.

Lemma 3. If $v \in \{b, c\}^*$ is written as in (31), then,

$$(32) \quad \hat{\mu}(v) = \text{diag}[\mu(a_1^{i_1} a_2^{i_2} \dots a_{k+1}^{i_{k+1}}), \dots, \mu(a_r^{i_1} a_{r+1}^{i_2} \dots a_{r+k}^{i_{k+1}})] \hat{\mu}(b^k).$$

For instance, when $r = 3$, (32) states that:

$$\hat{\mu}(c^2 b c^7 b c^9 b c b^2 c^{11}) = \begin{pmatrix} \mu(a_1^2 a_2^7 a_3^9 a_1 a_3^{11}) & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mu(a_2^2 a_3^7 a_1^9 a_2 a_1^{11}) & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mu(a_3^2 a_1^7 a_2^9 a_3 a_2^{11}) \end{pmatrix} \hat{\mu}(b^5).$$

Proof of Lemma 3. Consider the semigroup $\mathcal{D} \subset \mathcal{S}^{rn \times rn}$ of block diagonal matrices with r diagonal blocks of dimension n , together with the group \mathcal{R} generated by the matrix $B := \hat{\mu}(b)$ (B is invertible since $B^{-1} = B^T$). Since $B\mathcal{D}B^{-1} \subset \mathcal{D}$, for all $(D, R), (D', R') \in \mathcal{D} \times \mathcal{R}$, we have

$$(33) \quad DRD'R' = DRD'R^{-1}RR', \text{ where } DRD'R^{-1} \in \mathcal{D}, RR' \in \mathcal{R}.$$

(In other words, the semigroup of matrices of the form DR , where $(D, R) \in \mathcal{D} \times \mathcal{R}$, is a semidirect product of \mathcal{D} by \mathcal{R} .) Then, Formula (32) is proved by an immediate induction, thanks to (33), and to the observation that $D \mapsto B^{-1}DB$ acts on $D \in \mathcal{D}$ by cyclic permutation of diagonal blocks. \square

4. PROOF OF THE RESULTS

4.1. Proof of Reduction (1). Consider an instance of $\text{SREACH}(r, n)$, consisting of a linear representation (α, μ, β) of dimension n over \mathcal{S} , together with $\gamma \in \mathcal{S}$. Define the morphism $\hat{\mu}$ as in (30), together with

$$(34) \quad \alpha' = (\alpha, \mathbf{0}_{1n}, \dots, \mathbf{0}_{1n}) \in \mathcal{S}^{1 \times rn}, \beta' = \begin{pmatrix} \beta \\ \vdots \\ \beta \end{pmatrix} \in \mathcal{S}^{rn \times 1} .$$

Then, it follows readily from Lemma 3 that

$$\alpha' \hat{\mu}(v) \beta' = \alpha \mu(a_1^{i_1} a_2^{i_2} \dots a_{k+1}^{i_{k+1}}) \beta ,$$

again with a cyclic indexing of a_1, \dots, a_r . Therefore, $\alpha' \hat{\mu}(v) \beta'$ takes the same values when $v \in \{b, c\}^*$ as $\alpha \mu(w) \beta$ when $w \in \Sigma_r^*$, and $\text{SREACH}'(r, n)$ reduces to $\text{SREACH}'(2, rn)$. Using the equivalence (13), we get that $\text{SREACH}(r, n)$ reduces to $\text{SREACH}(2, rn)$, which shows (1). \square

4.2. Proof of Reduction (2). Consider an instance of $\text{VREACH}(r, n)$ consisting of a morphism $\mu : \Sigma_r^* \rightarrow \mathcal{S}^{n \times n}$ and vectors $\alpha, \eta \in \mathcal{S}^{1 \times n}$. Define $\hat{\mu}$ as in (30), α' as in (34), together with $\eta' = (\eta, \mathbf{0}_{1n}, \dots, \mathbf{0}_{1n}) \in \mathcal{S}^{1 \times rn}$. It follows from (32) that

$$(\exists w \in \Sigma_r^*, \alpha \mu(w) = \eta) \iff (\exists 0 \leq k \leq r-1, \exists v \in \{b, c\}^*, \alpha' \hat{\mu}(v) = \eta' \hat{\mu}(b)^k)$$

Thus, $\text{VREACH}'(r, n)$ reduces to $\text{VREACH}'(2, rn)$. Thanks to the equivalence (12), this shows that $\text{VREACH}(r, n)$ reduces to $\text{VREACH}(2, rn)$. \square

4.3. Proof of Reduction (3). We consider an instance of $\text{MREACH}(r, n)$ consisting of a morphism $\mu : \Sigma_r^* \rightarrow \mathcal{S}^{n \times n}$ together with a matrix $M \in \mathcal{S}^{n \times n}$. We shall split the proof in two cases.

4.3.1. Case $M = \mathbf{0}$. Then, we apply the reduction of [BT97, Th. 1] and [CK98, Th. 1], which is valid over any semiring. For completeness, we reprove this reduction. Consider the morphism $\hat{\mu} : \Sigma_r^* \rightarrow \mathcal{S}^{nr \times rn}$ built from μ as in (30). It follows readily from Lemma 3 that:

$$(35) \quad (\exists w \in \Sigma_r^+, \mu(w) = \mathbf{0}) \iff (\exists v \in \{b, c\}^+, \hat{\mu}(v) = \mathbf{0}) .$$

Therefore, $\text{MREACH}(r, n, M = \mathbf{0})$ reduces to $\text{MREACH}(2, rn, M = \mathbf{0})$.

