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# Over-constrained Weierstrass iteration and the nearest consistent system

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**Abstract:** We propose a generalization of the Weierstrass iteration for over-constrained systems of equations and we prove that the proposed method allows us to find the nearest system which has at least $k$ common roots and which is obtained via a perturbation of prescribed structure. In the univariate case we show the connection of our method to the optimization problem formulated by Karmarkar and Lakshman for the nearest GCD. In the multivariate case we generalize the expressions of Karmarkar and Lakshman, and give a simple iterative method to compute the optimum. The arithmetic complexity of the iteration is detailed.

**Key-words:** Over-constrained algebraic systems solving, approximated GCD, over-constrained Weierstrass method

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# Over-constrained Weierstrass iteration and the nearest consistent system

**Résumé :** On propose une généralisation de l'itération de Weierstrass pour les systèmes surcontraints et on prouve que la méthode proposée permet d'approcher le système le plus proche ayant $k$ racines obtenu par perturbation structurée. Dans le cas univarié, on connecte notre méthode avec un problème d'optimisation proposé pas Karmarkar et Lakshman pour le PGCD approché. Dans le cas multivarié, on généralise la formule de Karmarkar et Lakshman et on donne une méthode itérative très simple pour approcher la solution optimale. On aborde également les problèmes de complexité arithmétique.

**Mots-clés :** Résolution de systèmes algébriques surcontraints, PGCD approchés, méthode de Weierstrass surcontraintes

# 1 Introduction

In many physical and engineering applications one needs to solve over-constrained systems of equations, i.e. systems with more equations than unknowns, such that the existence of the solutions is guaranteed by some underlying physical property. However, the input system may be given only with limited accuracy due to measurement or rounding error, therefore the actual input may be inconsistent.

The work presented in this paper is concerned with the question of finding the "nearest" system with at least $k$ distinct common roots over $\mathbb{C}$. We introduce a generalization of the Gauss-Weierstrass method [24]. In the univariate case, the proposed iterative method allows computation of the nearest GCD of given degree, and is closely related to the formula of Karmarkar-Lakshman for the distance to the set of systems with at least $k$ common roots [19]. We show how to extend the iterative method to over-constrained systems of analytic functions. Using this extended construction we generalize the Karmarkar-Lakshman formula to the multivariate case.

More precisely, in the univariate case the problem we address in the paper is the following:

**Problem 1** *Given $f, g \in \mathbb{C}[x]$ and $k \in \mathbb{N}$, find a polynomial $h$ of degree $k$ such that there exist polynomials $\tilde{f}, \tilde{g} \in \mathbb{C}[x]$ such that $h$ divides both $\tilde{f}$ and $\tilde{g}$ and $f - \tilde{f}$ and $g - \tilde{g}$ has prescribed supports and minimal 2-norms.*

The method proposed here is based on a generalization of the so-called Weierstrass method (also called Durand-Kerner method [10, 20] or Dochev method [26, 13]) introduced in [24] (the method was generalized successively [2, 23, 22]). We show a link of this method to the work of Karmarkar and Lakshman [19]. Our main result in the univariate case is the following theorem:

**Theorem 2.10** *Let $f, g \in \mathbb{C}[x]$, $k > 0$, and $I, J \subset \mathbb{N}$. For fixed distinct $z_1, \ldots, z_k \in \mathbb{C}$ we can compute polynomials $f_{\mathbf{z}}, g_{\mathbf{z}} \in \mathbb{C}[x]$ such that the* **generalized Weierstrass iteration** *with supports $I$ and $J$, defined by*

$$z_i' := z_i - \frac{f_{\mathbf{z}}'(z_i)^* f(z_i) + g_{\mathbf{z}}'(z_i)^* g(z_i)}{|f_{\mathbf{z}}'(z_i)|^2 + |g_{\mathbf{z}}'(z_i)|^2} \quad i = 1, \ldots, k, \tag{1}$$

*is the Gauss-Newton iteration for finding $k$ roots of the nearest $\tilde{f}, \tilde{g}$ with at least $k$ common roots obtained from $f, g$ by the perturbation of coefficients corresponding to $I$ and $J$, respectively.*

In the multivariate case the problem we address is as follows:

**Problem 2** *Given an analytic function $\vec{f} = (f_1, \ldots, f_N) : \mathbb{C}^n \to \mathbb{C}^N$, $N > n$. Find perturbations $p_1, \ldots, p_N$ from a given finite dimensional vector space $\mathcal{P}$ such that $(f_1 -$*

$p_1, \ldots, f_N - p_N)$ *have at least* $k$ *distinct common roots in* $\mathbb{C}^n$ *and* $\|p_1\|_2^2 + \cdots + \|p_N\|_2^2$ *is minimal.*

Using a generalization of the Lagrange interpolation we obtain the following result, generalizing the formula of Karmarkar and Lakshman for the univariate nearest GCD to the multivariate case:

**Theorem 3.7** *Let* $\vec{f} = (f_1, \ldots, f_N)$ *be as above and let* $B_1, \ldots, B_N$ *be finite sets of analytic functions. Define the set* $\Omega_{\vec{B},k}(\vec{f}) := \left\{ \tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_N) \,:\, |\mathbf{V}(\tilde{f})| \geq k \text{ and } \forall i \; f_i - \tilde{f}_i \in \operatorname{span}_{\mathbb{C}}(B_i) \right\}$. *Then the distance of* $\vec{f}$ *to the set* $\Omega_{\vec{B},k}(\vec{f})$ *equals*

$$\min_{\vec{\mathbf{z}} \in \mathcal{R}_{\vec{B}}} \mathbf{f}_1^* M_{B_1}^{-1} \mathbf{f}_1(\vec{\mathbf{z}}) + \cdots + \mathbf{f}_N^* M_{B_N}^{-1} \mathbf{f}_N(\vec{\mathbf{z}}),$$

*assuming that the minimum exists. Here* $\mathbf{f}_i(\vec{\mathbf{z}}) := (f_i(\mathbf{z}_1), \ldots, f_i(\mathbf{z}_k)) \in \mathbb{C}^k$, $M_{B_i}(\vec{\mathbf{z}}) \in \mathbb{C}^{k \times k}$ *defined in (19), and* $\mathcal{R}_{\vec{B}} = \{\vec{\mathbf{z}} \in (\mathbb{C}^n)^k \,:\, \forall i \; \operatorname{rank}(M_{B_i}(\vec{\mathbf{z}})) = k\}$.

