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# *Improper choosability of graphs and maximum average degree*

Frédéric Havet — Jean-Sébastien Sereni

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## Improper choosability of graphs and maximum average degree

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**Abstract:** Improper choosability of planar graphs has been widely studied. In particular, Škrekovski investigated the smallest integer  $g_k$  such that every planar graph of girth at least  $g_k$  is  $k$ -improper 2-choosable. He proved [8] that  $6 \leq g_1 \leq 9$ ;  $5 \leq g_2 \leq 7$ ;  $5 \leq g_3 \leq 6$  and  $\forall k \geq 4, g_k = 5$ . In this paper, we study the greatest real  $M(k, l)$  such that every graph of maximum average degree less than  $M(k, l)$  is  $k$ -improper  $l$ -choosable. We prove that for  $l \geq 2$  then  $M(k, l) \geq l + \frac{lk}{l+k}$ . As a corollary, we deduce that  $g_1 \leq 8$  and  $g_2 \leq 6$ . We also provide an upper bound for  $M(k, l)$ . This implies that for any fixed  $l$ ,  $M(k, l) \xrightarrow[k \rightarrow \infty]{} 2l$ .

**Key-words:** improper colouring, choosability, maximum average degree, planar graph, girth

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## Choisissabilité impropre et degré maximum moyen

**Résumé :** La choisissabilité impropre des graphes planaires a été beaucoup étudiée. En particulier, Škrekovski a étudié le plus petit entier  $g_k$  tel que tout graphe planaire de maille au moins  $g_k$  soit  $k$ -impropre 2-choisissable. Il a prouvé [8] les résultats suivants :  $6 \leq g_1 \leq 9$ ;  $5 \leq g_2 \leq 7$ ;  $5 \leq g_3 \leq 6$  et  $\forall k \geq 4, g_k = 5$ . Dans cet article, nous introduisons le plus grand réel  $M(k, l)$  tel que tout graphe de degré moyen maximum strictement inférieur à  $M(k, l)$  soit  $k$ -impropre  $l$ -choisissable. Nous prouvons que  $\forall l \geq 2, M(k, l) + \frac{lk}{l+k}$ . Nous déduisons en corollaire que  $g_1 \leq 8$  et  $g_2 \leq 6$ . Nous donnons également une borne supérieure pour  $M(k, l)$  dont on déduit que, pour tout  $l$  fixé,  $M(k, l) \xrightarrow[k \rightarrow +\infty]{} 2l$ .

**Mots-clés :** coloration impropre, choisissabilité, degré moyen maximum, graphe planaire, maille

## 1 Introduction.

Let  $G$  be a graph. We note  $V(G)$  its vertex set and  $E(G)$  its edge set.

A *colouring* is an application from the vertex set into a set of colours  $S$ . If  $|S| = l$  we call it  $l$ -colouring. Let  $c$  be a colouring of  $G$ . The *impropriety* of a vertex  $v$  in  $G$  under  $c$ , denoted by  $im_G^c(v)$ , is the number of neighbours  $u$  of  $v$  in  $G$  such that  $c(u) = c(v)$ . The *impropriety* of  $c$  in  $G$  is  $im_G(c) = \max\{im_G^c(v) \mid v \in V(G)\}$ . A colouring is  $k$ -improper if its impropriety is at most  $k$  and a graph is  $k$ -improper  $l$ -colourable if it admits a  $k$ -improper  $l$ -colouring. The  $k$ -improper chromatic number of  $G$ , denoted by  $c_k(G)$ , is the smallest integer  $l$  such that  $G$  is  $k$ -improper  $l$ -colourable. Note that 0-improper colouring is the usual notion of proper colouring, so the 0-improper chromatic number is exactly the chromatic number usually denoted  $\chi(G)$ .

One can analogously generalize the notion of *choosability*. A *list-assignment* of a graph  $G$  is an application  $L$  which assigns to each vertex  $v \in V(G)$  a prescribed list of colours  $L(v)$ .  $L$  is an  $l$ -list-assignment provided each list is of size at least  $l$ .  $G$  is  $k$ -improper  $L$ -colourable if there exists a  $k$ -improper colouring  $c$  of  $G$  such that  $\forall v \in V(G), v \in L(v)$ . In this case,  $c$  is a  $k$ -improper  $L$ -colouring of  $G$ .  $G$  is  $k$ -improper  $l$ -choosable if it is  $k$ -improper  $L$ -colourable for every  $l$ -list-assignment  $L$ .

Colourings of planar graphs have been widely studied. In particular  $p_k$  and  $p_k^*$ , the smallest integers  $l$  such that every planar graph is  $k$ -improper  $l$ -colourable and  $k$ -improper  $l$ -choosable respectively, are known for almost all  $k$ . Indeed Thomassen showed in [9] that every planar graph is 5-choosable and there are planar graphs which are not 4-choosable [12] so  $p_0^* = 5$ . Every planar graph is 4-colourable [1, 2] and there are graphs which are not 1-improper 3-colourable, so  $p_0 = p_1 = 4$ . But we do not know the exact value of  $p_1^*$  which is either 4 or 5. However, it is conjectured that it is 4:

**Conjecture 1 (Eaton and Hull [3], Škrekovski [6])** *Every planar graph is 1-improper 4-choosable.*

As shown independently by Eaton and Hull [3] and Škrekovski [6], every planar graph is 2-improper 3-choosable and for every  $k$ , there are planar graphs which are not  $k$ -improper 2-colourable. Hence  $p_k = p_k^* = 3$  for any  $k \geq 2$ .

Moreover improper colourings of planar graphs have also been studied under some girth restrictions. The *girth* of graph is the smallest length of a cycle. The well-known theorem of Grötzsch [4, 11] states that every planar graph of girth at least 4 is 3-colourable. Voigt [13] showed a planar graph of girth 4 which is not 3-choosable and Thomassen [10] proved that every planar graph of girth at least 5 is 3-choosable. In [7], Škrekovski showed that every planar graph of girth at least 4 is 1-improper 3-choosable. In [8], Škrekovski investigated  $k$ -improper 2-choosability of planar graphs in relation with their girth. Denoting by  $g_k$  be the smallest integer such that every planar graph of girth at least  $g_k$  is  $k$ -improper 2-choosable, he proved that  $6 \leq g_1 \leq 9$ ,  $5 \leq g_2 \leq 7$ ,  $5 \leq g_3 \leq 6$  and  $\forall k \geq 4, g_k = 5$ . Hence the only unknown values are  $g_1, g_2$  and  $g_3$ .

In this paper, we study the  $k$ -improper  $l$ -choosability of graphs in relation with their maximum average degree.

**Definition 1** The maximal average degree of a graph  $G$  is:

$$Mad(G) := \max\left\{\frac{\sum_{v \in V(H)} d_H(v)}{|V(H)|}, H \text{ subgraph of } G\right\}.$$

The girth and the maximum average degree of a planar graph are related to each other:

**Theorem 1** *Let  $G$  be a planar graph of girth  $g$ .*

$$Mad(G) < \frac{2g}{g-2}.$$

**Proof.** We recall the Euler's formula for a planar graph  $H$ :  $|V(H)| - |E(H)| + |F(H)| = 2$  with  $|F(H)|$  the number of faces of  $H$ . Note that every subgraph  $H$  of  $G$  has girth at least  $g$ , so  $g|F(H)| \leq 2|E(H)|$ . Thus  $2g - g|V(H)| + g|E(H)| = g|F(H)| \leq 2|E(H)|$ . Hence  $\frac{2|E(H)|}{|V(H)|} \leq \frac{2g}{g-2} - \frac{4g}{(g-2)|V(H)|} < \frac{2g}{g-2}$  for every subgraph  $H$  of  $G$ .  $\square$

Let  $M(k, l)$  be the greatest real such that every graph of maximum average degree less than  $M(k, l)$  is  $k$ -improper  $l$ -choosable. Obviously,  $M(k_1, l) \leq M(k_2, l)$  if  $k_1 \leq k_2$ .

