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***Fault tolerant on-board networks with priorities***

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**N° 5363**

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Thème COM

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de recherche***



## Fault tolerant on-board networks with priorities

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Thème COM — Systèmes communicants  
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**Abstract:** We consider on-board networks in satellites interconnecting entering signals (inputs) to amplifiers (outputs). The connections are made via expensive switches with four links available. The paths connecting inputs to outputs should be link-disjoint. Some of the input signals, called priorities, must be connected to the amplifiers which provide the best quality of service (that is to some specific outputs). In practice, amplifiers are prone to fail and the faults cannot be repaired. Therefore, extra outputs have to be built into the network to ensure that every input can be routed to operational outputs. Given three integers,  $n$ ,  $p$ , and  $f$ , we would like to design a low cost network (where the network cost is proportional to the total number of switches) such that it is possible to route all  $n$  inputs to  $n$  operational amplifiers and to route the  $p$  priorities to the  $p$  best quality amplifiers for any set of  $f$  faulty and  $p$  best-quality amplifiers. Let  $R(n, p, f)$  be the minimum number of switches of such a network. We prove here that  $R(n, p, f) \leq \frac{n+f}{2} \lceil \log_2 p \rceil + \frac{5}{2}(n-p) + g(f)$  with  $g$  a function depending only on  $f$ . We then give exact values of  $R(n, p, f)$  for small  $p$  and  $f$ .

**Key-words:** network design, on-board network, fault tolerance, vulnerability

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## Réseaux embarqués tolérants aux pannes avec priorités

**Résumé :** Nous considérons les réseaux embarqués dans les satellites qui interconnectent des signaux d'entrée à des amplificateurs (sorties). Les connexions sont effectuées par l'intermédiaire de commutateurs à quatre liens. Les chemins connectant les entrées aux sorties ne peuvent pas partager un même lien. Certains signaux d'entrées, appelées priorités, doivent être connectés aux amplificateurs qui assurent la meilleure qualité de service (c'est-à-dire certaines sorties particulières). En pratique, les amplificateurs peuvent tomber en panne et ne peuvent être réparés. Ainsi, des sorties supplémentaires doivent être introduites afin de garantir un nombre suffisant de sorties opérationnelles. Etant donnés trois entiers  $n$ ,  $p$  et  $f$ , nous désirons construire le réseau le moins coûteux possible (le coût étant proportionnel au nombre de commutateurs) tel qu'il soit possible de diriger les  $n$  entrées vers  $n$  amplificateurs opérationnels et les  $p$  priorités vers les  $p$  amplificateurs de meilleure qualité pour tout choix de  $f$  amplificateurs en panne et  $p$  amplificateurs de meilleure qualité. Soit  $R(n, p, f)$  le nombre minimum de commutateurs d'un tel réseau. Nous montrons ici que  $R(n, p, f) \leq \frac{n+f}{2} \lceil \log_2 p \rceil + \frac{5}{2}(n-p) + g(f)$  avec  $g$  une fonction dépendant uniquement de  $f$ . Nous donnons ensuite les valeurs exactes de  $R(n, p, f)$  pour de petites valeurs de  $p$  et  $f$ .

**Mots-clés :** conception de réseaux, réseaux embarqués, tolérance aux pannes, vulnérabilité

## 1 Introduction

Modern telecommunication satellites are very complex to design. Components are often prone to failure, and so providing robustness at the lowest possible cost is an important issue for the manufacturers. A key component of telecommunication satellites is an interconnection network which allows to redirect signals received by the satellite to a set of amplifiers where the signals will be retransmitted. The network is made of expensive switches; so we want to minimise their number subject to the following conditions: each input (respectively output) is adjacent to exactly one link; each switch is adjacent to exactly four links; there are  $n$  inputs (signals) and  $n + f$  outputs (amplifiers); among the  $n + f$  outputs,  $f$  are allowed to fail (all failures are irreversible). All the input signals should be sent to operational outputs via link-disjoint paths. Designing such networks is a complex problem that was proposed by Alcatel Space Industries and partial solutions are given in [3] and [4]. Alcatel Space Industries proposed also to consider the case where each signal needs a specific amplifier; that leads to design a network realizing any permutation from the inputs to the outputs (see [1], [2]). Recently it asked the following new problem. Out of the total of  $n$  inputs,  $p$  are called priorities and must be connected to the amplifiers providing the best quality of service (that is to some specific outputs) and the other signals should be sent to other amplifiers. Note that the priority signals are given, but the amplifiers providing the quality of service change according the position of the satellite and so the network should be able to route the signals for any set of  $f$  failed outputs and route the  $p$  priorities to any set of  $p$  operational outputs.

This problem can be restated more formally as follows:

**Definition 1** An  $(n, p, f)$ -network  $G$  is a graph  $(V, E)$  where the vertex set  $V$  is partitioned into four subsets  $P, I, O$  and  $S$  called respectively the *priorities*, the *ordinary inputs*, the *outputs* and the *switches*, satisfying the following constraints:

- there are  $p$  priorities,  $n - p$  ordinary inputs and  $n + f$  outputs;
- each priority, each ordinary input and each output is connected to exactly one switch;
- switches have degree at most 4.

An  $(n, p, f)$ -network is a *repartitor* if for any disjoint subsets  $F$  and  $Q$  of  $O$  with  $|F| = f$  and  $|Q| = p$ , there exist  $n$  edge-disjoint paths in  $G$  such that  $p$  of them connect  $P$  to  $Q$  and the  $n - p$  others connects  $I$  to  $O \setminus (Q \cup F)$ . The set  $F$  corresponds to the set of failures and  $Q$  to the set of amplifiers providing the best quality of service. We denote by  $R(n, p, f)$  the minimum number of switches (i.e. cardinality of  $S$ ) of a  $(n, p, f)$ -repartitor. An  $(n, p, f)$ -repartitor with  $R(n, p, f)$  switches will be called minimum.

**Problem 2** Determine  $R(n, p, f)$  and construct minimum (or almost minimum) repartitors.

As indicated above, the problem with no priority (that is,  $p = 0$ ) has been considered in [3] and almost solved in [4]. In [3], it is shown that  $R(n, 0, 1) = R(n, 0, 2) = n$ . In [4], it is proved that  $\frac{3n}{2} - O(\frac{n}{f}) \leq R(n, 0, f) \leq \frac{3n}{2} + g(f)$  with  $g$  a function depending only on  $f$ . The minimum size repartitors is also bounded for the following small values of  $f$ :

$R(n, 0, 4) = n + \lceil \frac{n}{4} \rceil$ ,  $R(n, 0, 6) = n + \frac{n}{4} + \sqrt{\frac{n}{8}} + O(1)$ ,  $R(n, 0, 8) = n + \frac{n}{3} + \frac{2}{3}\sqrt{\frac{n}{3}} + O(\sqrt{n})$ ,  
 $R(n, 0, 10) = n + \frac{3n}{8} + \Theta(\sqrt{n})$  and  $R(n, 0, 12) = n + \frac{3n}{7} + O(\sqrt{n})$ .

In this paper, we study  $(n, p, f)$ -repartitors **when  $p$  is not zero**. We first give a general upper bound for  $R$ :

$$R(n, p, f) \leq \frac{n+f}{2} \lceil \log_2 p \rceil + \frac{5}{2}(n-p) + g(f).$$

We then give optimal or near optimal bounds on  $R(n, p, f)$  for small values of  $p$  and  $f$ :

$$\begin{aligned} R(n, 1, 0) &= n - 1 \\ R(n, 2, 0) &= \left\lceil \frac{3n}{2} - 3 \right\rceil, \\ R(n, 1, 1) &= \left\lceil \frac{3n-1}{2} \right\rceil, \\ R(n, 1, 2) &= \left\lceil \frac{3n-1}{2} \right\rceil, \\ R(n, 3, 0) &= \frac{3n}{2} + \frac{\sqrt{n}}{2} + \Theta(1), \\ R(n, 2, 1) &= \frac{3n}{2} + \frac{\sqrt{n}}{2} + \Theta(1), \\ R(n, 4, 0) &= \frac{7n}{4} + \Theta(1), \\ R(n, 3, 1) &= \frac{7n}{4} + \Theta(1). \end{aligned}$$

## 2 General upper bounds

The aim of this section is to show an upper bound on  $R(n, p, f)$ . We first give an inductive construction of  $(n, p, 0)$ -networks and then construct an  $(n, p, f)$ -repartitor from an  $(n-p, 0, f)$ -repartitor and an  $(n+f, p, 0)$ -repartitor.

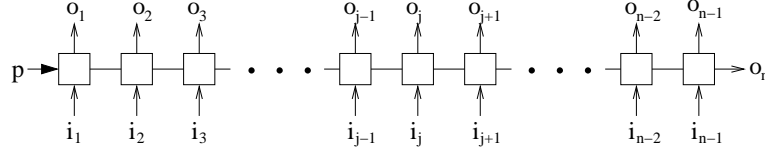
**Proposition 3** (i) If  $f \leq f'$  then  $R(n, p, f) \leq R(n, p, f')$ ;  
(ii)  $R(n, p, f) = R(n, n-p, f)$ .

**Conjecture 4** If  $p \leq p' \leq \frac{n}{2}$  then  $R(n, p, f) \leq R(n, p', f)$ .

**Proposition 5**

$$R(n, 1, 0) \leq n - 1$$

**Proof.** Let  $G$  be the graph with one priority  $p$ , ordinary input set  $\{i_1, i_2, \dots, i_{n-1}\}$ , output set  $\{o_1, o_2, \dots, o_n\}$  and switch set  $\{s_1, s_2, \dots, s_{n-1}\}$  such that  $(p, s_1, s_2, \dots, s_{n-1}, o_n)$  is a path and for  $l$ ,  $1 \leq l \leq n-1$ ,  $i_l$  and  $o_l$  are adjacent to  $s_l$ . (See Figure 1.) We claim that  $G$  is


 Figure 1: Minimum  $(n, 1, 0)$ -repartitor.

an  $(n, 1, 0)$ -repartitor. Indeed, suppose that  $o_j$  is the priority output. Since  $o_{n-1}$  and  $o_n$  are equivalent, we may suppose that  $j \neq n$ . The desired paths are  $P_0 = (p, s_1, s_2, \dots, s_j, o_j)$ ,  $P_j = (i_j, s_j, s_{j+1}, \dots, s_{n-1}, o_n)$  and  $P_l = (i_l, s_l, o_l)$ , for  $l \neq j$ . ■

**Theorem 6** For  $n \geq p \geq 1$ ,

- (i)  $R(2n, 2p, 0) \leq 2R(n, p, 0) + n - 1$ ,
- (ii)  $R(2n + 1, 2p, 0) \leq R(n + 1, p, 0) + R(n, p, 0) + n$ ,
- (iii)  $R(2n, 2p + 1, 0) \leq R(n, p + 1, 0) + R(n, p, 0) + n$ ,
- (iv)  $R(2n + 1, 2p + 1, 0) \leq R(n + 1, p + 1, 0) + R(n, p, 0) + n$ .

**Proof.** (i) Let  $G^1$  and  $G^2$  be two  $(n, p, 0)$ -repartitors. For  $i = 1, 2$ , let  $I^i$ ,  $P^i$ ,  $O^i = \{o_1^i, o_2^i, \dots, o_n^i\}$  and  $S^i$  be the ordinary input set, priority set, output set and switch set of  $G^i$ . Let  $H$  be a graph defined as follows: its ordinary input set is  $I = I^1 \cup I^2$ , its priority set is  $P = P^1 \cup P^2$ , its switch set is  $S^1 \cup S^2 \cup S$  with  $S = \{s_1, s_2, \dots, s_{n-1}\}$  and its output set is  $O = \{o_1, o_2, \dots, o_{n-1}\} \cup \{o'_1, o'_2, \dots, o'_{n-1}\} \cup \{o_n^1, o_n^2\}$ ;  $H$  contains the edges of  $G^1$  and  $G^2$  except the edges incident to  $o_j^1$  and  $o_j^2$  for  $1 \leq j \leq n - 1$ . Furthermore, for every  $1 \leq j \leq n - 1$ ,  $s_j$  is linked to  $o_j$ ,  $o'_j$ , the switch adjacent to  $o_j^1$  in  $G_1$  and the switch adjacent to  $o_j^2$  in  $G_2$ . (See Figure 2 (i))

We claim that  $H$  is a  $(2n, 2p, 0)$ -repartitor. Indeed, let  $Q$  be a  $2p$ -subset of  $O$ . Clearly, one can partition  $O$  in two  $n$ -sets  $O_1$  and  $O_2$  such that for  $i = 1, 2$ ,  $o_n^i \in O_i$ ,  $|Q \cap O_i| = p$  and  $\forall 1 \leq j \leq n - 1$ ,  $|O_i \cap \{o_j, o'_j\}| = 1$ . For  $i = 1, 2$ , let  $Q_i = Q \cap O_i$ . Since  $G^1$  and  $G^2$  are  $(n, p, 0)$ -repartitors, there are edge-disjoint paths joining  $P^i$  to  $Q_i$  and  $I^i$  to  $O_i \setminus Q_i$ . Hence, there are edge-disjoint paths joining  $P$  to  $Q$  and  $I$  to  $O \setminus Q$ .

The proofs of (ii), (iii) and (iv) are analogous and are omitted. (See Fig. 2) for the construction of  $H$ . ■

**Corollary 7**

$$R(n, p, 0) \leq \left\lfloor \frac{n}{2} \right\rfloor \lceil \log_2 p \rceil + n - p.$$

**Proof.** We proceed by induction on  $p$ . The inequality holds for  $p = 1$ , by Proposition 5.

Suppose that the inequality holds for every  $p' < p$ . Assume first that  $p$  and  $n$  are even. By Theorem 6 (i),

$$R(n, p, 0) \leq 2R(n/2, p/2) + n/2 - 1$$



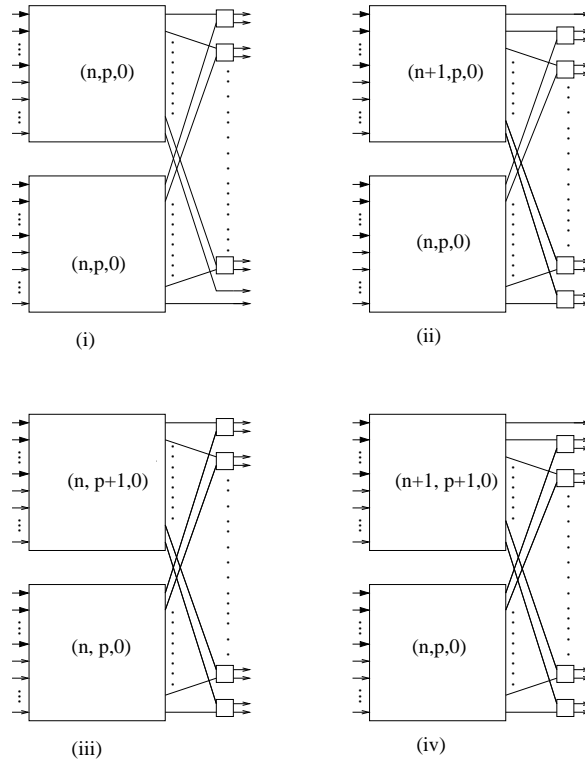


Figure 2: Construction of repartitors from smaller ones.

Thus, by induction hypothesis,

$$R(n, p, 0) \leq 2 \left\{ \left\lfloor \frac{n}{4} \right\rfloor \lceil \log_2(p/2) \rceil + n/2 - p/2 \right\} + n/2 - 1$$

Since  $2 \lfloor \frac{n}{4} \rfloor \leq \lfloor \frac{n}{2} \rfloor$ , we obtain

$$R(n, p, 0) \leq \left\lfloor \frac{n}{2} \right\rfloor \lceil \log_2(p/2) \rceil + n - p + n/2 - 1$$

Because  $\lceil \log_2(p/2) \rceil + 1 = \lceil \log_2(p) \rceil$  then

$$R(n, p, 0) \leq \left\lfloor \frac{n}{2} \right\rfloor \lceil \log_2 p \rceil + n - p - 1$$

If  $p$  or  $n$  is odd, we obtain the result analogously by Theorem 6 (ii), (iii) and (iv). ■

**Proposition 8** For  $p \leq n$ ,

$$R(n, p, f) \leq R(n - p, 0, f) + R(n + f, p, 0)$$

**Proof.** Let  $G_1$  be an  $(n - p, 0, f)$ -repartitor with output set  $O^1 = \{o_1^1, o_2^1, \dots, o_{n+f-p}^1\}$  and let  $G_2$  be an  $(n + f, p, 0)$ -repartitor with ordinary input set  $I^2 = \{i_1^2, i_2^2, \dots, i_{n+f-p}^2\}$ . Let  $G$  be the network obtained from the union of  $G_1$  and  $G_2$  by replacing each pair  $\{o_j^1, i_j^2\}$  by an edge between their neighboring switches. (See Figure 3). It remains to prove that

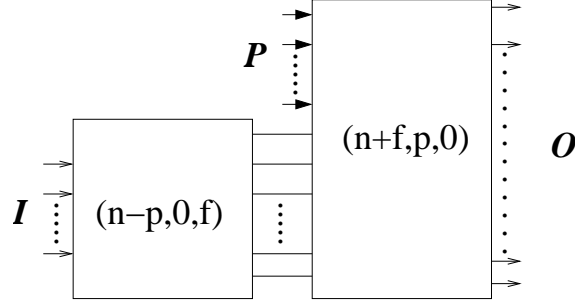


Figure 3: Construction of an  $(n, p, f)$ -repartitor.

$G$  is an  $(n, p, f)$ -repartitor. Let  $P = \{m_1, m_2, \dots, m_p\}$  the set of priorities of  $G_2$  be that of  $G$ ,  $I = \{i_1, i_2, \dots, i_{n-p}\}$  the input set of  $G_1$  be the ordinary input set of  $G$ , and  $O = \{o_1, o_2, \dots, o_{n+f}\}$  the output set of  $G_2$  be that of  $G$ . Let  $Q$  and  $F$  be two disjoint subsets of  $O$  with cardinality  $p$  and  $f$  respectively. Since  $G_2$  is operational there exists a set of edge-disjoint paths  $\{P_1, P_2, \dots, P_p\} \cup \{Q_1, Q_2, \dots, Q_{n+f-p}\}$  such that for  $1 \leq j \leq p$ ,  $P_j$  links  $m_j$  to an element of  $Q$  and for  $1 \leq j \leq n + f - p$ ,  $Q_j$  links  $i_j^2$  to an element of  $O \setminus Q$ . Let  $J$  be the set of indices  $j$  such that  $Q_j$  ends in a vertex of  $F$ . Set  $F^1 = \{o_j^1, j \in J\}$ . Since  $G_1$  is an  $(n - p, 0, f)$ -repartitor, there are edge-disjoint paths  $R_l$  in  $G_1$   $1 \leq l \leq n - p$  such that  $R_l$  links  $i_l$  to a vertex  $o_{\phi(l)}^1$  that is not in  $F^1$ . Now the union of the paths  $R_l$  and  $Q_{\phi(l)}$  induces a path  $P'_l$  in  $G$  which joins  $i_l$  to  $o_{\phi(l)} \in O \setminus (Q \cup F)$ . The paths  $P'_l$ ,  $1 \leq l \leq n - p$ , and the paths  $P_j$ ,  $1 \leq j \leq p$ , are obviously edge-disjoint. ■

**Corollary 9**

$$R(n, p, f) \leq \frac{n+f}{2} \lceil \log_2 p \rceil + \frac{5}{2}(n-p) + g(f)$$

**Proof.** By Proposition 8,  $R(n, p, f) \leq R(n - p, 0, f) + R(n + f, p, 0)$ . By Corollary 7,  $R(n + f, p, 0) \leq n + f - p + \frac{n+f}{2} \lceil \log_2 p \rceil$  and it is proved in [4] that  $R(n - p, 0, f) \leq \frac{3}{2}(n - p) + g(f)$ . ■

### 3 Lower bounds

To make computation easier and have equalities instead of inequalities, we will show that, without loss of generality, we can suppose that all the switches have degree 4 except possibly one. Let  $\epsilon(f) = 1$  if  $f$  is odd and 0 otherwise.

**Proposition 10** *There is a minimum  $(n, p, f)$ -repartitor with  $\epsilon(f)$  switch with degree 3 and all the others with degree 4.*

**Proof.** Let  $R$  be a minimum  $(n, p, f)$ -repartitor with the minimum number of switches with degree less than 4.

Obviously,  $R$  has no switches with degree 0 or 1 otherwise the network obtained by removing them is also a repartitor, which contradicts the minimality of  $R$ . Similarly,  $R$  has no switch of degree two. If a switch  $S$  had degree two, then we obtain a smaller valid repartitor by removing  $S$  and connecting its neighbours by an edge.

$R$  has at most one switch with degree 3, otherwise the network obtained from  $R$  by adding an edge between two switches with degree 3 is an  $(n, p, f)$ -repartitor with less switches with degree less than 4.

Since there is an even number of odd degree vertices in every graph,  $R$  has  $\epsilon(f)$  switches with degree 3. ■

Therefore, in the remaining of this section, we assume that every  $(n, p, f)$ -repartitor has  $\epsilon(f)$  switch with degree 3.

**Proposition 11** *In a minimum  $(n, p, f)$ -repartitor, there is no switch connected to two (or more) ordinary inputs and also there is no switch connected to two priorities.*

**Proof.** Let  $R$  be an  $(n, p, f)$ -repartitor containing a switch  $s$  connected to at least two ordinary inputs  $i_1$  and  $i_2$ . Let  $v_1$  and  $v_2$  be the two neighbours of  $s$  distinct from  $i_1$  and  $i_2$ . Then the  $(n, p, f)$ -network obtained from  $R$  by removing  $s$  and adding the two edges  $(i_1, v_1)$  and  $(i_2, v_2)$  is also a repartitor and  $R$  is not minimum, a contradiction. The same proof works if both  $i_1$  and  $i_2$  are priority inputs. ■

It follows that there are more switches than ordinary inputs.