Finally, the main result of the paper asserts that a generalization of the Weierstrass iteration gives an iterative method to solve Problem 2:

**Theorem 3.11** *Let* $\vec{f} = (f_1, \ldots, f_N)$ *and* $\vec{B} = (B_1, \ldots, B_N)$ *be as above. For fixed* $\vec{\mathbf{z}} = (\mathbf{z}_1, \ldots \mathbf{z}_k) \in \mathcal{R}_{\vec{B}}$ *we can compute* $\vec{f}_{\vec{\mathbf{z}}} = (f_{\vec{\mathbf{z}},1} \ldots, f_{\vec{\mathbf{z}},N}) : \mathbb{C}^n \to \mathbb{C}^N$ *such that if* $J_{\vec{\mathbf{z}}}(\mathbf{x})$ *is the* $N \times n$ *Jacobian matrix of* $\vec{f}_{\vec{\mathbf{z}}}(\mathbf{x})$, *then the* **generalized multivariate Weierstrass iteration**, *defined by*

$$\mathbf{z}_i' := \mathbf{z}_i - J_{\vec{\mathbf{z}}}(\mathbf{z}_i)^+ \vec{f}(\mathbf{z}_i) \;\; i = 1, \ldots, k,$$

*is the Gauss-Newton iteration for finding* $k$ *common roots of the nearest system* $\tilde{f} \in \Omega_{\vec{B},k}(\vec{f})$.

## 1.1 Related work

The computation of the GCD is a classical problem of symbolic computation and efficient algorithms are known to solve it ([6] and [3] for instance). The first approach to a problem similar to Problem 1 was proposed by Schönage in [25] where the input polynomials are known with infinite precision. Several later approaches were proposed where the polynomials are known with a bounded error. In [11, 7] the authors compute upper bounds on the degree of an $\epsilon$-GCD of two numerical polynomials using the singular values of a Sylverster resultant matrix. In [12], the authors give the exact degree of the $\epsilon$-GCD together with a certificate using a singular value decomposition of a subresultant matrix. In [18, 19, 5], the authors present the problem as a real optimization problem and propose numerical techniques in order to solve the optimization problem. Hinz et al. consider the nearest polynomial with constrained or real roots in the $l^2$ and $l^\infty$ norms in [17]. For a survey on other work related to the approximate GCD see [16, Section 2.12.3].

There are two main families of approaches in the literature to compute the solution of near-consistent multivariate over-constrained systems. One type of algorithm handles over-constrained polynomial systems with approximate coefficients by using a symbolic-numeric approach to reduce the problem to eigenvalue computation via *multiplication tables*. The first methods in the literature using reduction to eigenvalue problem include [1, 28, 21]. The existing methods to compute the multiplication tables use resultant matrices or Gröbner basis techniques, with complexity bound exponential in the number of variables.include

The other type of approaches formulate over-constrained systems as real optimization problems. Here we can only list a selected subset of the related literature. Giusti and Shost in [14] reduce the problem to the solution of a univariate polynomial. Dedieu and Shub give a heuristic predictor corrector method in [8]. They also prove alpha-theory for the Gauss-Newton method in [9]. Stetter in [27] studies the conditioning properties of near-consistent over-constrained systems. Ruatta in [24] generalizes the Weierstrass iteration for over-constrained systems and gives a heuristic predictor corrector method based on this iteration.

### 1.2 Notations

In all that follows, $\mathbb{C}$ denotes the field of complex numbers, $x$ is an indeterminate and we denote by $\mathbf{x} = (x_1, \ldots x_n)$ the vector of $n$ indeterminates for some $n \geq 1$. $\mathbb{C}[x]$ and $\mathbb{C}[\mathbf{x}]$ denote the rings of polynomials with complex coefficients in one and $n$ indeterminates, respectively. $\mathbb{C}[x]_m$ is the subspace of $\mathbb{C}[x]$ consisting of the polynomials of degree less or equal to $m \in \mathbb{N}$. For $I \subset \mathbb{N}$ a finite set, we denote $\mathbb{C}[x]_I$ the set of polynomials with support included in $I$, i.e.

$$\mathbb{C}[x]_I = \{p \in \mathbb{C}[x] \ : \ p(x) = \sum_{i \in I} p_i x^i, p_i \in \mathbb{C}\}. \tag{2}$$

For $F \subset \mathbb{C}[\mathbf{x}]$ and $\mathcal{R} \subseteq \mathbb{C}^n$ we denote by $\mathbf{V}_{\mathcal{R}}(F)$ the set of common roots of $F$ in $\mathcal{R}$. We denote indifferently $\|\|_2$ or $\|\|$ the $l^2$ norm of complex vectors which we call the 2-norm. For $f \in \mathbb{C}[x]$ we denote by $\|f\|$ the 2-norm of the vector of its coefficients. For $M \in \mathbb{C}^{k \times m}$, $\|M\|$ denotes the 2-norm of the vector of its entries. The 2-norm of a vector of polynomials is the vector of all their coefficients. For a matrix $M \in \mathbb{C}^{k \times n}$ we denote by $M^T$ its transpose matrix and $M^*$ the transpose of the conjugate of $M$, also called the adjoint of $M$. For $M \in \mathbb{C}^{k \times m}$ such that $\text{rank}(M) = m$ we denote $M^+ = M^*(MM^*)^{-1}$ its Moore-Penrose inverse.

## 2 Univariate case

In this section, we present a generalization of the Weierstrass iteration. First we present a version of the classical Lagrange interpolation method which is needed for the construction of the iterative method. Secondly, we show the link between this iteration function and the distance function to a set of coherent systems. Finally, we prove a simple coordinate-wise expression for the iteration function, and we give its arithmetic complexity.

## 2.1 Generalized Lagrange interpolation

In this subsection we introduced an optimization problem which generalizes the classical Lagrange interpolation problem and we give a solution to this problem using Moore-Penrose pseudo inverses.