4.3.2. Case $M \neq \mathbf{0}$. To any $1 \leq i \leq r$, we associate the morphism $\mu_i : \Sigma_r^* \rightarrow \mathcal{S}^{n \times n}$ obtained from μ by exchanging the matrices $\mu(a_i)$ and $\mu(a_r)$:

$$\mu_i(a_r) = \mu(a_i), \mu_i(a_i) = \mu(a_r), \text{ and } \mu_i(a_j) = \mu(a_j), \text{ for } j \notin \{i, r\} .$$

We define the morphism $\bar{\mu}_i : \Sigma_r^* \rightarrow \mathcal{S}^{rn \times rn}$ from μ_i as in (26), and we set

$$M' = \text{diag}(M, \mathbf{0}_{(r-1)n, (r-1)n}) .$$

We claim that

$$(36) \quad (\exists w \in \Sigma_r^+, \mu(w) = M) \iff (\exists 1 \leq i \leq r, \exists v \in \{b, c\}^+, \bar{\mu}_i(v) = M') .$$

Indeed, if $\mu(w) = M$ for some $w \in \Sigma_r^+$, we write $w = a_i z$ with $z \in \Sigma_r^*$. Let z' denote the word obtained from z by exchanging a_i and a_r . Then, we get from (29) that

$$\bar{\mu}_i \circ \delta(a_r z') = \begin{pmatrix} \mu(a_i z) & \mathbb{0}_{n, (r-1)n} \\ \mathbb{0}_{(r-1)n, n} & \mathbb{0}_{(r-1)n, (r-1)n} \end{pmatrix} = M' ,$$

which shows the “ \Rightarrow ” implication in (36). Conversely, let us assume that $\bar{\mu}_i(v) = M'$ for some $v \in \{b, c\}^+$. We can write (uniquely) as in (31), $v = c^{i_1} b \dots c^{i_k} b c^{i_{k+1}}$, with $k \geq 0$. Since $\bar{\mu}_i(c^r) = \mathbb{0}$ and $M' \neq \mathbb{0}$, v does not have c^r as a factor, i.e., $i_1, \dots, i_{k+1} \leq r-1$. If $i_{k+1} \neq 0$, using (28) and the expression of $\bar{\mu}_i(c)$, we get

$$\bar{\mu}_i(v) = \bar{\mu}_i(v') \bar{\mu}_i(c^{i_{k+1}}) = (\mathbb{0}_{nr, n} \quad *) = M'$$

and identifying the first diagonal block, we get $\mathbb{0} = M$, a contradiction. Therefore, $k \geq 1$ and $v = c^{i_1} b \dots c^{i_k} b = \delta(z)$, where $z = a_{i_1+1} \dots a_{i_k+1}$. Then, using (28) again, we get that $\mu_i(z) = M$, hence $\mu(z') = M$, where z' is obtained from z by exchanging a_i and a_r , which shows the “ \Leftarrow ” implication in (36). Then, the equivalence (36) shows that $\text{MREACH}(r, n, M \neq \mathbb{0})$ reduces to $\text{MREACH}(2, rn, M \neq \mathbb{0})$. \square

4.4. Proof of Reduction (4). Consider an instance of $\text{VREACH}(r, n)$, which consists of $\alpha, \eta \in \mathcal{S}^{1 \times n}$ and a morphism $\mu : \Sigma_r^* \rightarrow \mathcal{S}^{n \times n}$. We associate to this instance the $(n+1) \times (n+1)$ matrix

$$(37) \quad M_\eta = \begin{pmatrix} \mathbb{0}_{11} & \eta \\ \mathbb{0}_{n1} & \mathbb{0}_{nn} \end{pmatrix},$$

and the morphism $\mu' : \Sigma_{r+1}^* \rightarrow \mathcal{S}^{(n+1) \times (n+1)}$, defined by:

$$(38) \quad \mu'(a_{r+1}) = \begin{pmatrix} \mathbb{1} & \mathbb{0}_{1n} \\ \mathbb{0}_{n1} & \mathbb{0}_{nn} \end{pmatrix}, \text{ and } \mu'(a_i) = \begin{pmatrix} \mathbb{0}_{11} & \alpha\mu(a_i) \\ \mathbb{0}_{n1} & \mu(a_i) \end{pmatrix} \quad \forall 1 \leq i \leq r .$$

An immediate induction on the length of w shows that

$$(39) \quad \mu'(w) = \begin{pmatrix} \mathbb{0}_{11} & \alpha\mu(w) \\ \mathbb{0}_{n1} & \mu(w) \end{pmatrix}, \text{ and } \mu'(a_{r+1}w) = \begin{pmatrix} \mathbb{0}_{11} & \alpha\mu(w) \\ \mathbb{0}_{n1} & \mathbb{0}_{nn} \end{pmatrix} \quad \forall w \in \Sigma_r^+ .$$

We claim that if $\eta \neq \mathbb{0}$, then

$$(40) \quad (\exists w \in \Sigma_r^+, \alpha\mu(w) = \eta) \iff (\exists z \in \Sigma_{r+1}^+, \mu'(z) = M_\eta) .$$

The reduction $\text{VREACH}(r, n, \eta \neq \mathbb{0}) \rightarrow \text{MREACH}(r+1, n+1)$ will follow from (40).

