Arc-chromatic number of digraphs in which each vertex has bounded outdegree or bounded indegree

Stéphane Bessy, Etienne E. Birmelé, Frédéric Havet

To cite this version:

Stéphane Bessy, Etienne E. Birmelé, Frédéric Havet. Arc-chromatic number of digraphs in which each vertex has bounded outdegree or bounded indegree. [Research Report] RR-5364, INRIA. 2004, pp.20. inria-00070639

HAL Id: inria-00070639
https://inria.hal.science/inria-00070639
Submitted on 19 May 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Arc-chromatic number of digraphs in which each vertex has bounded outdegree or bounded indegree

S. Bessy — E. Birmelé — F. Havet

N° 5364
Novembre 2004
Arc-chromatic number of digraphs in which each vertex has bounded outdegree or bounded indegree

S. Bessy, E. Birmelé*, F. Havet

Thème COM — Systèmes communicants
Projet Mascotte

Rapport de recherche n° 5364 — Novembre 2004 — 20 pages

Abstract: A $k$-digraph is a digraph in which every vertex has outdegree at most $k$. A $(k \lor l)$-digraph is a digraph in which a vertex has either outdegree at most $k$ or indegree at most $l$. Motivated by function theory, we study the maximum value $\Phi(k)$ (resp. $\Phi^\lor(k, l)$) of the arc-chromatic number over the $k$-digraphs (resp. $(k \lor l)$-digraphs). El-Sahli [3] showed that $\Phi^\lor(k, k) \leq 2k+1$. After giving a simple proof of this result, we show some better bounds. We show $\max\{\log(2k+3), \theta(k+1)\} \leq \Phi(k) \leq \theta(2k)$ and $\max\{\log(2k+2l+4), \theta(k+1), \theta(l+1)\} \leq \Phi^\lor(k, l) \leq \theta(2k + 2l)$ where $\theta$ is the function defined by $\theta(k) = \min\{s : \left(\frac{s}{2}\right) \geq k\}$. We then study in more details properties of $\Phi$ and $\Phi^\lor$. Finally, we give the exact values of $\Phi(k)$ and $\Phi^\lor(k, l)$ for $l \leq k \leq 3$.

Key-words: arc-colouring, arc-chromatic number, function theory

* Laboratoire MAGE, Université de Haute-Alsace, 6, rue des frères Lumière, 68093 Mulhouse Cedex, France
Nombre arc-chromatique des digraphes dont les sommets ont degré externe ou degré interne borné

Résumé : Un $k$-digraphe est un digraphe dont les sommets ont degré externe au plus $k$. Un $(k \lor l)$-digraphe est un digraphe dont les sommets ont soit degré externe au plus $k$ soit degré interne au plus $l$. Motivés par la théorie des fonctions, nous étudions la valeur maximale $\Phi(k)$ (resp. $\Phi^V(k, l)$) du nombre arc-chromatique d’un $k$-digraphe (resp. $(k \lor l)$-digraphe). El-Sahili [3] a montré $\Phi^V(k, k) \leq 2k + 1$. Après avoir donné une preuve simple de ce résultat, nous donnons de meilleure borne. Nous prouvons $\max\{\log(2k + 3), \theta(k + 1)\} \leq \Phi(k) \leq \theta(2k)$ et $\max\{\log(2k + 2l + 4), \theta(k + 1), \theta(l + 1)\} \leq \Phi^V(k, l) \leq \theta(2k + 2l)$ avec $\theta$ la fonction définie par $\theta(k) = \min\{s : \left(\frac{s}{2}\right) \geq k\}$. Nous étudions ensuite plus en détails des propriétés de $\Phi$ et $\Phi^V$. Enfin, nous donnons les valeurs exactes de $\Phi(k)$ et $\Phi^V(k, l)$ pour $l \leq k \leq 3$.

Mots-clés : arc-coloration, nombre arc-chromatique, théorie des fonctions
1 Introduction

A directed graph or digraph $D$ is a pair $(V(D),E(D))$ of disjoint sets (of vertices and arcs) together with two maps tail : $E(D) \to V(D)$ and head : $E(D) \to V(D)$ assigning to every arc $e$ a tail, tail$(e)$, and a head, head$(e)$. The tail and the head of an arc are its ends. An arc with tail $u$ and head $v$ is denoted by $uv$; we say that $u$ dominates $v$ and write $u \to v$. We also say that $u$ and $v$ are adjacent. The order of a digraph is its number of vertices. In this paper, all the digraphs we consider are loopless, that is that every arc has its tail distinct from its head.

Let $D$ be a digraph. The line-digraph of $D$ is the digraph $L(D)$ such that $V(L(D)) = E(D)$ and an arc $a \in E(D)$ dominates an arc $b \in E(D)$ in $L(D)$ if and only if head$(a) = \text{tail}(b)$.

A vertex-colouring or colouring of $D$ is an application $c$ from the vertex-set $V(D)$ into a set of colours $S$ such that for any arc $uv$, $c(u) \neq c(v)$. The chromatic number of $D$, denoted $\chi(D)$, is the minimum number of colours of a colouring of $D$.

An arc-colouring of $D$ is an application $c$ from the arc-set $E(D)$ into a set of colours $S$ such that if the tail of an arc $e$ is the head of an arc $e'$ then $c(e) \neq c(e')$. Trivially, there is a one-to-one correspondence between arc-colourings of $D$ and colourings of $L(D)$. The arc-chromatic number of $D$, denoted $\chi_a(D)$, is the minimum number of colours of an arc-colouring of $D$. Clearly $\chi_a(D) = \chi(L(D))$.

A $k$-digraph is a digraph in which every vertex has outdegree at most $k$. A $(k \lor l)$-digraph is a digraph in which a vertex has either outdegree at most $k$ or indegree at most $l$.

For any digraph $D$ and set of vertices $V' \subseteq V(D)$, we denote by $D[V']$, the subdigraph of $D$ induced by the vertices of $V'$. For any subdigraph $F$ of $D$, we denote by $D - F$ the digraph $(V(D) \setminus V(F),E(D) \setminus E(F))$. For any arc-set $E' \subseteq E$, we denote by $D - E'$ the digraph $(V(D),E(D) \setminus E')$ and for any vertex $x \in V(D)$, we denote by $D - x$ the digraph induced by $V(D) \setminus \{x\}$.

Let $D$ be a $(k \lor l)$-digraph. We denote by $V^+(D)$, or $V^+$ if $D$ is clearly understood, the subset of the vertices of $D$ with outdegree at most $k$, and by $V^-(D)$ or $V^-$ the complementary of $V^+(D)$ in $V(D)$. Also $D^+$ (resp. $D^-$) denotes $D[V^+]$ (resp. $D[V^-]$).

In this paper, we study the arc-chromatic number of $k$-digraphs and $(k \lor l)$-digraphs. This is motivated by the following interpretation in function theory as shown by El-Sahili in [3].

Let $f$ and $g$ be two maps from a finite set $A$ into a set $B$. Suppose that $f$ and $g$ are nowhere coinciding, that is for all $a \in A$, $f(a) \neq g(a)$. A subset $A'$ of $A$ is $(f,g)$-independant if $f(A') \cap g(A') = \emptyset$. We are interested by finding the largest $(f,g)$-independant subset of $A$ and the minimum number of $(f,g)$-independant subsets to partition $A$. As shown by El-Sahili [3], this can be translated into an arc-colouring problem.

Let $D_{f,g}$ and $H_{f,g}$ be the digraphs defined as follows:

- $V(D_{f,g}) = B$ and $(b,b') \in E(D_{f,g})$ if there exists an element $a$ in $A$ such that $g(a) = b$ and $f(a) = b'$. Note that if for all $a$, $f(a) \neq g(a)$, then $D_{f,g}$ has no loop.
\[ V(H_{f,g}) = A \text{ and } (a, a') \in E(H_{f,g}) \text{ if } f(a) = g(a'). \]

We associate to each arc \((b, b')\) in \(D_{f,g}\) the vertex \(a\) of \(A\) such that \(g(a) = b\) and \(f(a) = b'\). Then \((a, a')\) is an arc in \(H_{f,g}\) if, and only if, \(\text{head}(a) = \text{tail}(a')\) (as arcs in \(D_{f,g}\)). Thus \(H_{f,g} = L(D_{f,g})\). Note that for every digraph \(D\), there exists maps \(f\) and \(g\) such that \(D = D_{f,g}\).