**Problem** [Generalized Lagrange interpolation] *Consider distinct complex numbers $z_1, \ldots, z_k \in \mathbb{C}$ and some arbitrary complex numbers $f_1, \ldots, f_k \in \mathbb{C}$. Fix $I \subset \mathbb{N}$ such that $|I| \geq k$. The generalized Lagrange interpolation problem consists of finding the minimal 2-norm polynomial $F \in \mathbb{C}[x]_I$ with support $I$ that satisfies:*

$$F(z_i) = f_i \text{ for } i = 1, \ldots, k. \tag{3}$$

We will need the following definition:

**Definitions 2.1** *Let $I = \{i_1, \ldots, i_p\} \subset \mathbb{N}$ such that $p \geq k$.*

- *For $\mathbf{z} = (z_1, \ldots, z_k) \in \mathbb{C}^k$ we define the $k \times k$ matrix $M_I(\mathbf{z})$ by:*

$$M_I(\mathbf{z}) = \left( \sum_{i \in I} (z_s z_t^*)^i \right)_{s,t=1,\ldots,k}. \tag{4}$$

- *Let $\mathbf{z} = (z_1, \ldots, z_k) \in \mathbb{C}^k$. We define the Vandermonde matrix associated with $\mathbf{z}$ and $I$ as following $k \times p$ matrix:*

$$V_I(\mathbf{z}) := \begin{pmatrix} z_1^{i_1} & \cdots & z_1^{i_p} \\ \vdots & \ddots & \vdots \\ z_k^{i_1} & \cdots & z_k^{i_p} \end{pmatrix}. \tag{5}$$

- *For $I \subset \mathbb{N}$ we define $\mathcal{R}_I := \left\{ (z_1, \ldots, z_k) \in \mathbb{C}^k \mid \operatorname{rank}(V_I(\mathbf{z})) = k \right\}$. For $I, J \subset \mathbb{N}$ we define $\mathcal{R}_{I,J} := \mathcal{R}_I \cap \mathcal{R}_J$.*

- *For $I, J \subset \mathbb{N}$ and $f, g \in \mathbb{C}[x]$ we define the set*

$$\Omega_{I,J,k}(f,g) := \left\{ (\tilde{f}, \tilde{g}) \mid f - \tilde{f} \in \mathbb{C}[x]_I, g - \tilde{g} \in \mathbb{C}[x]_J \text{ and } |\mathbf{V}_{\mathcal{R}_{I,J}}(\tilde{f}, \tilde{g})| \geq k \right\}.$$

  *Informally, $\Omega_{I,J,k}(f,g)$ is the set of systems with at least $k$ common roots which are obtained from $(f,g)$ via perturbation of the coefficients corresponding to $I$ and $J$, respectively. We may omit $(f,g)$ from $\Omega_{I,J,k}(f,g)$ if it is clear from the context.*

Next we introduce a family of polynomials which can be viewed as the generalization of the Lagrange polynomials.

**Definition 2.2** *Let $\mathbf{z} \in \mathcal{R}_I$ and let $V_I(\mathbf{z})$ be the generalized Vandermonde matrix associated with $\mathbf{z}$ and $I$. Define $\mathbf{x}_I = \left(x^{i_1}, \ldots, x^{i_p}\right)$ and denote by $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\} \subset \mathbb{C}^k$ the standard basis of $\mathbb{C}^k$. We define the* **generalized Lagrange polynomials** *with support in $I$ as follows:*

$$L_{I,i}(\mathbf{z}, x) := \mathbf{x}_I V_I(\mathbf{z})^+ \mathbf{e}_i \quad i = 1, \ldots, k. \tag{6}$$

Note that if $I = \{0, \ldots, k-1\}$ then $\{L_{I,i}(\mathbf{z}, x) \mid 1 \leq i \leq k\}$ are the classical Lagrange interpolation polynomials.

The following propositions assert that the generalized Lagrange polynomials allow us to find the minimal norm polynomial with prescribed support $I$ satisfying (3). We also highlight the connection between the 2-norms of the interpolation polynomials and the results of Karmarkar and Lakshman in [19].

**Proposition 2.3** *Let $I \subset \mathbb{N}$ with $p \geq k$ and $\mathbf{z} = (z_1, \ldots, z_k) \in \mathcal{R}_I$. Then for all $1 \leq i, j \leq k$, $L_{I,i}(\mathbf{z}, z_j) = \delta_{i,j}$.*

**Proof** From (6) we get that $L_{I,i}(\mathbf{z}, z_j) = \mathbf{e}_j^T V_I(\mathbf{z}) V_I(\mathbf{z})^+ \mathbf{e}_i$ for all $i, j \in \{1, \ldots, k\}$. Then we use that $V_I(\mathbf{z})$ has rank $k$ to get that $V_I(\mathbf{z})^+$ is the right inverse of $V_I(\mathbf{z})$. □

**Proposition 2.4** *Let $I \subset \mathbb{N}$, $\mathbf{z} \in \mathcal{R}_I$ and $\mathbf{f} = (f_1, \ldots, f_k) \in \mathbb{C}^k$. Define*

$$F(x) := \sum_{i=0}^{k} f_i L_{I,i}(\mathbf{z}, x). \tag{7}$$

*Then we have $F(x) \in \mathbb{C}[x]_I$ and*

$$F(z_j) = f_j, \forall j \in \{1, \ldots, k\}. \tag{8}$$

*Moreover,*

$$\|F\|^2 = \mathbf{f}^* M_I(\mathbf{z})^{-1} \mathbf{f} \tag{9}$$

*is minimal among the polynomials in $\mathbb{C}[x]_I$ satisfying (8).*

**Proof** Let $F(x)$ be as in (7). If we denote by $\mathbf{F} = (F_i)_{i \in I}$ the vector of coefficients of $F(x)$ then by the definition of the generalized Lagrange polynomials we have $\mathbf{F} = V_I(\mathbf{z})^+ \mathbf{f}$. It is easy to check that $\|\mathbf{F}\|^2 = \mathbf{f}^* M_I^{-1} \mathbf{f}$ using the fact that $M_I^{-1} = V_I(\mathbf{z})^{+*} V_I(\mathbf{z})^+$. On the other hand, $\mathbf{F}$ is the minimal 2-norm vector satisfying $V_I(\mathbf{z})\mathbf{F} = \mathbf{f}$, which follows from the properties of the Moore-Penrose pseudo inverse (see [15]). Finally, we note that $V_I(\mathbf{z})\mathbf{F} = \mathbf{f}$ is equivalent to (8). □

The above propositions allow us to state the main result of the subsection:

**Theorem 2.5** *Let $(f, g) \in \mathbb{C}[x]^2$, $I, J \subset \mathbb{N}$ and $\mathbf{z} \in \mathcal{R}_{I,J}$. We define the following polynomials in $\mathbb{C}[x]_I$ and $\mathbb{C}[x]_J$, respectively:*

$$F_I(\mathbf{z}, x) := \sum_{i=1}^{k} f(z_i) L_{I,i}(\mathbf{z}, x), \; G_J(\mathbf{z}, x) := \sum_{i=1}^{k} g(z_i) L_{J,i}(\mathbf{z}, x). \tag{10}$$

*Then*

$$(f(x) - F_I(\mathbf{z}, x), g(x) - G_J(\mathbf{z}, x)) \in \Omega_{I,J,k}(f, g).$$

*Moreover, if* $\min_{\mathbf{z} \in \mathcal{R}_{I,J}} \left( \mathbf{f}^* \, M_I(\mathbf{z})^{-1} \, \mathbf{f} + \mathbf{g}^* \, M_J(\mathbf{z})^{-1} \, \mathbf{g} \right)$ *exists and is reached at* $\zeta \in \mathcal{R}_{I,J}$ *then we have*

$$\|F_I(\zeta, x)\|^2 + \|G_J(\zeta, x)\|^2 = \min_{(\tilde{f}, \tilde{g}) \in \Omega_{I,J,k}} \left\{ \|f - \tilde{f}\|^2 + \|g - \tilde{g}\|^2 \right\}.$$

*Here* $\mathbf{f} = (f(z_1), \ldots, f(z_k)) \in \mathbb{C}^k$ *and* $\mathbf{g} = (g(z_1), \ldots, g(z_k)) \in \mathbb{C}^k$.

**Proof** The proof can be deduced easily from the proposition 2.4. □

## 2.2 Generalized Weierstrass iteration

In this section we give a generalization of the univariate over-constrained Weierstrass iteration of [24]. As with the Weierstrass iteration, our generalized version comes from the Gauss-Newton method applied to the so called Weierstrass map w.r.t. a pair $(f, g) \in \mathbb{C}[x]^2$.

Informally, for $(f, g) \in \mathbb{C}[x]^2$ the Weierstrass map $\mathcal{W}$ in [24] is a map defined on $\mathbb{C}^k$ with the property that $\mathcal{W}(z_1, \ldots, z_k) = 0$ if and only if $f(z_i) = g(z_i) = 0$ for $1 \leq i \leq k$. Applying the Gauss Newton method to the map $\mathcal{W}$ allows us to compute the solution $\mathbf{z} \in \mathbb{C}^k$ where $\|\mathcal{W}(\mathbf{z})\|_2$ is minimal.

The main contribution of this paper is the observation that the norm $\|\mathcal{W}(\mathbf{z})\|_2$ is closely related to the distance defined by Karmarkar and Lakshman in [19] and its generalization in Theorem 2.5 (see Remark 2.8 below). Using this observation, it is straightforward to generalize the Weierstrass map $\mathcal{W}$ so that its least square solution corresponds to the $k$ common roots of the closest system $(\tilde{f}, \tilde{g})$ which is obtained from $(f, g)$ via the perturbation of a prescribed subset of their coefficients.

First we give the definition of the generalized Weierstrass map using the generalized Lagrange polynomials defined in (6).

**Definition 2.6** *Let* $(f, g) \in \mathbb{C}[x]^2$, $k \geq 1$ *and* $I, J \subset \mathbb{N}$ *such that* $|I|, |J| \geq k$. *For a fixed* $\mathbf{z} \in \mathcal{R}_{I,J}$, *let* $F_I(\mathbf{z}, x) \in \mathbb{C}[x]_I$ *and* $G_J(\mathbf{z}, x) \in \mathbb{C}[x]_J$ *be the interpolation polynomials defined in (10). Then the map defined by*

$$\mathcal{W}_{I,J} : \begin{cases} \mathbb{C}^k \to \mathbb{C}[x]_I \oplus \mathbb{C}[x]_J \\ \mathbf{z} \mapsto (F_I(\mathbf{z}, x), G_J(\mathbf{z}, x)) \end{cases} \tag{11}$$

*is called the* **generalized Weierstrass map** *with supports* $I$ *and* $J$.
*We also define the vector of coordinates of* $\mathcal{W}_{I,J}(\mathbf{z})$ *in the bases of generalized Lagrange polynomials for* $I, J$ *and any fixed* $\mathbf{z} \in \mathbb{C}^k$:

$$W_{I,J}(\mathbf{z}) := (f(z_1), \ldots, f(z_k), g(z_1), \ldots, g(z_k)) \in \mathbb{C}^{2k} \tag{12}$$

In the next proposition we prove that the least square solution of the Weierstrass map and the optimization problem posed by Karmarkar and Lakshman in [19] are closely related.

**Proposition 2.7** *Let* $\mathbf{z} = (z_1, \ldots, z_k)$, $(f, g)$, *and* $\mathcal{W}_{I,J}$ *be as in Definition 2.6. Then*

*i.* $\mathcal{W}_{I,J}(\mathbf{z}) = 0$ *if and only if* $(z_1, \ldots, z_k)$ *are common roots of* $f$ *and* $g$.

*ii. Using the notation of theorem 2.5, for all* $\mathbf{z} \in \mathbb{C}^k$ *we have*

$$\|\mathcal{W}_{I,J}(\mathbf{z})\|^2 = \mathbf{f}^* M_I^{-1} \mathbf{f} + \mathbf{g}^* M_J^{-1} \mathbf{g}.$$

*iii.* $\min_{\mathbf{z} \in \mathcal{R}_{I,J}} \|\mathcal{W}_{I,J}(\mathbf{z})\|^2 = \min_{(\tilde{f},\tilde{g}) \in \Omega_{I,J,k}} \left\{ \|f - \tilde{f}\|^2 + \|g - \tilde{g}\|^2 \right\}$.