Let us assume that $\alpha\mu(w) = \eta$ for some $w \in \Sigma_r^+$. Then, it follows from (39) that $\mu'(z) = M_\eta$, where $z = a_{r+1}w$, which shows the “ \Rightarrow ” implication in (40). Conversely, let us assume that $\mu'(z) = M_\eta$ for some $z \in \Sigma_{r+1}^+$. We can write $z = w_1 a_{r+1} w_2 a_{r+1} \dots a_{r+1} w_{k+1}$, where $w_1, \dots, w_{k+1} \in \Sigma_r^*$ and $k \geq 0$. Since

$$\begin{pmatrix} \mathbb{0} & * \\ \mathbb{0} & * \end{pmatrix} \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} \end{pmatrix} = \mathbb{0} ,$$

whatever the values of the “*” entries are, and since $\mu'(z) = M_\eta \neq \mathbb{0}$, it follows that if $k \geq 1$, then w_1, \dots, w_k must be equal to the empty word. Therefore, since $\mu'(a_{r+1})^2 = \mu'(a_{r+1})$, we can assume that $k \leq 1$. If $k = 0$ we have $z = w_1$ and we readily check from (39)

that $\mu'(z) \neq M_\eta$, a contradiction. Therefore $k = 1$ and $z = a_{r+1}w$ for some $w \in \Sigma_r^*$. Since $\mu'(a_{r+1}) \neq M_\eta$, it follows that $z = a_{r+1}w$ for some $w \in \Sigma_r^+$. Then, we readily check from (39) that $\alpha\mu(w) = \eta$. This shows the “ \Leftarrow ” implication in (40).

It remains to consider the case when $\eta = \mathbf{0}$. Then, we introduce a trim unambiguous automaton \mathcal{A} recognizing $a_{r+1}\Sigma_r^*$. We can take for \mathcal{A} the minimal automaton of $a_{r+1}\Sigma_r^*$, which has two states, 1 and 2, a set of initial states $I = \{1\}$ and a set of final states $F = \{2\}$, with an associated morphism $\nu_{\mathcal{A}} : \Sigma_{r+1}^* \rightarrow \mathcal{S}^{2 \times 2}$ built as in §2.4:

$$\nu_{\mathcal{A}}(a_i) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \text{ for } 1 \leq i \leq r \text{ and } \nu_{\mathcal{A}}(a_{r+1}) = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

and

$$\mathbb{M}_{\mathcal{A}} = \{M'\} \text{ where } M' = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Let $\mu'' = \text{diag}(\mu', \nu_{\mathcal{A}})$ and $M''_\eta = \text{diag}(M_\eta, M')$. By Lemma 2, we have

$$(41) \quad \mu''(z) = M''_\eta \iff (z \in a_{r+1}\Sigma_r^* \text{ and } \mu'(z) = M_\eta).$$

But if $\mu'(z) = M_\eta = \mathbf{0}_{nn}$ and $z = a_{r+1}w$ with $w \in \Sigma_r^*$, w must be non-empty. Combining this observation with (39), we get that

$$(\exists z \in \Sigma_{r+1}^+, \mu''(z) = M''_\eta) \iff (\exists w \in \Sigma_r^+, \alpha\mu(w) = \eta = \mathbf{0}),$$

which shows that $\text{VREACH}(r, n, \eta = \mathbf{0}) \rightarrow \text{MREACH}(r+1, n+3)$. \square

4.5. Proof of Reduction (5). Consider an instance of $\text{SREACH}(r, n)$ given by a series $s \in \mathcal{S}^{\text{rat}}\langle\langle \Sigma_r \rangle\rangle$ with a linear representation (α, μ, β) of dimension n , and $\gamma \in \mathcal{S}$. By comparison to the proof of Reduction (4), we shall use a dual coding, and associate to this instance the $1 \times (n+1)$ matrices $\alpha' = (\alpha, \mathbf{0}_{1n})$ and $\eta_\gamma = (\mathbf{0}_{1n}, \gamma)$, and the morphism $\mu' : \Sigma_{r+1}^* \rightarrow \mathcal{S}^{(n+1) \times (n+1)}$, defined by:

$$(42) \quad \mu'(a_{r+1}) = \begin{pmatrix} \mathbf{0}_{nn} & \mathbf{0}_{n1} \\ \mathbf{0}_{1n} & \mathbf{1} \end{pmatrix}, \text{ and } \mu'(a_i) = \begin{pmatrix} \mu(a_i) & \mu(a_i)\beta \\ \mathbf{0}_{1n} & \mathbf{0}_{11} \end{pmatrix} \quad \forall 1 \leq i \leq r.$$

The dual version of (39) is:

$$(43) \quad \mu'(w) = \begin{pmatrix} \mu(w) & \mu(w)\beta \\ \mathbf{0}_{1n} & \mathbf{0}_{11} \end{pmatrix}, \text{ and } \mu'(wa_{r+1}) = \begin{pmatrix} \mathbf{0}_{nn} & \mu(w)\beta \\ \mathbf{0}_{1n} & \mathbf{0}_{11} \end{pmatrix} \quad \forall w \in \Sigma_r^+.$$

By dualizing the arguments of the proof of (40), we get that if $\gamma \neq \mathbf{0}$, then

$$(44) \quad (\exists w \in \Sigma_r^+, s(w) = \gamma) \iff (\exists z \in \Sigma_{r+1}^+, \alpha'\mu'(z) = \eta_\gamma).$$

The reduction $\text{SREACH}(r, n, \gamma \neq \mathbf{0}) \rightarrow \text{VREACH}(r+1, n+1)$ follows from (44).