It is easy to see that an \((f,g)\)-indecomposable subset of \(A\) is an indecomposable set in \(H_{f,g}\). In [2] El-Sahili proved the following:

**Theorem 1 (El-Sahili [2])** Let \(f \text{ and } g\) be two nowhere coinciding maps from a finite set \(A\) into a set \(B\). Then there exists an \((f,g)\)-indecomposable subset \(A'\) of cardinality at least \(|A|/4\).

Let \(f \text{ and } g\) be two nowhere coinciding maps from a finite set \(A\) into \(B\). We define \(\phi(f, g)\) as the minimum number of \((f,g)\)-indecomposable sets to partition \(A\). Then \(\phi(f, g) = \chi(H_{f,g}) = \chi(D_{f,g})\).

Let \(\Phi(k)\) (resp. \(\Phi^v(k, l)\)) be the maximum value of \(\phi(f, g)\) for two nowhere coinciding maps \(f\) and \(g\) from \(A\) into \(B\) such that for every \(z\) in \(B\), \(g^{-1}(z) \leq k\) (resp. each \(g^{-1}(z) \leq k\) or \(f^{-1}(z) \leq l\)). The condition \(f^{-1}(z)\) (resp. \(g^{-1}(z)\)) has at most \(k\) elements means that each vertex has indegree (resp. outdegree) at most \(k\) in \(D_{f,g}\). Hence \(\Phi(k)\) (resp. \(\Phi^v(k, l)\)) is the maximum value of \(\chi_a(D)\) for \(D\) a \(k\)-digraph (resp. \(k \lor l\)-digraph).

**Remark 2** Let \(f \text{ and } g\) be two nowhere coinciding maps from \(A\) into \(B\). Then \(A\) may be partitioned into \(\Phi(|A| - 1)\) \((f, g)\)-indecomposable sets.

The functions \(\Phi^v\) and \(\Phi\) are very close to each other:

**Proposition 3**

\[
\Phi(k) \leq \Phi^v(k, 0) \leq \cdots \leq \Phi^v(k, k) \leq \Phi(k) + 2
\]

**Proof.** The sole inequality that does not immediately follow the definitions is \(\Phi^v(k, k) \leq \Phi(k) + 2\). Let us prove it.

Let \(D\) be a \((k \lor k)\)-digraph. One can colour the arcs in \(D^+ \cup D^-\) with \(\Phi(k)\) colours. It remains to colour the arcs with tail in \(V^-\) and head in \(V^+\) with one new colour and the arcs with tail in \(V^+\) and head in \(V^-\) with a second new colour. \(\Box\)

Moreover, we conjecture that \(\Phi^v(k, k)\) is never equal to \(\Phi(k) + 2\).

**Conjecture 4**

\[
\Phi^v(k, k) \leq \Phi(k) + 1
\]

In [3], El-Sahili gave the following upper bound on \(\Phi^v(k, k)\):

**Theorem 5 (El-Sahili [3])** \(\Phi^v(k, k) \leq 2k + 1\)
In this paper, we first give simple proofs of Theorems 1 and 5. Then, in Section 3, we improve the upper bounds on $\Phi(k)$ and $\Phi^v(k, l)$. We show (Theorem 18) that $\Phi^v(k) \leq \theta(2k)$ if $k \geq 2$, and $\Phi^v(k, l) \leq \theta(2k + 2l)$ if $k + l \geq 3$, where $\theta$ is the function defined by $\theta(k) = \min\{s : \left\lfloor \frac{s}{k} \right\rfloor \geq k\}$. We also establish (Corollary 21) that $\Phi^v(k, l) \leq \theta(2k)$ if $\theta(2k) \geq 2l + 1$.

In Section 4, we study in more details the relations between $\Phi^v(k, l)$ and $\Phi(k)$. We conjecture that if $k$ is very large compared to $l$ then $\Phi^v(k, l) = \Phi(k)$. We prove that $\Phi^v(k, 0) = \Phi(k)$ and conjecture that $\Phi^v(k, 1) = \Phi(k)$ if $k \geq 1$. We prove that for a fixed $k$ either this latter conjecture holds or Conjecture 4 holds. This implies that $\Phi^v(k, 1) \leq \Phi(k) + 1$.

Finally, in Section 5, we give the exact values of $\Phi(k)$ and $\Phi^v(k, l)$ for $l \leq k \leq 3$. They are summarized in the following table:

<table>
<thead>
<tr>
<th>$\Phi^v(0, 0) = 1$</th>
<th>$\Phi^v(1, 0) = \Phi(1) = 3$</th>
<th>$\Phi^v(2, 0) = \Phi(2) = 4$</th>
<th>$\Phi^v(3, 0) = \Phi(3) = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi^v(1, 1) = 3$</td>
<td>$\Phi^v(2, 1) = 4$</td>
<td>$\Phi^v(3, 1) = 4$</td>
<td></td>
</tr>
<tr>
<td>$\Phi^v(2, 2) = 4$</td>
<td>$\Phi^v(3, 2) = 5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\Phi^v(3, 3) = 5$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2 Simple proofs of Theorems 1 and 5

Proof of Theorem 1. Let $D = D_{f,g}$. Let $(V_1, V_2)$ be a partition of $V(D)$ that maximizes the number of arcs $a$ with one end in $V_1$ and one end in $V_2$. It is well-known that $a \geq |E(D)|/2$. Now let $A_1$ be the set of arcs with head in $V_1$ and tail in $V_2$ and $A_2$ be the set of arcs with head in $V_2$ and tail in $V_1$. Then $A_1$ and $A_2$ corresponds to independant sets of $L(D)$ and $|A_1| + |A_2| = a$. Hence one of the $A_i$ has cardinality at least $a/2 = \frac{|E(D)|}{4}$. $\Box$

Before giving a short proof of Theorem 5, we precise few standard definitions.

Definition 6 A path is a non-empty digraph $P$ of the form

$$V(P) = \{v_0, v_1, \ldots, v_k\} \quad E(P) = \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\},$$

where the $v_i$ are all distinct. The vertices $v_0$ and $v_k$ are respectively called the origin and terminus of $P$.

A circuit is a non-empty digraph $C$ of the form

$$V(C) = \{v_0, v_1, \ldots, v_k\} \quad E(P) = \{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k, v_kv_0\},$$

where the $v_i$ are all distinct.

A digraph is strongly connected or strong if for every two vertices $u$ and $v$ there is a path with origin $u$ and terminus $v$. A maximal strong subdigraph of a digraph $D$ is called a strong component of $D$. A component $I$ of $D$ is initial if there is no arc with tail in $V(D) \setminus V(I)$.
and head in $V(I)$. A component $I$ of $D$ is terminal if there is no arc with tail in $V(I)$ and head in $V(D) \setminus V(I)$. A digraph is connected if its underlying graph is connected.

A digraph $D$ is $l$-degenerate if every subdigraph $H$ has a vertex of degree at most $l$.

The following lemma corresponding to the greedy colouring algorithm is a piece of folklore.

**Lemma 7** Every $l$-degenerate digraph is $(l+1)$-colourable.

**Proof of Theorem 5.** Let $D$ be a $(k \lor k)$-digraph. According to Lemma 7, it suffices to prove that $L(D)$ is $2k$-degenerate.

In every initial strong component $C$ there is a vertex with indegree at most $k$. Indeed if there is no such vertex then $(k+1)|C| \leq \sum_{v \in C} d^-(v) \leq \sum_{v \in C} d^+(v) \leq k|C|$. Analogously, in every terminal strong component there is a vertex with outdegree at most $k$.

Now, there is a path originating in a minimal component and terminating in a terminal one. Hence there is a path whose origin has indegree at most $k$ and whose terminus has outdegree at most $k$. Hence there is an arc $e$ whose tail has indegree at most $k$ and whose head has outdegree at most $k$. Thus $e$ has degree at most $2k$ in $L(D)$. $\square$

### 3 Lower and upper bounds for $\Phi$ and $\Phi^\vee$

We will now search for bounds on $\Phi$ since they also give bounds on $\Phi^\vee$.