We have that  $M(k, 1) = \frac{2k+2}{k+2}$  since a graph is  $k$ -improper 1-choosable if, and only if, it has maximum degree at most  $k$ .

In order to introduce our method which uses some discharging process, we first present it in Section 2 for improper 2-choosability: we prove that for every  $k \geq 0$ ,

$$4 - \frac{4}{k+2} \leq M(k, 2) \leq 4 - \frac{2k+4}{k^2+2k+2}.$$

As a corollary, we obtain the following upper bounds for  $g_k$  which are better than Škrekovski's ones:  $g_1 \leq 8$ ,  $g_2 \leq 6$ ,  $g_3 \leq 6$  and  $\forall k \geq 4, g_k = 5$ .

In Section 3 we extend the lower bound of Section 2 to any value of  $l$ : we prove that for every  $l \geq 2$  and  $k \geq 0$ ,

$$l + \frac{lk}{l+k} \leq M(k, l).$$

Last, we provide for any value of  $l$  and  $k$  a graph which is not  $k$ -improper  $l$ -choosable, and we deduce that  $M(k, l) \xrightarrow[k \rightarrow \infty]{} 2l$ .

## 2 Improper 2-choosability

### 2.1 Lower bound for $M(k, 2)$

In this subsection, we shall prove the following theorem:

**Theorem 2** *For all  $k \geq 0$ , all graphs of maximum average degree less than  $\frac{4k+4}{k+2}$  are  $k$ -improper 2-choosable.*

Note that if  $k = 0$  the result holds trivially. Indeed a graph with maximum average degree less than 2 contains no cycle and so is a forest. Hence it is 2-choosable. Furthermore  $M(0, 2) \leq 2$  since an odd cycle is not 2-colourable, so  $M(0, 2) = 2$ .

For bigger value of  $k$ , we will need the following preliminary definitions and results:

**Definition 2** If  $v \in V(G)$  then  $d_G(v)$  denotes the degree of  $v$  in the graph  $G$ . For all positive integer  $d$ , a vertex of degree equals to (resp. at most, resp. at least)  $d$  is called a  $d$ -vertex (resp.  $(\leq d)$ -vertex, resp.  $(\geq d)$ -vertex). For  $S \subseteq V(G)$  (resp.  $E \subseteq E(G)$ ) we denote by  $G - S$  (resp.  $G - E$ ) the induced subgraph of  $G$  obtained by removing the vertices (resp. edges) of  $S$  (resp.  $E$ ) from  $V(G)$  (resp.  $E(G)$ ). If  $S = \{v\}$  and  $E = \{uv\}$ , we shall note  $G - v = G - S$  and  $G - uv = G - E$ . The union (resp. intersection) of the graphs  $G_1$  and  $G_2$  is the graph  $G = G_1 \cup G_2$  (resp.  $G = G_1 \cap G_2$ ) such that  $V(G) = V(G_1) \cup V(G_2)$  (resp.  $V(G) = V(G_1) \cap V(G_2)$ ) and  $E(G) = E(G_1) \cup E(G_2)$  (resp.  $E(G) = E(G_1) \cap E(G_2)$ ).

A graph is said to be  $(k, 2)$ -minimal if it is not  $k$ -improper 2-choosable but each of its proper subgraphs is.

**Lemma 1** (**Škrekovski [8]**) *Let  $k \geq 1$  and let  $G$  be a  $(k, 2)$ -minimal graph. Then*

- (i)  $\delta \geq 2$ .
- (ii) Two  $(\leq k + 1)$ -vertices are not adjacent.

**Definition 3** Let  $D$  be a digraph. The outdegree (resp. indegree) of a vertex  $u$  in  $D$  is denoted by  $d_D^+(u)$  (resp.  $d_D^-(u)$ ). The degree of  $u$  is  $d_D(u) = d_D^-(u) + d_D^+(u)$ ; it is the degree of  $u$  in the underlying undirected graph.

If  $u$  and  $v$  are two of its vertices, a  $(u, v)$ -dipath is a directed path from  $u$  to  $v$ .

An *arborescence* is an oriented tree in which every path is directed from a vertex called the *root*. Note that in an arborescence every vertex except the root has indegree 1. The leaves of the arborescence are the vertices of outdegree 0. A vertex which is neither a leaf nor the root is an *internal vertex*. A *quasi-arborescence* is a directed graph obtained from an arborescence by identifying some leaves.

Let  $u$  be a vertex of a digraph  $D$ . The *outsection* of  $u$  in  $D$ , denoted  $A_D^+(u)$ , is the set of vertices  $v$  such that there is a  $(u, v)$ -dipath in  $D$ .

Let  $G$  be a  $(k, 2)$ -minimal graph. We partially orient  $G$  using the following process:

1. Orient each edge  $uv$  where  $v$  is a 2-vertex from  $u$  to  $v$ .
2. If  $k \geq 3$ , orient each edge  $uv$  where  $v$  is a 3-vertex from  $u$  to  $v$ .
3. While there is an unoriented edge  $uv$  where  $v$  an  $i$ -vertex with  $2 + k \leq i < \frac{3k}{2} + 2$  and outdegree  $i - 1$ , we orient it from  $u$  to  $v$ .

The digraph  $D$  induced by the oriented edges is called a *discharging digraph* of  $G$ .

The following proposition, whose proof is left to the reader, follows immediately from the definition of a discharging digraph.



**Proposition 1** *Let  $D$  be a discharging digraph of a  $(k, 2)$ -minimal graph.*

- *$D$  has no 2-circuit since two  $(\leq k + 1)$ -vertices are not adjacent by Lemma 1 (ii). So it has no circuit at all.*
- *If  $k \leq 2$ , only vertices of degree 2 or  $k + 2$  have indegree more than zero. If  $k \leq 3$ , only vertices of degree 2, 3 or  $k + 2$  have indegree more than zero.*
- *Every 2-vertex has indegree exactly 2 in  $D$  and if  $k \geq 3$ , every 3-vertex has indegree exactly 3.*
- *For every vertex  $u$ ,  $A_D^+(u)$  is a quasi-arborescence whose leaves have degree 2 (resp. 2 or 3) in  $G$  if  $k \leq 2$  (resp.  $k \geq 3$ ). In particular, the indegree of the leaves in  $A_D^+(u)$  is at most 2 (resp. 3).*

**Definition 4** A quasi-arborescence is a  $(k, 2)$ -quasi-arborescence if and only if:

- Every vertex has outdegree at most  $\max\{2, 2k - 1\}$ .
- Every leaf has indegree at most  $\min\{k, 3\}$ .

**Lemma 2** *Let  $k \geq 2$ . Let  $Q$  be a  $(k, 2)$ -quasi-arborescence rooted at  $u$  and  $L$  a 2-list-assignment of  $Q$ . Then any  $L$ -colouring of the leaves can be extended in a  $k$ -improper  $L$ -colouring of  $D$  such that  $u$  has improperty at most  $k - 1$ .*

**Proof.** By induction on the number of vertices of  $Q$ , the result being trivially true if  $|V(Q)| = 1$ .

Suppose now that  $|V(Q)| > 1$  and the result holds for smaller  $k$ -quasi-arborescences. Let  $v_1, \dots, v_s$  be the outneighbours of  $u$  in  $Q$ . Note that  $Q - u$  is the union of  $s$   $(k, 2)$ -quasi-arborescences  $Q_i$ ,  $1 \leq i \leq s$  rooted at  $v_i$  that are disjoint except possibly on their leaves.