It remains to consider the case when $\gamma = \mathbf{0}$. Then, we consider a trim unambiguous automaton \mathcal{A} with 2 states recognizing $\Sigma_r^*a_{r+1}$, together with the morphism $\nu_{\mathcal{A}} : \Sigma_{r+1}^* \rightarrow \mathcal{S}^{2 \times 2}$ built as in §2.4. We can assume that the initial state of \mathcal{A} is 1 and that its final state is 2. Let $\mu'' = \text{diag}(\mu', \nu_{\mathcal{A}})$, $\alpha'' = (\alpha, \mathbf{1}, \mathbf{1}, \mathbf{0})$ and $\eta'' = (\mathbf{0}_{1n}, \mathbf{0}, \mathbf{0}, \mathbf{1})$. By Lemma 2,

$$(45) \quad \alpha''\mu''(z) = \eta'' \iff (z \in \Sigma_r^*a_{r+1} \text{ and } (\alpha, \mathbf{1})\mu'(z) = (\mathbf{0}_{1n}, \mathbf{0})).$$

But if $(\alpha, \mathbb{1})\mu'(z) = (\mathbb{0}_{1n}, \mathbb{0})$ and $z = wa_{r+1}$ with $w \in \Sigma_r^*$, w must be non-empty. Combining this observation with (43), we get that

$$(\exists z \in \Sigma_{r+1}^+, \alpha''\mu''(z) = \eta'') \iff (\exists w \in \Sigma_r^+, s(w) = \gamma = \mathbb{0}) ,$$

which shows that $\text{SREACH}(r, n, \gamma = \mathbb{0}) \rightarrow \text{VREACH}(r+1, n+3)$. \square

4.6. Proof of Reduction (6). Consider an instance of $\text{SREACH}(r, n)$ given by a series $s \in \mathcal{S}^{\text{rat}}(\langle \Sigma_r \rangle)$ with a linear representation (α, μ, β) of dimension n , and $\gamma \in \mathcal{S}$. We associate to this instance the morphism $\mu' : \Sigma_{r+1}^* \rightarrow \mathcal{S}^{(n+2) \times (n+2)}$, with $\mu'(a_i)$ as in (22), for $1 \leq i \leq r$, and

$$\mu'(a_{r+1}) = \begin{pmatrix} \mathbb{1} & \mathbb{0}_{1n} & \mathbb{0}_{11} \\ \mathbb{0}_{n1} & \mathbb{0}_{nn} & \mathbb{0}_{n1} \\ \mathbb{0}_{11} & \mathbb{0}_{1n} & \mathbb{1} \end{pmatrix} .$$

Left and right multiplying (23) by $\mu'(a_{r+1})$, we get:

$$(46) \quad \mu'(a_{r+1}wa_{r+1}) = \begin{pmatrix} \mathbb{0}_{11} & \mathbb{0}_{1n} & s(w) \\ \mathbb{0}_{n1} & \mathbb{0}_{nn} & \mathbb{0}_{n1} \\ \mathbb{0}_{11} & \mathbb{0}_{1n} & \mathbb{0}_{11} \end{pmatrix} , \quad \forall w \in \Sigma_r^+ .$$

Let

$$M_\gamma = \begin{pmatrix} \mathbb{0}_{11} & \mathbb{0}_{1n} & \gamma \\ \mathbb{0}_{n1} & \mathbb{0}_{nn} & \mathbb{0}_{n1} \\ \mathbb{0}_{11} & \mathbb{0}_{1n} & \mathbb{0}_{11} \end{pmatrix} .$$

We claim that if $\gamma \neq \mathbb{0}$, then

$$(47) \quad (\exists w \in \Sigma_r^+, s(w) = \gamma) \iff (\exists z \in \Sigma_{r+1}^+, \mu'(z) = M_\gamma) .$$

The reduction $\text{SREACH}(r, n, \gamma \neq \mathbb{0}) \rightarrow \text{MREACH}(r+1, n+2)$ will follow from (47).