**Corollary 12**

$$R(n, p, f) \geq n - p$$

#### 3.1 Cut criterium

One main tool to obtain lower bounds is to use the following cut criterium which gives necessary conditions for a network to be a repartitor. Let  $W$  be a set of vertices. We denote by  $in(W)$  (resp.  $out(W)$ ,  $pr(W)$ ) the cardinality of the set  $In(W)$  (resp.  $Out(W)$ ,  $Pr(W)$ ) of ordinary inputs (resp. outputs, priorities) of  $W$ . We denote by  $deg(W)$  the number of edges with an endvertex in  $W$  and the other one not in  $W$ .

**Proposition 13** *Let  $G$  be a  $(n, p, f)$ -repartitor and  $W$  be a set of vertices of  $G$ .*

- (i) *If  $p + f \leq out(W)$  then  $deg(W) \geq in(W) - out(W) + 2p + f - pr(W)$ .*
- (ii) *If  $p \leq out(W) \leq p + f$  then  $deg(W) \geq in(W) + p - pr(W)$ .*
- (iii) *If  $out(W) \leq p$  then  $deg(W) \geq in(W) + |out(W) - pr(W)|$ .*
- (iv)  *$deg(W) \geq in(W) + pr(W) - \max(0, out(W) - f)$ .*

**Proof.**

(i) Suppose that  $p$  outputs of  $Out(W)$  are in  $Q$  and  $f$  others are faulty. Then  $in(W) - out(W) + p + f$  paths from an ordinary input to an output of  $O \setminus Q$  leave  $W$ . And  $p - pr(W)$  paths from priorities enter  $W$ . Since all these paths are edge-disjoint they go through different edges with an end in  $W$  and the other not in  $W$ . Thus,  $deg(W) \geq in(W) - out(W) + 2p + f - pr(W)$ .

(ii) Suppose that  $p$  outputs of  $Out(W)$  are in  $Q$  and  $out(W) - p$  others are faulty. Then  $in(W)$  paths from an ordinary input to an output leave  $W$  and  $p - pr(W)$  paths from priorities enter  $W$ . Thus,  $deg(W) \geq in(W) + p - pr(W)$ .

(iii) Suppose that  $out(W)$  outputs of  $Out(W)$  are in  $Q$ . Then  $in(W)$  paths from an ordinary input to an output leave  $W$ . If  $out(W) \geq pr(W)$  then  $out(W) - pr(W)$  paths from priorities enter  $W$  and if  $out(W) \leq pr(W)$  then  $pr(W) - out(W)$  paths from priorities leave  $W$ . Thus,  $deg(W) \geq in(W) + |out(W) - pr(W)|$ .

(iv) Suppose that  $\min(out(W), f)$  outputs of  $Out(W)$  are faulty. Then  $in(W) + pr(W) - \max(0, out(W) - f)$  paths leave  $W$ . ■

### 3.2 Graph $\tilde{G}$ associated to $G$

We can distinguish two kinds of switches in a minimum  $(n, p, f)$ -repartitor, by Proposition 11. An *ordinary switch* is a switch adjacent to an ordinary input. A *usual switch* is a switch that is not ordinary switch. Let  $\mathcal{S}_o$  (resp.  $\mathcal{S}_u$ ) be the set of ordinary (resp. usual) switches and  $s_o$  (resp.  $s_u$ ) their cardinality. Counting the ordinary inputs, we have  $s_o = n - p$ .

Hence **the total number of switches is  $N = n - p + s_u$ .**

Thus, to find a minimum  $(n, p, f)$ -repartitor, we need to find a repartitor with as few usual switches as possible.

In order to obtain lower bounds for  $s_u$ , the basic idea is to consider the sum  $\Sigma$  of the edges incident to each vertex of  $\mathcal{S}_u$ . Doing so, we count twice an edge joining two vertices of  $\mathcal{S}_u$ . To avoid this weight problem, we introduce a *link vertex*  $b_e$  in the middle of an edge  $e$  with both endvertices in  $\mathcal{S}_u$ . Let  $\tilde{G}$  be the graph obtained from  $G$  by replacing each edge  $e = (s, s')$  joining two usual switches by the path  $(s, b_e, s')$ . Then the sum  $\Sigma$  is exactly the number of edges of  $\tilde{G}$  between  $\mathcal{S}_u$  and  $G - \mathcal{S}_u$ .

Following an idea of [4], let us now consider the connected components of  $\tilde{G} - S_u$  called *blocks*. We distinguish two kinds of blocks, the *principal blocks* corresponding to connected components of  $G - S_u$  and *link blocks* reduced to a link vertex. Note that with our definition when a block contains a switch it also contains the inputs and outputs connected to it.

**Proposition 14** *For every block  $B$ ,  $\deg(B) \leq in(B) + 2 - out(B) - pr(B)$ .*

**Proof.** The statement holds trivially for link blocks. By definition, a principal block  $B$  has  $in(B)$  switches. Let  $e(B)$  be the number of edges connecting two switches of  $B$ . Since  $B$  is connected then  $e(B) \geq in(B) - 1$ . Furthermore, there are  $in(B)$  (resp.  $out(B)$ ,  $pr(B)$ ) edges linking inputs (resp. outputs, priorities) to switches. Thus,  $\deg(B) = \sum_{v \text{ switch}} [deg(v) - 2e(B) - in(B) - out(B) - pr(B)]$ . Since the degree of every switch is at most 4, we obtain  $\deg(B) \leq 4in(B) - 2e(B) - in(B) - out(B) - pr(B) \leq in(B) + 2 - out(B) - pr(B)$ . ■

**Remark 15** *Note that if equality holds in Proposition 14, then every switch has degree four.*

**Proposition 16** *Let  $B$  be a block of  $\tilde{G}$ .*

- (i)  $pr(B) \leq 1$ .
- (ii) If  $p \geq 2$  then  $out(B) \leq 1$ .
- (iii) If  $f \geq 1$ , then  $out(B) + pr(B) \leq 1$ .

**Proof.** (i) If  $B$  has two or more priorities, then by Proposition 14,  $\deg(B) \leq in(B) - out(B)$ . This contradicts Proposition 13 (iv).

(ii) Suppose that  $B$  contains two or more outputs. Then either  $out(B) \geq p + f$  and by Proposition 13.(i),  $\deg(B) \geq in(B) + 4 - out(B) - pr(B)$  contradicting Proposition 14; or  $p \leq out(B) \leq p + f$  and by Proposition 13.(ii),  $\deg(B) \geq in(B) + 2 - pr(B)$  contradicting Proposition 14; or  $p \geq out(B)$  and by Proposition 13.(iii),  $\deg(B) \geq in(B) + out(B) - pr(B)$  contradicting Proposition 14.

(iii) If  $B$  has one priority and one output then by Proposition 14,  $\deg(B) \leq in(B)$ . This contradicts Proposition 13.(iv) if  $f \geq 1$ . ■

In the remainder of this section, we assume that either  $p \geq 2$  or  $p = 1$  and  $f \geq 1$ .; indeed the case  $p = 1$ ,  $f = 0$  is already solved by Proposition 5 and Corollary 12.

We partition the blocks in the four following sets:

- $\mathcal{B}'_p$ , the set of blocks having one priority and one output;
- $\mathcal{B}_p$ , the set of blocks having one priority and no output;
- $\mathcal{B}_1$ , the set the blocks having one output and no priority;
- $\mathcal{B}_0$ , the set of blocks having no priority and no output including the link blocks;

**Remark 17** Note that if  $f \geq 1$ , then  $\mathcal{B}'_p$  is empty.

**Proposition 18** A block in  $\mathcal{B}'_p$  satisfies  $\deg(B) = \text{in}(B)$ . A block  $B \in \mathcal{B}_1 \cup \mathcal{B}_p$  satisfies  $\deg(B) = \text{in}(B) + 1$ .

**Proof.** It follows directly from Propositions 13.(iii) and 14. ■

**Remark 19** Note that in a block of  $\mathcal{B}'_p \cup \mathcal{B}_p \cup \mathcal{B}_1$  every switch has degree four by Remark 15.

**Proposition 20** A block  $B \in \mathcal{B}_0$  of a minimum  $(n, p, f)$ -repartitor satisfies  $\deg(B) = \text{in}(B) + 2$  unless it contains the only vertex switch with degree 3 in which case  $\deg(B) = \text{in}(B) + 1$ .

**Proof.** If  $B$  is a link block, then  $\deg(B) = 2 = \text{in}(B) + 2$ . Assume now that  $B$  contains switches. Let  $e(B)$  be the number of edges linking two switches of  $B$ .

By Proposition 13,  $\deg(B) \geq \text{in}(B)$  (a).

In the proof of Proposition 14, we have:  $\deg(B) = \sum_{v \text{ switch}} \deg(v) - 2e(B) - \text{in}(B)$ .

Thus, if one switch has degree 3 then:  $\deg(B) = 3\text{in}(B) - 2e(B) - 1$  (b).

Since  $e(B) \geq \text{in}(B) - 1$ , Equations (a) and (b) yield:  $\deg(B) = \text{in}(B) + 1$ .

If every switch has degree four we obtain:  $\deg(B) = 3\text{in}(B) - 2e(B)$  (c). Because  $e(B) \geq \text{in}(B) - 1$ , Equations (a) and (c) yield:  $\deg(B) = \text{in}(B) + 2$  or  $\deg(B) = \text{in}(B)$ . Now if  $\deg(B) = \text{in}(B)$ , then the graph obtained by removing the switches of  $B$  and linking one to one the ordinary inputs of  $B$  to the neighbours of  $B$  is also operational. This contradicts the minimality of  $G$ . So  $\deg(B) = \text{in}(B) + 2$ . ■

Let  $b_0, b_1, b_p$  and  $b'_p$  be the cardinality of  $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_p$  and  $\mathcal{B}'_p$ . We denote the number of ordinary inputs which are in a block of  $\mathcal{B}_0$  (resp.,  $\mathcal{B}_1, \mathcal{B}_p$ , and  $\mathcal{B}'_p$ ) by  $n_0$  (resp.,  $n_1, n_p$ , and  $n'_p$ ).

From previous definitions, propositions and remarks, one can easily derive few equations.

Let  $\epsilon'(f) = 0$  if  $f$  is even,  $\epsilon'(f) = 1$  if  $f$  is odd, and the switch of degree 3 is usual and  $\epsilon'(f) = -1$  otherwise.

**Proposition 21** We have the following equalities:

$$b_1 + b'_p = n + f \tag{1}$$

$$b'_p = 0 \quad \text{if } f \geq 1 \tag{2}$$

$$b_p + b'_p = p \tag{3}$$

$$n_1 + n_0 + n_p + n'_p = n - p \tag{4}$$

$$4s_u = 2n + f + 2b_0 - 2b'_p + \epsilon'(f) \tag{5}$$

**Proof.** (1) there is a one to one correspondence between blocks of  $\mathcal{B}_1 \cup \mathcal{B}'_p$  and the outputs they contain.

(2) is Remark 17.

(3) there is a one to one correspondence between blocks of  $\mathcal{B}_p \cup \mathcal{B}'_p$  and the priorities they contain.

(4) is obtained by noting that the  $n - p$  ordinary inputs are in blocks.

(5) counts the edges between usual switches and blocks. On one side, there are  $4s_u$  (minus  $\epsilon'(f)$  if the switch of degree 3 is a usual one). On the other side, we have by Proposition 18,  $n'_p$  edges to blocks of  $\mathcal{B}'_p$ ,  $n_p + b_p$  edges incident to blocks of  $\mathcal{B}_p$  and  $n_1 + b_1$  edges incident to blocks of  $\mathcal{B}_1$  and by Proposition 20,  $n_0 + 2b_0$  edges incident to blocks of  $\mathcal{B}_0$  (or  $n_0 + 2b_0 - 1$  if the switch of degree 3 is in a block of  $\mathcal{B}_0$ ). Thus  $4s_u = n'_p + n_p + b_p + n_1 + b_1 + n_0 + 2b_0 + \epsilon'(f)$ . Then by (1) (3) and (4), we obtain the result. ■

### 3.3 The lower bounds

**Theorem 22** *If  $f \geq 1$ , then*

$$R(n, p, f) \geq \frac{3n}{2} - p + \frac{f + \epsilon(f)}{4}$$

**Proof.** By Equations 5 and 2,  $s_u = n/2 + (f + 2b_0 + \epsilon'(f))/4$ . Now if  $\epsilon'(f) = -1$ , by Remark 19, the switch of degree 3 is in a block of  $\mathcal{B}_0$ ; hence  $b_0 \geq 1$ . Therefore  $s_u \geq n/2 + (f + \epsilon(f))/4$ . As  $N = n - p + s_u$ , we obtain  $R(n, p, f) \geq \frac{3n}{2} - p + \frac{f + \epsilon(f)}{4}$ . ■

**Theorem 23**

$$R(n, 2, 0) \geq \frac{3n}{2} - 3$$

**Proof.** By Equations 5,  $s_u \geq n/2 - b'_p/2 \geq n/2 - 1$ . Then  $R(n, 2, 0) \geq \frac{3n}{2} - 3$ . ■

From now on, we will suppose  $p \geq 3$  or  $p = 2$  and  $f \geq 1$ .

**Lemma 24** *For  $p \geq 3$  or  $p = 2$ ,  $f \geq 1$ , a usual switch is adjacent to at most 2 elements of  $\mathcal{B}_1$ .*

**Proof.** Suppose to the contrary that a usual switch  $S$  is adjacent to three blocks  $B_1, B'_1$  and  $B''_1$  of  $\mathcal{B}_1$ . Then  $W = S \cup B_1 \cup B'_1 \cup B''_1$  satisfies  $out(W) = 3$  and contradicts Proposition 13 (ii) if  $p = 2$  and  $f \geq 1$  or Proposition 13 (iii) if  $p \geq 3$  as  $deg(W) = in(W) + 1$  and  $pr(W) = 0$ . ■

For  $0 \leq i \leq 2$ , let  $\mathcal{S}_i$  be the set of usual switches adjacent to exactly  $i$  elements of  $\mathcal{B}_1$  and  $s_i$  its cardinality. By Lemma 24,  $(\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2)$  is a partition of  $\mathcal{S}_u$ . Moreover, together with Proposition 21, it yields the following corollary:

**Corollary 25** For  $p = 2$  and  $f \geq 1$  or  $p \geq 3$ ,

$$\sum_{B \in \mathcal{B}_1} \deg(B) = n_1 + b_1 = n_1 + n + f - b'_p = 2s_2 + s_1 \leq s_u + s_2 \quad (6)$$

**Lemma 26** If  $p \geq 3$ , then a block of  $\mathcal{B}'_p$  and a switch of  $\mathcal{S}_2$  cannot be adjacent.

**Proof.** Suppose that a block  $B'$  of  $\mathcal{B}'_p$  and a switch  $S_2$  of  $\mathcal{S}_2$  are adjacent. Let  $B_1$  and  $B'_1$  be the blocks of  $\mathcal{B}_1$  adjacent to  $S_2$ . Then  $W = B' \cup S_2 \cup B_1 \cup B'_1$  contradicts Proposition 13 (iii) as  $\deg(W) = \text{in}(W)$ ,  $\text{out}(W) = 3$  and  $\text{pr}(w) = 1$ . ■

It follows from Remark 17 and Lemma 26 that a switch of  $\mathcal{S}_2$  is adjacent to two blocks of  $\mathcal{B}_1$  and two blocks of  $\mathcal{B}_0 \cup \mathcal{B}_p$ . Let  $H$  be the multigraph whose vertices correspond to the blocks of  $\mathcal{B}_0 \cup \mathcal{B}_p$  and where two vertices are joined by  $\mu$  edges if the corresponding blocks share  $\mu$  neighbours in  $\mathcal{S}_2$ .

**Lemma 27** If  $p \geq 2$  and  $f \geq 1$ , then  $H$  has no cycle of length at most  $\lfloor \frac{f+3}{2} \rfloor$ . If  $p \geq 3$ , then  $H$  is a simple graph (has no cycle of length 2).

**Proof.** Suppose that in  $H$  there is a cycle  $C$  of length at most  $\lfloor \frac{f+3}{2} \rfloor$  or a cycle of length 2 if  $p \geq 3$ . Let  $W$  be the union of the blocks  $\mathcal{B}_0 \cup \mathcal{B}_p$  corresponding to the vertices of  $C$ , plus the switches of  $\mathcal{S}_2$  corresponding to edges of  $C$  and the blocks of  $\mathcal{B}_1$  adjacent to these switches. Then  $W$  contradicts Proposition 13 (i), (ii) or (iii), as  $\deg(W) = \text{in}(W) - \text{pr}(W)$ . ■

Since a simple graph on  $n$  vertices has at most  $n(n-1)/2$  edges, it follows:

**Corollary 28** If  $p \geq 2$  and  $f \geq 1$  or  $p \geq 3$ ,

$$s_2 \leq \frac{(b_0 + b_p)(b_0 + b_p - 1)}{2} \quad (7)$$

Turán's Theorem asserts that a simple graph on  $n$  vertices without cycle of length 3 has at most  $n^2/4$  edges. So we get:

**Corollary 29** For  $p \geq 2$  and  $f \geq 3$  then ,

$$s_2 \leq (b_0 + b_p)^2/4 \quad (8)$$

**Theorem 30**

$$R(n, 2, 2) \geq R(n, 2, 1) \geq \frac{3n}{2} + \frac{\sqrt{n+3}}{2} - 3$$

$$R(n, 3, 0) \geq \frac{3n}{2} + \frac{\sqrt{n-3}}{2} - \frac{9}{2}$$



**Proof.** Suppose  $p = 2$  and  $f = 1$ . From Inequality 7 and Equations 6, as  $b'_p = 0$  thus  $b_p = p = 2$ , we obtain:

$$\frac{(b_0 + 2)(b_0 + 1)}{2} \geq s_2 \geq n_1 + n + 1 - s_u$$

Replacing  $s_u$  by its value in Equation 5,

$$\begin{aligned} \frac{b_0^2 + 3b_0 + 2}{2} &\geq n_1 + n + 1 - n/2 - 1/4 - b_0/2 - \epsilon'(1)/4 \\ b_0^2 + 4b_0 + 4 &\geq n + 3 \\ b_0 + 2 &\geq \sqrt{n + 3} \end{aligned}$$

Writing this bound on  $b_0$  into Equation 5, we obtain  $s_u \geq \frac{n}{2} + \frac{\sqrt{n+3}}{2} - 1$ . Thus

$$R(n, 2, 1) \geq \frac{3n}{2} + \frac{\sqrt{n+3}}{2} - 3$$

The proof is analogous if  $p = 3$  and  $f = 0$ . We obtain  $\frac{(b_0+b_p)(b_0+b_p-1)}{2} \geq n_1 + n - b'_p - s_u$  and  $(b_0 + b_p)^2 \geq n - b'_p + b_p$ . Hence  $b_0 \geq \sqrt{n - b'_p + b_p} - b_p$ . So  $s_u \geq \frac{n}{2} + \frac{\sqrt{n-b'_p+b_p}}{2} - \frac{b_p+b'_p}{2} \geq \frac{n}{2} + \frac{\sqrt{n-3}}{2} - \frac{3}{2}$ .

Finally  $R(n, 3, 0) \geq \frac{3n}{2} + \frac{\sqrt{n-3}}{2} - \frac{9}{2}$ . ■

**Theorem 31**

$$\text{If } f \geq 3, \quad R(n, 2, f) \geq \frac{3n}{2} + \frac{\sqrt{n}}{\sqrt{2}} + \Omega(f)$$

**Proof.**

Suppose now that  $p = 2$  and  $f \geq 3$ . In the same way as above, it follows from Inequality 8 and Equations 6 and 5:

$$\begin{aligned} (b_0 + 2)^2 + 2b_0 &\geq 2n + 3f - \epsilon'(f) \\ b_0 + 3 &\geq \sqrt{2n + 3f - \epsilon'(f) + 5} \end{aligned}$$

Writing this bound on  $b_0$  into Equation 5, we obtain

$$s_u \geq \frac{n}{2} + \frac{1}{2}\sqrt{2n + 3f - \epsilon'(f) + 5} + \frac{f + \epsilon'(f)}{4} - \frac{3}{2}.$$

Thus

$$R(n, 2, f) \geq \frac{3n}{2} + \frac{1}{2}\sqrt{2n + 3f - \epsilon'(f) + 5} + \frac{f + \epsilon'(f)}{4} - \frac{7}{2}.$$

■

**Lemma 32** *If  $p \geq 4$  or  $p = 3$  and  $f \geq 1$ , then a block of  $\mathcal{B}_0 \cup \mathcal{B}_p$  is adjacent to at most one switch of  $\mathcal{S}_2$ .*

**Proof.** Suppose that a block  $B \in \mathcal{B}_0 \cup \mathcal{B}_p$  is adjacent to two switches  $S_2$  and  $S'_2$  of  $\mathcal{S}_2$ . Let  $B_1$  and  $C_1$  and  $B'_1$  and  $C'_1$  be the elements of  $\mathcal{B}_1$  adjacent to  $S_2$  and  $S'_2$  respectively. Then  $S_2 \cup S'_2 \cup B \cup B_1 \cup C_1 \cup B'_1 \cup C'_1$  contradicts Proposition 13 (iii) if  $p \geq 4$  or Proposition 13 (ii) if  $p = 3$  and  $f \geq 1$ . ( $\deg(W) = \text{in}(W) + 2 - pr(W)$ ,  $\text{out}(W) = 4$  and  $pr(W) \leq 1$ .) ■

Let  $\mathcal{B}'_0$  (resp.  $\mathcal{B}''_0$ ) be the set of elements of  $\mathcal{B}_0$  adjacent to exactly one (resp. no) switch in  $\mathcal{S}_2$  and let  $b'_0$  (resp.  $b''_0$ ) be its cardinality.