Let  $c$  be an  $L$ -colouring of the leaves of  $Q$ . Then by induction it can be extended in a  $k$ -improper  $L$ -colouring of each of the  $Q_i$  so that  $im(v_i) \leq k - 1$ . Since a leaf of  $Q$  has indegree at most  $\min\{k, 3\}$  and  $im_Q(x) = im_{Q_i}(x)$  for every vertex of  $Q_i$  which is not a leaf, then the union of these colourings is a  $k$ -improper  $L$ -colouring of  $Q$  such that  $im(v_i) \leq k - 1$ .

Now, one of the two colours of  $L(u)$ , say  $\alpha$ , is assigned to at most  $k - 1$  neighbours of  $u$  since  $s \leq 2k - 1$ . Thus setting  $c(u) = \alpha$ , we obtain the desired colouring.  $\square$

Obviously, the above result cannot be extended for  $k = 1$  because it is hopeless to extend every  $L$ -colouring of the leaves in a colouring such that the root has improperty 0. However, one can prove the following weaker result:

**Lemma 3** *Let  $Q$  be a  $(1, 2)$ -quasi-arborescence rooted at  $u$ ,  $L$  a 2-list-assignment of  $Q$  with  $L(u) = \{\alpha, \beta\}$  and  $c$  an  $L$ -colouring of  $S$  the set of leaves of  $Q$  with indegree 1. One the following holds:*

- (i)  $c$  may be extended in a 1-improper  $L$ -colouring of  $Q$  such that  $im(u) = 0$ ;
- (ii)  $c$  may be extended in two different 1-improper  $L$ -colourings of  $Q$ , one such that  $c(u) = \alpha$  and one such that  $c(u) = \beta$ .

**Proof.** We proceed by induction on the number of vertices of  $Q$ . Let  $v_1$  and  $v_2$  be two outneighbours of  $u$  in  $Q$ .  $Q - u$  is the union of two  $(1, 2)$ -quasi-arborescences  $Q_1$  and  $Q_2$ , rooted at  $v_1$  and  $v_2$  respectively, that are disjoint except possibly on their leaves. Let  $S'$  be the set of leaves in  $Q_1 \cap Q_2$  and  $L(u) = \{\alpha, \beta\}$ . We  $L$ -colour the leaves of  $Q_i$  that have indegree 1 in  $Q_i$ . By induction, each of the  $Q_i$  satisfies (i) or (ii).

If at least one of the  $Q_i$  satisfies (ii), then one can extend  $c$  to  $Q_1 \cup Q_2$  such that  $\{c(v_1), c(v_2)\} \neq L(u)$ , say  $\alpha \notin \{c(v_1), c(v_2)\}$ . Moreover for any vertex  $x$  not in  $V(Q_i) \setminus S'$ ,  $im_Q(x) = im_{Q_i}(x) \leq 1$ . If a vertex  $s' \in S'$  has property 2 then its two neighbours are coloured the same. So recolouring  $s'$  with the colour of  $L(s') \setminus \{c(s')\}$ , we get a 1-improper  $L$ -colouring of  $Q_1 \cup Q_2$ . Hence setting  $c(u) = \alpha$ , we get a 1-improper  $L$ -colouring of  $Q$  such that  $im(u) = 0$ . Thus  $Q$  satisfies (i).

Suppose now  $Q_1$  and  $Q_2$  both satisfy (i). Then, possibly with recolouring of vertices of  $S'$  as before, one can extend  $c$  into a 1-improper  $L$ -colouring of  $Q_1 \cup Q_2$  such that  $im(v_1) = im(v_2) = 0$ . If  $\{c(v_1), c(v_2)\} \neq L(u)$ , say  $\alpha \notin \{c(v_1), c(v_2)\}$  then setting  $c(u) = \alpha$ , we get a 1-improper  $L$ -colouring of  $Q$  such that  $im(u) = 0$ . Thus  $Q$  satisfies (i). If not then assigning to  $u$  the colours  $\alpha$  and  $\beta$ , we get the two 1-improper  $L$ -colourings of  $Q$  satisfying (ii).  $\square$

**Lemma 4** Let  $k \geq 3$ . Let  $D$  be a discharging digraph of a  $(k, 2)$ -minimal graph  $G$ .

- (i) Every  $i$ -vertex with  $4 \leq i \leq k + 1$  has outdegree zero.
- (ii) Every  $i$ -vertex with  $2 + k \leq i \leq 2k + 1$  has outdegree less than  $i$ .

**Proof.**

- (i) Suppose, for a contradiction, that  $v$  is a vertex contradicting the assertion and let  $u$  be an outneighbour of  $v$ . Note that  $u$  is a  $(\leq \frac{3k}{2} + 2)$ -vertex by definition of a discharging digraph.

Let  $L$  be a 2-list-assignment of  $G$ . Let  $S$  be the set of leaves of  $A_D^+(u)$ . By minimality, let  $c$  be a  $k$ -improper  $L$ -colouring of  $G - A_D^+(u)$ .

$A_D^+(u)$  is a  $(k, 2)$ -quasi-arborescence: since it is dominated by  $v$  in  $D$ ,  $u$  has outdegree less than  $\frac{3k}{2} + 1$  and so at most  $2k - 1$ . Thus by Lemma 2, we can extend  $c$  to  $G - vu$  so that  $im(u) \leq k - 1$ . Since the leaves have degree at most  $3 \leq k$ , the impropriety of the leaves is at most  $3 \leq k$ . So we obtain a  $k$ -improper  $L$ -colouring of  $G - uv$ .

If  $c(u) \neq c(v)$  or  $im_{G-uv}(v) \leq k - 1$  then  $c$  is a  $k$ -improper  $L$ -colouring of  $G$ . Otherwise all the  $k + 1$  neighbours of  $v$  are coloured the same so recolouring  $v$  with its other allowed colour yields a  $k$ -improper  $L$ -colouring of  $G$ .

Hence  $G$  is  $k$ -improper 2-choosable which is a contradiction.

(ii) Suppose, for a contradiction, that  $v$  is an  $i$ -vertex contradicting the assertion.

Let  $L$  be 2-list-assignment of  $G$  and  $c$  a  $k$ -improper  $L$ -colouring of  $G - v$ . There is a colour of  $L(v)$ , say  $\alpha$ , that is assigned to at most  $k$  neighbours of  $v$ . Let  $v_1, \dots, v_s$  be these neighbours.

Let  $G' = G - \bigcup_{j=1}^s A_D^+(v_j)$ . And set  $c' = c$  for every vertex of  $G'$  and every leaf of the  $A_D^+(v_j)$ . By Lemma 2 applied to each  $A_D^+(v_j)$  (which are disjoint except possibly on their leaves), we can extend  $c'$  into a  $k$ -improper  $L$ -colouring of  $G - v$  such that  $im(v_j) \leq k - 1$  for  $1 \leq j \leq s$ . Now by definition of  $c'$ , the only neighbours of  $v$  that may be assigned  $\alpha$  by  $c'$  are those of  $\{v_1, \dots, v_s\}$ . Hence setting  $c'(v) = \alpha$ , the  $L$ -colouring  $c'$  is  $k$ -improper.

Hence  $G$  is  $k$ -improper 2-choosable which is a contradiction.  $\square$

Analogously, one can prove the following lemma when  $k = 2$ .