Let us assume that $s(w) = \gamma$ for some $w \in \Sigma_r^+$. Then, it follows from (46) that $\mu'(z) = M_\gamma$, where $z = a_{r+1}wa_{r+1}$, which shows the “ \Rightarrow ” implication in (47). Conversely, let us assume that $\mu'(z) = M_\gamma$ for some $z \in \Sigma_{r+1}^+$. We can write

$$(48) \quad z = w_1 a_{r+1} w_2 a_{r+1} \dots a_{r+1} w_{k+1} ,$$

where $w_1, \dots, w_{k+1} \in \Sigma_r^*$ and $k \geq 0$. Since $\mu'(a_{r+1})^2 = \mu'(a_{r+1})$, if some positive power a_{r+1}^m appears as a factor of z , we may replace this power by a_{r+1} without changing $\mu'(z)$, which allows us to assume that when $k \geq 2$, all the w_2, \dots, w_k are non-empty words. We remark that

$$(49) \quad \begin{pmatrix} \mathbb{0} & * & * \\ \mathbb{0} & * & * \\ \mathbb{0} & \mathbb{0} & \mathbb{0} \end{pmatrix} \begin{pmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{0} & * & * \\ \mathbb{0} & * & * \\ \mathbb{0} & \mathbb{0} & \mathbb{0} \end{pmatrix} = \mathbb{0} ,$$

whatever the values of the “*” entries are. It follows from (49) and from $\mu'(z) = M_\gamma \neq \mathbb{0}$ that z has no factor of the form $wa_{r+1}w'$, with $w, w' \in \Sigma_r^+$. Therefore, in the factorization (48), $k \leq 2$ and at most one w_i is different from the empty word. If $k \leq 1$, we have $z = w_1 a_{r+1}$, or $z = a_{r+1} w_2$, or $z = w_1$, and in all these cases, we readily check from (23) that $\mu'(z) \neq M_\gamma$,

a contradiction. Therefore $k = 2$ and $z = a_{r+1}w_2a_{r+1}$ with $w_2 \in \Sigma_r^+$. Using (46), we get $s(w_2) = \gamma$. This shows the “ \Leftarrow ” implication in (47).

It remains to consider the case when $\gamma = 0$. Then, we consider a trim unambiguous automaton \mathcal{A} with 3 states recognizing $a_{r+1}\Sigma_r^*a_{r+1}$, together with the morphism $\nu_{\mathcal{A}} : \Sigma_{r+1}^* \rightarrow \mathcal{S}^{3 \times 3}$ and the set $M_{\mathcal{A}} \subset \mathcal{S}^{3 \times 3}$ built as in §2.4. Let $\mu'' = \text{diag}(\mu', \nu_{\mathcal{A}})$, and $M' = \{\text{diag}(M_\gamma, N) \mid N \in M_{\mathcal{A}}\}$. By Lemma 2,

$$(50) \quad \mu''(z) \in M' \iff (z \in a_{r+1}\Sigma_r^*a_{r+1} \text{ and } \mu'(z) = M_\gamma) .$$

But if $\mu'(z) = M_\gamma$ and $z = a_{r+1}wa_{r+1}$ with $w \in \Sigma_r^*$, w must be non-empty. Combining this observation with (46), we get that

$$(\exists z \in \Sigma_{r+1}^+, \mu''(z) \in M') \iff (\exists w \in \Sigma_r^+, s(w) = \gamma = 0) ,$$

which shows that $\text{SREACH}(r, n, \gamma = 0) \rightarrow \text{MREACH}(r + 1, n + 5)$. \square

4.7. Proof of Reduction (7). Consider an instance of $\text{MREACH}(r, n)$ consisting of a morphism $\mu : \Sigma_r^* \rightarrow \mathcal{S}^{n \times n}$ together with a matrix $M \in \mathcal{S}^{n \times n}$. We associate to this instance the morphism $\mu' : \Sigma_r^* \rightarrow \mathcal{S}^{rn \times rn}$ defined by $\mu' = \text{diag}(\mu, \dots, \mu)$. Let vec be the matrix to vector operation that develops a square matrix into a row vector by taking its rows one by one. Then we have

$$(51) \quad \text{vec}(\mu(a_i w)) = \text{vec}(\mu(a_i))\mu'(w), \text{ for all } 1 \leq i \leq r \text{ and } w \in \Sigma_r^* .$$

It follows from (51) that

$$(\exists w \in \Sigma_r^+, \mu(w) = M) \iff (\exists 1 \leq k \leq r, \exists v \in \Sigma_r^*, \text{vec}(\mu(a_k))\mu'(v) = \text{vec}(M)) .$$

Thus, $\text{MREACH}(r, n)$ reduces to $\text{VREACH}'(r, rn)$. Thanks to the equivalence (12), this shows that $\text{MREACH}(r, n)$ reduces to $\text{VREACH}(r, rn)$. \square

4.8. Proof of Corollary 4. We have the following chain of reductions:

$$\begin{array}{ccc} \text{CREACH}(n_P, 3, \gamma = 0) & \rightarrow & \text{SREACH}(n_P, 3, \gamma = 0) \\ & & \downarrow \\ \text{CREACH}(2, 3n_P + 2, \gamma = 0) & \leftarrow & \text{SREACH}(2, 3n_P, \gamma = 0) \end{array}$$

This follows from (21),(1), and (20) (and from the fact that the value of γ is preserved in (21),(1), and (20)). Since $\text{CREACH}(n_P, 3, \gamma = 0)$ is undecidable [Man74], it follows that $\text{CREACH}(2, 3n_P + 2, \gamma = 0)$, the zero corner problem for 2 matrices of dimension $3n_P + 2$, is undecidable. \square