**Corollary 33** *For  $p \geq 4$  or  $p = 3$  and  $f \geq 1$ ,*

$$b_0 + b_p \geq b'_0 + b_p \geq 2s_2 \quad (9)$$

**Theorem 34** *If  $p \geq 4$  or  $p = 3$  and  $f \geq 1$ ,*

$$R(n, p, f) \geq \frac{7n}{4} - \frac{7p}{4} + \frac{5f - \epsilon(f)}{8}$$

**Proof.** By Inequality 9 and Equations 6, we have:  $s_u \geq n + f - b'_p - (b_0 + b_p)/2$ . By Equation 5  $s_u = n/2 + (f + \epsilon'(f))/4 + (b_0 - b'_p)/2$ . Adding these two inequalities, we obtain:

$$2s_u \geq \frac{3n}{2} + \frac{5f + \epsilon'(f)}{4} - \frac{b_p + 3b'_p}{2}$$

Hence,  $R(n, p, f) \geq \frac{7n}{4} - \frac{7p}{4} + \frac{5f + \epsilon'(f)}{8}$ . ■

**Lemma 35** *If  $p \geq 5$  or  $p = 4$  and  $f \geq 1$ , every switch of  $\mathcal{S}_1$  is adjacent to at most one block of  $\mathcal{B}'_0$ .*

**Proof.** Suppose that  $S_1 \in \mathcal{S}_1$  is a switch adjacent to two blocks  $B_0$  and  $B'_0$  in  $\mathcal{B}'_0$ . Let  $S_2$  (resp.  $S'_2$ ) be the switch of  $\mathcal{S}_2$  adjacent to  $B_0$  (resp.  $B'_0$ ) and  $B_1$  and  $C_1$  (resp.  $B'_1$  and  $C'_1$ ) the two blocks of  $\mathcal{B}_1$  adjacent to  $S_2$  (resp.  $S'_2$ ). Let  $A_1$  be the block of  $\mathcal{B}_1$  adjacent to  $S_1$ . Then  $W = A_1 \cup S_1 \cup S_2 \cup B_0 \cup B_1 \cup C_1 \cup S'_2 \cup B'_0 \cup B'_1 \cup C'_1$  contradicts Proposition 13 as  $\deg(W) \leq \text{in}(W) + 3$  and  $\text{out}(W) = 5$ . ■

**Lemma 36** *If  $p \geq 6$  or  $p = 5$  and  $f \geq 1$ , every switch of  $\mathcal{S}_0$  is adjacent to at most two blocks of  $\mathcal{B}'_0$ .*

**Proof.** Suppose that  $S_0$  is a switch adjacent to three blocks  $B_0$ ,  $B'_0$  and  $B''_0$  in  $\mathcal{B}'_0$ . Let  $S_2$  (resp.  $S'_2$ ,  $S''_2$ ) be the switch of  $\mathcal{S}_2$  adjacent to  $B_0$  (resp.  $B'_0$ ,  $B''_0$ ) and  $B_1$  and  $C_1$  (resp.  $B'_1$  and  $C'_1$ ,  $B''_1$  and  $C''_1$ ) the two blocks of  $\mathcal{B}_1$  adjacent to  $S_2$  (resp.  $S'_2$ ,  $S''_2$ ). Then  $W = S_0 \cup S_2 \cup B_0 \cup B_1 \cup C_1 \cup S'_2 \cup B'_0 \cup B'_1 \cup C'_1 \cup S''_2 \cup B''_0 \cup B''_1 \cup C''_1$  contradicts Proposition 13 as  $\deg(W) \leq \text{in}(W) + 4$  and  $\text{out}(W) = 6$ . ■

**Proposition 37** *Let  $p \geq 2$ . A block  $B \in \mathcal{B}'_0$  has no switch of degree 3. Thus  $\deg(B) = \text{in}(B) + 2$ .*

**Proof.** Suppose that  $B$  has a switch with degree 3. Let  $S$  be the switch of  $\mathcal{S}_2$  adjacent to  $B$  and  $B_1$  and  $C_1$  be the two blocks of  $\mathcal{B}_1$  adjacent to  $S$ . Then  $W = B \cup S \cup B_1 \cup C_1$  contradicts Proposition 13-(iii), as  $\deg(W) = \text{in}(W) + 1$ . ■

Let  $S'_1$  be the set of elements of  $\mathcal{S}_1$  adjacent to exactly one block in  $\mathcal{B}'_0$  and let  $s'_1$  be its cardinality. Let  $n'_0$  denote the number of inputs in blocks of  $\mathcal{B}'_0$ .

**Corollary 38** *If  $p \geq 6$  or  $p = 5$  and  $f \geq 1$ ,*

$$n'_0 + b'_0 \leq s'_1 + 2s_0 \quad (10)$$

**Proof.** Let us count the number  $e$  of edges between  $\mathcal{B}'_0$  and  $S'_1 \cup \mathcal{S}_0$ .

On one side, every block  $B$  of  $\mathcal{B}'_0$  is adjacent to exactly one element of  $\mathcal{S}_2$  and no in  $\mathcal{S}_1 \setminus S'_1$ . Thus, by Proposition 37, there are  $\text{in}(B) + 1$  edges from  $B$  to  $S'_1 \cup \mathcal{S}_0$ . Thus  $e = n'_0 + b'_0$ .

On the other side, there are  $s'_1$  (resp. at most  $2s_0$ ) edges joining  $S'_1$  (resp.  $\mathcal{S}_0$ ) to  $\mathcal{B}'_0$  according to the definition of  $S'_1$  and Lemma 35 (resp. Lemma 36). Thus  $e \leq s'_1 + 2s_0$ . ■

**Lemma 39** *If  $p \geq 4$ , a block of  $\mathcal{B}'_p$  is not adjacent to a switch of  $\mathcal{S}'_1$ .*

**Proof.** Suppose  $B'$  is a block of  $\mathcal{B}'_p$  adjacent to  $S'$  in  $\mathcal{S}'_1$ . Let  $A_1$  (resp.  $B$ ) be the block of  $\mathcal{B}_1$  (resp.  $\mathcal{B}'_0$ ) adjacent to  $S'$ . Let  $S_2$  be the switch of  $\mathcal{S}_2$  adjacent to  $B$  and  $B_1$  and  $C_1$  the two blocks of  $\mathcal{B}_1$  adjacent to  $S_2$ . Then  $W = B' \cup S' \cup A_1 \cup B \cup S_2 \cup B_1 \cup C_1$  contradicts Proposition 13 (iii), as  $\deg(W) = \text{in}(W) + 1$ ,  $\text{out}(W) = 4$  and  $\text{pr}(W) = 1$ . ■

**Lemma 40** *If  $p \geq 6$  or  $p = 5$  and  $f \geq 1$ , every element of  $\mathcal{B}''_0 \cup \mathcal{B}_p$  is adjacent to at most one element of  $\mathcal{S}'_1$ .*

**Proof.** If  $B \in \mathcal{B}''_0 \cup \mathcal{B}_p$  is adjacent to two switches  $S_1$  and  $S'_1$  of  $\mathcal{S}'_1$ . Let  $B_1$  and  $B_0$  (resp.  $B'_1$  and  $B'_0$ ) be the blocks of  $\mathcal{B}_1$  and  $\mathcal{B}'_0$  adjacent to  $S_1$  (resp.  $S'_1$ ). Let  $W_0$  (resp.  $W'_0$ ) be the union of the switch of  $\mathcal{S}_2$  adjacent to  $B_0$  (resp.  $B'_0$ ) and its two adjacent blocks of  $\mathcal{B}_1$ . Then  $W = B \cup B_1 \cup B_0 \cup W_0 \cup B'_1 \cup B'_0 \cup W'_0$  contradicts Proposition 13 (ii) or (iii), as  $\deg(W) = \text{in}(W) + 4 - \text{pr}(W)$ ,  $\text{out}(W) = 6$  and  $\text{pr}(W) \leq 1$ . ■

**Corollary 41** *If  $p \geq 6$  or  $p = 5$  and  $f \geq 1$ ,*

$$b''_0 + b_p \geq 2s'_1 \quad (11)$$

**Theorem 42** *If  $p \geq 6$  or  $p = 5$  and  $f \geq 1$ ,*

$$R(n, p, f) \geq \frac{15n}{8} - \frac{15p}{8} + \frac{13f - \epsilon(f)}{16}$$

**Proof.** We have  $s_u = s_2 + s_1 + s_0$ , then by Equations 6,

$$s_u = n + n_1 + f - b'_p - s_2 + s_0$$

$$2s_u \geq 2n + 2f - 2b'_p - 2s_2 + 2s_0$$

By Inequalities 10 and 9,  $2s_0 \geq n'_0 + 2s_2 - s'_1 - b_p$ , thus

$$2s_u \geq 2n + 2f + n'_0 - s'_1 - b_p - 2b'_p$$

Then by Inequality 11,

$$2s_u \geq 2n + 2f + n'_0 - b''_0/2 - 3b_p/2 - 2b'_p$$

$$4s_u \geq 4n + 4f - b''_0 - 3b_p - 4b'_p$$

Since  $b_0 = b'_0 + b''_0$ , by Equation 5 and Inequality 9,  $2s_u \geq n + (f + \epsilon'(f))/2 - b'_p + b''_0 + 2s_2 - b_p$ , so

$$6s_u \geq 5n + 2s_2 + \frac{9f + \epsilon'(f)}{2} - 4b_p - 5b'_p$$

Therefore, by Equations 6,

$$8s_u \geq 7n + \frac{13f + \epsilon'(f)}{2} - 4b_p - 7b'_p$$

■

## 4 Constructions

### 4.1 Useful constructions

In this subsection, we give general constructions how to compose several repartitors to a larger repartitor. The first one is in the same vein as Proposition 8. The second one constructs an  $(n - 1, p, f)$ -repartitor from an  $(n, p, f)$ -repartitor under some condition.

**Proposition 43** For  $0 \leq f' \leq f$ ,

$$R(n, p, f) \leq R(n + f', p + f', f - f') + R(p, p, f')$$

**Proof.** Let  $G_1$  be a  $(p, p, f')$ -repartitor and  $G_2$  be a  $(n + f', p + f', f - f')$ -repartitor. For  $i = 1, 2$ , let  $I_i$  (resp.  $P_i, O_i, S_i$ ) be the sets of ordinary inputs (resp. priorities, outputs, switches) of  $G_i$  (note that  $I_1$  is empty); and let  $P_2 = \{p_1, p_2, \dots, p_{p+f'}\}$  and  $O_1 = \{o_1, o_2, \dots, o_{p+f'}\}$ . Let  $G$  be the network defined as follows:

$$V(G) = P_1 \cup S_1 \cup I_2 \cup S_2 \cup O_2;$$

two vertices of  $V(G)$  are joined by an edge if and only if  $(u, v) \in E(G_1) \cup E(G_2)$  or there exists some  $i$  such that  $(u, o_i) \in E(G_1)$  and  $(p_i, v) \in E(G_2)$ ; the ordinary input set of  $G$  is  $I_2$ , its priority set  $P_1$ , its output set  $O_2$  and its switch set  $S_1 \cup S_2$ .

The proof that  $G$  is an  $(n, p, f)$ -repartitor is similar to that of Proposition 8. ■

**Definition 44** Let  $G$  be a network. An edge  $(s, s')$  is said to be *linking* if there is an ordinary input  $i$  connected to  $s$  and an output  $o$  connected to  $s'$ .

The *L-contraction* of a linking edge  $(s, s')$  consists in contracting the edge  $(s, s')$  in a vertex  $t$  and deleting the input connected to  $s$  and the output connected to  $s'$ . See Figure 4

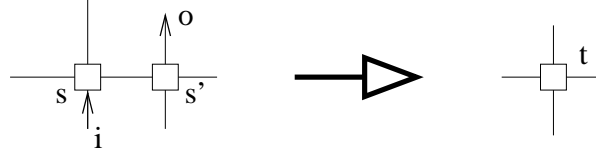


Figure 4: L-contraction

**Proposition 45** Let  $G$  be an  $(n, p, f)$ -repartitor. Then the network obtained by the L-contraction of a linking arc is an  $(n - 1, p, f)$ -repartitor.

**Proof.** Let  $(s, s')$  be a linking edge of  $G$  and  $i$  the input linked to  $s$  and  $o$  the output linked to  $s'$ . Let  $G'$  be the network obtained by the L-contraction of  $(s, s')$  in  $t$ .

Let  $Q$  and  $F$  be two disjoint subsets of outputs of  $G'$  with  $|Q| = p$  and  $|F| = k$ . Since  $G$  is a repartitor, there exists a set  $\mathcal{P}$  of  $n$  edge disjoint paths in  $G$ , connecting the  $p$  priorities to  $Q$  and the  $n - p$  ordinary inputs to  $O \setminus (Q \cup F)$ .

Case 1: A path  $P \in \mathcal{P}$  connect  $i$  to  $o$ . Then replacing in the  $n - 1$  paths of  $\mathcal{P} \setminus P$  the vertex  $s, s'$  or the succession of these two by  $t$ , we obtain the required set of  $n - 1$  paths in  $G'$ .

Case 2: A path  $P_1$  of  $\mathcal{P}$  connect an ordinary input  $i' \neq i$  to  $o$  and a path  $P_2$  of  $\mathcal{P}$  connect  $i$  to an output  $o'$ . Let  $P_1 = (i', s'_1, s'_2, \dots, s'_j, s', o)$  and  $P_2 = (i, s, s_1, s_2, \dots, s_l, o')$ . Let  $W'$  be the walk  $(i', s'_1, s'_2, \dots, s'_j, s', s, s_1, s_2, \dots, s_l, o')$  and  $P'$  the path from  $i'$  to  $o'$  contained in  $W'$ . Let  $\mathcal{P}' = (\mathcal{P} \setminus \{P_1, P_2\}) \cup \{P'\}$ . Then replacing in the  $n - 1$  paths of  $\mathcal{P}'$  the vertex  $s, s'$  or the succession of these two by  $t$ , we obtain the required set of  $n - 1$  paths in  $G'$ . ■

## 4.2 Minimum $(n, 1, 0)$ -repartitors.

**Theorem 46**

$$R(n, 1, 0) = n - 1$$

**Proof.** By Proposition 5,  $R(n, 1, 0) \leq n - 1$ . By Corollary 12,  $R(n, 1, 0) \geq n - 1$ . ■

We can, in fact, completely describe the set of minimum  $(n, 1, 0)$ -repartitors.

Let  $T$  be a binary tree with  $n$  leaves and  $n - 1$  internal nodes one being the root  $r$ . Let us define the graph  $G_T$  associated to  $T$  in the following way: every leaf of  $T$  is an output of  $G_T$ ; the internal nodes of  $T$  are the switches of  $G_T$ ; each internal node of  $T$  is adjacent to an ordinary input and furthermore the root is adjacent to the priority (See Figure 5.).

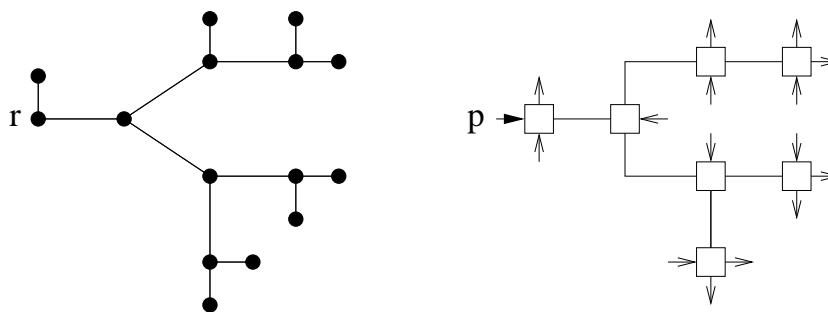


Figure 5: A binary tree with 8 leaves and the corresponding minimum  $(8, 1, 0)$ -repartitor.

**Proposition 47** *A  $(n, 1, 0)$ -repartitor is minimum if and only if it is associated to a binary tree with  $n$  leaves.*

**Proof.** It is easy to see that the graph associated to a binary tree is a minimum  $(n, 1, 0)$ -repartitor.

Let  $N$  be a minimum  $(n, 1, 0)$ -repartitor. It has  $n - 1$  switches and each of them is adjacent to exactly one ordinary input. Moreover  $n$  outputs and one priority are connected to the switches. Thus, there are exactly  $(4(n - 1) - 2n)/2 = n - 2$  edges connecting a switch to another. Since  $N$  is connected then it is a tree. Let  $T$  be the subtree of  $N$  induced by the switches and the outputs. It is clearly a binary tree. Indeed every output has degree one and every switch has degree 3 except the one that is connected to the priority that has degree 2 (and thus is the root). ■

### 4.3 Minimum $(n, 2, 0)$ -repartitors.

**Theorem 48**

$$R(n, 2, 0) = \left\lceil \frac{3n}{2} - 3 \right\rceil$$

**Proof.** Theorem 23 yields  $R(n, 2, 0) \geq \lceil \frac{3n}{2} - 3 \rceil$ . Since  $R(n, 1, 0) = n - 1$ , by the first inequality of Theorem 6,  $R(n, 2, 0) \leq \lceil \frac{3n}{2} - 3 \rceil$ . ■

**Remark 49** Note that minimum  $(n, 2, 0)$ -repartitors are not necessarily formed from two minimum  $(n/2, 1, 0)$ -repartitors with the construction of Theorem 6. Indeed, let  $R$  be a minimum  $(n, 2, 0)$ -repartitor and  $o$  and  $o'$  two distinct outputs incident to the switches  $s$  and  $s'$  respectively. Let  $R'$  be the  $(n + 2, 2, 0)$ -network obtained by removing  $o$  and  $o'$  and adding three switches  $t, t'$  and  $u$  such that  $t$  is adjacent to  $u, s$  and an ordinary input and an output,  $t'$  is adjacent to  $u, s'$  and an ordinary input and an output, and  $u$  is adjacent to two outputs. It is easy to check that  $R$  is a minimum  $(n + 2, 2, 0)$ -repartitor. And since  $t$

and  $t'$  are adjacent to an input and an output,  $R'$  may not be obtained via the mentioned construction.

#### 4.4 Minimum $(n, 1, 1)$ -repartitors and $(n, 1, 2)$ -repartitors

**Theorem 50**

$$R(n, 1, 1) = \left\lceil \frac{3n-1}{2} \right\rceil$$

**Proof.** Theorem 22 yields  $R(n, 1, 1) \geq \lceil \frac{3n-1}{2} \rceil$ . Now from, Proposition 43,  $R(n, 1, 1) \leq R(n+1, 2, 0) + R(2, 1, 0)$ . By Theorem 48,  $R(n+1, 2, 0) = \lceil \frac{3n+3}{2} - 3 \rceil$ , and by Theorem 46,  $R(2, 1, 0) = 1$ . Thus  $R(n, 1, 1) \leq \lceil \frac{3n}{2} - \frac{1}{2} \rceil$ . ■

**Theorem 51**

$$R(n, 1, 2) = \left\lceil \frac{3n-1}{2} \right\rceil$$

**Proof.** Theorem 22 yields  $R(n, 1, 2) \geq \lceil \frac{3n}{2} - \frac{1}{2} \rceil$ .

The network depicted Figure 6 has  $\lceil \frac{3n}{2} - \frac{1}{2} \rceil$  switches. Checking that this network is an

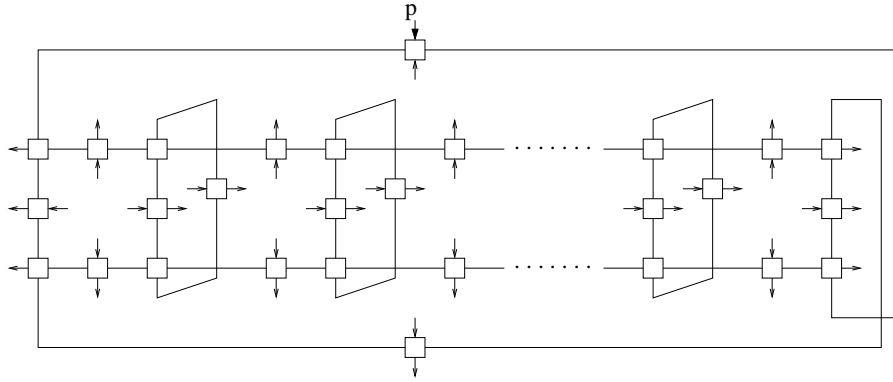


Figure 6: Minimum  $(n, 1, 2)$ -repartitor

$(n, 1, 2)$ -repartitor is not difficult but tedious since we must investigate a certain amount of configurations and exhibit each time the corresponding edge-disjoint paths. The complete proof is left in appendix A. ■

#### 4.5 $(n, 3, 0)$ -repartitors

**Theorem 52**

$$R(n, 3, 0) = \frac{3n}{2} + \frac{\sqrt{n}}{2} + c$$

where  $-5 \leq c \leq 9/4$ .