**Lemma 5** *Let  $D$  be a discharging digraph of a  $(2, 2)$ -minimal graph  $G$ .*

(i) *The outdegree of a 3-vertex is zero.*

(ii) *If  $v$  is an  $i$ -vertex with  $i \in \{4; 5\}$  then its outdegree is less than  $i$ .*

**Lemma 6** *Let  $D$  be a discharging of a  $(1, 2)$ -minimal graph  $G$ . There is no 3-vertex with outdegree 3 in  $D$ .*

**Proof.** Suppose, for a contradiction, that  $v$  is a 3-vertex with outdegree 3. Let  $u$  be an outneighbour of  $v$ . Let  $Q_1 = A_D^+(u)$ ,  $Q_2 = A_{D-vu}^+(v)$ ,  $S$  be the set of leaves of  $A_D^+(v)$  with indegree 1 in  $A_D^+(v)$  and  $S'$  the set of leaves with indegree 2 in  $A_D^+(v)$ .

Let  $L$  be a 2-list-assignment of  $G$ . By minimality of  $G$ , let  $c$  be a 1-improper  $L$ -colouring of  $G - A_D^+(v)$ . Vertices not in  $S$  have no neighbour in  $G - A_D^+(v)$  and every vertex of  $S$  has exactly one neighbour in  $G - A_D^+(v)$ . Extend  $c$  to  $S \cup S'$  by assigning to each vertex of  $S$  a colour of its list not assigned to its neighbour in  $G - A_D^+(v)$  and any colour of its list to a vertex of  $S'$ .

Now  $Q_1$  and  $Q_2$  satisfy either (i) or (ii) of Lemma 3. If one of them satisfies (ii), then possibly with recolouring of vertices of  $S'$  one can extend  $c$  into a 1-improper  $L$ -colouring of  $G - vu$  such that  $c(v) \neq c(u)$ . Hence  $c$  is a 1-improper  $L$ -colouring of  $G$ .

If  $Q_1$  and  $Q_2$  satisfies both (i), then possibly with recolouring of vertices of  $S'$  one can extend  $c$  into a 1-improper  $L$ -colouring of  $G - vu$  such that  $im(v) = im(u) = 0$ . Hence  $c$  is a 1-improper  $L$ -colouring of  $G$ .

So  $G$  is 1-improper 2-choosable which is a contradiction.  $\square$

**Proof of Theorem 2.** Let  $G$  be a  $(k, 2)$ -minimal graph and  $D$  a discharging digraph of  $G$ . We start with a charge  $w(v) = d(v)$  on each vertex and we apply the following discharging rule: every vertex gives  $\frac{k}{k+2}$  to each of its outneighbours.

Let us examine the new charge  $w'(v)$  of a vertex  $v$ :

- If  $v$  is a 2-vertex, it has indegree 2 so its new charge is  $w'(v) = 2 + \frac{2k}{k+2} = \frac{4k+4}{k+2}$ .
- If  $v$  is a 3-vertex and  $k \geq 3$ , it has indegree 3 so its new charge is  $w'(v) = 3 + 3 \times \frac{k}{k+2} = \frac{6k+6}{k+2} > \frac{4k+4}{k+2}$ . If  $v$  is a 3-vertex and  $k = 2$  then it has outdegree 0 by Lemma 5 and indegree 0 by construction so  $w'(v) = 3$ .
- If  $4 \leq d(v) \leq k+1$ , ( $k \geq 3$ ), then by Lemma 4 (i),  $v$  has outdegree zero so its charge is  $d(v) \geq 4 > \frac{4k+4}{k+2}$ .
- If  $k+2 \leq d(v) < \frac{3k}{2} + 2$  then either  $v$  has outdegree at most  $d(v) - 2$  and so its new charge is at least  $d(v) - (d(v) - 2) \times \frac{k}{k+2} = \frac{2d(v)}{k+2} + \frac{2k}{k+2} \geq 2 + \frac{2k}{k+2} = \frac{4k+4}{k+2}$ , or by Lemmas 4, 5 and 6, it has outdegree  $d(v) - 1$ . In this case, by definition of a discharging digraph,  $v$  has indegree 1 so its new charge is:

$$d(v) - (d(v) - 1) \times \frac{k}{k+2} + \frac{k}{k+2} = d(v) - (d(v) - 2) \times \frac{k}{k+2} \geq \frac{4k+4}{k+2}.$$

- If  $\frac{3k}{2} + 2 \leq d(v) \leq 2k+1$ , ( $k \geq 2$ ), then by Lemmas 4 and 5,  $v$  has outdegree at most  $d(v) - 1$ . So  $w'(v) \geq d(v) - (d(v) - 1) \times \frac{k}{k+2} = \frac{2d(v)}{k+2} + \frac{k}{k+2} \geq \frac{3k+4+k}{k+2} = \frac{4k+4}{k+2}$ .
- If  $d(v) \geq 2k+2$ , then  $w'(v) \geq d(v)(1 - \frac{k}{k+2}) = \frac{2d(v)}{k+2} \geq \frac{4k+4}{k+2}$ .

$$\text{Hence } Mad(G) \geq \frac{1}{|V|} \sum_{v \in V} d(v) = \frac{1}{|V|} \sum_{v \in V} w'(v) \geq \frac{4k+4}{k+2}.$$

□

**Corollary 1** *Let  $G$  be a planar graph of girth  $g$ .*

1. *If  $g \geq 8$  then  $G$  is 1-improper 2-choosable, so  $g_1 \leq 8$ .*
2. *If  $g \geq 6$  then  $G$  is 2-improper 2-choosable, so  $g_3 \leq g_2 \leq 6$ .*
3. *If  $g \geq 5$  then  $G$  is 4-improper 2-choosable, so  $g_k \leq 4$  for  $k \geq 5$ .*

## 2.2 Upper bound for $M(k, 2)$

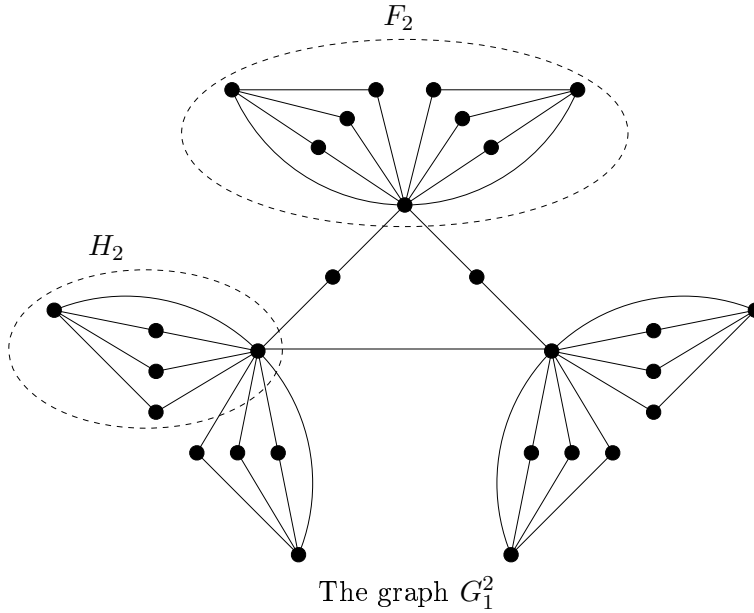
Let us fix  $k \geq 1$ . In this section, we shall construct a family of graphs  $(G_n^k)_{n \geq 1}$  such that for all  $n \geq 1$ :

- $G_n^k$  is not  $k$ -improper 2-colourable.
- $Mad(G_n^k) = \frac{2n(4k^2 + 6k + 4) + 4k^2 + 6k + 2}{2n(k^2 + 2k + 2) + (k+1)^2}$ .

Hence we will deduce:

**Theorem 3** For all  $k \geq 1$ ,  $M(k, 2) \leq \frac{4k^2 + 6k + 4}{k^2 + 2k + 2} = 4 - \frac{2k + 4}{k^2 + 2k + 2}$ .