Theorem 1 and an easy induction yields $\chi_a(D) \leq \log_{4/3} |D|$. However there exists better upper bounds provided by Poljak and Rödl [5]. For sake of completeness and in order to introduce useful tools, we provide a proof of Theorem 11.

**Definition 8** We denote by $\overline{\Pi}_k$ the complementary of the hypercube of dimension $k$, that is the digraph with vertex-set all the subsets of $\{1, \ldots, k\}$ and with arc-set $\{xy : x \nsubseteq y\}$.

A homomorphism $h : D \to D'$ is a mapping $h : V(D) \to V(D')$ such that for every arc $xy$ of $D$, $h(x)h(y)$ is an arc of $D'$.

Let $c$ be an arc-colouring of a digraph $D$ into a set of colours $S$. For any vertex $x$ of $D$, we denote by $Col^\pm(x)$ or simply $Col^\pm(x)$ the set of colours assigned to the arcs with tail $x$. We define $Col^-(x) = S \setminus Col^+(x)$. Note that $Col^-(x)$ contains (but may be bigger than) the set of colours assigned to the arcs with head $x$. The cardinality of $Col^+(x)$ (resp. $Col^-(x)$) is denoted by $\text{col}^+(x)$ (resp. $\text{col}^-(x)$).

**Theorem 9** For every digraph $D$, $\chi_a(D) = \min \{k : D \to \overline{\Pi}_k\}$.

**Proof.** Assume that $D$ admits an arc-colouring with $\{1, \ldots, k\}$. It is easy to check that $Col^+$ is a homomorphism from $D$ to $\overline{\Pi}_k$.

Conversely, suppose that there exists a homomorphism $h$ from $D$ to $\overline{\Pi}_k$. Assign to each arc $xy$ an element of $h(y) \setminus h(x)$, which is not empty. This provides an arc-colouring of $D$. $\square$
**Definition 10** The complete digraph of order $n$, denoted $K_n$, is the digraph with vertex-set $\{v_1, v_2, \ldots, v_n\}$ and arc-set $\{v_iv_j : i \neq j\}$.

The transitive tournament of order $n$, denoted $TT_n$, is the digraph with vertex-set $\{v_1, v_2, \ldots, v_n\}$ and arc-set $\{v_iv_j : i < j\}$.

The following corollary of Theorem 9 provides bounds on the arc-chromatic number of a digraph according to its chromatic number.

**Theorem 11 (Poljak and Rödl [5])** For every digraph $D$,

$$\lceil \log(\chi(D)) \rceil \leq \chi_a(D) \leq \theta(\chi(D)).$$

**Proof.** By definition of the chromatic number, $D \rightarrow \overrightarrow{K}_{\chi(D)}$. As the subsets of $\{1, \ldots, k\}$ with cardinality $\left\lceil \frac{k}{2} \right\rceil$ induce a complete digraph on $\left\lceil \frac{k}{2} \right\rceil$ vertices in $\overrightarrow{P}_{k}$, we obtain a homomorphism from $D$ to $\overrightarrow{P}_{\theta(\chi(D))}$. So $\chi_a(D) \leq \theta(\chi(D))$.

By Theorem 9, we have $D \rightarrow \overrightarrow{P}_{\chi_a(D)}$. As $\chi(\overrightarrow{P}_{\chi_a(D)}) = 2^{\chi_a(D)}$, we obtain $D \rightarrow \overrightarrow{K}_{2^{\chi_a(D)}}$.

These bounds are tight since the lower one is achieved by transitive tournaments and the upper one by complete digraphs by Sperner’s Lemma (see [6]). However, the lower bound may be increased if the digraph has no sink (vertex with outdegree 0) or/and no source (vertex with indegree 0).

**Theorem 12** Let $D$ be a digraph.

(i) If $D$ has no sink then $\log(\chi(D) + 1) \leq \chi_a(D)$.

(ii) If $D$ has no source and no sink then $\log(\chi(D) + 2) \leq \chi_a(D)$.

**Proof.** The proof is identical to the proof of Theorem 11. But if a digraph has no source (resp. no sink) then for every $v$, $Col^+(v) \neq S$ (resp. $Col^+(v) \neq \emptyset$).

Again these two lower bounds are also tight. Let $Q_n$ (resp. $W_n$) be the tournament of order $n$ obtained from $TT_n$ by reversing the arc $v_1v_n$ (resp. $v_2v_n$). One can easily check that $\chi_a(W_n) = \lceil \log(n + 1) \rceil = \lceil \log(\chi(W_n) + 1) \rceil$ and $\chi_a(Q_n) = \lceil \log(n + 2) \rceil = \lceil \log(\chi(Q_n) + 2) \rceil$.

**Proposition 13** Every $k$-digraph is $2k$-degenerate.

**Proof.** Every subdigraph of a $k$-digraph is also a $k$-digraph. Hence it suffices to prove that every $k$-digraph has a vertex with degree at most $2k$. Since the sum of outdegrees equals the sum of indegrees, there is a vertex with indegree at most $k$ and thus with degree at most $2k$.

RR n° 5364
Corollary 14
\[ \max \{ \log(2k + 3), \theta(k + 1) \} \leq \Phi(k) \leq \theta(2k + 1) \]

**Proof.** The upper bound follows from Theorem 11, Proposition 13 and Lemma 7. The lower bound comes from a regular tournament on \(2k + 1\) vertices \(T_{2k+1}\) and the complete digraph on \(k + 1\) vertices \(\tilde{K}_{k+1}\). Indeed \(\chi(T_{2k+1}) = 2k + 1\), so \(\chi_a(T_{2k+1}) \geq \log(2k + 3)\) by Theorem 12 and \(\chi_a(\tilde{K}_{k+1}) = \theta(k + 1)\). \(\square\)

Corollary 15
\[ \max \{ \log(2k + 2l + 4), \theta(k + 1), \theta(l + 1) \} \leq \Phi^V(k, l) \leq \theta(2k + 2l + 2) \]

**Proof.** The upper bound follows Theorem 11 and Proposition degen since every \((k \lor l)\)-digraph \(D\) is \(2k + 2l + 2\)-colourable \((D^+\) is \(2k\)-degenerate and \((D+\) is \(2l\)-degenerate and so \((2l + 1)\)-colourable\). The lower bound comes from \(\tilde{K}_{k+1}, \tilde{K}_{l+1}\) and a tournament \(T\) composed of a regular tournament on \(2l + 1\) vertices dominating a regular tournament on \(2k + 1\) vertices. Indeed \(\chi_a(\tilde{K}_{k+1}) = \theta(k + 1), \chi_a(\tilde{K}_{l+1}) = \theta(l + 1)\) and \(T\) has no source, no sink and chromatic number \(2k + 2l + 2\), by Theorem 12, \(\chi_a(T) \geq \log(2k + 2l + 4)\). \(\square\)

We can obtain a slightly better upper bound on \(\Phi\). Bounds on \(\Phi^V\) follow.

**Definition 16** For any integer \(k \geq 1\), let \(T^+_k(k \geq 1)\) be the complete digraph on \(t^+_1, \ldots, t^+_{2k+1}\) minus the arcs \(\{t^+_1t^+_2, t^+_1t^+_3, \ldots, t^+_1t^+_{2k+1}\}\).

**Lemma 17** Let \(k \geq 1\) be an integer. If \(D\) is a \(k\)-digraph, then there exists a homomorphism \(h^+\) from \(D\) to \(T^+_k\) such that if \(h^+(x) = t^+_i\) then \(d^+(x) = k\).

**Proof.** For a fixed \(k\), we prove it by induction on \(|V(D)|\).

First, suppose that there exists in \(D\) a vertex \(x\) with \(d^-(x) < k\). Then, \(d^+(x) + d^-(x) < 2k\). By induction on \(D - x\), there is a homomorphism \(h\) from \(D - x\) to \(T^+_k\) such that if \(h^+(v) = t^+_i\) then \(d^+_{D-x}(v) = k\). Hence, \(h^+(y) \neq t^+_i\) for every inneighbour \(y\) of \(x\), because \(d^+_{D-x}(y) < k\). As \(x\) has at most \(2k - 1\) neighbours, we find \(i \in \{2, \ldots, 2k + 1\}\) such that no neighbour \(y\) of \(x\) satisfies \(h^+(y) = t^+_i\). So, \(h^+(x) = t^+_i\) extends \(h^+\) to a homomorphism from \(D\) to \(T^+_k\).