Let  $G_l$  be the graph with vertex set  $\{a_{i,j}, 1 \leq i \leq 3, 1 \leq j \leq l\}$  whose edge set is the union of the three paths  $P_i = (a_{i,1}, a_{i,2}, \dots, a_{i,l})$  for  $1 \leq i \leq 3$  and the  $l$  cycles  $C_j = (a_{1,j}, a_{2,j}, a_{3,j}, a_{1,j})$  for  $1 \leq j \leq l$ . See Figure 7.

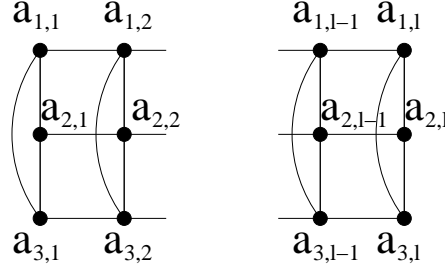


Figure 7: The graph  $G_l$ .

So  $G_l$  has  $3l$  vertices and  $6l - 3$  edges.

From  $G_l$ , we construct the network  $H_l$  as follows.

To each vertex  $a_{i,j}$ ,  $1 \leq i \leq 3$ ,  $1 \leq j \leq l$ , we associate a switch  $s_{i,j}$ . Each switch  $s_{i,0}$ ,  $1 \leq i \leq 3$ , is connected to a priority input  $p_i$  and each switch  $s_{i,l}$ ,  $1 \leq i \leq 3$ , is connected to some output  $o_i$ . We replace each edge  $e$  of  $G_l$  by a path  $P(e)$  consisting of  $n(e)$  switches with some ordinary input connected to each of them. In the bipartite graph  $\tilde{H}_l$  defined previously, it corresponds to associate to each edge  $e$  a block  $B_0(e)$  of  $\mathcal{B}_0$ . The size  $n(e)$  of the paths  $P(e)$  is  $in(B_0(e))$  and determined as follows: In  $\tilde{H}_l$ , we connect two blocks  $B_0(e)$  and  $B_0(f)$  to a switch  $s_{e,f}$ , if the corresponding edges  $e$  and  $f$  are not adjacent. In  $H_l$  it corresponds to connect an ordinary switch of  $P(e)$  to an ordinary switch of  $P(f)$  via  $s_{e,f}$  (in such a way that each vertex has degree four). Furthermore  $s_{e,f}$  is connected to two outputs  $o_{e,f}$  and  $o'_{e,f}$ . (In  $\tilde{H}_l$ ,  $s_{e,f}$  is in  $\mathcal{S}_2$ .)

If  $l \geq 3$ , an edge  $e$  of  $G_l$  is connected to 6 other edges except the edges of  $E_1 = \{(a_{i,1}, a_{i,2}), 1 \leq i \leq 3\} \cup \{(a_{i,l-1}, a_{i,l}), 1 \leq i \leq 3\}$ , which are connected to 5 other edges and the edges of  $E_2 = \{(a_{i,1}, a_{i+1,1}), 1 \leq i \leq 3\} \cup \{(a_{i,l}, a_{i+1,l}), 1 \leq i \leq 3\}$ , which are connected to 4 edges. Hence the number  $n(e)$  of switches of  $P(e)$  is defined as follows:  $n(e) = 6l - 8$  if  $e \in E_2$ ,  $n(e) = 6l - 9$  if  $e \in E_1$  and  $n(e) = 6l - 10$  otherwise.

In summary,  $H_l$  has for  $l \geq 3$ :

- 3 priority inputs,
- $n_l - 3 = 6(6l - 8) + 6(6l - 9) + (6l - 15)(6l - 10) = 36l^2 - 78l + 48$  ordinary inputs,
- $n_l$  outputs,
- $n_l - 3$  ordinary switches (those of the  $P(e)$ ),



- $(n_l - 3)/2$  switches of  $\mathcal{S}_2$  (the  $s_{e,f}$ ), and
- $3l$  switches of  $\mathcal{S}_0$  (the  $s_{i,j}$ ).

Hence  $H_l$  has  $N_l = \frac{3n_l}{2} + 3l - \frac{9}{2}$  switches. As  $n_l = 36l^2 - 78l + 51 = (6l - \frac{13}{2})^2 + \frac{35}{4}$ .  
Hence  $N_l = \frac{3n_l}{2} + \frac{1}{2}\sqrt{n_l - \frac{35}{4}} - \frac{5}{4} \leq \frac{3n_l}{2} + \frac{1}{2}\sqrt{n_l} - \frac{5}{4}$ .

Note that all the formulae are also valid for  $l = 2$ . Indeed in that case  $n(e) = 4$  for any of the 9 edges and so  $n_l - 3 = 36$ .

**Theorem 53**  $H_l$  is a repartitor.

We will reduce the proof of this theorem to the existence of some specific walks in the graph  $G_l$  slightly modified.

**Definition 54** Let  $G'_l$  be the graph obtained from  $G_l$  by adding the set  $R_3 = \{a_{1,l+1}, a_{2,l+1}, a_{3,l+1}\}$  and the three edges  $(a_{i,l}, a_{i,l+1})$ ,  $1 \leq i \leq 3$ .

Let  $E_3 = \{e_1, e_2, e_3\}$  be a set of distinct edges of  $G'_l$ . An  $E_3$ -good set of walks is a set of three edge disjoint walks  $\{W_1, W_2, W_3\}$  such that  $W_i$  contains  $e_i$  and the set of initial vertices of the  $W_i$  is  $\{a_{1,1}, a_{2,1}, a_{3,1}\}$  and the set of terminal vertices of the  $W_i$  is  $R_3$ .

**Lemma 55** If  $G'_l$  admits an  $E_3$ -good set of walks for any set  $E_3$  of three distinct edges such that two of them are not adjacent, then  $H_l$  is a repartitor.

**Proof.** Let us define an *assignment* as a mapping  $\phi$  from the output set of  $H_l$  into the edges of  $G'_l$  such that :

- $\phi(o_i) = (a_{i,l}, a_{i,l+1})$  for  $i \in \{1, 2, 3\}$ ;
- for the two outputs  $o_{e,f}$  and  $o'_{e,f}$  adjacent to the switch  $s_{e,f}$ , then  $\phi(o_{e,f}) = e$  and  $\phi(o'_{e,f}) = f$  or  $\phi(o_{e,f}) = f$  and  $\phi(o'_{e,f}) = e$ .

Let  $Q = \{q_1, q_2, q_3\}$  be any set of three outputs of  $H_l$  and let us denote  $\{o_1, o_2, o_3\}$  by  $O_3$ . To prove that  $H_l$  is a repartitor, we have to find the desired paths in  $H_l$ .

First let us show that there exists an assignment  $\phi$  such that the three edges  $\phi(q_i)$  are distinct and two of them are not adjacent.

- If  $|Q \cap O_3| \geq 2$ , let  $q_1 = o_{i_1}$  and  $q_2 = o_{i_2}$ . Then for any assignment  $\phi$ ,  $\phi(q_1) = e_1 = (a_{i_1,l}, a_{i_1,l+1})$  and  $\phi(q_2) = e_2 = (a_{i_2,l}, a_{i_2,l+1})$ . Hence  $e_1$  and  $e_2$  are not adjacent and distinct from  $\phi(e_3)$  since only  $q_1$  (resp.  $q_2$ ) may be mapped on  $e_1$  (resp.  $e_2$ ) by an assignment.
- If  $|Q \cap O_3| = 1$ , let  $q_1 = o_{i_1}$  and  $q_2$  (resp.  $q_3$ ) be connected to  $s_{e_2, f_2}$  (resp.  $s_{e_3, f_3}$ ). As all the edges incident to  $\phi(q_1) = e_1 = (a_{1,l}, a_{1,l+1})$  are adjacent, one of  $\{e_2, f_2\}$  is not adjacent to  $e_1$ . Let  $\phi(q_2)$  be this edge and let  $\phi(q_3)$  be an edge of  $\{e_3, f_3\} \setminus \{e_2\}$ . Then  $\phi$  satisfies the requirement.

- If  $|Q \cap O_3| = 0$ , let  $q_i$ ,  $1 \leq i \leq 3$ , be connected to  $s_{e_i, f_i}$ . By definition  $|\{e_i, f_i\} \cap \{e_j, f_j\}| \leq 1$ .

Suppose first that there is  $i \neq j$  such that  $|\{e_i, f_i\} \cap \{e_j, f_j\}| = 1$ . Without loss of generality, we may assume that  $e_1 = e_2$ . Then let  $\phi(q_1) = e_1$  and  $\phi(q_2) = f_2$ .  $e_1 = e_2$  and  $f_2$  are not adjacent. Let  $\phi(q_3)$  be an edge of  $\{e_3, f_3\} \setminus \{e_2, f_2\}$ . (Such an edge exists since  $|\{e_2, f_2\} \cap \{e_3, f_3\}| \leq 1$ .)

Suppose now that all the six edges  $e_1, f_1, e_2, f_2, e_3, f_3$  are distinct. If one edge of  $\{e_1, f_1\}$  is not adjacent to one edge of  $\{e_2, f_2\}$ , say  $e_1$  is not adjacent to  $e_2$ . Then let  $\phi(q_i) = e_i$ . Otherwise the four edges  $e_1, f_1, e_2, f_2$  form a 4-cycle. Since there is no diagonal of a 4-cycle in  $G'_l$ ,  $e_3$  is adjacent to at most one edge of  $\{e_1, f_1\}$ , say  $e_3$  is not adjacent to  $e_1$ . Then setting  $\phi(q_i) = e_i$ , we obtain the desired assignment.

Let us now exhibit the  $n = 36l^2 - 78l + 51$  required edge disjoint paths, 3 of them joining the priorities to  $Q$  and the  $n - 3$  remaining joining the ordinary inputs to outputs.

Consider an assignment  $\phi$  as above and let  $E_3 = \{\phi(q_1), \phi(q_2), \phi(q_3)\}$ . By hypothesis, there exists an  $E_3$ -good set of walks  $\{W_1, W_2, W_3\}$  such that  $W_i$  contains  $\phi(q_i)$ .

Let  $o$  be an output not in  $O_3$ . Let  $e = \phi(o)$ ,  $s_{e, f}(o)$  is the switch adjacent to  $o$ ,  $s_e(o)$  the vertex of  $P(e)$  adjacent to  $s_{e, f}(o)$  and  $i(o)$  the input adjacent to  $s_e(o)$ . The *default path* associated to  $o$  is  $P(o) = (i(o), s_e(o), s_{e, f}(o), o)$ .

If  $o \notin Q \cup O_3$ , we choose its default path for joining the ordinary input  $i(o)$  to  $o$ .

To each  $W_i$  corresponds a walk  $Q_i$  in  $H(l)$  obtained by replacing an edge of  $G'_l$  by the corresponding path  $P(e)$  and the edge  $\{a_{i, l}, a_{i, l+1}\}$  by the output  $o_i$ .

If  $q_i \in O_3$  then choose  $Q_i$  which joins a priority to the priority output  $q_i$ .

If  $q_i \notin O_3$ , let  $s_i = s_e(q_i)$  be the switch that is in both  $Q_i$  and the default path of  $q_i$ . Let  $D_i$  (resp.  $F_i$ ) be the subpath of  $Q_i$  starting in  $a_{j, 0}$  (resp. at  $s_i$ ) and ending at  $s_i$  (resp. in  $O_3$ ). Then choose  $(p_{j_i}, D_i, s_{e, f}(q_i), q_i)$  which joins a priority to a priority output and  $(i_{q_i}, F_i)$  which joins an ordinary input to a non-priority output. All the paths we have constructed are edge-disjoint. ■

**Proof of Theorem 53.** By Lemma 55, it suffices to prove that for any set  $E_3$  of three edges of  $G'_l$  with two non adjacent edges, there is an  $E_3$ -good set of walks.

For  $1 \leq i \leq 3$  and  $1 \leq j \leq k \leq l+1$ , we will denote by  $L_i[j, k]$  the path  $(a_{i, j}, a_{i, j+1}, \dots, a_{i, k})$ .

By symmetry of  $G'_l$ , it suffices to prove it for the following ten cases. All these cases and the corresponding covering walks are depicted Figure 8. The walk  $W_1$  (resp.  $W_2, W_3$ ) is drawn as a dotted (resp. bold, thin) line. The grey (resp. black, white) ellipses represent the possible positions of  $e_1$  (resp.  $e_2, e_3$ ).

- (1)  $e_1 = (a_{1, i_1}, a_{1, i_1+1})$ ,  $e_2 = (a_{1, i_2}, a_{1, i_2+1})$  and  $e_3 = (a_{1, i_3}, a_{1, i_3+1})$  with  $0 \leq i_1 < i_2 < i_3 \leq l$ . The covering walks are  $W_1 = (L_1[1, i_2], a_{3, i_2}, L_2[i_2, l+1])$ ,  $W_2 = (L_2[1, i_2], L_1[i_2, i_3], L_3[i_3, l+1])$ , and  $W_3 = (L_3[1, i_3], a_{2, i_3}, L_1[i_3, l+1])$ .
- (2)  $e_1 = (a_{1, i_1}, a_{1, i_1+1})$ ,  $e_2 = (a_{1, i_2}, a_{1, i_2+1})$  and  $e_3 = (a_{3, i_3}, a_{3, i_3+1})$  with  $0 \leq i_1 < i_2 \leq l$ . The covering walks are  $W_1 = (L_1[1, i_2], L_2[i_2, l+1])$ ,  $W_2 = (L_2[1, i_2], a_{3, i_2}, L_1[i_2, l+1])$ , and  $W_3 = (L_3[1, l+1])$ .

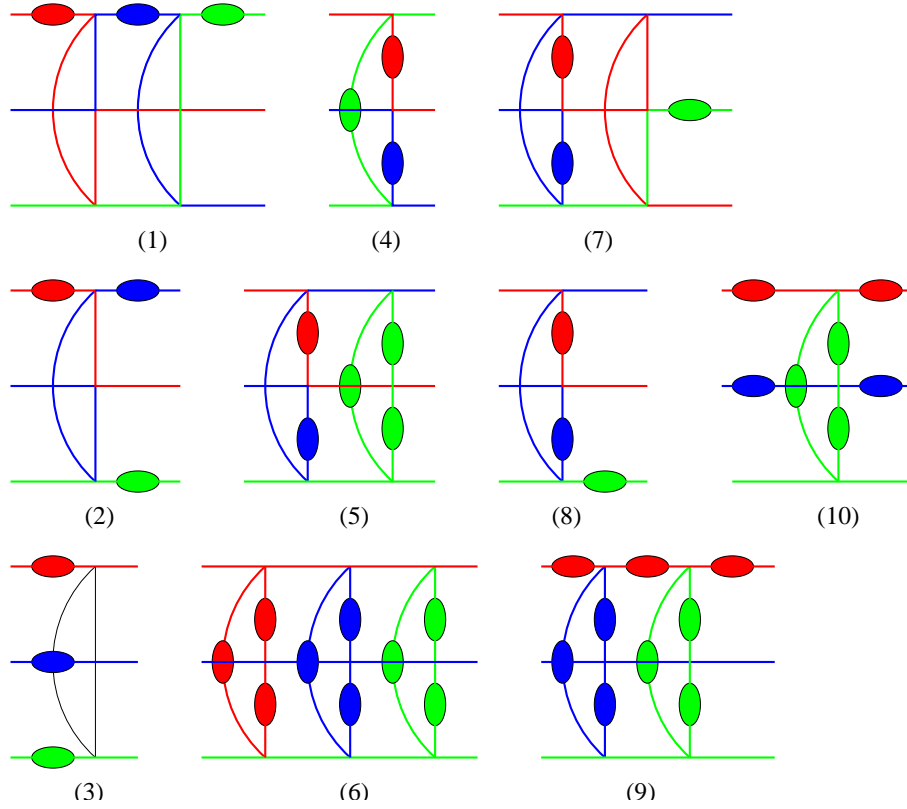


Figure 8: The covering walks in each case.

- (3)  $e_1 = (a_{1,i_1}, a_{1,i_1+1})$ ,  $e_2 = (a_{2,i_2}, a_{2,i_2+1})$  and  $e_3 = (a_{3,i_3}, a_{3,i_3+1})$ . The covering walks are  $W_1 = (L_1[1, l+1])$ ,  $W_2 = (L_2[1, l+1])$ , and  $W_3 = (L_3[1, l+1])$ .
- (4)  $e_1 = (a_{1,i}, a_{2,i})$ ,  $e_2 = (a_{2,i}, a_{3,i})$  and  $e_3 = (a_{3,i}, a_{1,i})$ .  $W_1 = (L_1[1, i], L_2[i, l+1])$ ,  $W_2 = (L_2[1, i], L_3[i, l+1])$ , and  $W_3 = (L_3[1, i], L_1[i, l+1])$ .
- (5)  $e_1 = (a_{1,i_1}, a_{2,i_1})$ ,  $e_2 = (a_{2,i_1}, a_{3,i_1})$  and  $e_3 \in \{(a_{1,i_3}, a_{2,i_3}), (a_{2,i_3}, a_{3,i_3}), (a_{3,i_3}, a_{1,i_3})\}$  with  $i_3 \neq i_1$ .  $W_1 = (L_1[1, i_1], L_2[i_1, l+1])$ ,  $W_2 = (L_2[1, i_1], a_{3,i_1}, L_1[i_1, l+1])$ , and  $W_3 = (L_3[1, i_3], a_{1,i_3}, a_{2,i_3}, L_3[i_3, l+1])$ .
- (6)  $e_1 \in \{(a_{1,i_1}, a_{2,i_1}), (a_{2,i_1}, a_{3,i_1}), (a_{3,i_1}, a_{1,i_1})\}$ ,  $e_2 \in \{(a_{1,i_2}, a_{2,i_2}), (a_{2,i_2}, a_{3,i_2}), (a_{3,i_2}, a_{1,i_2})\}$  and  $e_3 \in \{(a_{1,i_3}, a_{2,i_3}), (a_{2,i_3}, a_{3,i_3}), (a_{3,i_3}, a_{1,i_3})\}$  with  $i_1 < i_2 < i_3$ .  $W_1 = (L_1[1, i_1], a_{2,i_1}, a_{3,i_1}, L_1[i_1, l+1])$ ,  $W_2 = (L_2[1, i_2], a_{3,i_2}, a_{1,i_2}, L_2[i_2, l+1])$ , and  $W_3 = (L_3[1, i_3], a_{1,i_3}, a_{2,i_3}, L_3[i_3, l+1])$ .

- (7)  $e_1 = (a_{1,i_1}, a_{2,i_1})$ ,  $e_2 = (a_{2,i_1}, a_{3,i_1})$  and  $e_3 = (a_{2,i_3}, a_{2,i_3+1})$  with  $i_1 < i_3$ .  
 $W_1 = (L_1[1, i_1], L_2[i_1, i_3], a_{1,i_3}, L_3[i_3, l+1])$ ,  $W_2 = (L_2[1, i_1], a_{3,i_1}, L_1[i_1, l+1])$ , and  
 $W_3 = (L_3[1, i_3], L_2[i_3, l+1])$ .
- (8)  $e_1 = (a_{1,i_1}, a_{2,i_1})$ ,  $e_2 = (a_{2,i_1}, a_{3,i_1})$  and  $e_3 = (a_{3,i_3}, a_{3,i_3+1})$ .  $W_1 = (L_1[1, i_1], L_2[i_1, l+1])$ ,  $W_2 = (L_2[1, i_1], a_{3,i_1}, L_1[i_1, l+1])$ , and  $W_3 = (L_3[1, l+1])$ .
- (9)  $e_1 = (a_{1,i_1}, a_{1,i_1+1})$ ,  $e_2 \in \{(a_{1,i_2}, a_{2,i_2}), (a_{2,i_2}, a_{3,i_2}), (a_{3,i_2}, a_{1,i_2})\}$  and  $e_3 \in \{(a_{1,i_3}, a_{2,i_3}), (a_{2,i_3}, a_{3,i_3}), (a_{3,i_3}, a_{1,i_3})\}$  with  $i_2 < i_3$ .  $W_1 = (L_1[1, l+1])$ ,  $W_2 = (L_2[1, i_2], a_{3,i_2}, a_{1,i_2}, L_2[i_2, l+1])$ , and  $W_3 = (L_3[1, i_3], a_{1,i_3}, a_{2,i_3}, L_3[i_3, l+1])$ .
- (10)  $e_1 = (a_{1,i_1}, a_{1,i_1+1})$ ,  $e_2 = (a_{2,i_2}, a_{2,i_2+1})$  and  $e_3 \in \{(a_{1,i_3}, a_{2,i_3}), (a_{2,i_3}, a_{3,i_3}), (a_{3,i_3}, a_{1,i_3})\}$ .  
 $W_1 = (L_1[1, l+1])$ ,  $W_2 = (L_2[1, l+1])$ , and  $W_3 = (L_3[1, i_3], a_{1,i_3}, a_{2,i_3}, L_3[i_3, l+1])$ .

■

**Proof of Theorem 52** By Theorem 30,  $R(n, 3, 0) \geq \frac{3n}{2} + \frac{\sqrt{n}}{2} - 5$ .

Let us now define the  $(n, 3, 0)$ -network  $G_3(n)$  inductively as follows :  
 For  $l \geq 2$ , let  $n_l = 36l^2 - 78l + 51$ . If  $n = n_l$  for some integer  $l$ , then  $G_3(n) = H_l$ . Otherwise let  $n_{l-1} < n < n_l$ . If  $n$  is odd then  $G_3(n)$  is obtained from  $G_3(n+2)$  by removing a switch  $s_{e,f}$  and its two adjacent outputs, and deleting the two switches  $s(e) \in P(e)$  and  $s(f) \in P(f)$  adjacent to  $s_{e,f}$  and their adjacent inputs, and adding an edge joining the two neighbours of  $s(e)$  (resp.  $s(f)$ ) in the input path  $P(e)$  (resp.  $P(f)$ ).  
 If  $n$  is even then  $G_3(n)$  is obtained from  $G_3(n+1)$  by an L-contraction of a linking edge  $\{s(e), s_{e,f}\}$ .