We denote by  $H_k$  the graph composed of two adjacent vertices  $u$  and  $v$  also connected by  $k + 1$  disjoint paths of length 2. Take  $k$  copies of  $H_k$  and create the graph  $F_k$  by identifying the vertex  $v$  of each copy. Note that  $F_k$  has one vertex of degree  $k(k + 2)$ ,  $k$  vertices of degree  $k + 2$  and  $k(k + 1)$  vertices of degree 2. Now we take  $2n + 1$  copies of  $F_k$  and we join the vertices  $v$  of each copy creating a cycle of size  $2n + 1$ . At last we make a subdivision of all the edges of the cycle but one so as to obtain the graph  $G_n^k$ .



**Lemma 7**  $G_n^k$  is not  $k$ -improper 2-colourable.

**Proof.** First remark that in any  $k$ -improper 2 colouring of  $H_k$ ,  $v$  has impropriety at least 1. Indeed  $v$  is a  $(k + 2)$  vertex in  $H_k$ , so if it has impropriety zero then its  $k + 2$  neighbours are coloured the same, but this is impossible since  $u$  is a neighbour of  $v$  adjacent to the  $k + 1$  remaining neighbours. Hence in any  $k$ -improper colouring of  $F_k$ ,  $v$  has impropriety  $k$ . So in order to colour the whole graph, we must properly colour the subdivided cycle with 2 colours, which is impossible.  $\square$

**Lemma 8** The maximum average degree of  $G_n^k$  is  $M_n^k = \frac{(2n+1)(4k^2+6k+4)-2}{(2n+1)(k^2+2k+2)-1}$ .

**Proof.** As it is easily seen, the maximum average degree of  $G$  is its average degree, which is:

$$\frac{(2n+1)[(1 \times k(k+2) + 2) + (k \times (k+2)) + (k(k+1) \times 2)] + (2n) \times 2}{(2n+1)(1+k+k(k+1)) + 2n} = M_n^k.$$

□

### 3 Improper $l$ -choosability, $l \geq 2$

#### 3.1 Lower bound for $M(k, l)$

In this subsection, we shall prove the following theorem:

**Theorem 4** *For all  $l \geq 2$  and all  $k \geq 0$ , all graphs of maximum average degree less than  $\frac{l(l+2k)}{l+k}$  are  $k$ -improper  $l$ -choosable.*

The result of the theorem is trivial if  $k = 0$  since a graph of maximum average degree less than  $l$  is  $(l-1)$ -degenerate (i.e. each of its subgraph has a vertex of degree at most  $l-1$ ). Hence it is  $l$ -choosable. For bigger values of  $k$ , we will need some preliminary results.

**Definition 5** A graph is said to be  $(k, l)$ -minimal if it is not  $k$ -improper  $l$ -choosable but every of its proper subgraph is.

**Lemma 9** *Let  $G$  be a graph,  $L$  a list-assignment and  $c$  an  $L$ -colouring. If a vertex  $v$  has impropriety at least  $d(v) - |L(v)| + 2$  under  $c$ , then there exists an  $L$ -colouring  $c'$  of  $G$  such that  $c'(u) = c(u)$  if  $u \neq v$  and  $im_{c'}(v) = 0$ .*

**Proof.** Let  $c(v) = \alpha$ . Then  $v$  has at most  $d(v) - (d(v) - |L(v)| + 2) = |L(v)| - 2$  neighbours that are not coloured with  $\alpha$ . Hence there exists a colour  $\beta \in L(v)$  that does not colour any neighbour of  $v$ . So setting  $c(v) = \beta$  we obtain the desired colouring. □

We now prove a generalization of Lemma 1.

**Lemma 10** *Let  $k \geq 1$  and let  $G$  be a  $(k, l)$ -minimal graph. Then:*

- (i)  $\delta \geq l$ .
- (ii) Two  $(\leq l + k - 1)$ -vertices are not adjacent.

**Proof.**

- (i) Let  $L$  be an  $l$ -list-assignment and suppose  $v$  is a  $(\leq l - 1)$ -vertex. By minimality let  $c$  be a  $k$ -improper  $L$ -colouring of  $G - v$ . As  $v$  has at most  $l - 1$  neighbours in  $G$ , there exists a colour, say  $\alpha$ , that is not assigned to any neighbour of  $v$ . Hence colouring  $v$  with  $\alpha$  yields a  $k$ -improper  $L$ -colouring of  $G$ .

Hence  $G$  is  $k$ -improper  $l$ -choosable, a contradiction.

- (ii) Let  $L$  be an  $l$ -list-assignment and suppose, for a contradiction, that  $u$  and  $v$  are two neighbours of degree at most  $l + k - 1$ . By minimality, let  $c$  be a  $k$ -improper  $L$ -colouring of  $G - \{uv\}$ . Then  $c$  is an  $L$ -colouring of  $G$  such that each vertex has improperity at most  $k$ , except possibly  $u$  and  $v$  which may have improperity  $k + 1$ . But in this case we use Lemma 9 to recolour these vertices and obtain a  $k$ -improper  $L$ -colouring of  $G$ .

Hence  $G$  is  $k$ -improper  $l$ -choosable, a contradiction.

□

**Definition 6** Let  $G$  be a  $(k, l)$ -minimal graph. We partially orient  $G$  using the following process:

1. Orient each edge  $uv$  where  $v$  is a  $(\leq l + k - 1)$ -vertex from  $u$  to  $v$ .
2. While there is an  $i$ -vertex  $v$  with  $l + k \leq i < l + k + \frac{k}{7}$  having outdegree exactly  $i - l + 1$  and indegree 0, we orient one of its unoriented incident edges  $uv$  from  $u$  to  $v$ .

The digraph  $D$  induced by the oriented edges is called a *discharging digraph* of  $G$ .

The following remark follows from the definition of a discharging digraph.

**Remark 1**

- Only vertices of degree less than  $l + k + \frac{k}{7}$  can have indegree more than zero.
- For  $i \leq l + k - 1$ , every  $i$ -vertex has indegree exactly  $i$  in  $D$ .

**Definition 7** A quasi-arborescence rooted at  $u$  is a  $(k, l)$ -quasi-arborescence if and only if:

- Every vertex has outdegree at most  $\max\{2, 2k - 1\}$ .
- Every leaf has indegree at most  $l + k - 1$

Now we generalize Lemmas 2 and 3.

**Lemma 11** Let  $k \geq 2$  and let  $Q$  be a  $(k, l)$ -quasi-arborescence rooted at  $u$ . Let  $L$  be a list-assignment of  $Q$  such that  $|L(v)| \geq \max\{1, d_Q(v) - k + 1\}$  if  $v$  is a leaf and  $|L(v)| \geq 2$  otherwise. We denote by  $S$  the set of leaves that have indegree at least  $k + 1$  in  $Q$  (and hence a colour-list of size at least 2). Any  $L$ -colouring of the leaves extends in an  $L$ -colouring of  $Q$  such that:

- $im(u) \leq k - 1$ .
- $\forall v \notin S, im(v) \leq k$ .

Furthermore, possibly by recolouring some vertices of  $S$ , this  $L$ -colouring of  $G$  can be made  $k$ -improper.

**Proof.** By induction on the number of vertices of  $Q$ , the result being trivially true if  $|V(Q)| = 1$ .

Suppose now that  $|V(Q)| > 1$  and the result holds for smaller  $(k, l)$ -quasi-arborescences. Let  $v_1, \dots, v_s$  be the outneighbours of  $u$  in  $Q$ . Note that  $Q - u$  is the union of  $s$   $(k, l)$ -quasi-arborescences  $Q_i$  rooted at  $v_i$ ,  $1 \leq i \leq s$ , that are disjoint except possibly on their leaves. We start by  $L$ -colouring all the leaves of  $Q$ .