**Proof** (i) $\mathcal{W}_{I,J}(\mathbf{z}) = 0$ if and only if $F_I(\mathbf{z}, x) = G_I(\mathbf{z}, x) = 0$ for all $x \in \mathbb{C}$. This implies that $f(z_i) = F_I(\mathbf{z}, z_i) = 0$ and $g(z_i) = G_I(\mathbf{z}, z_i) = 0$ for all $1 \leq i \leq k$. On the other hand, assume that $z_1, \ldots z_k$ are common roots of $f$ and $g$. Since $F_I$ and $G_J$ are the minimal 2-norm polynomials interpolating $(f(z_1), \ldots, f(z_k)) = 0$ and $(g(z_1), \ldots, g(z_k)) = 0$, $F_I$ and $G_J$ must both be the zero polynomial.
(ii) follows from the definition of $\mathcal{W}_{I,J}$ in (11), the definition of $F_I(\mathbf{z}, x)$ and $G_I(\mathbf{z}, x)$ in (10) and from (9).
(iii) follows from (ii) and from Theorem 2.5. □

**Remark 2.8** As a special case of the above proposition, we get that the least squares solution of the univariate over-constrained Weierstrass map $\mathcal{W}$ defined in [24] gives the common roots of the closest system with $k$ common roots, and obtained via the perturbation of the coefficients corresponding to $I = J = \{0, 1, \ldots, k-1\}$, i.e. the terms of $f$ and $g$ of degree less than $k$.

We will use the Gauss-Newton method to find a least square solution of $\mathcal{W}_{I,J}(\mathbf{z})$. Each iteration of the Gauss-Newton method is defined by

$$\mathbf{z}' = \mathbf{z} - D\mathcal{W}_{I,J}^{+}(\mathbf{z}) W_{I,J}(\mathbf{z}), \tag{13}$$

where $W_{I,J}(\mathbf{z}) \in \mathbb{C}^{2k}$ is defined in (12) and $D\mathcal{W}_{I,J}(\mathbf{z})$ is the Jacobian matrix of $\mathcal{W}_{I,J}$ at $\mathbf{z}$. To be more precise, in order to make the expression in (13) well defined, we need to restrict the image space of $D\mathcal{W}_{I,J}(\mathbf{z})$ to a subspace of $\mathbb{C}[x]_I \oplus \mathbb{C}[x]_J$ depending on $\mathbf{z}$, so that this restriction makes $D\mathcal{W}_{I,J}(\mathbf{z})$ a $2k \times k$ matrix and its pseudo-inverse a $k \times 2k$ matrix. This restriction is defined in the next proposal.

The rest of this subsection is devoted to the coordinate-wise expression of the Gauss-Newton iteration map in (13), using the basis of generalized Lagrange polynomials. We will define the generalized Weierstrass iteration to be the coordinate-wise expression for this map.

**Proposition 2.9** *Let $f$, $\mathbf{z}$, $I$, and $F_I(\mathbf{z},x)$ be as in Definition 2.6. For a fixed $\mathbf{z} \in \mathbb{C}^k$, define the linear subspace*

$$\mathcal{L}_I := \langle L_{I,1}, \ldots, L_{I,k} \rangle \subset \mathbb{C}[x]_I, \tag{14}$$

*generated by the generalized Lagrange polynomials. Let $\pi$ be the projection $\pi : \mathbb{C}[x]_I \to \mathcal{L}_I$. Then for all $1 \leq i \leq k$ we have*

$$\pi\left(\frac{\partial F_I(\mathbf{z},x)}{\partial z_i}\right) = (f'(z_i) - F_I'(\mathbf{z},z_i))\, L_{I,i}(\mathbf{z},x). \tag{15}$$

**Proof** Implicitly differentiating the equations

$$F_I(\mathbf{z},z_j) = f(z_j) \;\; j = 1,\ldots,k$$

by $z_i$ we get

$$\frac{\partial F_I(\mathbf{z},x)}{\partial z_i}\Big|_{x=z_j} + \delta_{i,j}\frac{\partial F_I(\mathbf{z},x)}{\partial x}\Big|_{x=z_j} = \delta_{i,j}\frac{\partial f(x)}{\partial x}\Big|_{x=z_j}.$$

This proves that the projection of $\frac{\partial F_I(\mathbf{z},x)}{\partial z_i}$ to $\mathcal{L}_I$ is equal to the expression in the claim. □

Proposition 2.9 implies that the Gauss-Newton method applied to the generalized Weierstrass map $\mathcal{W}_{I,J}$ can be given coordinate-wise. The following theorem, the main result of this section, summarizes the above results and defines the generalized Weierstrass iteration by the coordinate-wise expressions for the Gauss-Newton method.

**Theorem 2.10** *Let $(f,g)$, $k$, $\mathbf{z} = (z_1,\ldots,z_k)$, $I$, $J$, $F_I(\mathbf{z},x)$, and $G_J(\mathbf{z},x)$ be as in Definition 2.6. Define*

$$f_{\mathbf{z}}(x) := f(x) - F_I(\mathbf{z},x), \;\; g_{\mathbf{z}}(x) := g(x) - G_J(\mathbf{z},x).$$

*Assume that none of the $z_i$'s are common roots of the derivatives $f_{\mathbf{z}}'(x)$ and $g_{\mathbf{z}}'(x)$. Then the* **generalized Weierstrass iteration** *with supports $I$ and $J$, defined by*

$$z_i' := z_i - \frac{f_{\mathbf{z}}'(z_i)^* f(z_i) + g_{\mathbf{z}}'(z_i)^* g(z_i)}{|f_{\mathbf{z}}'(z_i)|^2 + |g_{\mathbf{z}}'(z_i)|^2} \;\; i = 1,\ldots,k, \tag{16}$$

*is the Gauss-Newton iteration for finding $k$ common roots of the closest system to $(f,g)$ in $\Omega_{I,J,k}(f,g)$.*

## 2.3 Arithmetic complexity of the iteration

In this subsection we are interested in the arithmetic complexity of doing one iteration of the generalized Weierstrass iteration. In all that follows $\mathcal{E}(d,k)$ denotes the multi-point evaluation of degree $d$ at $k$ points (see [4]).

**Proposition 2.11** *Let* $I = \{i_1, \ldots, i_m\}$, $D = \max(I \cup \{\deg(f)\})$, $J = \{j_1, \ldots, j_{m'}\}$, *and* $D' = \max(J \cup \{\deg(g)\})$. *Then the number of arithmetic operations needed to compute the value of the iteration function (16) at a point in* $\mathbb{C}^k$ *is* $\mathcal{O}\left(k^2 * (m + m')\right) + \mathcal{O}\left(m * \mathcal{E}\left(D, k\right) + m' * \mathcal{E}\left(D', k\right)\right)$.