Since  $H_l$  is a repartitor, then if  $n$  is odd  $G_3(n)$  is obviously an  $(n, 3, 0)$ -repartitor and if  $n$  is even, by Proposition 45,  $G_3(n)$  is an  $(n, 3, 0)$ -repartitor.  $G_3(n)$  has  $N = N_l - \frac{3}{2}(n - n_l) + \varepsilon/2 = \frac{3n}{2} + \frac{\sqrt{n_l}}{2} - 5/4 + \varepsilon/2$  switches where  $\varepsilon = 1$  if  $n$  is even. But as  $n > n_{l-1}$ ,  $\sqrt{n_l} - \sqrt{n} < \sqrt{n_l} - \sqrt{n_{l-1}} \leq 6$ . So  $N \leq \frac{3n}{2} + \frac{\sqrt{n}}{2} + \frac{9}{4}$ . ■

**Theorem 56**

$$R(n, 2, 1) = \frac{3n}{2} + \frac{\sqrt{n+1}}{2} + c$$

where  $-3 \leq c \leq \frac{23}{4}$ .

**Proof.** From Proposition 43, we obtain:  $R(n, 2, 1) \leq R(n+1, 3, 0) + R(2, 2, 1)$ . By Theorem 52,  $R(n+1, 3, 0) \leq \frac{3n}{2} + \frac{\sqrt{n+1}}{2} + \frac{15}{4}$ . And by Proposition 3,  $R(2, 2, 1) = R(2, 0, 1)$  and  $R(2, 0, 1) = 2$  (see [3]). Hence  $R(n, 2, 1) \leq \frac{3n}{2} + \frac{\sqrt{n+1}}{2} + \frac{23}{4}$ . Theorem 30 gives the other inequality. ■

## 4.6 $(n, 4, 0)$ -networks

The aim of this subsection is to prove the following theorem :

**Theorem 57**

$$R(n, 4, 0) = \frac{7n}{4} + c$$

where  $-7 \leq c \leq \frac{89}{4}$ .

We will construct a network  $H(m)$ , which is a  $(16m + 4, 4, 0)$ -repartitor.

We start with a basis brick  $B$  (see Figure 9). It contains 12 switches  $a_l, b_l, c_l$  for  $1 \leq l \leq 4$ .

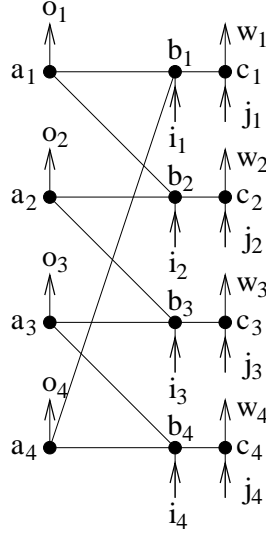


Figure 9: A brick

$a_l$  is connected to  $b_l$  and  $b_{l+1}$  (the indices being taken modulo 4) and  $b_l$  is connected to  $c_l$ . There are 8 inputs: for  $1 \leq l \leq 4$ ,  $i_l$  connected to  $b_l$  and  $j_l$  connected to  $c_l$ . There are 8 outputs: for  $1 \leq l \leq 4$ ,  $o_l$  connected to  $a_l$  and  $w_l$  connected to  $c_l$ .

Let  $G(m)$  be the network obtained by concatenating  $m$  bricks  $B^k$ ,  $1 \leq k \leq m$ , that is by joining  $c_l^k$  to  $a_l^{k+1}$  for  $1 \leq k \leq m-1$  and  $1 \leq l \leq 4$ .

Let  $F(m)$  be the network obtained by taking two copies of  $G(m)$ ,  $G(m)$  and  $\overline{G}_m$ , deleting the  $8m$  outputs  $w_l^k$  and  $\overline{w}_l^k$ , adding  $4m$  switches  $s_l^k$  connected to  $c_l^k$ ,  $\overline{c}_l^k$  and two new outputs  $r_l^k$  and  $\overline{r}_l^k$ .

Finally let  $IS$  be an *input selector* with 4 inputs  $p_l$ ,  $1 \leq l \leq 4$ , and 8 outputs  $\alpha_l$  and  $\overline{\alpha}_l$ ,  $1 \leq l \leq 4$ , such that for any set of four outputs, there are four edge-disjoint paths joining the inputs to these outputs.

Let  $OS$  be an *output selector* with 8 inputs  $\gamma_l$  and  $\overline{\gamma}_l$ ,  $1 \leq l \leq 4$  and 4 outputs  $t_l$ ,  $1 \leq l \leq 4$ , such that for any set of four inputs of the form  $C_1 \cup C_2$  and any partition  $T_1 \cup T_2$  of  $T = \{t_1, t_2, t_3, t_4\}$  with  $|C_1| = |T_1|$ , there are four edge-disjoint paths,  $|C_1|$  of them joining  $C_1$  to  $T_1$  and  $|C_2|$  of them joining  $C_2$  to  $T_2$ .

Then  $H(m)$  is obtained by concatenating  $IS$ ,  $F(m)$  and  $OS$  that is by identifying  $\alpha_l$  (resp.  $\bar{\alpha}_l$ ) with  $a_l^1$  (resp.  $\bar{a}_l^1$ ) and  $c_l^m$  (resp.  $\bar{c}_l^m$ ) with  $\gamma_l$  (resp.  $\bar{\gamma}_l$ ). The network  $H(m)$  is depicted in Figure 10

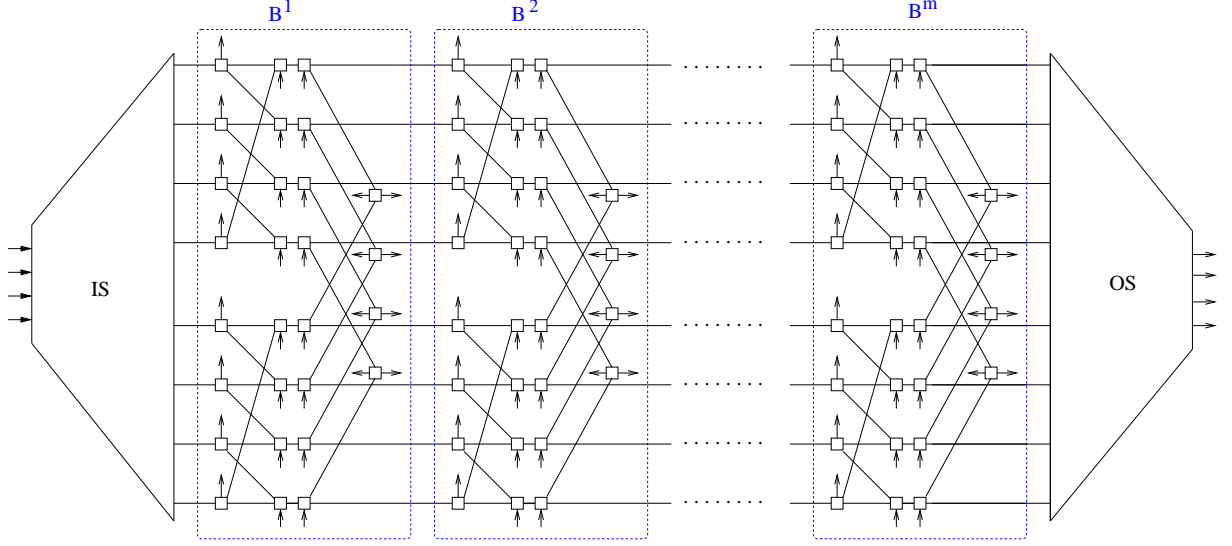


Figure 10: The network  $H(m)$ .

$H(m)$  has 4 priorities  $p_l$ ,  $1 \leq l \leq 4$ ,  $16m$  ordinary inputs  $i_l^k, \bar{i}_l^k, j_l^k, \bar{j}_l^k$  for  $1 \leq k \leq m$  and  $1 \leq l \leq 4$  and  $n = 16m + 4$  outputs,  $o_l^k, \bar{o}_l^k, r_l^k, \bar{r}_l^k$  for  $1 \leq k \leq m$  and  $1 \leq l \leq 4$  and  $t_l$ ,  $1 \leq l \leq 4$ . It has  $28m + N(IS) + N(OS)$  switches where  $N(IS)$  (resp.  $N(OS)$ ) is the minimum number of switches of an input (resp. output) selector. An input selector is nothing else than a  $(4, 0, 4)$ -repartitor and it is proved in [4] that  $R(4, 0, 4) = 5$ , so  $N(IS) = 5$ . An optimum input selector is drawn Figure 11 left. The network depicted Figure 11 right is an output selector. Indeed, it is constructed from a Waksman network (see [1, 2]), which is able to realize any permutation of the 8 inputs into the 8 outputs, by removing the useless outputs, switches and links (dotted on the figure). Hence  $N(OS) \leq 13$ .

**Theorem 58**  $H(m)$  is a  $(16m + 4, 4, 0)$ -repartitor.

**Proof.** In what follows we will denote  $O = \bigcup_{k,l} o_l^k$ ,  $\bar{O} = \bigcup_{k,l} \bar{o}_l^k$ ,  $R = \bigcup_{k,l} r_l^k$ ,  $\bar{R} = \bigcup_{k,l} \bar{r}_l^k$ ,  $W = \bigcup_{k,l} w_l^k$ ,  $\bar{W} = \bigcup_{k,l} \bar{w}_l^k$ .

Let  $Q$  be any set of four outputs of  $H(m)$ . Let  $Q_1 = Q \cap (O \cup \bar{O} \cup R \cup \bar{R})$  and  $Q_2 = Q \cap T$  and  $q_i = |Q_i|$ ,  $1 \leq i \leq 2$ .

To prove that  $H(m)$  is a repartitor it suffices to find in  $F(m)$  for any set  $Q = Q_1 \cup Q_2$  a set of  $16m + 4$  edge disjoint paths such that:

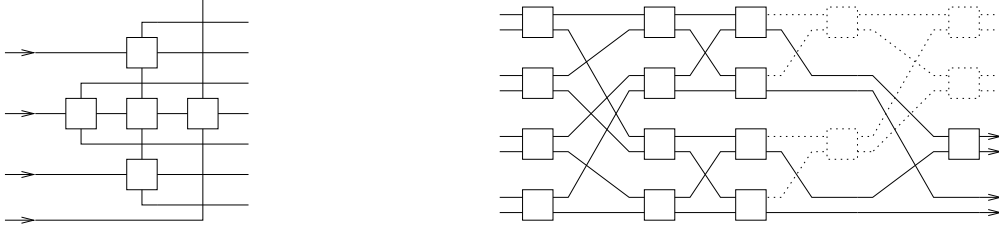


Figure 11: An optimum input selector and an output selector.

$q_1$  of them join a set  $A_1$  of  $q_1$  elements of  $\bigcup_l \{a_l^1, \bar{a}_l^1\}$  to  $Q_1$ ,  $q_2$  of them join a set  $A_2$  of  $q_2$  elements of  $\bigcup_l \{a_l^1, \bar{a}_l^1\} \setminus A_1$  to a set  $C_2$  of  $q_2$  elements of  $\bigcup_l \{c_l^m, \bar{c}_l^m\}$ , and the  $16m$  other paths join the ordinary inputs to the set of outputs  $(O \cup \bar{O} \cup R \cup \bar{R}) \setminus Q_1$  and a set  $C_1$  of  $q_1$  elements of  $\bigcup_l \{c_l^m, \bar{c}_l^m\} \setminus C_2$ .

Indeed  $IS$  being an input selector, we can connect the 4 priorities of  $H(m)$  to the 4 outputs identified to  $A_1 \cup A_2$  with edge disjoint paths; then one can link  $A_1$  to  $Q_1$  and  $A_2$  to  $C_2$  and the ordinary inputs either to outputs of  $(O \cup \bar{O} \cup R \cup \bar{R}) \setminus Q_1$  or vertices of  $C_1$ ; at last,  $OS$  being an output selector, we can find edge-disjoint paths from  $C_2$  to  $Q_2$ , and from  $C_1$  to  $T \setminus Q_2$ . Hence we have the  $16m + 4$  required disjoint paths of  $H(m)$ , 4 of them joining the  $p_j$  to  $Q$  and the  $16m$  other joining the ordinary inputs to the outputs not in  $Q$ .

In fact, we can reduce the problem of finding the required set of paths in  $F(m)$  to that of finding in  $G(m)$  a  $Q_0$ -good path set defined as follows :

Let  $Q_0$  be any set of at most 4 outputs of  $G(m)$ ,  $Q_0 \subseteq O \cup W$  and  $|Q_0| = q_0 \leq 4$ . A  $Q_0$ -good path set consists in  $8m + 4$  edge-disjoint paths such that

- 4 of them link the vertices of  $\bigcup_{1 \leq l \leq 4} a_l^1$  to  $Q_0$  and a set  $D_1$  of  $4 - q_0$  vertices of  $\bigcup_{1 \leq l \leq 4} c_l^m$ , and
- the  $8m$  other link the  $8m$  ordinary inputs of  $G(m)$  to  $(O \cup W) \setminus Q_0$  and  $D_2 = \bigcup_{1 \leq l \leq 4} c_l^m \setminus D_1$ .

Let us define an *assignment* as an one-to-one mapping from  $R \cup \bar{R}$  to  $W \cup \bar{W}$  by associating to  $r_l^k$  one of  $w_l^k$  and  $\bar{w}_l^k$  and to  $\bar{r}_l^k$  the other one.

Let  $Q_r = Q \cap (R \cup \bar{R})$ . For any assignment  $\phi$ , let  $Q_\phi = (Q \cap O) \cup (\phi(Q_r) \cap W)$  and  $\bar{Q}_\phi = (Q \cap \bar{O}) \cup (\phi(Q_r) \cap \bar{W})$ .

If  $G(m)$  contains a  $Q_\phi$ -good path set and  $\bar{G}(m)$  a  $\bar{Q}_\phi$ -good path set, then we obtain the required set of edge-disjoint paths of  $F(m)$  by taking the paths in  $G(m)$  and  $\bar{G}(m)$  and replacing for  $1 \leq k \leq m$  and  $1 \leq l \leq 4$ , the last edge  $(c_l^k, w_l^k)$  (resp.  $(\bar{c}_l^k, \bar{w}_l^k)$ ) of the path ending in  $w_l^k$  (resp.  $\bar{w}_l^k$ ) by the subpath  $(c_l^k, s_l^k, \phi^{-1}(w_l^k))$  (resp.  $(\bar{c}_l^k, s_l^k, \phi^{-1}(\bar{w}_l^k))$ ). Note that we actually get more paths than required ; some of them joining some  $a_l^1$  or  $\bar{a}_l^1$  to some  $c_l^m$  or  $\bar{c}_l^m$  are not needed.

Finally note that we have many choices of possible assignments and we can always choose an assignment  $\phi$  such that  $Q_\phi$  (resp.  $\bar{Q}_\phi$ ) contains:

- either 2 outputs of  $W$  (resp.  $\bar{W}$ ) and none of  $O$  (resp.  $\bar{O}$ ),
- or 1 output of  $W$  (resp.  $\bar{W}$ ) and at most 2 of  $O$  (resp.  $\bar{O}$ ),
- or no output of  $W$  (resp.  $\bar{W}$ ).

Indeed choose  $\phi$  such that:

- if  $|Q_r| = 4$ , then  $|\phi(Q_r) \cap W| = |\phi(Q_r) \cap \bar{W}| = 2$ ;
- if  $|Q_r| = 3$ , and  $|Q \cap O| = 1$  then  $|\phi(Q_r) \cap W| = 1$  and  $|\phi(Q_r) \cap \bar{W}| = 2$ ;
- if  $|Q_r| = 3$ , and  $|Q \cap \bar{O}| = 1$  then  $|\phi(Q_r) \cap W| = 2$  and  $|\phi(Q_r) \cap \bar{W}| = 1$ ;
- if  $|Q_r| = 2$ , then  $|\phi(Q_r) \cap W| = |\phi(Q_r) \cap \bar{W}| = 1$ ;
- if  $|Q_r| = 1$ , and  $|Q \cap O| > |Q \cap \bar{O}|$  then  $|\phi(Q_r) \cap W| = 0$  and  $|\phi(Q_r) \cap \bar{W}| = 1$ ;
- if  $|Q_r| = 1$ , and  $|Q \cap O| < |Q \cap \bar{O}|$  then  $|\phi(Q_r) \cap W| = 1$  and  $|\phi(Q_r) \cap \bar{W}| = 0$ .

In summary, to prove that  $H(m)$  is a repartitor it suffices to prove the following lemma :

**Lemma 59** *For any set  $Q_0$  of outputs of  $G(m)$  such that either  $|Q_0 \cap W| = 2$  and  $|Q_0 \cap O| = 0$ , or  $|Q_0 \cap W| = 1$  and  $|Q_0 \cap O| \leq 2$ , or  $|Q_0 \cap W| = 0$ , there exists a  $Q_0$ -good path set in  $G(m)$ .*

**Proof.** We call *default path* for  $w_i^k$ , (resp.  $o_i^k$ ) the path  $(j_i^k, c_i^k, w_i^k)$  (resp.  $(i_{i+1}^k, b_{i+1}^k, a_i^k, o_i^k)$ ). Let  $L_l$  be the path  $(a_l^1, b_l^1, c_l^1, a_l^2, b_l^2, c_l^2, \dots, a_l^m, b_l^m, c_l^m)$  and let us denote by  $\bar{L}_l[x, y]$  the subpath of  $L_l$  with endvertices  $x$  and  $y$ . Let  $O_l = \{o_i^k | 1 \leq k \leq m\}$  and  $W_l = \{w_i^k | 1 \leq k \leq m\}$ .

By symmetry of  $G_m$ , we only need to prove it for the following cases :

- (i)  $|Q_0 \cap (O_l \cup W_l)| \leq 1$  for  $1 \leq l \leq 4$ ,
- (ii)  $Q_0 \cap (O \cup W) = \{w_1^{k_1}, w_1^{k_2}\}$  with  $k_1 < k_2$ ,
- (iii) If  $Q_0 \cap W = \{w_1^{k_1}\}$  and  $Q_0 \cap O = \{o_1^{k_2}\}$ .
- (iv) If  $Q_0 \cap W = \{w_1^{k_0}\}$  and  $Q_0 \cap O = \{o_{l_1}^{k_1}, o_{l_2}^{k_2}\}$  with  $l_1 = 1$  or  $l_1 = l_2$ .
- (v) If  $Q_0 \cap (W \cup O) = \{o_1^{k_1}, o_1^{k_2}, o_{l_3}^{k_3}, o_{l_4}^{k_4}\}$ .

We only give here the proof in cases (i), (ii), (iii) and (v) with  $l_3 = l_4 = 1$ , the other cases being very similar.

- (i) - if  $|Q_0 \cap (O_l \cup W_l)| = 0$ , take the path  $L_l$ ;
- if  $Q_0 \cap (O_l \cup W_l) = w_i^{k_0}$ , take the paths  $(L_l[a_l^1, c_i^{k_0}], w_i^{k_0})$  and  $(j_i^{k_0}, L[c_i^{k_0}, c_l^m])$ .



- if  $Q_0 \cap (O_l \cup W_l) = o_l^{k_0}$ , take the paths  $(L_l[a_l^1, a_l^{k_0}], o_l^{k_0})$  and  $(i_{l+1}^{k_0}, b_{l+1}^{k_0}, L[a_l^{k_0}, c_l^m])$ .

These paths plus the default paths associated to the outputs not in  $Q_0$  form a  $Q_0$ -good path set.

(ii) Consider the paths  $(L_1[a_1^1, c_1^{k_1}], w_1^{k_1}), (j_1^{k_1}, L_1[c_1^{k_1}, a_1^{k_2}], o_1^{k_2}), (L_2[a_2^1, b_2^{k_2}], L_1[a_1^{k_2}, c_1^{k_2}], w_1^{k_2}), (i_2^{k_2}, L_2[b_2^{k_2}, c_2^m]), (j_1^{k_2}, L_1[c_1^{k_2}, c_1^m]), L_3$  and  $L_4$ . These paths plus the default paths associated to the outputs not in  $Q_0 \cup \{o_1^{k_2}\}$  form a  $Q_0$ -good path set.

(iii) Consider the paths  $(L_1[a_1^1, c_1^{k_1}], w_1^{k_1}), (j_1^{k_1}, L_1[c_1^{k_1}, c_1^m]), (L_2[a_2^1, b_2^{k_2}], a_1^{k_2}, o_1^{k_2}), (i_2^{k_2}, L_2[b_2^{k_2}, c_2^m]), L_3$  and  $L_4$ . These paths plus the default paths associated to the outputs not in  $Q_0$  form a  $Q_0$ -good path set.