By induction we extend this colouring to an  $L$ -colouring of each of the  $Q_i$  such that  $im(v_i) \leq k - 1$ . Note that  $im_Q(x) = im_{Q_i}(x) \leq k$  for every vertex of  $Q_i$  which is not a leaf and  $im_Q(x) \leq k$  for each leaf not in  $S$ . One of the two colours of  $L(u)$ , say  $\alpha$ , is assigned to at most  $k - 1$  neighbours of  $u$  since  $deg(u) \leq 2k - 1$ . Hence setting  $c(u) = \alpha$ , we obtain the first desired colouring.

Now, we can recolour each leaf  $f$  of  $S$  with impropriety at least  $k + 1$  using Lemma 9 since  $d_Q(f) - |L(f)| + 2 \leq d_Q(f) - d_Q(f) + k - 1 + 2 = k + 1$ . This concludes the proof.  $\square$

The above result cannot be extended for  $k = 1$ . However one can prove the following:

**Lemma 12** *Let  $Q$  be a  $(1, l)$ -quasi-arborescence rooted at  $u$ ,  $L$  be a list-assignment of  $Q$  such that  $|L(v)| \geq 2$  if  $v$  is not a leaf, and  $|L(v)| \geq d_Q(v)$  otherwise. We denote by  $S$  the set of leaves with indegree at least 2. Let  $c$  be an  $L$ -colouring of the leaves. One of the followings holds:*

- (i)  *$c$  can be extended in an  $L$ -colouring of  $Q$  such that  $im(u) = 0$  and  $im(v) \leq 1$  if  $v \notin S$ ;*
- (ii)  *$c$  can be extended in two different  $L$ -colourings of  $Q$   $c_1$  and  $c_2$  such that  $c_1(v) = c_2(v)$  if  $v \neq u$  and  $im^{c_i}(v) \leq 1$  if  $v \notin S$ .*

Furthermore, possibly by recolouring vertices of  $S$ , all these  $L$ -colourings can be made 1-improper.

Moreover, if  $|L(u)| \geq 3$  then (i) holds.

**Proof.** By induction on the number of vertices, the result being obvious if  $|V(Q)| = 1$ .

$Q - u$  is the union of two  $(1, l)$ -quasi-arborescences  $Q_1$  and  $Q_2$  rooted at  $v_1$  and  $v_2$  respectively. They are disjoint except possibly on their leaves. Let  $c$  be an  $L$ -colouring of the leaves of  $Q$ . By induction we extend  $c$  to  $Q_1$  and  $Q_2$ . Note that for each vertex  $v$  of  $Q - S$   $im_Q(v) = im_{Q_i}(v) \leq 1$ .

If at least one of the  $Q_i$  satisfies (ii), or if  $|L(u)| \geq 3$ , we can suppose that  $\{c(v_1), c(v_2)\} \neq L(u)$  and hence we extend  $c$  into an  $L$ -colouring of  $Q$  fulfilling (i).



If both  $Q_1$  and  $Q_2$  satisfy (i), then either  $c(v_1) = c(v_2)$  and hence setting  $c(u) \in L(u) \setminus \{c(v_1)\}$  yields an  $L$ -colouring of  $Q$  that satisfies (i); or colouring  $u$  with two colours of its list gives the two desired colourings of (ii).

Now we can recolour with impropriety zero each leaf  $f \in S$  that has impropriety at least 2 in  $Q$  using Lemma 9, since  $d_Q(f) - |L(f)| + 2 \leq 2$ . This concludes the proof.  $\square$

Using these results, we can say more about the structure of a discharging digraph. The following lemma generalizes Proposition 1.

**Lemma 13** *Let  $D$  be a discharging digraph of a  $(k, l)$ -minimal graph  $G$ .*

- (i) *Every vertex  $u$  with  $l + k \leq d(u) \leq l + 2k - 1$  has outdegree at most  $d(u) - l + 1$ . In particular,  $D$  is acyclic.*
- (ii) *For every vertex  $u$ ,  $A_D^+(u)$  is a  $(k, l)$ -quasi-arborescence. In particular, the indegree of the leaves in  $A_D^+(u)$  is at most  $l + k - 1$ .*

**Proof.** (ii) follows easily from (i). So, let us prove (i).

Let  $L$  be an  $l$ -list-assignment of  $G$ . First,  $D$  has no 2-circuit since two  $(\leq l + k - 1)$ -vertices are not adjacent by Lemma 10. Note also that in order to create a circuit in  $D$ , it is necessary to create a vertex  $u$  of outdegree at least  $d(u) - l + 2$ . Now suppose, for a contradiction, that  $D$  contains a vertex  $u$  of outdegree at least  $d(u) - l + 2$  and let  $D'$  be the digraph obtained just after having created the first such vertex  $u$ . Let  $u \rightarrow v$  be the last edge that is oriented in  $D'$ .  $u$  has  $d(u) - l + 2$  outneighbours (including  $v$ ) while  $v$  has  $d(v) - l + 1$  outneighbours. We distinguish two cases depending whether the orientation of  $uv$  creates a circuit (which is necessary the first), or not.

**First Case:** the orientation of  $uv$  creates a circuit  $C$ . Let  $w$  be the inneighbour of  $u$  in  $C$ . We define  $Q_1 = A_{D' - uv}^+(v)$ ,  $Q_2 = A_{D' - uv}^+(u)$  and  $Q = Q_1 \cup Q_2$ . Note that  $Q_1$  and  $Q_2$  are  $(k, l)$ -quasi-arborescences which are disjoint, except possibly on some leaves. In particular the outdegree in  $D'$  of every internal vertex  $x$  of  $Q$  is at most  $d_G(x) - l + 1$ . More precisely every internal vertex  $x \neq w$  satisfies  $d_{D'}^+(x) = d_G(x) - l + 1$  while  $d_{D'}^+(w) = d_G(w) - l$  and for all every vertex  $v$   $d_{D'}^-(v) = 1$ . Recall that  $d_Q(w) = d^+(w) + d^-(w)$ . Let  $F$  be the set of leaves in  $Q$ ,  $S$  the set of leaves that have indegree at least  $k + 1$  in  $Q$  and  $\bar{S} = F \setminus S$ . We define  $\bar{Q} = Q - \bar{S}$ . By minimality, let  $c$  be a  $k$ -improper  $L$ -colouring of  $G' = G - \bar{Q}$ . Let  $f \in S$ : if  $f$  has impropriety at least  $k - d_Q(f) + 1$ , then using Lemma 9 we recolour it with impropriety 0 since  $d_{G'}(f) - |L(f)| + 2 = d_G(f) - d_Q(f) - l + 2 \leq l + k - 1 - d_Q(f) - l + 2 = k - d_Q(f) + 1$ . Now, let  $L_1$  be the following list-assignment of  $Q_1$ :  
 $L_1(x) = L(x) \setminus \{\alpha \mid \exists z \in N_{G-Q_1}(x), c(z) = \alpha\}$  if  $x \notin \bar{S}$ , and  $L_1(x) = \{c(x)\}$  otherwise.  
 Note that if  $x \neq w$  is an internal vertex then:

$$|L_1(x)| \geq l - (d_G(x) - d_{Q_1}(x)) = l - d_G(x) + d_G(x) - l + 1 + 1 = 2$$

and since  $d^+(w) = d_G(w) - l$  but  $u$  is yet uncoloured:

$$|L_1(w)| \geq l - (d_G(w) - d_{Q_1}(w)) + 1 = l - d_G(w) + d_G(w) - l + 1 + 1 = 2.$$

For the root  $v$ ,  $d^-(v) = 0$  but  $u$  is uncoloured yet so:

$$|L_1(v)| \geq l - (d_G(v) - d_{Q_1}(v)) + 1 = l - d_G(v) + d_G(v) - l + 1 + 1 = 2,$$

and for a leaf  $f \in S$ :

$$|L_1(f)| \geq l - d_G(f) + d_Q(f) \geq l - (l + k - 1) + d_Q(f) = d_Q(f) - k + 1.$$

Thus we may apply Lemmas 11 and 12. To do so, we  $L_1$ -colour all the leaves in  $Q$ .