**Proof** The matrix $V_I(\mathbf{z})$ can be computed in $m * \mathcal{E}(D, k)$ operations. The matrix $V_I^+(\mathbf{z})$ can be computed in $\mathcal{O}\left(k^2 * m\right)$ arithmetic operations using classical linear algebra. Then the $k$ polynomials $L'_{I,i}(\mathbf{z}, x)$, $i \in \{1, \ldots, k\}$ can be computed in $\mathcal{O}(m)$ arithmetic operations, using (6). Finally, we need a constant number multi-point evaluation to get the expression on the right hand side of (16) using $\mathcal{O}\left(\mathcal{E}\left(D, k\right) + \mathcal{E}\left(D', k\right)\right)$ arithmetic operations. Putting it all together, we have the estimate given in the proposition. □

**Remark 2.12** For numerical stability, it is better to use the singular value decomposition of $V_I(\mathbf{z})$ to find the Moore-Penrose pseudo inverse $V_I(\mathbf{z})^+$, instead of the algebraic formula. It doesn't change the number of arithmetic operations used in the above process.

The following is a simpler bound if the polynomials are dense:

**Corollary 2.13** *If* $I = \{1, \ldots, m\}$ *and* $J = \{1, \ldots, m'\}$, *then the number of arithmetic operation used to compute the iteration function (16) is* $\mathcal{O}\left(k^2 * (m + m')\right)$.

# 3 Multivariate Case

In this section, we describe the generalization of the results of the previous section to the multivariate setting. In the multivariate case we extend our construction to over-constrained systems of analytic functions as input, not only polynomials. Since the set of over-constrained systems of analytic functions with at least $k$ common roots is infinite dimensional, we will restrict our objective to find the closest such system which is obtained via some perturbation from a finite dimensional "perturbation space", given by a finite basis of analytic functions. For our general construction we will need to generalize the Lagrange interpolation to finding elements in the perturbation space with prescribed evaluations and minimal 2-norms.

**Definition 3.1** *We denote by* $\mathbb{C}_n^\infty$ *the set of analytic functions* $\mathbb{C}^n \to \mathbb{C}$. *Let* $\vec{f} = (f_1, \ldots, f_N) \in (\mathbb{C}_n^\infty)^N$ *for some* $N > n$. *For each* $i = 1, \ldots, N$ *let* $B_i := \{b_{i,1}, \ldots, , \ldots, b_{i,m_i}\} \subset \mathbb{C}_n^\infty$ *linearly independent over* $\mathbb{C}$. *We call* $\mathcal{P} := \bigoplus_{i=1}^N \mathrm{span}_{\mathbb{C}}(B_i)$ *the* **perturbation space** *with basis* $\vec{B} := (B_1, \ldots, B_N)$.

We address the following problem:

**Problem**: *Given* $\vec{f} = (f_1, \ldots, f_N)$ *and* $\vec{B} = (B_1, \ldots, B_N)$ *as above. Find* $(p_1, \ldots, p_N) \in \mathcal{P}$ *such that* $(f_1 - p_1, \ldots, f_N - p_N)$ *has at least* $k$ *distinct common roots in* $\mathbb{C}^n$ *and* $\|p_1\|_{B_1}^2 + \cdots \|p_N\|_{B_N}^2$ *is minimal. Here* $\|p_i\|_{B_i}$ *denotes the 2-norm of the coefficients of* $p_i$ *in the* $\mathbb{C}$*-basis* $B_i$.

Let us define the generalized Vandermonde matrix and determinant associated with a set of basis functions $B$:

**Definition 3.2** *Let $\vec{\mathbf{z}} = (\mathbf{z}_1, \ldots, \mathbf{z}_k) \in (\mathbb{C}^n)^k$. For $B = \{b_1, \ldots, b_m\} \subset \mathbb{C}_n^\infty$ we define the* **generalized Vandermonde matrix** *associated with $B$ to be the $k \times m$ matrix with entries*

$$V_B(\vec{\mathbf{z}})_{i,j} := b_j(\mathbf{z}_i).$$

*We denote*

$$\mathcal{R}_B := \left\{ \vec{\mathbf{z}} \in (\mathbb{C}^n)^k \mid \operatorname{rank}\left(V_B\left(\vec{\mathbf{z}}\right)\right) = k \right\},$$

*and for $\vec{B} = (B_1, \ldots, B_N)$ we we use the notation $\mathcal{R}_{\vec{B}} := \bigcap_{i=1}^N \mathcal{R}_{B_i}$.*

**Remark 3.3** We can choose the bases $B_1, \ldots, B_N$ of the perturbation space freely as long as the set $\mathcal{R}_{\vec{B}}$ is open and everywhere dense, or it includes the possible roots we are searching for.

Now we can define the generalized multivariate Lagrange polynomials :

**Definition 3.4** *Let $B = \{b_1, \ldots, b_m\} \subset \mathbb{C}_n^\infty$. For $\mathbf{x} \in \mathbb{C}^n$ denote $\mathbf{x}_B = [b_1(\mathbf{x}), \ldots, b_m(\mathbf{x})]$. Let $\mathbf{e_1} \ldots \mathbf{e_k}$ be the standard basis of $\mathbb{C}^k$. Let $\vec{\mathbf{z}} \in \mathcal{R}_B$. We define the* **generalized Lagrange polynomials** *associated with $B$ as $L_{B,i}(\vec{\mathbf{z}}, \mathbf{x}) := \mathbf{x}_B V_B(\vec{\mathbf{z}})^+ \mathbf{e_i}$ for $i = 1, \ldots, k$.*

**Remark 3.5** If $m = k$ and $B = \{\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_k}\}$ for some $\alpha_i \in \mathbb{N}^n$, then the generalized Vandermonde matrix is a square matrix and the above formula is the one given by Ruatta in [24] for the Lagrange interpolation basis.

The following proposition is a straightforward generalization of Propositions 2.3 and 2.4.