Suppose now that every vertex \(v\) of \(D\) has indegree at least \(k\). Since \(\sum_{v \in V(D)} d^-(v) = \sum_{v \in V(D)} d^+(v) \leq k|V(D)|\), every vertex has indegree and outdegree \(k\). Hence, by Brooks Theorem (see [1]) either \(D\) is \(2k\)-colourable and \(D \rightarrow T^+_k\{\{t^+_1, \ldots, t^+_{2k+1}\}\}\), or \(D\) is a regular tournament on \(2k + 1\) vertices. In this latter case, label the vertices of \(D\) with \(v_1, v_2, \ldots, v_{2k+1}\) such that \(N^-(v_1) = \{v_2, \ldots, v_{k+1}\}\). Then \(h^+\) defined by \(h^+(v_i) = t^+_i\) is the desired homomorphism. \(\square\)
Theorem 18 Let \( k \) and \( l \) be two positive integers.

(i) If \( k \geq 2 \), then \( \Phi(k) \leq \theta(2k) \).

(ii) If \( k + l \geq 3 \), then \( \Phi^l(k,l) \leq \theta(2k+2l) \).

Proof. (i) If \( k = 2 \), the result follows Corollary 14 since \( \theta(4) = \theta(5) = 4 \). Suppose now that \( k \geq 3 \). Let \( D \) be a \( k \)-digraph. By Lemma 17 there is a homomorphism from \( D \) to \( T_k^+ \). We will provide a homomorphism \( g \) from \( T_k^+ \) to \( \overline{H}_{\theta(2k)} \).

Fix \( S_1, \ldots, S_{2k} \), \( 2k \) subsets of \( \{1, \ldots, \theta(2k)\} \) with cardinality \( \lfloor \theta(2k)/2 \rfloor \) and \( S \) a subset of \( \{1, \ldots, 2k\} \) with cardinality \( \lfloor \theta(2k)/2 \rfloor - 1 \). Without loss of generality, the \( S_i \) containing \( S \) are \( S_1, \ldots, S_l \) with \( l \leq \theta(2k)/2 + 1 \leq k \). (One can easily check that \( \theta(2k)/2 + 1 \leq k \) provided \( k \geq 3 \).) Now, set \( g(t_1^+) = S \) and \( g(t_i^+) = S_i \) for \( 1 \leq i \leq 2k \). It is straightforward to check that \( g \) is a homomorphism.

(ii) Let \( D \) be a \((k \lor l)\)-digraph. By Lemma 17, there exists a homomorphism \( h^+ \) from \( D^+ \) to \( T_k^+ \) such that \( h^+(x) = t_1^+ \) then \( d^+(x) = k \) and, by symmetry, a homomorphism \( h^- \) from \( D^- \) to \( T_k^- \), the complete digraph on \( \{t_1^- \ldots, t_{2k}^- \} \) minus the arcs \( \{t_1^-, t_2^-, t_3^-, \ldots, t_{2k}^- \} \), such that \( h^-(x) = t_1^- \) then \( d^+(x) = l \). We now provide a homomorphism from \( D \) to \( \overline{H}_{\theta(2k+2l)} \).

Fix \( S^+ \) and \( S^- \), two subsets of \( \{1, \ldots, \theta(2k+2l)\} \) with cardinality \( \lfloor \theta(2k+2l)/2 \rfloor - 1 \) for \( S^+ \) and \( \lfloor \theta(2k+2l)/2 \rfloor + 1 \) for \( S^- \) such that \( S^+ \not\subset S^- \). (This is possible since \( \theta(2k+2l) \geq 4 \), because \( k + l \geq 2 \).) Set \( N = \{U \subset \{1, \ldots, \theta(2k+2l)\} : |U| = \lfloor \theta(2k+2l)/2 \rfloor \} \). We want a partition of \( N \) into two parts \( A \) and \( B \) with \( |A| \geq 2k \) and \( |B| \geq 2l \), such that \( S^+ \) is included in most \( k \) sets of \( A \) and \( S^- \) contains at most \( l \) sets of \( B \). Let \( N_{S^+} \) (resp. \( N_{S^-} \)) be the set of elements of \( N \) containing \( S^+ \) (resp. contained in \( S^- \)). We have \( |N_{S^+}| = \lfloor \theta(2k+2l)/2 \rfloor - 1 \) and \( |N_{S^-}| = \lfloor \theta(2k+2l)/2 \rfloor + 1 \); because \( k + l \geq 3 \), it follows \( |N_{S^+}| \leq k \) and \( |N_{S^-}| \leq k + l \). Moreover, the sets \( N_{S^+} \) and \( N_{S^-} \) are disjoint and \( |N_{S^-}| \leq k + l \). Let us sort the elements of \( N \) beginning with those of \( N_{S^-} \) and ending with those of \( N_{S^+} \). Let \( A \) be the \( 2k \) first sets in this sorting and \( B \) what remains (\( |B| \geq 2l \)). We claim that \( A \) contains at most \( k \) elements of \( N_{S^+} \). If not, then \(|A| > \lfloor (2k+2l)/2 \rfloor - |N_{S^+}| + k \). We obtain \( 2k > 2k + 2l - |N_{S^+}| + k \) which contradicts \( |N_{S^+}| \leq k + l \). With same argument, \( B \) contains at most \( l \) elements of \( N_{S^-} \).

Finally, set \( A = \{A_1, \ldots, A_{2k}\} \) such that none of \( A_{k+1}, \ldots, A_{2k} \) contains \( S^+ \) and \( \{B_1, \ldots, B_{2l}\} \) \( 2l \) sets of \( S^- \) such that none of \( B_{k+1}, \ldots, B_{2l} \) is contained in \( S^- \).

Let us define \( h : D \rightarrow \overline{H}_{2k+2l} \). If \( x \in V^+ \) and \( h^+(x) = t_i^+ \) then \( h(x) = S^+ \) if \( i = 1 \) and \( h(x) = A_{i-1} \) otherwise. If \( x \in V^- \) and \( h^-(x) = t_i^- \) the \( h(x) = S^- \) if \( i = 1 \) and \( h(x) = B_{i-1} \) otherwise. Let us check that \( h \) is a homomorphism. Let \( xy \) be an arc of \( D \). \( T_i^+ \) is a subdigraph of \( \overline{H}_{2k+2l} \{A_1, \ldots, A_{2k}, S\} \) and \( T_i^- \) is a subdigraph of \( \overline{H}_{2k+2l} \{B_1, \ldots, B_{2l}, S^-\} \).

So, \( h(x)(y) = h(x) \) is an arc of \( \overline{H}_{2k+2l} \) if \( x \) and \( y \) are both in \( V^+ \) or both in \( V^- \). Suppose now that \( x \in V^+ \) and \( y \in V^- \), then \( h^+(x) \neq t_i^+ \) because \( d^+(x) < k \) and \( h(x) \neq S^+ \). Similarly, \( h^-(y) \neq S^- \). Thus \( h(x) \) and \( h(y) \) are elements of \( N \), so \( h(x)(y) \in E(\overline{H}_{2k+2l}) \). Finally, suppose that \( x \in V^- \) and \( y \in V^+ \). Then \( h(x)(y) \in E(\overline{H}_{2k+2l}) \) because no element of \( \{B_1, \ldots, B_{2l}, S^-\} \) is a subset of an element of \( \{A_1, \ldots, A_{2k}, S\} \). \( \square \)
Remark 19  Note that the homomorphism provided in (i) has for image subsets of \( \{1, \ldots, \theta(2k)\} \) with cardinality at most \( \lfloor \theta(2k)/2 \rfloor \). So, using the method developed in Theorem 9, we provide an arc-colouring of a \( k \)-digraph \( D \) with \( \theta(2k) \) colours which satisfies \( \text{col}^+(x) \leq \lfloor \theta(2k)/2 \rfloor \), so \( \text{col}^-(x) \geq \lceil \theta(2k)/2 \rceil \), for every vertex \( x \) of \( D \).

We will now improve the bound (ii) of Theorem 18 when \( l \) is very small compared to \( k \).