Suppose first  $k \geq 2$ . By Lemma 11, we obtain an  $L_1$ -colouring  $c_1$  of  $Q_1$  such that  $im_{Q_1}^{c_1}(v) \leq k - 1$ . Note that  $c_1$  extends  $c$  into an  $L$ -colouring of  $G - Q_2$  such that each vertex has impropriety at most  $k$  except possibly some vertices of  $S$ . Furthermore,  $im_{G-Q_2}(v) \leq k - 1$ . We define a list-assignment  $L_2$  of  $Q_2$  by  $L_2(u) = L(u) \setminus \{\alpha \mid \exists z \neq v \in N_{G-Q_2}(u), c(z) = \alpha\}$ ,  $L_2(x) = \{c(x)\}$  if  $x$  is a leaf and  $L_2(x) = L(x) \setminus \{\alpha \mid \exists z \in N_{G-Q_2}(x), c(z) = \alpha\}$  otherwise. Note that we have  $|L_2(u)| \geq 2$ . We now apply Lemma 11 so as to get an  $L_2$ -colouring of  $Q_2$  and hence an  $L$ -colouring of  $G$ . Every vertex not in  $S \cup \{u, v\}$  has impropriety at most  $k$ . If  $x \in \{u, v\}$  then:  $im_G(x) \leq im_{G-Q_2}(x) + 1 \leq k - 1 + 1 = k$  since there cannot be in  $L_2(u)$  the colour of a neighbour of  $u$  in  $G - (Q_2 - v)$ . If  $f \in S$  has impropriety at least  $k + 1$ , then we recolour it with impropriety 0 using Lemma 9 since  $d_G(f) - |L(f)| + 2 \leq l + k - 1 - l + 2 = k + 1$ . Thus we obtain a  $k$ -improper  $L$ -colouring of  $G$ .

Suppose now  $k = 1$ . Applying Lemma 12, we obtain an  $L_1$ -colouring of  $G - Q_2$  such that every vertex not in  $S$  has impropriety at most 1, and either  $v$  has impropriety 0 (i), or it has impropriety 1 and we can indifferently colour it with two colours of its list (ii). Note that if  $v$  has one neighbour distinct from  $u$  which is an internal vertex in  $Q_2$  then  $|L_1(v)| \geq 3$  so we may suppose that  $v$  fulfils (i). Defining  $L_2$  as before, we can apply Lemma 12 to  $Q_2$  so as to obtain an  $L_2$ -colouring of  $Q_2$  and hence an  $L$ -colouring of  $G$  such that  $u$  fulfils (i) or (ii). Now, every vertex not in  $S \cup \{u, v\}$  has impropriety at most 1. If  $v$  satisfies (i), then either  $u$  also satisfies (i) or  $u$  satisfies (ii) but in this case we may suppose  $u$  and  $v$  are coloured differently so in all cases they have impropriety at most 1 in  $G$ . If  $v$  satisfies (ii), then the only neighbour of  $v$  in  $Q_2$  is  $u$ . Hence we may safely suppose that  $u$  and  $v$  are coloured differently, so they have impropriety at most 1 in  $G$ .

Finally, we can recolour each leaf of  $S$  that has impropriety at least 2 by using Lemma 9 and thus we obtain a 1-improper  $L$ -colouring of  $G$ .

Hence  $G$  is  $k$ -improper  $l$ -choosable, a contradiction.

**Second Case:** there is no circuit in  $D'$ . Then  $Q = A_{D'}^+(u)$  is a quasi-arborescence. Moreover each internal vertex  $v$  has outdegree at most (and hence exactly)  $d(v) - l + 1$ . Let  $v_1, \dots, v_s$  be the outneighbours of  $u$ , we define  $Q_j = A_{D'}^+(v_j)$ ,  $1 \leq j \leq s$ . The  $Q_i$  are  $(k, l)$ -quasi-arborescences that are disjoint except possibly on their leaves. Let  $F$  be the set of leaves in  $Q$ ,  $S$  the set of leaves with indegree at least  $k + 1$  in  $Q$  and  $\bar{S} = F \setminus S$ . We define  $\dot{Q} = Q - \bar{S}$ . Let  $L$  be an  $l$ -list-assignment of  $G$ . By minimality, let  $c$  be a  $k$ -improper  $L$ -colouring of  $G' = G - \dot{Q}$ . Let  $f$  be a leaf in  $\bar{S}$ . If  $f$  has impropriety at least

$k - d_Q(f) + 1$ , we recolour it with improperty 0 using Lemma 9 since:  $d_{G'}(f) - |L(f)| + 2 \leq d_G(f) - d_Q(f) - l + 2 \leq l + k - 1 - d_Q(f) - l + 2 = k - d_Q(f) + 1$ .

For each vertex  $v \in Q$ , we define  $L'(v) = L(v) \setminus \{\alpha \mid \exists w \in N_G(v), c(w) = \alpha\}$  if  $v \notin \bar{S}$  and  $L'(v) = \{c(v)\}$  otherwise. Note that for an internal vertex  $v$ :

$$|L'(v)| \geq l - (d_G(v) - d_Q(v)) = l - d_G(v) + d_Q(v) - l + 1 + 1 = 2.$$

For a leaf  $f \in S$ :

$$|L'(f)| \geq l - d_G(f) + d_Q(f) \geq l - (l + k - 1) + d_Q(f) = d_Q(f) - k + 1.$$

Suppose first  $k \geq 2$ . We  $L'$ -colour all the leaves, use Lemma 11 so as to extend it into an  $L'$ -colouring of each of the  $Q_i$ , and possibly with recolouring some leaves in  $S$  we get a  $k$ -improper  $L$ -colouring of  $G - u$  such that  $im(v_j) \leq k - 1$ ,  $1 \leq j \leq s$ .

Now  $|L'(u)| \geq |L(u)| - d(u) + d_{D'}^+(u) = l - d(u) + d(u) - l + 2 \geq 2$ . And  $u$  has  $d^+(u) = d(u) - l + 2 \leq 2k + 1$  outneighbours in  $D'$ . Thus there is a colour of  $L'(u)$ , say  $\alpha$ , that is assigned to at most  $k$  outneighbours of  $u$ . Thus setting  $c(u) = \alpha$  yields a  $k$ -improper  $L$ -colouring of  $G$  by definition of  $L'$ .

Suppose now  $k = 1$ . We  $L'$ -colour all the leaves, use Lemma 12 so as to extend it in an  $L'$ -colouring of each of the  $Q_i$ , and possibly with recolouring some leaves in  $S$  we get a 1-improper  $L$ -colouring of  $G - u$  such that for each  $v_j$  either  $im(v_j) = 0$  or  $v_j$  can safely be recoloured with another colour of  $L'(v_j)$ .

The same calculation as above shows there exists a colour of  $L'(u)$ , say  $\alpha$ , that is assigned to at most 1 neighbour of  $u$ , say  $v_i$ . We set  $c(u) = \alpha$ . If  $v_i$  satisfies the first condition, we have a 1-improper  $L$ -colouring of  $G$ . If  $v_i$  satisfies the second condition then we may suppose that  $c(u) \neq c(v)$  and thus we also have a 1-improper  $L$ -colouring of  $G$ .