**Proposition 3.6** *Let $f \in \mathbb{C}_n^\infty$, $B \subset \mathbb{C}_n^\infty$, $|B| = m$ linearly independent over $\mathbb{C}$, and let $\mathcal{P} = \operatorname{span}_{\mathbb{C}}(B)$. Fix $\vec{\mathbf{z}} \in \mathcal{R}_B$. Then $L_{B,i}\left(\vec{\mathbf{z}}, \mathbf{z}_j\right) = \delta_{i,j}$ for all $i, j = 1, \ldots, k$. Furthermore, for any $\mathbf{f} := (f_1, \ldots, f_k) \in \mathbb{C}^k$, define $p(\vec{\mathbf{z}}, \mathbf{x}) = \sum_{i=1}^k f_i L_{B,i}\left(\vec{\mathbf{z}}, \mathbf{x}\right) \in \mathcal{P}$. Then*

$$p(\vec{\mathbf{z}}, \mathbf{z}_j) = f_j \text{ for all } j \in \{1, \ldots, k\}. \tag{17}$$

*Moreover,*

$$\|p\|_B^2 = \mathbf{f}^* M_B\left(\vec{\mathbf{z}}\right)^{-1} \mathbf{f} \tag{18}$$

*is minimal among the polynomials in $\mathcal{P}$ satisfying (17). Here*

$$M_B\left(\vec{\mathbf{z}}\right) := V_B\left(\vec{\mathbf{z}}\right) V_B\left(\vec{\mathbf{z}}\right)^* = \left( \sum_{b \in B} b(\mathbf{z}_i) b(\mathbf{z}_j)^* \right)_{i,j \in \{1,\ldots,k\}}. \tag{19}$$

The next theorem gives a generalization of the expressions of Karmarkar and Lakshman in [19] for the multivariate case. This is one of the main results of the paper.

**Theorem 3.7** *Let $N > n \in \mathbb{N}$, $\vec{f} = (f_1, \ldots, f_N) \in (\mathbb{C}_n^\infty)^N$, $\vec{B} = (B_1, \ldots, B_N)$ and $\mathcal{P}$ be as in Definition 3.1. Define $\mathbf{f}_i(\vec{\mathbf{z}}) := (f_i(\mathbf{z}_1), \ldots, f_i(\mathbf{z}_k)) \in \mathbb{C}^k$ and let $M_{B_i}(\vec{\mathbf{z}})$ be as in (19) for $i = 1, \ldots, N$. Then, when*

$$\min_{\vec{\mathbf{z}} \in \mathcal{R}_{\vec{B}}} \mathbf{f}_1^* M_{B_1}^{-1} \mathbf{f}_1 (\vec{\mathbf{z}}) + \cdots + \mathbf{f}_N^* M_{B_N}^{-1} \mathbf{f}_N (\vec{\mathbf{z}}) \tag{20}$$

*exists, it is equal to*

$$\min_{\tilde{f} \in \Omega_{\vec{B},k}(\vec{f})} \|f_1 - \tilde{f}_1\|_{B_1}^2 + \ldots + \|f_N - \tilde{f}_N\|_{B_N}^2. \tag{21}$$

*Here the minimum is taken within the set $\Omega_{\vec{B},k}(f)$ defined by*

$$\Omega_{\vec{B},k}(\vec{f}) := \left\{ \tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_N) \,:\, |\mathbf{V}_{\mathcal{R}_{\vec{B}}}(\tilde{f})| \geq k \text{ and } \forall i \; f_i - \tilde{f}_i \in \mathrm{span}_{\mathbb{C}}(B_i) \right\}.$$

**Proof** For a fixed $\vec{\mathbf{z}} \in \mathcal{R}_{\vec{B}}$ define $p_i(\vec{\mathbf{z}}, \mathbf{x}) := \sum_{j=1}^k f_i(\mathbf{z}_j) L_{B_i,j}(\vec{\mathbf{z}}, \mathbf{x}) \in \mathrm{span}_{\mathbb{C}}(B_i)$ for all $i = 1, \ldots, N$. Assume that the minimum in (20) exists and is taken at $\vec{\zeta} = (\zeta_1, \ldots, \zeta_k) \in \mathcal{R}_{\vec{E}}$. Note that for all $i \in \{1, \ldots, N\}$, if $\tilde{f}_i$ vanishes on $\zeta_1, \ldots, \zeta_k$ and $f_i - \tilde{f}_i \in \mathrm{span}_{\mathbb{C}}(B_i)$, then, by Proposition 3.6, $\|f_i - \tilde{f}_i\| \geq \|p_i(\vec{\zeta}, \mathbf{x})\|$. This implies that

$$\Big( f_1(\mathbf{x}) - p_1(\vec{\zeta}, \mathbf{x}), \ldots, f_N(\mathbf{x}) - p_N(\vec{\zeta}, \mathbf{x}) \Big) \in \Omega_{\vec{B},k}(\vec{f})$$

must minimize (21). The equality of (20) and (21) follows from

$$\|p_1(\vec{\zeta}, \mathbf{x})\|_{B_1}^2 + \cdots + \|p_N(\vec{\zeta}, \mathbf{x})\|_{B_N}^2 = \mathbf{f}_1^* M_{B_1}^{-1} \mathbf{f}_1(\vec{\zeta}) + \cdots + \mathbf{f}_N^* M_{B_N}^{-1} \mathbf{f}_N(\vec{\zeta}).$$

□

Next we define the multivariate generalization of the Weierstrass map :

**Definition 3.8** *Let $f_1, \ldots, f_N \in \mathbb{C}_n^\infty$, $\vec{B} = (B_1, \ldots, B_N)$ and $\mathcal{P}$ be as above. The generalized Weierstrass map is defined as follows:*

$$\mathcal{W}_{\vec{B}} : \left\{ \begin{array}{ccc} \mathcal{R}_{\vec{B}} & \longrightarrow & \mathcal{P} \\ \\ \vec{\mathbf{z}} & \longmapsto & \begin{pmatrix} p_1(\vec{\mathbf{z}}, \mathbf{x}) \\ \vdots \\ p_N(\vec{\mathbf{z}}, \mathbf{x}) \end{pmatrix} \end{array} \right. , \tag{22}$$

*where*

$$p_i(\vec{\mathbf{z}}, \mathbf{x}) := \sum_{j=1}^k f_i(\mathbf{z}_j) L_{B_i,j}(\vec{\mathbf{z}}, \mathbf{x}) \quad i = 1, \ldots, N.$$

The next proposition is a straightforward generalization of Proposition 2.7 :