(v) Suppose that  $l_3 = l_4 = 1$ . Without loss of generality, we may assume that  $k_1 < k_2 < k_3 < k_4$ . Consider the paths  $(L_1[a_1^1, a_1^{k_1}], o_1^{k_1}), (L_2[a_2^1, b_2^{k_2}], a_1^{k_2}], o_1^{k_2}), (L_3[a_3^1, b_3^{k_3}], a_2^{k_3}, b_2^{k_3}, a_1^{k_3}], o_1^{k_3}), (L_4[a_4^1, b_4^{k_4}], a_3^{k_4}, b_3^{k_4}, a_2^{k_4}, b_2^{k_4}, a_1^{k_4}], o_1^{k_4}), (i_2^{k_1}, b_2^{k_1}, L_1[a_1^{k_1}, c_1^m]), (i_2^{k_2}, L_2[b_2^{k_2}, a_2^{k_3}], o_2^{k_3}), (i_2^{k_3}, L_2[b_2^{k_3}, a_2^{k_4}], o_2^{k_4}), (i_3^{k_3}, L_3[b_3^{k_3}, a_3^{k_4}], o_3^{k_4}), (i_2^{k_4}, L_2[b_2^{k_4}, c_2^m]), (i_3^{k_4}, L_3[b_3^{k_4}, c_3^m]), and  $(i_4^{k_4}, L_4[b_4^{k_4}, c_4^m])$ .$

These paths plus the default paths associated to the outputs not in  $Q_0 \cup \{o_2^{k_3}, o_2^{k_4}, o_3^{k_4}\}$  form a  $Q_0$ -good path set.

**Proof of Theorem 57.** By Theorem 34,  $R(n, 4, 0) \geq \frac{7n}{4} - 7$ . For  $n = 16m + 4$ , we have constructed a  $(n, 4, 0)$ -repartitor  $H(m)$  with  $28m + 18 = \frac{7n}{4} + 11$  switches. If  $n$  is not of the form  $16m + 4$ , let  $n = 16m + 4 - h$  with  $1 \leq h \leq 15$ . Let  $K(n)$  be the network obtained from  $H(m)$  by the L-contraction of  $h$  linking edges. That is always possible as the edges  $(b_l^k, a_{l+1}^k)$  or  $(c_l^k, s_l^k)$  are linking. By Proposition 45,  $K(n)$  is an  $(n, 4, 0)$ -repartitor. And  $K(n)$  has  $28m + 18 - h = \frac{7n}{4} + 11 + \frac{3h}{4} \leq \frac{7n}{4} + \frac{89}{4}$  switches. ■

**Theorem 60**

$$R(n, 3, 1) = \frac{7n}{4} + c$$

where  $-\frac{19}{4} \leq c \leq 27$ .

**Proof.** From Proposition 43, we obtain:  $R(n, 3, 1) \leq R(n + 1, 4, 0) + R(3, 3, 1)$ . By Theorem 57,  $R(n + 1, 4, 0) \leq \frac{7n}{4} + 24$ . And  $R(3, 3, 1) = R(3, 0, 1) = 3$ . Hence  $R(n, 3, 1) \leq \frac{7n}{4} + 27$ . Theorem 34 gives the other inequality. ■

## 5 Conclusion

In this paper, we obtained general upper bounds for  $R(n, p, f)$  and lower bounds and exact values for small  $p$  and  $f$ . It will be interesting to see if the lower bound for  $p = 6$  and  $f = 0$  is attained and to study the case  $p = 1, 2, 3, 4$  and any  $f$ .

Another interesting problem is to know more on the behaviour of  $R(n, p, f)$  and to answer Conjecture 4. According to this work, it holds for small  $n$ ,  $p$  and  $f$ . In view of this conjecture, the case  $p = n/2$  looks promising.

In a more recent paper, Havet [5] gives a better asymptotic upper bound:  $R(n, p, f) \leq \frac{69}{2}n + \frac{35}{2}f - 33p + O(\log(n + f))$  if  $p \leq \frac{n-f}{2}$ ,  $R(n, p, f) \leq 18n + 34f + O(\log(n + f))$ , if  $\frac{n-f}{2} \leq p \leq \frac{n+f}{2}$  and  $R(n, p, f) \leq \frac{3}{2}n + \frac{35}{2}f + 33p + O(\log(n + f))$  if  $p \geq \frac{n+f}{2}$ .

## References

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## A Proof of Theorem 51

Let us exhibit a  $(n, 1, 2)$ -repartitor with  $\lceil \frac{3n}{2} - \frac{1}{2} \rceil$  switches.

We will now prove that the depicted network is a  $(n, 1, 2)$ -repartitor by examining all the possible cases. By symmetry, we may assume that the best quality output is either one of the  $o(c_k)$ , one of the  $o(e_k)$ ,  $o(d)$ ,  $o(a_1)$  or  $o(b_1)$ .

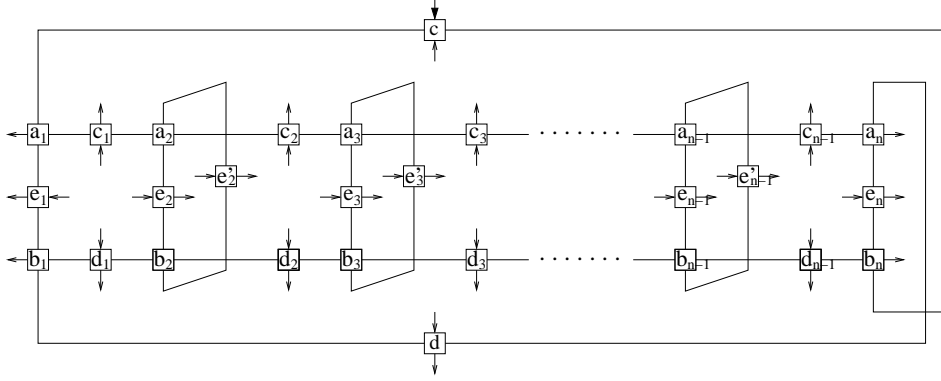
The priority is denoted  $p$  and for every switch  $s$   $i(s)$  (resp.  $o(s)$ ) denotes the ordinary input (resp. output) adjacent to  $s$  if it exists. A *default path* is a path of type  $i(s), s, o(s)$ . In each case, we only write paths which are not default ones.

The subpaths of  $(a_1, c_1, a_2, c_2, \dots, c_{n-1}, a_n)$  are denoted  $[x, y]$  where  $x$  and  $y$  are its two endvertices. Analogously, subpaths of  $(b_1, b_1, b_2, d_2, \dots, d_{n-1}, b_n)$  are denoted  $[x, y]$  where  $x$  and  $y$  are its two endvertices.

Suppose first that  $Q = \{o(c_k)\}$ .

1.  $F = \{o(c_k), o(c_l)\}$ .

$$\begin{array}{l} (p, c, [a_1, c_j], o(c_j)) \\ (i(c_j), c_j, a_{j+1}, e_{j+1}, [b_{j+1}, b_1], o(b_1)) \\ \text{If } j < k < l, \quad (i(c_k), [c_k, a_l], e_l, [b_l, b_n], o(b_n)) \\ (i(c_l), [c_l, a_n], o(a_n)) \\ (i(c), c, b_n, e_n, a_n, d, b_1, e_1, a_1, o(a_1)) \end{array}$$

Figure 12: Minimum  $(n, 1, 2)$ -repartitor

- $(p, c, a_1, e_1, [b_1, b_j], e_j, a_j, c_j, o(c_j))$   
 $(i(c_k), [c_k, a_1], o(a_1))$   
 If  $k < j < l$ ,  $(i(c_j), c_j, a_{j+1}, e_{j+1}, [b_{j+1}, b_n], o(b_n))$   
 $(i(c_l), [c_l, a_n], o(a_n))$   
 $(i(c), c, b_n, e_n, a_n, d, b_1, o(b_1))$   
 $(p, c, a_1, e_1, [b_1, b_j], e_j, a_j, c_j, o(c_j))$   
 $(i(c_j), [c_j, a_n], o(a_n))$   
 If  $k < j < l$ ,  $(i(c_k), [c_k, a_1], o(a_1))$   
 $(i(c_l), c_l, a_l, e_l, [b_l, b_1], o(b_1))$   
 $(i(c), c, b_n, o(b_n))$
2.  $F = \{o(c_k), o(e'_l)\}$ ,  $l \neq 1$ ,  $l \neq n$ .
- $(p, c, [a_1, c_j], o(c_j))$   
 $(i(c_j), c_j, e_j, [b_j, b_1], o(b_1))$   
 If  $j < k$ ,  $j \leq l$ ,  $(i(e'_l), e'_l, [b_l, b_n], o(b_n))$   
 $(i(c_k), [c_k, a_n], o(a_n))$   
 $(i(c), c, b_n, e_n, a_n, d, b_1, e_1, a_1, o(a_1))$   
 $(p, c, [a_1, c_j], o(c_j))$   
 $(i(c_j), c_j, e_j, [b_j, b_n], o(b_n))$   
 If  $j < k$ ,  $j \geq l$ ,  $(i(e'_l), e'_l, [b_l, b_1], o(b_1))$   
 $(i(c_k), [c_k, a_n], o(a_n))$   
 $(i(c), c, b_n, e_n, a_n, d, b_1, e_1, a_1, o(a_1))$

$$\begin{aligned}
& (p, c, a_1, e_1, [b_1, b_j], e_j, a_j, c_j, o(c_j)) \\
& (i(c_j), [c_j, a_n], o(a_n)) \\
\text{If } j > k, j \leq l, & (i(e'_l), e'_l, [b_l, b_n], o(b_n)) \\
& (i(c_k), [c_k, a_1], o(a_1)) \\
& (i(c), c, b_n, e_n, a_n, d, b_1, o(b_1)) \\
& (p, c, [b_n, b_j], e_j, a_j, c_j, o(c_j)) \\
& (i(c_j), [c_j, a_n], o(a_n)) \\
\text{If } j > k, j \geq l, & (i(e'_l), e'_l, [b_l, b_1], o(b_1)) \\
& (i(c_k), [c_k, a_1], o(a_1)) \\
& (i(c), c, a_1, e_1, b_1, d, a_n, e_n, b_n, o(b_n))
\end{aligned}$$

3. If  $F = \{o(c_k), o(e_n)\}$ ,

$$\begin{aligned}
& (p, c, [a_1, c_j], o(c_j)) \\
& (i(c_j), c_j, a_{j+1}, e_{j+1}, [b_{j+1}, b_1], o(b_1)) \\
\text{If } j < k, & (i(c_k), [c_k, a_n], o(a_n)) \\
& (i(e_n), e_n, b_n, o(b_n)) \\
& (i(c), c, a_1, o(a_1)) \\
& (p, c, [b_n, b_j], e_j, a_j, c_j, o(c_j)) \\
& (i(c_j), [c_j, a_n], o(a_n)) \\
\text{If } j > k, & (i(c_k), [c_k, a_1], o(a_1)) \\
& (i(e_n), e_n, b_n, o(b_n)) \\
& (i(c), c, a_1, e_1, b_1, o(b_1))
\end{aligned}$$

4. If  $F = \{o(c_k), o(e_1)\}$ ,

$$\begin{aligned}
& (p, c, [a_1, c_j], o(c_j)) \\
& (i(c_j), c_j, a_{j+1}, e_{j+1}, [b_{j+1}, b_1], o(b_1)) \\
\text{If } j < k, & (i(c_k), [c_k, a_n], o(a_n)) \\
& (i(e_1), e_1, a_1, o(a_1)) \\
& (i(c), c, b_n, o(b_n)) \\
& (p, c, [b_n, b_j], e_j, a_j, c_j, o(c_j)) \\
& (i(c_j), [c_j, a_n], o(a_n)) \\
\text{If } j > k, & (i(c_k), c_k, a_{k+1}, e_{k+1}, [b_{k+1}, b_1], o(b_1)) \\
& (i(e_1), e_1, b_1, d, a_n, e_n, b_n, o(b_n)) \\
& (i(c), c, a_1, o(a_1))
\end{aligned}$$

5. If  $F = \{o(e_k), o(e'_l)\}$ ,  $k \leq l$ .

$$\begin{aligned}
& (p, c, [a_1, c_j], o(c_j)) \\
& (i(c_j), [c_j, a_n], o(a_n)) \\
\text{If } l \neq n, & (i(e_k), e_k, [b_k, b_1], o(b_1)) \\
& (i(e_l), e_l, [b_l, b_n], o(b_n)) \\
& (i(c), c, b_n, e_n, a_n, d, b_1, o(b_1))
\end{aligned}$$

- $(p, c, [b_n, b_{n-1}], e'_{n-1}, [a_{n-1}, c_j], o(c_j))$   
 $(i(c_j), [c_j, a_1], o(a_1))$
- If  $l = n, j < k$ ,
  $(i(e_k), e_k, [b_k, b_1], o(b_1))$   
 $(i(e_n), e_n, b_n, o(b_n))$   
 $(i(c), c, a_1, e_1, b_1, d, a_n, o(a_n))$
- $(p, c, [b_n, b_j], e'_j, a_j, c_j, o(c_j))$   
 $(i(c_j), [c_j, a_n], o(a_n))$
- If  $l = n, j \geq k$ ,
  $(i(e_k), e_k, [b_k, b_1], o(b_1))$   
 $(i(e_n), e_n, b_n, o(b_n))$   
 $(i(c), c, a_1, o(a_1))$
6. If  $F = \{o(c_k), o(d_l)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$   
 $(i(c_j), c_j, a_{j+1}, e_{j+1}, [b_{j+1}, b_1], o(b_1))$
- If  $j < k, j < l$ ,
  $(i(c_k), [c_k, a_n], o(a_n))$   
 $(i(d_l), [d_l, b_n], o(b_n))$   
 $(i(c), c, b_n, e_n, a_n, d, b_1, e_1, a_1, o(a_1))$
- $(p, c, [a_1, c_j], o(c_j))$   
 $(i(c_j), c_j, a_{j+1}, e_{j+1}, [b_{j+1}, b_n], o(b_n))$
- If  $j < k, j \geq l$ ,
  $(i(c_k), [c_k, a_n], o(a_n))$   
 $(i(d_l), [d_l, b_1], o(b_1))$   
 $(i(c), c, b_n, e_n, a_n, d, b_1, e_1, a_1, o(a_1))$
- $(p, c, b_n, e_n, [a_n, c_j], o(c_j))$   
 $(i(c_j), c_j, a_j, e_j, [b_j, b_n], o(b_n))$
- If  $j > k, j \geq l$ ,
  $(i(c_k), [c_k, a_1], o(a_1))$   
 $(i(d_l), [d_l, b_1], o(b_1))$   
 $(i(c), c, a_1, e_1, b_1, d, a_n, o(a_n))$
- $(p, c, b_n, e_n, [a_n, c_j], o(c_j))$   
 $(i(c_j), c_j, a_j, e_j, [b_j, b_1], o(b_1))$
- If  $j > k, j \leq l$ ,
  $(i(c_k), [c_k, a_1], o(a_1))$   
 $(i(d_l), [d_l, b_n], o(b_n))$   
 $(i(c), c, a_1, e_1, b_1, d, a_n, o(a_n))$
7. If  $F = \{o(e_k), o(d_l)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$   
 $(i(c_j), [c_j, a_n], o(a_n))$
- If  $k > l$ ,
  $(i(e_k), e_k, [b_k, b_n], e_n, a_n, d, b_1, e_1, a_1, o(a_1))$   
 $(i(d_l), [d_l, b_1], o(b_1))$   
 $(i(c), c, b_n, o(b_n))$

$$\begin{aligned} & (p, c, [a_1, c_j], o(c_j)) \\ & (i(c_j), [c_j, a_n], o(a_n)) \\ \text{If } 1 < k \leq l, & (i(e_k), e_k, [b_k, b_1], o(b_1)) \\ & (i(d_l), [d_l, b_n], e_n, a_n, d, b_1, e_1, a_1, o(a_1)) \\ & (i(c), c, b_n, o(b_n)) \end{aligned}$$

$$\begin{aligned} & (p, c, [a_1, c_j], o(c_j)) \\ & (i(c_j), [c_j, a_n], o(a_n)) \\ \text{If } k = 1, & (i(e_1), e_1, a_1, o(a_1)) \\ & (i(d_l), [d_l, b_n], e_n, a_n, d, b_1, o(b_1)) \\ & (i(c), c, b_n, o(b_n)) \end{aligned}$$

8. If  $F = \{o(d_k), o(d_l)\}$ ,  $k \leq l$ ,

$$\begin{aligned} & (p, c, [a_1, c_j], o(c_j)) \\ & (i(c_j), [c_j, a_n], o(a_n)) \\ & (i(d_k), [d_k, b_1], o(b_1)) \\ & (i(d_l), [d_l, b_n], o(b_n)) \\ & (i(c), c, b_n, e_n, a_n, d, b_1, e_1, a_1, o(a_1)) \end{aligned}$$

9. If  $F = \{o(d_k), o(d)\}$ ,

$$\begin{aligned} & (p, c, [a_1, c_j], o(c_j)) \\ & (i(c_j), c_j, a_{j+1}, e_{j+1}, [b_{j+1}, b_1], o(b_1)) \\ \text{If } j < k, & (i(c_k), [c_k, a_n], o(a_n)) \\ & (i(d), d, b_1, e_1, a_1, o(a_1)) \\ & (i(c), c, b_n, o(b_n)) \\ & (p, c, b_n, e_n, [a_n, c_j], o(c_j)) \\ & (i(c_j), c_j, a_j, e_j, [b_j, b_n], o(b_n)) \\ \text{If } j > k, & (i(c_k), [c_k, a_1], o(a_1)) \\ & (i(d), d, a_n, o(a_n)) \\ & (i(c), c, a_1, e_1, b_1, o(b_1)) \end{aligned}$$

10. If  $F = \{o(e_k), o(d)\}$ ,

$$\begin{aligned} & (p, c, [a_1, c_j], o(c_j)) \\ & (i(c_j), [c_j, a_n], o(a_n)) \\ & (i(e_k), e_k, [b_k, b_1], e_1, a_1, o(a_1)) \\ & (i(d), d, b_1, o(b_1)) \\ & (i(c), c, b_n, o(b_n)) \end{aligned}$$

11. If  $F = \{o(d_k), o(d)\}$ ,

$$\begin{aligned} & (p, c, [a_1, c_j], o(c_j)) \\ & (i(c_j), [c_j, a_n], o(a_n)) \\ & (i(d_k), [d_k, b_1], e_1, a_1, o(a_1)) \\ & (i(d), d, b_1, o(b_1)) \\ & (i(c), c, b_n, o(b_n)) \end{aligned}$$

12. If  $F = \{o(c_k), o(a_1)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$   
 If  $j < k$ ,  $(i(c_j), c_j, a_{j+1}, e_{j+1}, [b_{j+1}, b_1], o(b_1))$   
 $(i(c_k), [c_k, a_n], o(a_n))$   
 $(i(c), c, b_n, o(b_n))$
- $(p, c, b_n, e_n, [a_n, c_j], o(c_j))$   
 If  $j > k$ ,  $(i(c_j), c_j, a_j, e'_j, [b_j, b_n], o(b_n))$   
 $(i(c_k), c_k, a_{k+1}, e_{k+1}, [b_{k+1}, b_1], o(b_1))$   
 $(i(c), c, a_1, e_1, b_1, d, a_n, o(a_n))$
13. If  $F = \{o(e_k), o(a_1)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$   
 $(i(c_j), [c_j, a_n], o(a_n))$   
 $(i(e_k), e_k, [b_k, b_1], o(b_1))$   
 $(i(c), c, b_n, o(b_n))$
14. If  $F = \{o(d_k), o(a_1)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$   
 $(i(c_j), [c_j, a_n], o(a_n))$   
 $(i(d_k), [d_k, b_1], o(b_1))$   
 $(i(c), c, b_n, o(b_n))$
15. If  $F = \{o(d), o(a_1)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$   
 $(i(c_j), [c_j, a_n], o(a_n))$   
 $(i(d), [d, b_1], o(b_1))$   
 $(i(c), c, b_n, o(b_n))$
16. If  $F = \{o(c_k), o(b_1)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$   
 If  $j < k$ ,  $(i(c_j), c_j, a_{j+1}, e_{j+1}, [b_{j+1}, b_n], o(b_n))$   
 $(i(c_k), [c_k, a_n], o(a_n))$   
 $(i(c), c, b_n, e_n, a_n, d, b_1, e_1, a_1, o(a_1))$
- $(p, c, b_n, e_n, [a_n, c_j], o(c_j))$   
 If  $j > k$ ,  $(i(c_j), c_j, a_j, e_j, [b_j, b_n], o(b_n))$   
 $(i(c_k), [c_k, a_1], o(a_1))$   
 $(i(c), c, a_1, e_1, b_1, d, a_n, o(a_n))$
17. If  $F = \{o(e_k), o(b_1)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$   
 $(i(c_j), [c_j, a_n], o(a_n))$   
 $(i(e_k), e_k, [b_k, b_1], e_1, a_1, o(a_1))$   
 $(i(c), c, b_n, o(b_n))$

18. If  $F = \{o(d_k), o(b_1)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$
  - $(i(c_j), [c_j, a_n], o(a_n))$
  - $(i(d_k), [d_k, b_1], e_1, a_1, o(a_1))$
  - $(i(c), c, b_n, o(b_n))$
19. If  $F = \{o(d), o(b_1)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$
  - $(i(c_j), [c_j, a_n], o(a_n))$
  - $(i(d), d, b_1, e_1, a_1, o(a_1))$
  - $(i(c), c, b_n, o(b_n))$
20. If  $F = \{o(c_k), o(a_n)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$
  - If  $j < k$ ,
    - $(i(c_j), c_j, a_{j+1}, e_{j+1}, [b_{j+1}, b_1], o(b_1))$
    - $(i(c_k), c_k, a_k, e'_k, [b_k, b_n], o(b_n))$
    - $(i(c), c, b_n, e_n, a_n, d, b_1, e_1, a_1, o(a_1))$
  - If  $j > k$ ,
    - $(p, c, [b_n, b_j], e_j, a_j, c_j, o(c_j))$
    - $(i(c_j), [c_j, a_n], e_n, b_n, o(b_n))$
    - $(i(c_k), [c_k, a_1], o(a_1))$
    - $(i(c), c, a_1, e_1, b_1, o(b_1))$
21. If  $F = \{o(e_k), o(a_n)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$
  - $(i(c_j), [c_j, a_n], d, b_1, o(b_1))$
  - $(i(e_k), e_k, [b_k, b_1], e_1, a_1, o(a_1))$
  - $(i(c), c, b_n, o(b_n))$
22. If  $F = \{o(d_k), o(a_n)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$
  - $(i(c_j), [c_j, a_n], d, b_1, o(b_1))$
  - $(i(d_k), [d_k, b_1], e_1, a_1, o(a_1))$
  - $(i(c), c, b_n, o(b_n))$
23. If  $F = \{o(d), o(a_n)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$
  - $(i(c_j), [c_j, a_n], d, b_1, o(b_1))$
  - $(i(d), d, b_1, e_1, a_1, o(a_1))$
  - $(i(c), c, b_n, o(b_n))$
24. If  $F = \{o(a_1), o(a_n)\}$ ,