Hence  $G$  is  $k$ -improper  $L$ -choosable, a contradiction □

**Proof of Theorem 4.** Let  $G$  be a  $(k, l)$ -minimal graph and  $D$  a discharging digraph of  $G$ . We start with a charge  $w(v) = d(v)$  on each vertex and we apply the following discharging rule: every vertex gives  $\frac{k}{l+k}$  to each of its outneighbours.

Let us examine the new charge  $w'(v)$  of a vertex  $v$ :

- If  $d(v) \leq l + k - 1$  it has indegree  $d(v)$  so its new charge is  $w'(v) = d(v) + \frac{d(v)k}{l+k} \geq l + \frac{lk}{l+k}$ .
- If  $l + k \leq d(v) < l + k + \frac{k}{7}$  then either  $v$  has outdegree at most  $d(v) - l$  and so its new charge is at least  $d(v) - (d(v) - l) \times \frac{k}{l+k} = \frac{ld(v)}{l+k} + \frac{lk}{l+k} \geq l + \frac{lk}{l+k}$ , or by Lemma 13, it has outdegree  $d(v) - l + 1$ . In this case, by definition of a discharging digraph,  $v$  has indegree 1 so its new charge is:

$$w'(v) = d(v) - (d(v) - l + 1) \times \frac{k}{l+k} + \frac{k}{l+k} = d(v) - (d(v) - l) \times \frac{k}{l+k} \geq l + \frac{k}{l+k}.$$

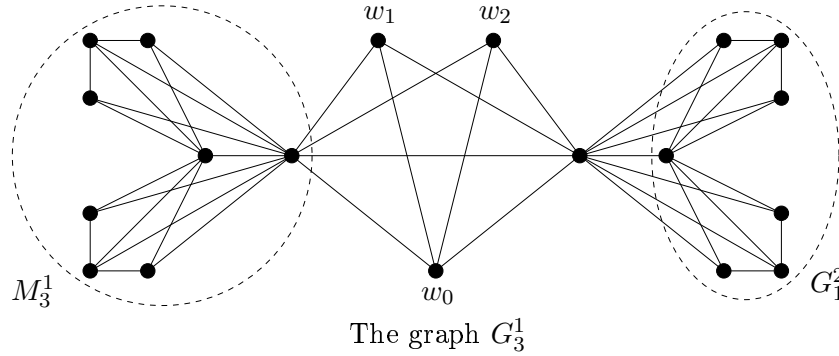
- If  $l + k + \frac{k}{l} \leq d(v) \leq l + 2k - 1$ , then by Lemma 13,  $v$  has outdegree at most  $d(v) - l + 1$ . So  $w'(v) \geq d(v) - (d(v) - l + 1) \times \frac{k}{l+k} = \frac{ld(v)}{l+k} + \frac{kl-k}{l+k} \geq \frac{l^2+2kl}{l+k} = l + \frac{kl}{l+k}$ .
- If  $d(v) \geq l + 2k$ , then  $w'(v) \geq d(v)(1 - \frac{k}{l+k}) = \frac{ld(v)}{l+k} \geq \frac{l^2+2kl}{l+k} = l + \frac{kl}{l+k}$ .

$$\text{Hence } Mad(G) \geq \frac{1}{|V|} \sum_{v \in V} d(v) = \frac{1}{|V|} \sum_{v \in V} w'(v) \geq l + \frac{kl}{l+k}.$$

□

### 3.2 Upper bound for $M(k, l)$

In this section we shall construct for all  $l \geq 2$  and all  $k \geq 1$ , a graph  $G_l^k$  which is not  $k$ -improper  $l$ -colourable. So its maximum average degree will give an upper bound for  $M(k, l)$ . To construct  $G_2^k$ , take  $k+1$  copies of  $H_k$  (defined in Subsection 2.2) and identify their vertex  $v$ . We define  $G_l^k$ ,  $l \geq 3$ , inductively. First we create the graph  $M_l^k$  by taking  $k$  copies of  $G_{l-1}^k$  and adding a vertex  $w$  which we join to every other vertices. Then we take  $l-1$  copies  $M^1, \dots, M^{l-1}$  of  $M_l^k$  and we join all the vertices  $w_1, \dots, w_{l-1}$  (so that they form a complete graph of size  $l-1$ ). Now, we add  $k+2$  vertices  $z_0, z_1, \dots, z_{k+1}$  each joined to each of the  $w_i$ ,  $1 \leq i \leq l-1$ . Last we add the edges  $z_0 z_i$  for  $1 \leq i \leq k+1$ .



**Lemma 14** For all  $l \geq 2$  and all  $k \geq 1$ , the graph  $G_l^k$  is not  $k$ -improper  $l$ -colourable.

**Proof.** The result is clear for  $G_2^k$ . Suppose the result is true for  $l-1 \geq 2$  and let us prove it for  $G_l^k$ . First note that in any  $k$ -improper  $l$ -colouring of  $M_l^k$ , the vertex  $w_i$  has property  $k$ . Indeed,  $w_i$  has a neighbour of its colour in each copy of  $G_{l-1}^k$  since otherwise  $G_{l-1}^k$  would be  $k$ -improper  $(l-1)$ -colourable. Hence each of the  $w_i$ ,  $1 \leq i \leq l-1$ , cannot have any neighbour of its colour in  $G_l^k - M^i$ . In particular, as the subgraph induced by  $w_1, \dots, w_{l-1}$

is complete, all the  $z_i$ ,  $0 \leq i \leq k+1$ , must be coloured the same. But then  $w_0$  must have improperly  $k+1$ .  $\square$

**Lemma 15** *The number of vertices of  $G_l^k$  is:*

$$n_l^k = 2l + (l+1)k + \sum_{i=2}^l \frac{(l-1)!}{(l-i)!} k^i.$$

*In particular, it is a polynomial in  $k$  of degree  $l$  and dominant coefficient  $(l-1)!$ .*

**Proof.**  $n_l^k$  satisfies:  $n_2^k = k^2 + 3k + 3$  and  $\forall l \geq 3, n_l^k = (k \times n_{l-1}^k + 1) \times (l-1) + k + 2$ .  $\square$

Let  $s_l^k$  denotes the sum of the degrees of the vertices in  $G_l^k$ . We have the following result:

**Lemma 16**  *$s_l^k$  is a polynomial in  $k$  of degree  $l$  whose dominant coefficient is  $2l!$ .*

**Proof.**  $s_l^k$  satisfies:  $s_2^k = 4k^2 + 10k + 6$  and  $s_l^k = (l-1)(k \times s_{l-1}^k + 2k \times n_{l-1}^k + l + k) + (l+1)k + 2l$  if  $l \geq 3$ . Hence it is a polynomial in  $k$  of degree  $l$ . Furthermore, denoting by  $c_l^k$  its dominant coefficient, we have:  $c_2^k = 4$  and  $\forall l \geq 3, c_l^k = (l-1) \times c_{l-1}^k + 2k \times (l-1)!$ . Thus  $c_l^k = 2l!$ .  $\square$

**Proposition 2**  $\lim_{k \rightarrow \infty} Mad(G_l^k) = 2l$ .

**Proof.** It is clear that the maximum average degree of  $G_l^k$  is its average degree. Then by Lemmas 15 and 16, we have:

$$\lim_{k \rightarrow \infty} Mad(G_l^k) = 2 \frac{l!}{(l-1)!} = 2l.$$

$\square$

**Corollary 2** *For any fixed  $l$ ,  $\lim_{k \rightarrow +\infty} M(k, l) = 2l$ .*

**Proof.** It follows from Theorem 4 and Proposition 2.  $\square$

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