4.9. Proof of Proposition 1. We shall combine a slight modification of the proof of Paterson [Pat70] with the idea of Bournez and Branicky [BB02] of using the Modified Post Correspondence Problem. Recall that the Modified Post Correspondence Problem (MPCP) can be stated as: given a finite set of pairs of words $\{(u_i, v_i) \mid 1 \leq i \leq r\}$ over a finite alphabet, is there a finite sequence $1 \leq i_2, \dots, i_k \leq r$ such that $u_1 u_{i_2} \dots u_{i_k} = v_1 v_{i_2} \dots v_{i_k}$? Of course, the MPCP is undecidable for any value of r for which the Post Correspondence Problem is undecidable. We shall assume, without loss of generality, that the alphabet is $\Sigma = \{1, \dots, n\}$. Let b denote any integer (strictly) greater than n , and for any $w \in \Sigma^*$, let

$[w]_b$ denote the integer obtained by interpreting the word w in base b (we set $[\]_b = 0$ in the case of the empty word), and let $|w|$ denote the length of a word w . Paterson associated to any $u, v \in \Sigma^*$, the matrix

$$W(u, v) = \begin{pmatrix} b^{|u|} & 0 & 0 \\ 0 & b^{|v|} & 0 \\ [u]_b & [v]_b & 1 \end{pmatrix}$$

and observed that

$$(52) \quad \forall u', v' \in \Sigma^*, ([u']_b \ [v']_b \ 1) W(u, v) = ([u'u']_b \ [v'v']_b \ 1) .$$

To any instance $I = \{(u_i, v_i) \mid 1 \leq i \leq r\}$ of the MPCP over the alphabet Σ , we associate the vector $\alpha = ([u_1]_b, [v_1]_b, 1) \in \mathbb{Z}^{1 \times 3}$, and the morphism $\mu : \Sigma_{r+1}^* \rightarrow \mathbb{Z}^{3 \times 3}$, such that $\mu(a_i) = W(u_i, v_i)$ for $1 \leq i \leq r$, and $\mu(a_{r+1}) = T$, where

$$T = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

It follows readily from (52) and from the form of T that for all $1 \leq i_2, \dots, i_k \leq r$,

$$(53) \quad \alpha W(u_{i_2}, v_{i_2}) \cdots W(u_{i_k}, v_{i_k}) T = ([u]_b - [v]_b \ [v]_b - [u]_b \ 0) \\ \text{where } u = u_1 u_{i_2} \cdots u_{i_k} \text{ and } v = v_1 v_{i_2} \cdots v_{i_k} .$$

We claim that

$$(54) \quad \text{Instance } I \text{ has a solution} \iff \exists w \in \Sigma_{r+1}^+, \alpha \mu(w) = 0 .$$

Indeed, the “ \Rightarrow ” implication in (54) follows readily from (53). Conversely, let us assume that $\alpha \mu(w) = 0$ for some $w \in \Sigma_{r+1}^+$, that we choose of minimal length. We can write $w = w_1 a_{r+1} w_2 \dots a_{r+1} w_{k+1}$, where $w_1, \dots, w_{k+1} \in \Sigma_r^*$. Since the matrices $\mu(a_i) = W(u_i, v_i)$ all are invertible, for $1 \leq i \leq r$, $\mu(w_{k+1})$ is invertible, so that by minimality of w , w_{k+1} must be equal to the empty word. We also note that all the matrices $W(u_i, v_i)$ are of the form

$$(55) \quad \begin{pmatrix} p & 0 & 0 \\ 0 & s & 0 \\ q & t & 1 \end{pmatrix}$$

where $p, s \geq 1$ and $q, t \geq 0$ and that the matrices of this form yield a semigroup. Since

$$T \begin{pmatrix} p & 0 & 0 \\ 0 & s & 0 \\ q & t & 1 \end{pmatrix} T = (p + s) T$$

we conclude, using again the minimality of w , that w_2, \dots, w_k must be equal to the empty word. Thus, $w = w_1 a_{r+1}^m$, for some $m \geq 0$. Since $T^2 = 2T$, the minimality of w yields $m \leq 1$. If $m = 0$, then, $w = w_1 \in \Sigma_r^*$, and $\alpha \mu(w) = (*, *, 1) \neq 0$, a contradiction. Thus $m = 1$, so that $w = w_1 a_{r+1}$ where $w_1 = a_{i_2} \dots a_{i_k}$ for some $1 \leq i_2, \dots, i_k \leq r$. Then it follows from (53) that $[u_1 u_{i_2} \dots u_{i_k}]_b = [v_1 v_{i_2} \dots v_{i_k}]_b$, which shows that i_2, \dots, i_k solves Instance I of the MPCP. We have proved the “ \Leftarrow ” implication in (54). \square

4.10. Proof of Corollaries 2 and 3. Consider an instance of $\text{MREACH}(r, n)$ over \mathbb{Z}_{\max} , consisting of a morphism $\mu : \Sigma_r^* \rightarrow \mathbb{Z}_{\max}^{n \times n}$ and a matrix $M \in \mathbb{Z}_{\max}^{n \times n}$. Let $\nu = \text{diag}(\mathbf{1}, \mu)$, and $M' = \text{diag}(\mathbf{1}, M)$. Since

$$\nu(w) \sim M' \iff \mu(w) = M ,$$

$\text{MREACH}(r, n)$ over \mathbb{Z}_{\max} , which is undecidable when $r = 2$ by Theorem 3, reduces to the projective matrix reachability problem for r matrices of dimension $n + 1$. This shows Corollary 2.