Lemma 20  Let \( D \) be a \((k \lor l)\)-digraph and \( D^1 \) the subdigraph of \( D \) induced by the arcs with tail in \( V^+ \). If there exists an arc-colouring of \( D^1 \) with \( m \geq 2l+1 \) colours such that for every \( x \) in \( V^+ \), \( \text{col}^-(x) \geq l+1 \) then \( \chi_a(D) = m \).

Proof.  We will extend the colouring as stated into an arc-colouring of \( D \).

Let us now first extend this colouring to the arcs of \( D^- \). Since \( \sum_{v \in V^-} d_{D^-}^+(v) = \sum_{v \in V^-} d_{D^-}^-(v) \leq |V^-| \), there is a vertex \( v_1 \in V^- \) such that \( d_{D^-}^+(v_1) \leq l \). And so on, by induction, there is an ordering \((v_1, v_2, \ldots, v_p)\) of the vertices of \( D^- \) such that, for every \( 1 \leq i \leq p \), \( v_i \) dominates at most \( l \) vertices in \( \{ v_j: j > i \} \). Let us colour the arcs of \( D^- \) in decreasing order of their head; that is first colour the arcs with head \( v_p \) then those with head \( v_{p-1} \), and so on. This is possible since at each stage \( i \), an arc \( uv_i \) has at most \( 2l < m \) forbidden colours (\( l \) ingoing \( u \) and \( l \) outgoing \( v_i \) to a vertex in \( \{ v_j: j > i \} \)).

It remains to colour the arcs with tail in \( V^- \) and head in \( V^+ \). Let \( v^-v^+ \) be such an arc. Since \( \text{col}^-(v^+) \geq l+1 \) and \( d^-(v^-) \leq l \), there is a colour \( \alpha \) in \( \text{Col}^- (v^+ \) that is assigned to no arc ingoing \( v^- \). Hence, assigning \( \alpha \) to \( v^-v^+ \), we extend the arc-colouring to \( v^-v^+ \). \( \square \)

Corollary 21  If \( \theta(2k) \geq 2l+1 \), then \( \phi^V(k, l) \leq \theta(2k) \).

Proof.  The digraph \( D^1 \), as defined in Lemma 20, is a \( k \)-digraph. The result follows directly from Remark 19 and Lemma 20. \( \square \)

4 Relations between \( \Phi(k) \) and \( \Phi^V(k, l) \)

Conjecture 22  Let \( l \) be a positive integer. There exists an integer \( k_l \) such that if \( k \geq k_l \) then \( \Phi^V(k, l) = \Phi(k) \).

We now prove Conjecture 22, for \( l = 0 \), showing that \( k_0 = 1 \).

Theorem 23  If \( k \geq 1 \),
\[
\Phi^V(k, 0) = \Phi(k).
\]

Proof.  Let \( D = (V, A) \) be a \((k \lor 0)\)-digraph. Let \( V_0 \) be the set of vertices with indegree 0. Let \( D' \) be the digraph obtained from \( D \) by splitting each vertex \( v \) of \( V_0 \) into \( d^+(v) \) vertices with outdegree 1. Formally, for each vertex \( v \in V_0 \) incident to the arcs \( vw_1, \ldots, vw_{d^+(v)} \),
replace $v$ by $\{v_1, v_2, \ldots, v_{d^+(v)}\}$ and $vw_i$ by $v_iw$, $1 \leq i \leq d^+(v)$. By construction, $D'$ is a $k$-digraph and $L(D) = L(D')$. So $\chi_a(D) = \chi_a(D') \leq \Phi(k)$. \hfill \Box

We conjecture that if $l = 1$, Conjecture 22 holds with $k_1 = 1$.

**Conjecture 24** If $k \geq 1$,

$$\Phi^v(k, 1) = \Phi(k)$$

**Theorem 25** If $\Phi(k) = \Phi(k - 1)$ or $\Phi(k) = \Phi(k + 1)$ then $\Phi^v(k, 1) = \Phi(k)$.

**Proof.** By Lemma 20, it suffices to prove that if $\Phi(k) = \Phi(k - 1)$ or $\Phi(k) = \Phi(k + 1)$ every $k$-digraph admits an arc-colouring with $\Phi(k)$ colours such that for every vertex $x$, $\text{col}^-(x) \geq 2$.

Suppose that $\Phi(k) = \Phi(k - 1)$. Let $D$ be a $k$-digraph and $D'$ be a $(k - 1)$-digraph such that $\chi_a(D') = \Phi(k)$. Let $C$ be the digraph constructed as follows: for every vertex $x \in V(D)$ add a copy $D'(x)$ of $D'$ such that every vertex of $D'(x)$ dominates $x$. Then $C$ is a $k$-digraph, so it admits an arc-colouring with $\Phi(k)$ colours. Note that $c$ is also an arc-colouring of $D$ which is a subdigraph of $C$. Let us prove that for every vertex $x \in V(D)$, $\text{col}^-(x) \geq 2$.

Suppose, reductio ad absurdum, that there is a vertex $x \in V(D)$ such that $\text{col}^-(x) \leq 1$. Since there are arcs ingoing $x$ in $C$ (those from $V(D'(x))$), then $\text{col}^-(x)$ is a singleton $\{\alpha\}$. Now every arc $vx$ with $v \in D'(x)$ is coloured $\alpha$ so any arc $uv \in E(D'(x))$ is not coloured $\alpha$. Hence $c$ is an arc-colouring with $\Phi(k) - 1$ colours which is a contradiction.

The proof is analogous if $\Phi(k) = \Phi(k + 1)$ with $D'$ a $k$-digraph such that $\chi_a(D') = \Phi(k)$. Then $C$ is a $(k + 1)$-digraph and we get the result in the same way. \hfill \Box

The next theorem shows that for a fixed integer $k$, one of the Conjectures 24 and 4 holds.

**Theorem 26** Let $k$ be an integer. Then $\Phi^v(k, 1) = \Phi(k)$ or $\Phi^v(k, k) \leq \Phi(k) + 1$.

**Proof.** Suppose that $\Phi^v(k, 1) \neq \Phi(k)$. Let $C$ be a $(k \lor 1)$-digraph such that $\chi_a(C) = \Phi(k) + 1$ and $C^1$ the subdigraph of $C$ induced by the arcs with tail in $V^+(C)$. By Lemma 20, for every arc-colouring of $C^1$ with $\Phi(k)$ colours there exists a vertex $x$ of $C^+$ with $\text{col}^-(x) \leq 1$.

Let $D$ be a $(k \lor k)$-digraph. Let $D^1$ (resp. $D^2$) be the subdigraph of $D$ induced by the arcs with tail in $V^+(D)$ (resp. head in $V^-(D)$). Let $E'$ be the set of arcs of $D$ with tail in $V^-$ and head in $V^+$. Let $F^1$ be the digraph constructed from $C^1$ as follows: for every vertex $x \in V^+(C)$, add a copy $D^+(x)$ of $D^+$ and the arcs $\{u(x)x, uv \in E(D), u \in V^+(D)$, and $v \in V^-(D)\}$. Then $F^1$ is a $k$-digraph so it admits an arc-colouring $c_1$ with $\{1, \ldots, \Phi(k)\}$. Now there is a vertex $x \in V^+(C)$ such that $\text{col}^-(x) \leq 1$. So all the arcs from $D^+(x)$ to $x$ are coloured the same. Free to permute the labels, we may assume they are coloured 1. Since $F^1[V^+(D) \cup x]$ has the same line-digraph than $D^1$, the arc-colourings of $F^1[V^+(D) \cup x]$ is in one-to-one correspondence with the arc-colourings of $D^1$. So $D^1$ admits an arc-colouring $c^1$ with $\{1, \ldots, \Phi(k)\}$ such that every arc with head in $V^-$ is coloured 1.