- $(p, c, [a_1, c_j], o(c_j))$   
 $(i(c_j), c_j, a_{j+1}, e_{j+1}, [b_{j+1}, b_1], o(b_1))$   
 $(i(c), c, b_n, o(b_n))$
25. If  $F = \{o(b_1), o(a_n)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$   
 $(i(c_j), c_j, a_{j+1}, e_{j+1}, [b_{j+1}, b_1], e_1, a_1, o(a_1))$   
 $(i(c), c, b_n, o(b_n))$
26. If  $F = \{o(c_k), o(b_n)\}$ ,
- If  $j < k$ ,
  $(p, c, [a_1, c_j], o(c_j))$   
 $(i(c_j), c_j, a_{j+1}, e_{j+1}, [b_{j+1}, b_1], o(b_1))$   
 $(i(c_k), [c_k, a_n], o(a_n))$   
 $(i(c), c, b_n, e_n, a_n, d, b_1, e_1, a_1, o(a_1))$
- If  $j > k$ ,
  $(p, c, [b_n, b_j], e_j, a_j, c_j, o(c_j))$   
 $(i(c_j), [c_j, a_n], o(a_n))$   
 $(i(c_k), [c_k, a_1], o(a_1))$   
 $(i(c), c, a_1, e_1, b_1, o(b_1))$
27. If  $F = \{o(e_k), o(b_n)\}$ ,
- If  $k \neq n$ ,
  $(p, c, [a_1, c_j], o(c_j))$   
 $(i(c_j), [c_j, a_n], o(a_n))$   
 $(i(e_k), e_k, [b_k, b_1], e_1, a_1, o(a_1))$   
 $(i(c), c, b_n, e_n, a_n, d, b_1, o(b_1))$
- If  $k = n$ ,
  $(p, c, [a_1, c_j], o(c_j))$   
 $(i(c_j), [c_j, a_n], o(a_n))$   
 $(i(e_n), e_n, a_n, d, b_1, e_1, a_1, o(a_1))$   
 $(i(c), c, [b_n, b_1], o(b_1))$
28. If  $F = \{o(d_k), o(b_n)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$   
 $(i(c_j), [c_j, a_n], o(a_n))$   
 $(i(d_k), [d_k, b_1], e_1, a_1, o(a_1))$   
 $(i(c), c, b_n, e_n, a_n, d, b_1, o(b_1))$
29. If  $F = \{o(d), o(b_n)\}$ ,
- $(p, c, [a_1, c_j], o(c_j))$   
 $(i(c_j), [c_j, a_n], o(a_n))$   
 $(i(d), d, b_1, e_1, a_1, o(a_1))$   
 $(i(c), c, b_n, e_n, a_n, d, b_1, o(b_1))$
30. If  $F = \{o(a_1), o(b_n)\}$ ,

$$\begin{aligned} & (p, c, [a_1, c_j], o(c_j)) \\ & (i(c_j), [c_j, a_n], o(a_n)) \\ & (i(c), c, [b_n, b_1], o(b_1)) \end{aligned}$$

31. If  $F = \{o(b_1), o(b_n)\}$ ,

$$\begin{aligned} & (p, c, [a_1, c_j], o(c_j)) \\ & (i(c_j), [c_j, a_n], o(a_n)) \\ & (i(c), c, [b_n, b_1], e_1, a_1, o(a_1)) \end{aligned}$$

32. If  $F = \{o(a_n), o(b_n)\}$ ,

$$\begin{aligned} & (p, c, [a_1, c_j], o(c_j)) \\ & (i(c_j), [c_j, a_n], d, b_1, o(b_1)) \\ & (i(c), c, [b_n, b_1], e_1, a_1, o(a_1)) \end{aligned}$$

Suppose that  $Q = \{o(e_k)\}$ . By symmetry, we may assume that we are in one of the following cases:

1. If  $F = \{o(c_k), o(c_l)\}$ ,  $k < l$ ,

$$\begin{aligned} & (p, c, [b_n, b_j], e_j, o(e_j)) \\ & (i(e_j), e_j, a_j, e'_j, [b_j, b_1], o(b_1)) \\ \text{If } j \neq 1, j \neq n, & \quad (i(c_k), [c_k, a_1], o(a_1)) \\ & \quad (i(c_l), [c_l, a_n], o(a_n)) \\ & \quad (i(c), c, a_1, e_1, b_1, d, a_n, e_n, b_n, o(b_n)) \end{aligned}$$

$$\begin{aligned} & (p, c, a_1, e_1, o(e_1)) \\ & (i(e_1), e_1, b_1, o(b_1)) \\ \text{If } j = 1, & \quad (i(c_k), [c_k, a_1], o(a_1)) \\ & \quad (i(c_l), [c_l, a_n], o(a_n)) \\ & \quad (i(c), c, b_n, o(b_n)) \end{aligned}$$

$$\begin{aligned} & (p, c, b_n, e_n, o(e_n)) \\ \text{If } j = n, & \quad (i(e_n), e_n, a_n, o(a_n)) \quad (i(c_k), [c_k, a_1], o(a_1)) \\ & \quad (i(c_l), c_l, a_l, e_l, [b_l, b_n], o(b_n)) \\ & \quad (i(c), c, a_1, e_1, b_1, o(b_1)) \end{aligned}$$

2. If  $F = \{o(c_k), o(e'_l)\}$ ,

$$\begin{aligned} & (p, c, [b_n, b_j], e_j, o(e_j)) \\ & (i(e_j), e_j, a_j, e'_j, [b_j, b_1], o(b_1)) \\ \text{If } j \neq 1, j \neq l, k \geq l, & \quad (i(c_k), [c_k, a_n], o(a_n)) \\ & \quad (i(e'_l), e'_l, [a_l, a_1], o(a_1)) \\ & \quad (i(c), c, b_n, o(b_n)) \end{aligned}$$

$$\begin{aligned}
& (p, c, [b_n, b_j], e_j, o(e_j)) \\
& (i(e_j), e_j, a_j, e'_j, [b_j, b_1], o(b_1)) \\
\text{If } j \neq 1, j \neq l, k < l, & (i(c_k), [c_k, a_1], o(a_1)) \\
& (i(e'_l), e'_l, [a_l, a_n], o(a_n)) \\
& (i(c), c, b_n, o(b_n)) \\
& (p, c, a_1, e_1, o(e_1)) \\
& (i(e_1), e_1, b_1, o(b_1)) \\
\text{If } j = 1, k \geq l, & (i(c_k), [c_k, a_n], o(a_n)) \\
& (i(e'_l), e'_l, [a_l, a_1], o(a_1)) \\
& (i(c), c, b_n, o(b_n)) \\
& (p, c, a_1, e_1, o(e_1)) \\
& (i(e_1), e_1, b_1, o(b_1)) \\
\text{If } j = 1, k < l, & (i(c_k), [c_k, a_1], o(a_1)) \\
& (i(e'_l), e'_l, [a_l, a_n], o(a_n)) \\
& (i(c), c, b_n, o(b_n)) \\
& (p, c, [a_1, a_j], e_j, o(e_j)) \\
& (i(e_j), e_j, [b_j, b_1], o(b_1)) \\
\text{If } j = l, \text{ w.l.o.g. } j \leq l, & (i(e'_j), e'_j, [b_j, b_n], o(b_n)) \\
& (i(c_k), [c_k, a_n], o(a_n)) \\
& (i(c), c, b_n, e_n, a_n, d, b_1, e_1, a_1, o(a_1))
\end{aligned}$$

3. If  $F = \{o(c_k), o(d_l)\}$ ,

$$\begin{aligned}
& (p, c, a_1, e_1, o(e_1)) \\
& (i(e_1), e_1, b_1, o(b_1)) \\
\text{If } j = 1, & (i(c_k), [c_k, a_1], o(a_1)) \\
& (i(d_l), [d_l, b_n], o(b_n)) \\
& (i(c), c, b_n, e_n, a_n, o(a_n)) \\
& (p, c, [a_1, a_j], e_j, o(e_j)) \\
& (i(e_j), e_j, [b_j, b_1], o(b_1)) \\
\text{If } j \leq k, j \leq l, j \neq 1, & (i(c_k), [c_k, a_n], o(a_n)) \\
& (i(d_l), [d_l, b_n], o(b_n)) \\
& (i(c), c, b_n, e_n, a_n, d, b_1, e_1, a_1, o(a_1)) \\
& (p, c, a_1, e_1, [b_1, b_j], e_j, o(e_j)) \\
& (i(e_j), e_j, [a_j, a_n], o(a_n)) \\
\text{If } k < j \leq l, & (i(c_k), [c_k, a_1], o(a_1)) \\
& (i(d_l), [d_l, b_n], o(b_n)) \\
& (i(c), c, b_n, e_n, a_n, d, b_1, o(b_1))
\end{aligned}$$

By symmetry, we have the result in the other cases.

4. If  $F = \{o(c_k), o(d)\}$ ,

- $$\begin{aligned}
& (p, c, a_1, e_1, o(e_1)) \\
& (i(e_1), e_1, b_1, o(b_1)) \\
\text{If } j = 1, & \quad (i(c_k), [c_k, a_1], o(a_1)) \\
& (i(d), d, a_n, o(a_n)) \\
& (i(c), c, b_n, o(b_n)) \\
& (p, c, [a_1, a_j], e_j, o(e_j)) \\
& (i(e_j), e_j, [b_j, b_1], o(b_1)) \\
\text{If } 1 < j \leq k, & \quad (i(c_k), [c_k, a_n], o(a_n)) \\
& (i(d), d, b_1, e_1, a_1, o(a_1)) \\
& (i(c), c, b_n, o(b_n)) \\
& (p, c, a_1, e_1, [b_1, b_j], e_j, o(e_j)) \\
& (i(e_j), e_j, [a_j, a_n], o(a_n)) \\
\text{If } j > k, & \quad (i(c_k), [c_k, a_1], o(a_1)) \\
& (i(d), d, b_1, o(b_1)) \\
& (i(c), c, b_n, o(b_n))
\end{aligned}$$
5.  $F = \{o(e'_k), o(d)\}$ , w.l.o.g.  $j \leq k$ ,
- $$\begin{aligned}
& (p, c, [a_1, a_j], e_j, o(e_j)) \\
& (i(e_j), e_j, [b_j, b_1], o(b_1)) \\
\text{If } j = 1, & \quad (i(e_k), e_k, [a_k, a_1], o(a_1)) \\
& (i(d), d, a_n, o(a_n)) \\
& (i(c), c, b_n, o(b_n)) \\
& (p, c, [a_1, a_j], e_j, o(e_j)) \\
& (i(e_j), e_j, [b_j, b_1], o(b_1)) \\
\text{If } j \neq 1, & \quad (i(e_k), e_k, [a_k, a_n], o(a_n)) \\
& (i(d), d, b_1, e_1, a_1, o(a_1)) \\
& (i(c), c, b_n, o(b_n))
\end{aligned}$$
6. If  $F = \{o(c_k), o(a_1)\}$ ,
- $$\begin{aligned}
& (p, c, [a_1, a_j], e_j, o(e_j)) \\
\text{If } k \geq j, & \quad (i(e_j), e_j, [b_j, b_1], o(b_1)) \\
& (i(c_k), [c_k, a_n], o(a_n)) \\
& (i(c), c, b_n, o(b_n)) \\
& (p, c, [b_n, b_j], e_j, o(e_j)) \\
\text{If } k < j, & \quad (i(e_j), e_j, a_j, e'_j, [b_j, b_1], o(b_1)) \\
& (i(c_k), [c_k, a_n], o(a_n)) \\
& (i(c), c, a_1, e_1, b_1, d, a_n, o(a_n))
\end{aligned}$$
7. If  $F = \{o(e'_k), o(a_1)\}$ ,

- $(p, c, [a_1, a_j], e_j, o(e_j))$   
 If  $k \geq j$ ,  $(i(e_j), e_j, [b_j, b_1], o(b_1))$   
 $(i(e'_k), e'_k, [a_k, a_n], o(a_n))$   
 $(i(c), c, b_n, o(b_n))$
- $(p, c, [b_n, b_j], e_j, o(e_j))$   
 If  $1 < k < j < n$ ,  $(i(e_j), e_j, [a_j, a_n], o(a_n))$   
 $(i(e'_k), e'_k, [b_k, b_1], o(b_1))$   
 $(i(c), c, a_1, e_1, b_1, d, a_n, e_n, b_n, o(b_n))$
- $(p, c, [a_1, a_j], e_j, o(e_j))$   
 If  $k = 1$ ,  $(i(e_j), e_j, [b_j, b_n], e_n, a_n, o(a_n))$   
 $(i(e_1), e_1, b_1, o(b_1))$   
 $(i(c), c, b_n, o(b_n))$
- $(p, c, b_n, e_n, o(e_n))$   
 If  $j = n$ ,  $(i(e_n), e_n, a_n, o(a_n))$   
 $(i(e'_k), e'_k, [b_k, b_1], o(b_1))$   
 $(i(c), c, [a_1, a_{n-1}, e_{n-1}, b_{n-1}, d_{n-1}, b_n, o(b_n))$
8. If  $F = \{o(d_k), o(a_1)\}$ ,
- $(p, c, [a_1, a_j], e_j, o(e_j))$   
 If  $j \leq k$ ,  $(i(e_j), e_j, [b_j, b_1], o(b_1))$   
 $(i(d_k), [d_k, b_n], o(b_n))$   
 $(i(c), c, b_n, e_n, a_n, o(a_n))$
- $(p, c, [b_n, b_j], e_j, o(e_j))$   
 If  $k < j < n$ ,  $(i(e_j), e_j, [a_j, a_n], o(a_n))$   
 $(i(d_k), [d_k, b_1], o(b_1))$   
 $(i(c), c, a_1, e_1, b_1, d, a_n, e_n, b_n, o(b_n))$
- $(p, c, b_n, e_n, o(e_n))$   
 If  $j = n$ ,  $(i(e_n), e_n, a_n, o(a_n))$   
 $(i(d_k), [d_k, b_n], o(b_n))$   
 $(i(c), c, a_1, e_1, b_1, o(b_1))$
9. If  $F = \{o(d), o(a_1)\}$ ,
- $(p, c, [a_1, a_j], e_j, o(e_j))$   
 $(i(e_j), e_j, [b_j, b_1], o(b_1))$   
 $(i(d), d, a_n, o(a_n))$   
 $(i(c), c, b_n, o(b_n))$
10. If  $F = \{o(a_1), o(b_n)\}$ , by symmetry,  $j \neq n$ ,
- $(p, c, [a_1, a_j], e_j, o(e_j))$   
 $(i(e_j), e_j, [b_j, b_1], o(b_1))$   
 $(i(c), c, b_n, e_n, a_n, o(a_n))$

11. If  $F = \{o(c_k), o(b_1)\}$ ,
- $(p, c, a_1, e_1, o(e_1))$   
 If  $j = 1$ ,  $(i(e_1), e_1, [b_1, b_n], o(b_n))$   
 $(i(c_k), [c_k, a_1], o(a_1))$   
 $(i(c), c, b_n, e_n, a_n, o(a_n))$
- $(p, c, [a_1, a_j], e_j, o(e_j))$   
 If  $1 < j \leq k$ ,  $(i(e_j), e_j, [b_j, b_n], o(b_n))$   
 $(i(c_k), [c_k, a_n], o(a_n))$   
 $(i(c), c, b_n, e_n, a_n, d, b_1, e_1, a_1, o(a_1))$
- $(p, c, [b_n, b_j], e_j, o(e_j))$   
 If  $k < j < n$ ,  $(i(e_j), e_j, [a_j, a_n], o(a_n))$   
 $(i(c_k), [c_k, a_1], o(a_1))$   
 $(i(c), c, a_1, e_1, b_1, d, a_n, e_n, b_n, o(b_n))$
- $(p, c, b_n, e_n, o(e_n))$   
 If  $j = n$ ,  $(i(e_n), e_n, a_n, o(a_n))$   
 $(i(c_k), [c_k, a_1], o(a_1))$   
 $(i(c), c, a_1, e_1, [b_1, b_n], o(b_n))$
12. If  $F = \{o(e'_k), o(b_1)\}$ ,
- $(p, c, [b_n, b_j], e_j, o(e_j))$   
 If  $1 < k \leq j < n$ ,  $(i(e_j), e_j, [a_j, a_n], e_n, b_n, o(b_n))$   
 $(i(e'_k), e'_k, [a_k, a_1], o(a_1))$   
 $(i(c), c, a_1, e_1, b_1, d, a_n, o(a_n))$
- $(p, c, [b_n, b_j], e_j, o(e_j))$   
 If  $k = 1$ ,  $(i(e_j), e_j, [a_j, a_n], e_n, b_n, o(b_n))$   
 $(i(e_1), e_1, b_1, d, a_n, o(a_n))$   
 $(i(c), c, a_1, o(a_1))$
- $(p, c, b_n, e_n, o(e_n))$   
 If  $j = n$ ,  $(i(e_n), e_n, a_n, o(a_n))$   
 $(i(e'_k), e'_k, [b_k, b_n], o(b_n))$   
 $(i(c), c, a_1, o(a_1))$
- $(p, c, [b_n, b_j], e_j, o(e_j))$   
 If  $1 < j \leq k < n$ ,  $(i(e_j), e_j, [a_j, a_1], o(a_1))$   
 $(i(e'_k), e'_k, [a_k, a_n], o(a_n))$   
 $(i(c), c, a_1, e_1, b_1, d, a_n, e_n, b_n, o(b_n))$
- $(p, c, a_1, e_1, o(e_1))$   
 If  $j = 1$ ,  $(i(e_1), e_1, b_1, d, a_n, o(a_n))$   
 $(i(e'_k), e'_k, [a_k, a_1], o(a_1))$   
 $(i(c), c, b_n, o(b_n))$

$$\text{If } k = n, \begin{array}{l} (p, c, [b_n, b_j], e_j, o(e_j)) \\ (i(e_j), e_j, [a_j, a_n], o(a_n)) \\ (i(e_n), e_n, b_n, o(b_n)) \\ (i(c), c, a_1, o(a_1)). \end{array}$$

13. If  $F = \{o(d_k), o(b_1)\}$ ,

$$\text{If } 1 < j < n, \begin{array}{l} (p, c, [a_1, a_j], e_j, o(e_j)) \\ (i(e_j), b_j, e'_j, [a_j, a_n], o(a_n)) \\ (i(d_k), [d_k, b_n], o(b_n)) \\ (i(c), c, b_n, e_n, [a_n, a_1]o(a_1)) \end{array}$$

$$\text{If } j = 1, \begin{array}{l} (p, c, a_1, e_1, o(e_1)) \\ (i(e_1), e_1, b_1, d, a_n, o(a_n)) \\ (i(d_k), [d_k, b_n], o(b_n)) \\ (i(c), c, b_n, e_n, [a_n, a_1], o(a_1)) \end{array}$$

$$\text{If } j = n, \begin{array}{l} (p, c, b_n, e_n, o(e_n)) \\ (i(e_n), e_n, a_n, o(a_n)) \\ (i(d_k), [d_k, b_n], o(b_n)) \\ (i(c), c, a_1, o(a_1)) \end{array}$$

14. If  $F = \{o(d), o(b_1)\}$ ,

$$\text{If } j \neq 1, \text{ and } j \neq n, \begin{array}{l} (p, c, [b_n, b_j], e_j, o(e_j)) \\ (i(e_j), e_j, [a_j, a_n], o(a_n)) \\ (i(d), d, a_n, e_n, b_n, o(b_n)) \\ (i(c), c, a_1, o(a_1)) \end{array}$$

$$\text{If } j = 1, \begin{array}{l} (p, c, a_1, e_1, o(e_1)) \\ (i(e_1), e_1, [b_1, b_n], o(b_n)) \\ (i(d), d, [a_n, a_1], o(a_1)) \\ (i(c), c, b_n, o(b_n)) \end{array}$$

$$\text{If } j = n, \begin{array}{l} (p, c, b_n, e_n, o(e_n)) \\ (i(e_n), e_n, a_n, o(a_n)) \\ (i(d), d, [b_1, b_n], o(b_n)) \\ (i(c), c, a_1, o(a_1)) \end{array}$$

15. If  $F = \{o(a_1), o(b_1)\}$ ,

$$\text{If } j \neq n, \begin{array}{l} (p, c, [a_1, a_j], e_j, o(e_j)) \\ (i(e_j), e_j, [b_j, b_n], o(b_n)) \\ (i(c), c, b_n, e_n, a_n, o(a_n)) \end{array}$$

$$\text{If } j = n, \begin{array}{l} (p, c, b_n, e_n, o(e_n)) \\ (i(e_n), e_n, a_n, o(a_n)) \\ (i(c), c, a_1, e_1, [b_1, b_n], o(b_n)) \end{array}$$