**Proposition 3.9** *Let $\vec{f} = (f_1, \ldots, f_N) \in (\mathbb{C}_n^\infty)^N$, $\vec{B} = (B_1, \ldots, B_N)$ be as above. Then for all $\vec{\mathbf{z}} \in \mathcal{R}_{\vec{B}}$ we have $\mathcal{W}_{\vec{B}}(\vec{\mathbf{z}}) = 0$ if and only if $\{\mathbf{z}_1, \ldots, \mathbf{z}_k\}$ are common roots of $f_1, \ldots, f_N$. Moreover, using the notation of Theorem 3.7, we have*

$$\min_{\vec{\mathbf{z}} \in \mathcal{R}_{\vec{B}}} \|\mathcal{W}_{\vec{B}}(\vec{\mathbf{z}})\| = \min_{\tilde{f} \in \Omega_{\vec{B},k}(\vec{f})} \|f_1 - \tilde{f}_1\|_{B_1}^2 + \ldots + \|f_N - \tilde{f}_N\|_{B_N}^2. \tag{23}$$

Similarly to the univariate case, we propose to compute the least square solution of $\mathcal{W}_{\vec{B}}$ using Gauss-Newton iteration. Again, we will restrict the image space of the Jacobian $D\mathcal{W}_{\vec{B}}$ to the subspace of the perturbation space $\mathcal{P}$ generated by the generalized Lagrange polynomials. Thus, the size of the restricted Jacobian matrix is $(Nk) \times (nk)$. The rest of the section is devoted to the proof that this Jacobian matrix and its pseudo-inverse both have block diagonal structure, with block size $N \times n$ and $n \times N$ respectively. Once this is proved, we define the generalized multivariate Weierstrass iterator for each $\mathbf{z}_i$ separately.

First we prove that the partial derivatives of the interpolation functions have simple structures in the multivariate Lagrange basis.

**Proposition 3.10** *Let $f \in \mathbb{C}_n^\infty$, $B \subset \mathbb{C}_n^\infty$, $\mathcal{P}$, $L_{B,i}(\vec{\mathbf{z}}, \mathbf{x})$ and $p(\vec{\mathbf{z}}, \mathbf{x})$ be as in Proposition 3.6. For a fixed $\vec{\mathbf{z}} = (\mathbf{z}_1, \ldots, \mathbf{z}_k) \in \mathcal{R}_E$, we define the linear subspace*

$$\mathcal{L}_B := \langle L_{B,1}, \ldots, L_{B,k} \rangle \subset \mathcal{P} \tag{24}$$

*generated by the generalized Lagrange polynomials. Let $\pi$ be the projection $\pi : \mathcal{P} \to \mathcal{L}_B$. Then for all $1 \leq i \leq k$ and $1 \leq j \leq n$ we have*

$$\pi\left(\frac{\partial p(\vec{\mathbf{z}}, \mathbf{x})}{\partial z_{i,j}}\right) = \left[\frac{\partial f}{\partial x_j}(\mathbf{z}_i) - \frac{\partial p}{\partial x_j}(\vec{\mathbf{z}}, \mathbf{z}_i)\right] L_{B,i}(\vec{\mathbf{z}}, \mathbf{x}),$$

*where $\mathbf{z}_i = (z_{i,1}, \ldots, z_{i,n})$.*

**Proof** Implicitly differentiating the equations

$$p(\vec{\mathbf{z}}, \mathbf{z}_l) = f(\mathbf{z}_l) \quad l = 1, \ldots k$$

we get

$$\frac{\partial p(\vec{\mathbf{z}}, \mathbf{x})}{\partial z_{i,j}}\Big|_{x = z_l} = \begin{cases} \frac{\partial f}{\partial x_j}(\mathbf{z}_l) - \frac{\partial p}{\partial x_j}(\vec{\mathbf{z}}, \mathbf{z}_l) & i = l \\ 0 & i \neq l \end{cases}.$$

This implies the claim, using the fact that the evaluations at $\mathbf{z}_1, \ldots, \mathbf{z}_k$ uniquely determine the coordinates of the elements of $\mathcal{L}_B$ in the generalized Lagrange basis. □

In the next theorem we define the generalized multivariate Weierstrass iteration and summarize the results above.

**Theorem 3.11** *Let $\vec{f} = (f_1, \ldots, f_N)$ and $\vec{B} = (B_1, \ldots, B_N)$ be as above. Let $(p_1(\vec{\mathbf{z}}, \mathbf{x}), \ldots, p_N(\vec{\mathbf{z}}, \mathbf{x})) \in \mathcal{P}$ be as in Theorem 3.7. Fix $\vec{\mathbf{z}} = (\mathbf{z}_1, \ldots \mathbf{z}_k) \in \mathcal{R}_{\vec{B}}$. Define*

$$\vec{f}_{\vec{\mathbf{z}}}(\mathbf{x}) := (f_1(\mathbf{x}) - p_1(\vec{\mathbf{z}}, \mathbf{x}), \ldots, f_N(\mathbf{x}) - p_N(\vec{\mathbf{z}}, \mathbf{x})) .$$

*Let $J_{\vec{\mathbf{z}}}(\mathbf{x})$ be the $N \times n$ Jacobian matrix of $\vec{f}_{\vec{\mathbf{z}}}(\mathbf{x})$. Assume that* $\operatorname{rank}(J_{\vec{\mathbf{z}}}(\mathbf{z}_i)) = n$ *for all $i = 1, \ldots, k$. Then the* **generalized multivariate Weierstrass iteration**, *defined by*

$$\mathbf{z}_i' := \mathbf{z}_i - J_{\vec{\mathbf{z}}}(\mathbf{z}_i)^+ \vec{f}(\mathbf{z}_i) \quad i = 1, \ldots, k, \tag{25}$$

*is the Gauss-Newton iteration for finding $k$ common roots of the closest system $\tilde{f} \in \Omega_{\vec{B},k}(\vec{f})$ to $\vec{f}$.*

In the next theorem we give the arithmetic complexity of each step of the generalized multivariate Weierstrass iteration.

**Theorem 3.12** *Let $\mathcal{E}_n(D)$ be the arithmetic complexity of an evaluation of a polynomial of degree less or equal to $D$ in $n$ variables. Assume that the degree of the input polynomials is less or equal to $D \in \mathbb{N}$, then the number of arithmetic operations used to compute one iteration of the above algorithm is $\mathcal{O}(k * n * N * \mathcal{E}_n(D)) + \mathcal{O}(k * n^2 * N)$.*

**Proof** The first term come from the computation of the $k$ specialized "restricted Jacobian" matrices (which are $n \times N$) and the computation of the generalized Lagrange polynomial (see the approach of the univariate case). The other term comes from the computation of the Moore-Penrose pseudo-inverses in (25). □

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