Analogously, $D^2$ admits an arc-colouring $c^2$ with $\{1, \ldots, \Phi(k)\}$ such that every arc with head in $V^-$ is coloured 1. The union of $c_1$ and $c_2$ is an arc-colouring of $D - E'$ with
\{1, \ldots, \Phi(k)\}. Hence assigning \Phi(k) + 1 to every arc of \(E'\), we obtain an arc-colouring of \(D\) with \(\Phi(k) + 1\) colours.

\[\text{Corollary 27} \quad \Phi'(k, 1) \leq \Phi(k) + 1.\]

Note that since \(\Phi(k)\) is bounded by \(\theta(2k)\), the condition \(\Phi'(k, 1) = \Phi(k)\) or \(\Phi'(k, k) \leq \Phi(k) + 1\) is very often true. Indeed, we conjecture that it it always true and that \(\Phi\) behaves “smoothly”.

\[\text{Conjecture 28} \quad (i) \text{ If } k \geq 1, \Phi(k + 1) \leq \Phi(k) + 1.
(ii) \text{ If } k \geq 1, \Phi(k + 2) \leq \Phi(k) + 1.
(iii) \Phi(k_1 k_2) \leq \Phi(k_1) + \Phi(k_2).\]

Note that (ii) implies (i) and Conjecture 24.

The arc-set of a \((k_1 + k_2)\)-digraph \(D\) may trivially be partitioned into two sets \(E_1\) and \(E_2\) such that \((V(D), E_1)\) is a \(k_1\)-digraph and \((V(D), E_2)\) is a \(k_2\)-digraph. So \(\Phi(k_1 + k_2) \leq \Phi(k_1) + \Phi(k_2)\). In particular, \(\Phi(k + 1) \leq \Phi(k) + \Phi(1) = \Phi(k) + 3\). Despite we were not able to prove Conjecture 28(i), we now improve the above trivial result.

\[\text{Theorem 29} \quad \text{If } k \geq 1 \text{ then, } \Phi(k + 1) \leq \Phi(k) + 2.\]

**Proof.** Let \(D\) be a \((k + 1)\)-digraph. Free to add arcs, we may assume that \(d^+(v) = k + 1\) for every \(v \in V(D)\). Let \(T_1, \ldots, T_p\) be the terminal components of \(D\). Each \(T_i\) contains a circuit \(C_i\) which has a chord. Indeed consider a maximal path \(P\) in \(T_i\) and two arcs with tail its terminus and head in \(P\), by maximality. One can extend \(\bigcup C_i\) into a subdigraph \(F\) spanning \(D\) such that \(d^+_F(v) = 1\) for every \(v \in V(D)\) and the sole circuits are the \(C_i, 1 \leq i \leq p\). \(F\) is the union of \(p\) connected components \(F_1, \ldots, F_p\), each \(F_i\) being the union of \(C_i\) and inarborences \(A^+_1, \ldots, A^+_p\) with roots \(r^+_1, \ldots, r^+_p\) in \(C_i\) such that \((V(C), A^+_1 \setminus \{r^+_1\}, \ldots, V(A^+_p) \setminus \{r^+_p\})\) is a partition of \(V(D)\).

Now \(D - F\) is a \(k\)-digraph. We colour the arcs of \(D - F\) with \(\Phi(k)\) colours. Let \(\alpha\) and \(\beta\) be two new colours. Let us colour the arcs of \(F\). Let \(1 \leq i \leq p\). If \(C_i\) is an even circuit then \(F_i\) is bipartite and its arcs may be coloured by \(\alpha\) and \(\beta\). If \(C_i\) is an odd circuit, consider its chord \(xy\) in \(E(D - F)\). In the colouring of \(D - F, Col^+(x) \not\subset Col^+(y)\) thus there is an arc \(x'y'\) of \(E(C)\) such that \(Col^+(x') \not\subset Col^+(y')\). Hence we may assign to \(x'y'\) a colour of \(Col^+(x') \setminus Col^+(y')\) Now \(F_i - x'y'\) is bipartite and its arcs may be coloured by \(\alpha\) and \(\beta\). \qed
5 \( \Phi \) and \( \Phi^V \) for small value of \( k \) or \( l \).

5.1 \( \Phi(1), \Phi^V(1, 0) \) and \( \Phi^V(1, 1) \).

Theorem 30

\[ \Phi^V(1, 1) = \Phi^V(1, 0) = \Phi(1) = 3 \]

Proof. By Theorem 5, \( \Phi^V(1, 1) \leq 3 \). The 3-circuit is its own line-digraph and is not 2-colourable. \( \square \)

5.2 \( \Phi(2) \) and \( \Phi^V(2, l) \), for \( l \leq 2 \).

The aim of this subsection is to prove Theorem 35, that is \( \Phi(2) = \Phi^V(2, 0) = \Phi^V(2, 1) = \Phi^V(2, 2) = 4 \). Therefore, we first exhibit a 2-digraph which is not 3-arc-colourable. Then we show that \( \Phi^V(2, 2) \leq 4 \).

Definition 31 For any integer \( k \geq 1 \), the **rotative tournament** on \( 2k + 1 \) vertices, denoted \( R_{2k+1} \), is the tournament with vertex-set \( \{v_1, \ldots, v_{2k+1}\} \) and arc-set \( \{v_i, v_j, j - i \pmod {2k+1} \in \{1, \ldots, k\}\} \).

Proposition 32 The tournament \( R_3 \) is not 3-arc-colourable. So \( \Phi(2) \geq 4 \).

Proof. Suppose that \( R_3 \) admits a 3-arc-colouring \( c \) in \( \{1, 2, 3\} \). Then, for any two vertices \( x \) and \( y \), \( Col^+(x) \neq Col^+(y) \) and \( 1 \leq col^+(x) \leq 2 \). Hence there is a vertex, say \( v_1 \), such that \( col^+(v_1) = 1 \), say \( Col^+(v_1) = \{1\} \). Then \( Col^+(v_2) \) and \( Col^+(v_3) \) are subsets of \( \{2, 3\} \) and \( Col^+(v_2) \not\subseteq Col^+(v_3) \). It follows that \( col^+(v_3) = 1 \). Repeating the argument for \( v_3 \), we obtain \( col^+(v_5) = 1 \) and then \( col^+(v_i) = 1 \), for every \( 1 \leq i \leq 5 \), which is a contradiction. \( \square \)

In order to prove that \( \Phi^V(2, 2) \leq 4 \), we need to show that every \( (2 \lor 2) \)-digraph admits homomorphism \( h \) into \( \overrightarrow{H}_4 \). In order to exhibit such a homomorphism, we first show that there is a homomorphism \( h^+ \) from \( D^+ \) into a subdigraph \( S^+ \) of \( \overrightarrow{H}_4 \) and a homomorphism \( h^- \) from \( D^- \) into a subdigraph \( S^- \) of \( \overrightarrow{H}_4 \) with specific properties allowing us to extend \( h^+ \) and \( h^- \) into a homomorphism \( h \) from \( D \) into \( \overrightarrow{H}_4 \).

Definition 33 Let \( S^+_2 \) be the digraph with vertex-set \( \{s^+_1, \ldots, s^+_6\} \) with arc-set \( \{s^+_i, s^+_j \mid i \neq j\} \setminus \{s^+_2, s^+_4, s^+_5, s^+_6\} \).

Lemma 34 Let \( D \) be a 2-digraph. There exists a homomorphism \( h^+ \) from \( D \) to \( S^+_2 \) such that the vertices \( x \) with \( h^+(x) \in \{s^+_2, s^+_4, s^+_6\} \) have outdegree 2.

Proof. Let us prove it by induction on \( |V(D)| \). If \( d^+(x) \leq 1 \) for every vertex \( x \) of \( D \) then \( D \) is 3-colourable and \( D \rightarrow S^+_2 \{[s^+_1, s^+_3, s^+_5]\} \). So, we assume that there exists a vertex \( x \) with outdegree 2. By induction hypothesis, there is a homomorphism \( h^+ : D - x \rightarrow S^+_2 \) with
the required condition. Note that every inneighbour of \( x \) has outdegree at most 1 in \( D - x \) and thus can not have image \( s_i^+, s_j^+ \) or \( s_i^- \) by \( h^+ \). Denote by \( y \) and \( z \) the outneighbours of \( x \). The set \( \{ h^+(y) , h^+(z) \} \) does not intersect one set of \( \{ s_i^+, s_j^+ \} \), \( \{ s_i^+ , s_j^+ \} \) and \( \{ s_i^+ , s_j^+ \} \), \( \{ s_i^+ , s_j^+ \} \). Then, setting \( h^+(x) = s_2^+ \), we extend \( h^+ \) into a homomorphism from \( D \) to \( S_2^+ \) with the required condition.

\[ \square \]

**Theorem 35**

\[ \Phi(2) = \Phi^V(2,0) = \Phi^V(2,1) = \Phi^V(2,2) = 4 \]

**Proof.**

By Proposition 32, \( 4 \leq \Phi(2) \leq \Phi^V(2,0) \leq \Phi^V(2,1) \leq \Phi^V(2,2) \).