Consider now an instance of $\text{VREACH}(r, n)$ over \mathbb{Z}_{\max} , consisting of a morphism $\mu : \Sigma_r^* \rightarrow \mathbb{Z}_{\max}^{n \times n}$ and vectors $\alpha, \eta \in \mathbb{Z}_{\max}^{1 \times n}$. Let $\nu = \text{diag}(\mathbf{1}, \mu)$ as above, $\alpha' = (\mathbf{1}, \alpha) \in \mathbb{Z}_{\max}^{1 \times (n+1)}$, and $\eta' = (\mathbf{1}, \eta) \in \mathbb{Z}_{\max}^{1 \times (n+1)}$. Since

$$\alpha' \nu(w) \sim \eta' \iff \alpha \mu(w) = \eta ,$$

$\text{VREACH}(r, n)$ over \mathbb{Z}_{\max} , which is undecidable when $r = 2$ by Theorem 3, reduces to the projective vector reachability problem for r matrices of dimension $n + 1$. This shows Corollary 3.

4.11. Proof of Theorem 4. We shall use the following stronger form of the separation property.

Lemma 4. *A semiring \mathcal{S} is (effectively) separated by morphisms of finite image if, and only if, there are (effective) maps $B \mapsto \mathcal{S}_B$ and $B \mapsto \pi_B$ which to any finite subset B of \mathcal{S} , associate a finite semiring \mathcal{S}_B and a semiring morphism π_B from \mathcal{S} to \mathcal{S}_B , such that $\pi_B^{-1}(\pi_B(y)) = \{y\}$ for all $y \in B$.*

Proof. The “if” part is trivial. Conversely, assume that a semiring \mathcal{S} is separated by morphisms of finite image, and let $B = \{b_1, \dots, b_k\}$ be a finite subset of \mathcal{S} . For all $1 \leq i \leq k$, there is a finite semiring \mathcal{S}_{b_i} and a semiring morphism π_{b_i} from \mathcal{S} to \mathcal{S}_{b_i} such that $\pi_{b_i}^{-1}(\pi_{b_i}(b_i)) = \{b_i\}$. Let $\mathcal{S}_B = \mathcal{S}_{b_1} \times \dots \times \mathcal{S}_{b_k}$ denote the *Cartesian product* of $\mathcal{S}_{b_1}, \dots, \mathcal{S}_{b_k}$, that is, the Cartesian product of the underlying sets equipped with entrywise sum and product, and consider the semiring morphism $\pi_B : \mathcal{S} \rightarrow \mathcal{S}_B$, $\pi_B(y) = (\pi_{b_i}(y))_{1 \leq i \leq k}$. We have $\pi_B^{-1}(\pi_B(b_i)) = \{b_i\}$ for all $1 \leq i \leq k$. This shows the “only if” part. Finally, we note that effective aspects are preserved in the above construction. \square

We now prove Theorem 4. By Theorem 1, it suffices to show that the matrix reachability problem is decidable over \mathcal{S} . Let us consider an instance of $\text{MREACH}(r, n)$ over \mathcal{S} , consisting of a morphism $\mu : \Sigma_r^* \rightarrow \mathcal{S}^{n \times n}$ and a matrix $M \in \mathcal{S}^{n \times n}$. Define $B = \{M_{ij} \mid 1 \leq i, j \leq n\}$. We know that there is a finite semiring \mathcal{S}_B and a semiring morphism π_B from \mathcal{S} to \mathcal{S}_B such that $\pi_B^{-1}(\pi_B(y)) = \{y\}$ for all $y \in B$. We extend π_B to a map from $\mathcal{S}^{n \times n}$ to $\mathcal{S}_B^{n \times n}$ by making π_B act on each entry. We note that the problem:

$$(56) \quad \exists w \in \Sigma_r^+, \pi_B \circ \mu(w) = \pi_B(M) ?$$

is decidable. Indeed, using the effective part of Lemma 4, we can compute the matrices $\pi_B(X) \in \mathcal{S}_B^{n \times n}$, for $X \in \mu(\Sigma_r) \cup \{M\}$, and we know the addition and multiplication tables

of the (finite) semiring \mathcal{S}_B . Therefore, we can compute the finite semigroup $\pi_B \circ \mu(\Sigma_r^+)$, and we can test whether it contains $\pi_B(M)$.

We finally show that

$$(57) \quad (\exists w \in \Sigma_r^+, \mu(w) = M) \iff (\exists w \in \Sigma_r^+, \pi_B \circ \mu(w) = \pi_B(M)) .$$

Clearly, if $\exists w \in \Sigma_r^+$ such that $\mu(w) = M$, then $\pi_B \circ \mu(w) = \pi_B(M)$. Conversely, assume that $\pi_B \circ \mu(w) = \pi_B(M)$ for some $w \in \Sigma_r^+$. Then, $(\mu(w))_{ij} \in \pi_B^{-1}(\pi_B(M_{ij})) = \{M_{ij}\}$ for all $1 \leq i, j \leq n$ and therefore $\mu(w) = M$, which shows (57). It follows readily from (57) that the matrix reachability problem is decidable over \mathcal{S} . \square