16. If  $F = \{o(b_1), o(b_n)\}$ ,
- $$\begin{aligned} & (p, c, [b_n, b_j], e_j, o(e_j)) \\ & (i(e_j), e_j, [a_j, a_n], o(a_n)) \\ & (i(c), c, a_1, o(a_1)) \end{aligned}$$
17. If  $F = \{o(b_1), o(a_n)\}$ ,
- $$\begin{aligned} & (p, c, [a_1, a_j], e_j, o(e_j)) \\ \text{If } j \neq 1, \text{ and } j \neq n, & (i(e_j), e_j, [b_j, b_n], o(b_n)) \\ & (i(c), c, b_n, e_n, a_n, d, b_1, e_1, a_1, o(a_1)) \\ \\ & (p, c, a_1, e_1, o(e_1)) \\ \text{If } j = 1, & (i(e_1), e_1, [b_1, b_n], o(b_n)) \\ & (i(c), c, b_n, e_n, [a_n, a_1], o(a_1)) \\ \\ & (p, c, [a_1, a_n], e_n, o(e_n)) \\ \text{If } j = n, & (i(e_n), e_n, b_n, o(b_n)) \\ & (i(c), c, [b_n, b_1], e_1, a_1, o(a_1)) \end{aligned}$$

Suppose that  $Q = \{o(d)\}$ . By symmetry, we are in one of the following cases :

- $$\begin{aligned} & (p, c, a_1, e_1, b_1, d, o(d)) \\ & (i(d), d, a_n, e_n, b_n, o(b_n)) \end{aligned}$$
1. If  $F = \{o(c_k), o(c_l)\}$ ,  $k < l$ ,
- $$\begin{aligned} & (i(c_k), [c_k, a_1], o(a_1)) \\ & (i(c_l), [c_l, a_n], o(a_n)) \\ & (i(c), c, [b_n, b_1], o(b_1)) \end{aligned}$$
2. If  $F = \{o(c_k), o(e_l)\}$
- $$\begin{aligned} & (p, c, a_1, e_1, b_1, d, o(d)) \\ & (i(d), d, a_n, o(a_n)) \\ \text{If } l \neq 1, & (i(c_k), [c_k, a_1], o(a_1)) \\ & (i(e_l), e_l, [b_l, b_1], o(b_1)) \\ & (i(c), c, b_n, o(b_n)) \\ \\ & (p, c, b_n, e_n, a_n, d, o(d)) \\ & (i(d), d, b_1, o(b_1)) \\ \text{If } l = 1, & (i(c_k), [c_k, a_n], o(a_n)) \\ & (i(e_1), e_1, [b_1, b_n], o(b_n)) \\ & (i(c), c, a_1, o(a_1)) \end{aligned}$$
3. If  $F = \{o(c_k), o(e_l)\}$
- $$\begin{aligned} & (p, c, a_1, e_1, b_1, d, o(d)) \\ & (i(d), d, a_n, o(a_n)) \\ & (i(c_k), [c_k, a_1], o(a_1)) \\ & (i(d_l), [d_l, b_1], o(b_1)) \\ & (i(c), c, b_n, o(b_n)) \end{aligned}$$



4. If  $F = \{o(e_k), o(e'_l)\}$ ,  $k \leq l$

$$\begin{aligned} & (p, c, a_1, e_1, b_1, d, o(d)) \\ & (i(d), d, a_n, o(a_n)) \\ \text{If } k \neq 1, & (i(e_k), e_k, [a_k, a_1], o(a_1)) \\ & (i(e'_l), e'_l, [b_l, b_1], o(b_1)) \\ & (i(c), c, b_n, o(b_n)) \end{aligned}$$

By symmetry, we have the result if  $l \neq n$ .

$$\begin{aligned} & (p, c, [b_n, b_1], d, o(d)) \\ & (i(d), d, a_n, o(a_n)) \\ \text{If } k = 1 \text{ and } l = n, & (i(e_1), e_1, b_1, o(b_1)) \\ & (i(e_n), e_n, b_n, o(b_n)) \\ & (i(c), c, a_1, o(a_1)) \end{aligned}$$

5. If  $F = \{o(c_k), o(a_1)\}$

$$\begin{aligned} & (p, c, a_1, e_1, b_1, d, o(d)) \\ & (i(d), d, a_n, o(a_n)) \\ & (i(c_k), c_k, a_{k+1}, e_{k+1}, [b_{k+1}, b_1], o(b_1)) \\ & (i(c), c, b_n, o(b_n)) \end{aligned}$$

6. If  $F = \{o(e_k), o(a_1)\}$

$$\begin{aligned} & (p, c, a_1, e_1, b_1, d, o(d)) \\ & (i(d), d, a_n, o(a_n)) \\ \text{If } k \neq 1, & (i(e_k), e_k, [b_k, b_1], o(b_1)) \\ & (i(c), c, b_n, o(b_n)) \\ & (p, c, b_n, e_n, a_n, d, o(d)) \\ \text{If } k = 1, & (i(d), d, b_1, o(b_1)) \\ & (i(e_1), e_1, [b_1, b_n], o(b_n)) \\ & (i(c), c, [a_1, a_n], o(a_n)) \end{aligned}$$

7. If  $F = \{o(e_k), o(a_1)\}$

$$\begin{aligned} & (p, c, a_1, e_1, b_1, d, o(d)) \\ & (i(d), d, a_n, o(a_n)) \\ & (i(d_k), [d_k, b_1], o(b_1)) \\ & (i(c), c, b_n, o(b_n)) \end{aligned}$$

8. If  $F = \{o(b_n), o(a_1)\}$

$$\begin{aligned} & (p, c, a_1, e_1, b_1, d, o(d)) \\ & (i(d), d, a_n, o(a_n)) \\ & (i(c), c, [b_n, b_1], o(b_1)) \end{aligned}$$

9. If  $F = \{o(c_k), o(b_1)\}$

$$\begin{aligned} &(p, c, a_1, e_1, b_1, d, o(d)) \\ &(i(d), d, a_n, o(a_n)) \\ &(i(c_k), [c_k, a_1], o(a_1)) \\ &(i(c), c, b_n, o(b_n)) \end{aligned}$$

10. If  $F = \{o(e_k), o(b_1)\}$

$$\begin{aligned} &(p, c, a_1, e_1, b_1, d, o(d)) \\ \text{If } k \neq 1, & \begin{aligned} &(i(d), d, a_n, o(a_n)) \\ &(i(e_k), e_k, [a_k, a_1], o(a_1)) \\ &(i(c), c, b_n, o(b_n)) \end{aligned} \end{aligned}$$

$$\begin{aligned} &(p, c, b_n, e_n, a_n, d, o(d)) \\ \text{If } k = 1, & \begin{aligned} &(i(d), d, [b_1, b_n], o(b_n)) \\ &(i(e_1), e_1, [a_1, a_n], o(a_n)) \\ &(i(c), c, a_1, o(a_1)) \end{aligned} \end{aligned}$$

11. If  $F = \{o(d_k), o(b_1)\}$

$$\begin{aligned} &(p, c, a_1, e_1, b_1, d, o(d)) \\ &(i(d), d, a_n, o(a_n)) \\ &(i(d_k), [d_k, b_n], o(b_n)) \\ &(i(c), c, b_n, e_n, [a_n, a_1], o(a_1)) \end{aligned}$$

12. If  $F = \{o(a_n), o(b_1)\}$

$$\begin{aligned} &(p, c, a_1, e_1, b_1, d, o(d)) \\ &(i(d), d, [a_n, a_1], o(a_1)) \\ &(i(c), c, b_n, o(b_n)) \end{aligned}$$

13. If  $F = \{o(a_n), o(b_1)\}$

$$\begin{aligned} &(p, c, a_1, e_1, b_1, d, o(d)) \\ &(i(d), d, a_n, o(a_n)) \\ &(i(c), c, b_n, o(b_n)) \end{aligned}$$

14. If  $F = \{o(b_1), o(b_n)\}$

$$\begin{aligned} &(p, c, a_1, e_1, b_1, d, o(d)) \\ &(i(d), d, a_n, o(a_n)) \\ &(i(c), c, b_n, e_n, [a_n, a_1], o(a_1)) \end{aligned}$$

Suppose that  $Q = \{o(a_1)\}$ .

In each case, the path from  $p$  to  $o(a_1)$  is  $(p, c, a_1, o(a_1))$ . We now list the remaining non-default paths.

1. If  $F = \{o(c_k), o(c_l)\}$ ,  $k < l$ ,

$$\begin{aligned} &(i(c_k), [c_k, a_1], e_1, b_1, o(b_1)) \\ &(i(c_l), [c_l, a_n], o(a_n)) \\ &(i(c), c, b_n, o(b_n)) \end{aligned}$$

2. If  $F = \{o(c_k), o(e_l)\}$ ,  
 $(i(c_k), [c_k, a_n], o(a_n))$   
 $(i(e_l), e_l, [b_l, b_1], o(b_1))$   
 $(i(c), c, b_n, o(b_n))$
3. If  $F = \{o(c_k), o(d_l)\}$ ,  
 $(i(c_k), [c_k, a_n], o(a_n))$   
 $(i(d_l), [d_l, b_1], o(b_1))$   
 $(i(c), c, b_n, o(b_n))$
4. If  $F = \{o(e_k), o(e_l)\}$ ,  
 $(i(e_k), e_k, [a_k, a_n], o(a_n))$   
 $(i(e'_l), e'_l, [b_l, b_1], o(b_1))$   
 $(i(c), c, b_n, o(b_n))$
5. If  $F = \{o(e_k), o(d_l)\}$ ,  
 $(i(e_k), e_k, [a_k, a_n], o(a_n))$   
 $(i(d_l), [d_l, b_1], o(b_1))$   
 $(i(c), c, b_n, o(b_n))$
6. If  $F = \{o(d_k), o(d_l)\}$ ,  $k < l$   
 $(i(d_k), [d_k, b_1], o(b_1))$   
 $(i(d_l), [d_l, b_n], e_n, a_n, o(a_n))$   
 $(i(c), c, b_n, o(b_n))$
7. If  $F = \{o(c_k), o(a_n)\}$ ,  
 $(i(c_k), [c_k, a_1], e_1, b_1, o(b_1))$   
 $(i(c), c, b_n, o(b_n))$
8. If  $F = \{o(c_k), o(b_1)\}$ ,  
 $(i(c_k), [c_k, a_n], o(a_n))$   
 $(i(c), c, b_n, o(b_n))$
9. If  $F = \{o(c_k), o(b_n)\}$ ,  
 $(i(c_k), [c_k, a_1], e_1, b_1, o(b_1))$   
 $(i(c), c, b_n, e_n, a_n, o(a_n))$
10. If  $F = \{o(e_k), o(a_n)\}$ ,  
 $(i(e_k), e_k, [b_k, b_1], o(b_1))$   
 $(i(c), c, b_n, o(b_n))$
11. If  $F = \{o(e_k), o(b_1)\}$ ,  
 $(i(e_k), e_k, [a_k, a_n], o(a_n))$   
 $(i(c), c, b_n, o(b_n))$

12. If  $F = \{o(e_k), o(b_n)\}$ ,
  - $(i(e_k), e_k, [a_k, a_n], o(a_n))$
  - $(i(c), c, [b_n, b_1], o(b_1))$
13. If  $F = \{o(d_k), o(a_n)\}$ ,
  - $(i(d_k), [d_k, b_1], o(b_1))$
  - $(i(c), c, b_n, o(b_n))$
14. If  $F = \{o(d_k), o(b_1)\}$ ,
  - $(i(d_k), [d_k, b_n], e_n, a_n, o(a_n))$
  - $(i(c), c, b_n, o(b_n))$
15. If  $F = \{o(d_k), o(b_n)\}$ ,
  - $(i(d_k), [d_k, b_1], o(b_1))$
  - $(i(c), c, b_n, e_n, a_n, o(a_n))$
16. If  $F \subset \{b_1, a_n, b_n\}$ , it is easy there is a path from  $i(c)$  to the element of  $\{b_1, a_n, b_n\} \setminus F$  since the network minus the link  $ca_1$  is connected.

Suppose that  $Q = \{o(b_1)\}$ .

1. If  $F = \{o(c_k), o(c_l)\}$ ,  $k < l$ 
  - $(p, c, a_1, e_1, b_1, o(b_1))$
  - $(i(c_k), [c_k, a_1], o(a_1))$
  - $(i(c_l), [c_l, a_n], o(a_n))$
  - $(i(c), c, b_n, o(b_n))$
2. If  $F = \{o(c_k), o(e_l)\}$ ,
  - $(p, c, a_1, e_1, b_1, o(b_1))$
  - If  $l \neq 1$ ,
    - $(i(c_k), [c_k, a_1], o(a_1))$
    - $(i(e_l), e_l, [b_l, b_n], e_n, a_n, o(a_n))$
    - $(i(c), c, b_n, o(b_n))$
  - $(p, c, b_n, e_n, a_n, d, b_1, o(b_1))$
  - If  $l = 1$ ,
    - $(i(c_k), [c_k, a_n], o(a_n))$
    - $(i(e_1), e_1, [b_1, b_n], o(b_n))$
    - $(i(c), c, a_1, o(a_1))$
3. If  $F = \{o(c_k), o(d_l)\}$ ,
  - $(p, c, a_1, e_1, b_1, o(b_1))$
  - $(i(c_k), [c_k, a_1], o(a_1))$
  - $(i(d_l), [d_l, b_n], e_n, a_n, o(a_n))$
  - $(i(c), c, b_n, o(b_n))$

4. If  $F = \{o(e_k), o(e'_l)\}$ ,  $k \leq l$ ,
- $(p, c, a_1, e_1, b_1, o(b_1))$
- If  $k \neq 1$ ,
- $(i(e_k), e_k, [a_k, a_1], o(a_1))$   
 $(i(e'_l), e'_l, [a_l, a_n], o(a_n))$   
 $(i(c), c, b_n, o(b_n))$
- If  $k = 1$  and  $l \neq n$ ,
- $(p, c, b_n, e_n, a_n, d, b_1, o(b_1))$   
 $(i(e_1), e_1, [a_1, a_n], o(a_n))$   
 $(i(e'_l), e'_l, [b_l, b_n], o(b_n))$   
 $(i(c), c, a_1, o(a_1))$
- If  $k = 1$  and  $l = n$ ,
- $(p, c, [b_n, b_1], o(b_1))$   
 $(i(e_1), e_1, [a_1, a_n], o(a_n))$   
 $(i(e_n), e_n, b_n, o(b_n))$   
 $(i(c), c, a_1, o(a_1))$
5. If  $F = \{o(e_k), o(d_l)\}$ ,
- $(p, c, a_1, e_1, b_1, o(b_1))$
- If  $k \neq 1$  and  $k \neq n$ ,
- $(i(e_k), e_k, [a_k, a_1], o(a_1))$   
 $(i(d_l), [d_l, b_n], e_n, a_n, o(a_n))$   
 $(i(c), c, b_n, o(b_n))$
- If  $k = 1$ ,
- $(p, c, b_n, e_n, a_n, d, b_1, o(b_1))$   
 $(i(e_k), e_k, [a_k, a_n], o(a_n))$   
 $(i(d_l), [d_l, b_n], o(b_n))$   
 $(i(c), c, a_1, o(a_1))$
- If  $k = n$ ,
- $(p, c, a_1, e_1, b_1, o(b_1))$   
 $(i(e_n), a_n, o(a_n))$   
 $(i(d_l), d_l, b_{l+1}, e_{l+1}, [a_{l+1}, a_1], o(a_1))$  or  $(i(d_l), d_l, b_l, e_l, [a_l, a_1], o(a_1))$   
 $(i(c), c, b_n, o(b_n))$
6. If  $F = \{o(d_k), o(d_l)\}$ ,  $k < l$ ,
- $(p, c, a_1, e_1, b_1, o(b_1))$   
 $(i(d_k), d_k, b_{k+1}, e_{k+1}, [a_{k+1}, a_1], o(a_1))$   
 $(i(d_l), [d_l, b_n], e_n, a_n, o(a_n))$   
 $(i(c), c, b_n, o(b_n))$
7. If  $F = \{o(c_k), o(d)\}$ ,
- $(p, c, a_1, e_1, b_1, o(b_1))$   
 $(i(c_k), [c_k, a_1], o(a_1))$   
 $(i(d), d, a_n, o(a_n))$   
 $(i(c), c, b_n, o(b_n))$
8. If  $F = \{o(e_k), o(d)\}$ ,

$$\text{If } k \neq 1, \begin{array}{l} (p, c, a_1, e_1, b_1, o(b_1)) \\ (i(e_k), e_k, [a_k, a_1], o(a_1)) \\ (i(d), d, a_n, o(a_n)) \\ (i(c), c, b_n, o(b_n)) \end{array}$$

$$\text{If } k = 1, \begin{array}{l} (p, c, b_n, e_n, a_n, d, b_1, o(b_1)) \\ (i(e_1), e_1, [b_1, b_n], o(b_n)) \\ (i(d), d, a_n, o(a_n)) \\ (i(c), c, a_1, o(a_1)) \end{array}$$

9. If  $F = \{o(d_k), o(d)\}$ ,
 
$$\begin{array}{l} (p, c, a_1, e_1, b_1, o(b_1)) \\ (i(d_k), d_k, b_{k+1}, e_{k+1}, [a_{k+1}, a_1], o(a_1)) \\ (i(d), d, a_n, o(a_n)) \\ (i(c), c, b_n, o(b_n)) \end{array}$$
10. If  $F = \{o(c_k), o(a_1)\}$ ,
 
$$\begin{array}{l} (p, c, a_1, e_1, b_1, o(b_1)) \\ (i(c_k), [c_k, a_n], o(a_n)) \\ (i(c), c, b_n, o(b_n)) \end{array}$$
11. If  $F = \{o(e_k), o(a_1)\}$ ,
 
$$\begin{array}{l} (p, c, a_1, e_1, b_1, o(b_1)) \\ (i(e_k), e_k, [a_k, a_n], o(a_n)) \\ (i(c), c, b_n, o(b_n)) \end{array}$$
12. If  $F = \{o(e_k), o(a_1)\}$ ,
 
$$\begin{array}{l} (p, c, a_1, e_1, b_1, o(b_1)) \\ (i(d_k), [d_k, b_n], e_n, a_n, o(a_n)) \\ (i(c), c, b_n, o(b_n)) \end{array}$$
13. If  $F = \{o(d), o(a_1)\}$ ,
 
$$\begin{array}{l} (p, c, a_1, e_1, b_1, o(b_1)) \\ (i(d), d, a_n, o(a_n)) \\ (i(c), c, b_n, o(b_n)) \end{array}$$
14. If  $F = \{o(c_k), o(a_n)\}$ ,
 
$$\begin{array}{l} (p, c, a_1, e_1, b_1, o(b_1)) \\ (i(c_k), [c_k, a_1], o(a_1)) \\ (i(c), c, b_n, o(b_n)) \end{array}$$
15. If  $F = \{o(e_k), o(a_n)\}$ ,
 
$$\begin{array}{l} (p, c, b_n, e_n, a_n, d, b_1, o(b_1)) \\ (i(e_k), e_k, [b_k, b_n], o(b_n)) \\ (i(c), c, a_1, o(a_1)) \end{array}$$

16. If  $F = \{o(d_k), o(a_n)\}$ ,
- $(p, c, b_n, e_n, a_n, d, b_1, o(b_1))$
  - $(i(d_k), [d_k, b_n], o(b_n))$
  - $(i(c), c, a_1, o(a_1))$
17. If  $F = \{o(d), o(a_n)\}$ ,
- $(p, c, a_1, e_1, b_1, o(b_1))$
  - $(i(d), d, [a_n, a_1], o(a_1))$
  - $(i(c), c, b_n, o(b_n))$
18. If  $F = \{o(c_k), o(b_n)\}$ ,
- $(p, c, a_1, e_1, b_1, o(b_1))$
  - $(i(c_k), [c_k, a_1], o(a_1))$
  - $(i(c), c, b_n, e_n, a_n, o(a_n))$
19. If  $F = \{o(e_k), o(b_n)\}$ ,
- $(p, c, a_1, e_1, b_1, o(b_1))$
  - If  $k \neq 1$  and  $k \neq n$ ,  $(i(e_k), e_k, [a_k, a_1], o(a_1))$
  - $(i(c), c, b_n, e_n, a_n, o(a_n))$
  - $(p, c, b_n, e_n, a_n, d, b_1, o(b_1))$
  - If  $k = 1$ ,  $(i(e_1), e_1, a_1, o(a_1))$
  - $(i(c), c, b_n, e_n, a_n, o(a_n))$
  - $(p, c, a_1, e_1, b_1, o(b_1))$
  - If  $k = n$ ,  $(i(e_n), e_n, a_n, o(a_n))$
  - $(i(c), c, b_n, d_{n-1}, b_{n-1}, e_{n-1}, [a_{n-1}, a_1], o(a_1))$
20. If  $F = \{o(d_k), o(b_n)\}$ ,
- $(p, c, a_1, e_1, b_1, o(b_1))$
  - $(i(d_k), [d_k, b_1], d, [a_n, a_1], o(a_1))$
  - $(i(c), c, b_n, e_n, a_n, o(a_n))$
21. If  $F = \{o(d), o(b_n)\}$ ,
- $(p, c, a_1, e_1, b_1, o(b_1))$
  - $(i(d), d, [a_n, a_1], o(a_1))$
  - $(i(c), c, b_n, e_n, a_n, o(a_n))$
22. If  $F \subset \{a_1, a_n, b_n\}$ , it is easy there is a path from  $i(c)$  to the element of  $\{a_1, a_n, b_n\} \setminus F$  since the network minus the path  $(p, c, a_1, e_1, b_1, o(b_1))$  is connected.



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