Let us prove that \( \Phi^V(2,2) \leq 4 \). Let \( D \) be a \( (2 \land 2) \)-digraph. We will provide a homomorphism from \( D \) to \( \overline{H}_4 \).

Let \( S_2^- \) be the dual of \( S_2^+ \), that is the digraph on \( \{ s_i^- , \ldots , s_6^- \} \) with arc-set \( \{ s_i^- s_j^- : i \neq j \} \setminus \{ s_i^+ s_j^- : i \neq j \} \). By Lemma 34, there is a homomorphism \( h^+ : D^+ \rightarrow S_2^+ \) such that if \( h^+(x) \in \{ s_2^+ , s_4^+ , s_6^+ \} \) then \( d^+_{h^+}(x) = 2 \). Symmetrically, there exists a homomorphism \( h^- : D^- \rightarrow S_2^- \) such that if \( h^-(x) \in \{ s_2^- , s_4^- , s_6^- \} \) then \( d^-_{h^-}(x) = 2 \).

Let \( S_2 \) be the digraph obtained from the disjoint union of \( S_2^+ \) and \( S_2^- \) by adding the arcs of \( \{ s_i^- s_j^+ : 1 \leq i \leq 6, 1 \leq j \leq 6 \} \cup \{ s_i^+ s_j^- : i = 1, 3, 5, j = 1, 3, 5 \} \). The mapping \( h : D \rightarrow S_2 \) defined by \( h(x) = h^+(x) \) if \( x \in V^+ \) and \( h(x) = h^-(x) \) if \( x \in V^- \) is a homomorphism. Indeed if \( xy \) is an arc of \( D \) with \( x \in V^+ \) and \( y \in V^- \), conditions on \( h^+ \) and \( h^- \) imply that \( h(x) = h^+(x) \in \{ s_i^+ , s_j^+ , s_6^+ \} \) and \( h(y) = h^-(y) \in \{ s_1^- , s_3^- , s_5^- \} \). To conclude, Figure 1 provides a homomorphism \( g \) from \( S_2 \) to \( \overline{H}_4 \). The non-oriented arcs on the figure corresponds to circuits of length 2 and all the arcs from \( S_2^- \) to \( S_2^+ \) are not represented.

\[ \square \]

![Figure 1: The homomorphism \( g \) from \( S_2 \) to \( \overline{H}_4 \).](image)
5.3 Φ(3) and Φ^V(3, l), for l ≤ 3.

Theorem 36

\[ Φ(3) = Φ^V(3, 0) = Φ^V(3, 1) = 4 \]

Proof. 4 ≤ Φ(2) ≤ Φ(3) ≤ Φ^V(3, 0) ≤ Φ^V(3, 1) ≤ θ(6) = 4 by Corollary 21. □

In the remaining of this subsection, we shall prove Theorem 35, that is Φ^V(3, 2) = Φ^V(3, 3) = 5. Therefore, we first exhibit a (3 ⊕ 2)-digraph which is not 4-arc-colourable. Then we show that Φ^V(3, 3) ≤ 5.

Definition 37 Let \( G^- \) be the digraph obtained from the rotative tournament on five vertices \( R_5 \), with vertex set \( \{v_1^-, \ldots, v_5^-\} \) and arc-set \( \{v_i^- v_j^- : j \equiv i \mod 5 \in \{1, 2\}\} \) and five copies of the 3-circuits \( R_3^1, \ldots, R_3^5 \) by adding, for \( 1 \leq i \leq 5 \), the arcs \( vv_i^- \), for \( v \in R_5 \).

Let \( G^+ \) be the digraph obtained from the rotative tournament on seven vertices \( R_7 \), with vertex set \( \{v_1^+, \ldots, v_7^+\} \) and arc-set \( \{v_i^+ v_j^+ : j \equiv i \mod 7 \in \{1, 2, 3\}\} \) and seven copies of the rotative tournament of \( R_3^1, \ldots, R_3^5 \) by adding, for \( 1 \leq i \leq 7 \), the arcs \( vv_i^+ \), for \( v \in R_7 \).

Finally, let \( G \) be the \((3 \oplus 2)\)-digraph obtained from the disjoint union of \( G^- \) and \( G^+ \) by adding all the arcs of the form \( v^- v^+ \) with \( v^- \in V(G^-) \) and \( v^+ \in V(G^+) \). See Figure 2.

Proposition 38 The digraph \( G \) is not 4-arc-colourable. So \( Φ(3, 2) \geq 5 \).

Proof. Suppose for a contradiction that \( G \) admits an arc-colouring \( c \in \{1, 2, 3, 4\} \).

Let \( v^+ \) be a vertex of \( G^+ \) and \( v^- \) a vertex of \( G^- \). Then since \( v^- v^+ \) is an arc, \( Col^+(v^+) \neq Col^+(v^-) \). We will show:

1. there are two 2-subsets \( S \) of \( \{1, 2, 3, 4\} \) such that a vertex \( v^- \in G^- \) satisfies \( Col^+(v^-) = S \);

2. there are five 2-subsets \( S \) of \( \{1, 2, 3, 4\} \) such that a vertex \( v^+ \in G^+ \) satisfies \( Col^+(v^+) = S \).

This gives a contradiction since there are only six 2-subsets in \( \{1, 2, 3, 4\} \).

Let us first show (1). Every vertex of \( G^- \) satisfies \( col^+ \geq 2 \) otherwise all the arcs of \( R_7 \) in \( G^+ \) must be coloured with three colours, a contradiction to Theorem 12. Hence, since in \( R_5 \) all the \( Col^+ \) are distinct and not \( \{1, 2, 3, 4\} \), a vertex of \( R_5 \), say \( v_1^- \), has \( col^+ = 2 \), say \( Col^+(v_1^-) = \{1, 2\} \). Consider now the vertices of \( R_3^1 \). None of them has \( Col^+ = \{1, 2, 3\} \) or \( Col^+ = \{1, 2, 4\} \) since they are dominated by \( v_1^- \). Moreover they all have different \( Col^+ \) since \( R_3^1 \) is a tournament. Hence one of them, say \( v \), satisfies \( col^+(v) = 2 \). Now \( Col^+(v) \neq Col^+(v_1^-) \) since \( v_1^- \rightarrow v \).

Let us now prove (2). Let \( S = \{2\text{-subsets } S \text{ such that } \exists v^+ \in G^+, Col^+(v^+) = S\} \) and suppose that \( |S| \leq 4 \). Every vertex of \( G^+ \) has \( col^+ \leq 2 \) otherwise all the arcs of \( R_5 \) in \( G^- \) must be coloured with three colours, a contradiction to Proposition 32. Now, all the vertices
Figure 2: The non 4-arc-colourable $(3 \lor 2)$-digraph $G$. 
of $R_7$ have distinct and non-empty $\text{Col}^+$. So at least three vertices of $R_7$ have $\text{col}^+ = 2$ and $|S| \geq 3$. Thus, without loss of generality, we are in one of these three following cases:

(a) $S \subset \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}$ and $\text{Col}^+(v^+_1) = \{1, 2\}$;

(b) $S \subset \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 4\}$ and $\text{Col}^+(v^+_1) = \{2, 3\}$;

(c) $S \subset \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 4\}$ and $\text{Col}^+(v^+_1) = \{1, 4\}$.

Let $x_1, \ldots, x_5$ be the vertices of $R_5^1$ such that $x_ix_j$ is an arc if and only if $j = i + 1 \mod 5$ or $j = i + 2 \mod 5$ and $\mathcal{F} = \{\text{Col}^+(x_i) : 1 \leq i \leq 5\}$. Recall that $|\mathcal{F}| = 5$ since $R_5^1$ is a tournament and that every element $S$ of $\mathcal{F}$ is not included in $\text{Col}^+(v^+_1)$ since $x_i \rightarrow v^+_1$ for every $1 \leq i \leq 5$.