Theorem 4 can be thought of as an extension of Krob's [Kro94, Proposition 2.2], which shows that if $s \in \mathbb{N}_{\min}^{\text{rat}}(\langle \Sigma_r \rangle)$ (resp. $s \in \mathbb{N}_{\max}^{\text{rat}}(\langle \Sigma_r \rangle)$), for all $\gamma \in \mathbb{N}_{\min}$ (resp. $\gamma \in \mathbb{N}_{\max}$), $\{w \in \Sigma_r^* \mid s(w) = \gamma\}$ is a constructible rational language. We can in fact restate Theorem 4 in the following more precise way:

Theorem 5. *Let \mathcal{S} denote a semiring that is effectively separated by morphisms of finite image, let $\alpha, \eta \in \mathcal{S}^{1 \times n}$, $\mu : \Sigma_r^* \rightarrow \mathcal{S}^{n \times n}$ a morphism, $\beta \in \mathcal{S}^{n \times 1}$, $M \in \mathcal{S}^{n \times n}$, and $\gamma \in \mathcal{S}$. Then, the following sets all are constructible rational languages:*

$$(58) \quad \{w \in \Sigma_r^* \mid \alpha \mu(w) \beta = \gamma\} ,$$

$$(59) \quad \{w \in \Sigma_r^* \mid \alpha \mu(w) = \eta\} ,$$

$$(60) \quad \{w \in \Sigma_r^* \mid \mu(w) = M\} .$$

Proof. It follows from the proof of (57) that:

$$(61) \quad \{w \in \Sigma_r^* \mid \mu(w) = M\} = (\pi_B \circ \mu)^{-1}\{\pi_B(M)\} .$$

Now, recall that the Kleene-Schützenberger theorem shows that a language of Σ_r^* is rational if and only if it can be written as $\kappa^{-1}(F)$, where κ is a morphism from Σ_r^* to a finite monoid P , and F is a subset of P . Taking $P = \mathcal{S}_B^{n \times n}$, $F = \{\pi_B(M)\}$, and $\kappa = \pi_B \circ \mu$, it follows from (61) that $\{w \in \Sigma_r^* \mid \mu(w) = M\}$ is rational, and this rational language is constructible since P, F and κ can be effectively computed. This argument can be readily adapted to the languages (58),(59). For instance, in the case of (58), we can take a finite semiring \mathcal{S}_γ together with a morphism $\pi_\gamma : \mathcal{S} \rightarrow \mathcal{S}_\gamma$ such that $\pi_\gamma^{-1}(\pi_\gamma(\gamma)) = \{\gamma\}$, and note that $\{w \in \Sigma_r^* \mid \alpha \mu(w) \beta = \gamma\} = \kappa^{-1}(F)$ where $P = \mathcal{S}_\gamma^{n \times n}$, $F = \{U \in P \mid \pi_\gamma(\alpha) U \pi_\gamma(\beta) = \pi_\gamma(\gamma)\}$, and $\kappa = \pi_\gamma \circ \mu$. The adaptation in the case of (59) is similar. \square

4.12. Proof of Proposition 2. Let \mathcal{S} be any of the semirings \mathbb{N}_{\min} , \mathbb{N}_{\max} , $\bar{\mathbb{N}}_{\max}$, \mathcal{L} , \mathbb{N} , $\bar{\mathbb{N}}$, and let $\gamma \in \mathcal{S}$ be arbitrary. If γ is a natural number, let us denote by n any natural number strictly greater than γ . If γ is not a natural number choose n arbitrarily (for example $n = 1$). Consider the quotient of \mathcal{S} by the congruence which identifies all the integers greater than or equal to n . (We call *congruence* an equivalence relation which preserves the semiring structure.) Let us denote by \mathcal{S}_γ the resulting finite semiring equipped with the quotient laws, and by π_γ the canonical morphism from \mathcal{S} to \mathcal{S}_γ . Then we have that $\pi_\gamma^{-1}(\pi_\gamma(\gamma)) = \{\gamma\}$, which shows Proposition 2. \square

4.13. **Case $r = 1$.** When $r = 1$, the decidability of the reachability problems follows readily from known results. For instance, the cyclicity theorem for reducible max-plus matrices shows that if A is a $n \times n$ matrix with entries in the semiring \mathbb{Z}_{\max} there are positive integers c, N , such that for all $1 \leq i, j \leq n$, there are scalars $\lambda_0, \dots, \lambda_{c-1}$ (depending on i, j) such that for all $0 \leq l \leq c - 1$,

$$(62) \quad \forall n \geq N, \quad (A^{(n+1)c+l})_{ij} = \lambda_l (A^{nc+l})_{ij} ,$$

and the integers c, N together with the scalars λ_l can be effectively computed. This cyclicity theorem, which is taken from [Gau92, VI,1.1.10], where it is proved more generally for matrices with entries in the semiring $(\mathbb{R} \cup \{-\infty\}, \max, +)$, is an immediate consequence of the characterization of max-plus rational series in one letter as merge of ultimately geometric series, see [Mol88], [Gau92, VI,1.1.8] (or [Gau94]) and [KB94]. It follows that the matrix, vector, and scalar reachability problems in \mathbb{Z}_{\max} are decidable when $r = 1$.

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