Case (a): We have $\mathcal{F} = \{\{3\}, \{4\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$. So we may assume that $\text{Col}^+(x_1) = \{3\}$. Now because $x_1 \rightarrow x_2$, $x_1 \rightarrow x_3$ and $x_2 \rightarrow x_3$, $\text{Col}^+(x_1) \not\subseteq \text{Col}^+(x_2)$, $\text{Col}^+(x_1) \not\subseteq \text{Col}^+(x_2)$, and $\text{Col}^+(x_2) \not\subseteq \text{Col}^+(x_3)$. It follows that $\text{Col}^+(x_2) = \{1, 4\}$ and $\text{Col}^+(x_3) = \{4\}$. Hence, none of $\text{Col}^+(x_4)$ and $\text{Col}^+(x_5)$ is $\{3, 4\}$ since $x_3 \rightarrow x_4$ and $x_3 \rightarrow x_5$, a contradiction.

Case (b): We have $\mathcal{F} = \{\{1\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}\}$. So we may assume that $\text{Col}^+(x_1) = \{1\}$. Since $x_1 \rightarrow x_2$, $\text{Col}^+(x_1) \not\subseteq \text{Col}^+(x_2)$, so $\text{Col}^+(x_2) = \{4\}$. Similarly, $\text{Col}^+(x_3) = \{4\}$ which is a contradiction.

Case (c): We have $\mathcal{F} = \{\{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$. So we may assume that $\text{Col}^+(x_1) = \{2\}$. It follows that $\text{Col}^+(x_2) = \{1, 3\}$ and $\text{Col}^+(x_3) = \{3\}$. Hence, none of $\text{Col}^+(x_4)$ and $\text{Col}^+(x_5)$ is $\{2, 3\}$ since $x_3 \rightarrow x_4$ and $x_3 \rightarrow x_5$, a contradiction. \hfill \Box

We will now prove that $\Phi^V(3, 3) \leq 5$. As in the proof of Theorem 35, in order to exhibit a homomorphism from a $(3 \lor 3)$-digraph $D$ to $\overline{H}_5$, we first show that there are two homomorphisms, $h^+$ from $D^+$ into a subdigraph $S_3^+$ of $\overline{H}_5$ and $h^-$ from $D^-$ into another subdigraph $S_3^-$ of $\overline{H}_5$, with specific properties.

**Definition 39** Let $S_3^+$ be the complete digraph with vertex-set $\{s^+_1, \ldots, s^+_5\}$. Let $S_3^-$ be the digraph with vertex-set $\{s^-_1, \ldots, s^-_5\}$ and arc-set $\{s^-_i s^-_j : i \neq j\} \backslash \{s^-_1 s^-_5, s^-_2 s^-_1\}$.

**Lemma 40** Let $D$ be a 3-digraph. There exists a homomorphism $h^+$ from $D$ to $S_3^+$ such that the vertices $x$ with $h^+(x) \in \{s^+_1, s^+_2\}$ have outdegree 3.

**Proof.** Let us prove it by induction on $n = |V(D)|$. If there exists a vertex $x$ with $d^+(x) + d^-(x) \leq 4$ then we obtain the desired homomorphism $h^+$ from $D - x$ to $S_3^+$ and extend it with a suitable choice of $h^+(x)$ in $\{s^+_1, \ldots, s^+_5\}$.

Assume now that $d^+(x) + d^-(x) \geq 5$ for every $x$. Let $n_i$ be the number of vertices with outdegree $i$. Clearly, $n = n_0 + n_1 + n_2 + n_3$. Moreover, we have:

$$3n \geq \sum_{x \in V} d^+(x) + \sum_{x \in V} d^-(x) = \sum_{d^+(x) = 0} d^+(x) + \sum_{d^+(x) = 1} d^-(x) + \sum_{d^+(x) = 2} d^+(x) + \sum_{d^+(x) = 3} d^-(x)$$
Then, by assumption:

\[ 3n \geq 5n_0 + 4n_1 + 3n_2 + \sum_{d^-(x) = 3} d^-(x) \]

If there is no vertex with outdegree 3, then \( D \) is 5-colourable and there is an homomorphism \( h^+ \) from \( D \) to \( S_3^+ \). Suppose now that there exists a vertex with outdegree 3. Then, there exists a vertex with outdegree 3 and indegree at most 3. If not, \( d^-(x) \geq 4 \) for every \( x \) with \( d^+(x) = 3 \) and the previous inequality implies \( 3n \geq 5n_0 + 4n_1 + 3n_2 + 4n_3 \) with \( n_3 \neq 0 \), what contradicts \( n = n_0 + n_1 + n_2 + n_3 \).

Finally, let \( x \) be a vertex with outdegree 3 and indegree at most 3. By induction hypothesis, there is a homomorphism \( h^+ \) from \( D - x \) to \( S_3^+ \) with the required property. As \( x \) has at most 6 neighbours, we extend \( h^+ \) with a suitable choice for \( h^+(x) \) in \( \{ s_1^+, \ldots, s_5^+ \} \).

**Lemma 41** Let \( D \) be a digraph with maximal indegree at most 3. There exists a homomorphism \( h^- \) from \( D \) to \( S_3^- \) such that the vertices \( x \) with \( h^-(x) \in \{ s_6^-, \ldots, s_9^- \} \) have indegree 3.

**Proof.** We prove the result by induction on \( |V(D)| \). If every vertex have indegree at most 2 then, by the dual form of the Lemma 17, there exists a homomorphism from \( D \) to \( S_3^- \). Now, let \( x \) be a vertex with indegree 3. Let \( y_1, y_2 \) and \( y_3 \) be the outneighbours of \( x \). By induction, there is a homomorphism \( h^- \) from \( D - x \) to \( S_3^- \) with the required property. In particular, as the vertex \( y_i \), \( 1 \leq i \leq 3 \), has indegree at most 2 in \( D - x \), we have \( h^-(y_i) \in \{ s_6^-, \ldots, s_9^- \} \). So, as \( x \) has 3 inneighbours, we can extend \( h^- \) with a suitable choice for \( h^-(x) \) in \( \{ s_6^-, \ldots, s_9^- \} \).

**Theorem 42**

\[ \Phi^V(3, 2) = \Phi^V(3, 3) = 5 \]

**Proof.** By Proposition 38, \( 5 \leq \Phi^V(3, 2) \leq \Phi^V(3, 3) \). We will prove that \( \Phi^V(3, 3) \leq 5 \). Let \( D \) be a \( (3 \lor 3) \)-digraph, we will provide a homomorphism from \( D \) to \( \overrightarrow{H}_5 \).

By Lemma 40, there is a homomorphism \( h^+ : D^+ \to S_3^+ \) such that if \( h^+(x) \in \{ s_1^+, s_2^+ \} \) then \( d_{D^+}^+(x) = 3 \). Moreover by Lemma 41, there is a homomorphism \( h^- : D^- \to S_3^- \) such that if \( h^-(x) \in \{ s_6^-, \ldots, s_9^- \} \), then \( d_{D^-}^+(x) = 3 \).

Let \( S_3 \) be the digraph obtained from the disjoint union of \( S_3^+ \) and \( S_3^- \) by adding the arcs of \( \{ s_i^-s_j^+ : 1 \leq i \leq 9, 1 \leq j \leq 7 \} \cup \{ s_i^+s_j^- : i = 1, \ldots, 5; j = 1, \ldots, 8 \} \). The mapping \( h : D \to S_3 \) defined by \( h(x) = h^+(x) \) if \( x \in V^+ \) and \( h(x) = h^-(x) \) if \( x \in V^- \) is a homomorphism. Indeed if \( xy \) is an arc of \( D \) with \( x \in V^+ \) and \( y \in V^- \), conditions on \( h^+ \) and \( h^- \) imply that \( h(x) = h^+(x) \in \{ s_i^+, \ldots, s_5^+ \} \) and \( h(y) = h^-(y) \in \{ s_6^-, \ldots, s_9^- \} \). To conclude, Figure 3 provides a homomorphism \( g \) from \( S_3 \) to \( \overrightarrow{H}_5 \). Inside \( S_3^- \) and \( S_3^+ \), only the arcs which are not in a circuit of length 2 are represented, every pair of not adjacent vertices are, in fact, linked by two arcs, one in each way.
Figure 3: The homomorphism $g$ from $S_3$ to $\overline{P}_5$. 
References


