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*Arc-chromatic number of digraphs in which each vertex has bounded outdegree or bounded indegree*

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Thème COM



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## Arc-chromatic number of digraphs in which each vertex has bounded outdegree or bounded indegree

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**Abstract:** A  $k$ -digraph is a digraph in which every vertex has outdegree at most  $k$ . A  $(k \vee l)$ -digraph is a digraph in which a vertex has either outdegree at most  $k$  or indegree at most  $l$ . Motivated by function theory, we study the maximum value  $\Phi(k)$  (resp.  $\Phi^\vee(k, l)$ ) of the arc-chromatic number over the  $k$ -digraphs (resp.  $(k \vee l)$ -digraphs). El-Sahili [3] showed that  $\Phi^\vee(k, k) \leq 2k + 1$ . After giving a simple proof of this result, we show some better bounds. We show  $\max\{\log(2k + 3), \theta(k + 1)\} \leq \Phi(k) \leq \theta(2k)$  and  $\max\{\log(2k + 2l + 4), \theta(k + 1), \theta(l + 1)\} \leq \Phi^\vee(k, l) \leq \theta(2k + 2l)$  where  $\theta$  is the function defined by  $\theta(k) = \min\{s : \binom{s}{\lceil \frac{s}{2} \rceil} \geq k\}$ . We then study in more details properties of  $\Phi$  and  $\Phi^\vee$ . Finally, we give the exact values of  $\Phi(k)$  and  $\Phi^\vee(k, l)$  for  $l \leq k \leq 3$ .

**Key-words:** arc-colouring, arc-chromatic number, function theory

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## Nombre arc-chromatique des digraphes dont les sommets ont degré externe ou degré interne borné

**Résumé :** Un  $k$ -digraphe est un digraphe dont les sommets ont degré externe au plus  $k$ . Un  $(k \vee l)$ -digraphe est un digraphe dont les sommets ont soit degré externe au plus  $k$  soit degré interne au plus  $l$ . Motivés par la théorie des fonctions, nous étudions la valeur maximale  $\Phi(k)$  (resp.  $\Phi^\vee(k, l)$ ) du nombre arc-chromatique d'un  $k$ -digraphe (resp.  $(k \vee l)$ -digraphe). El-Sahili [3] a montré  $\Phi^\vee(k, k) \leq 2k + 1$ . Après avoir donné une preuve simple de ce résultat, nous donnons de meilleure borne. Nous prouvons  $\max\{\log(2k + 3), \theta(k + 1)\} \leq \Phi(k) \leq \theta(2k)$  et  $\max\{\log(2k + 2l + 4), \theta(k + 1), \theta(l + 1)\} \leq \Phi^\vee(k, l) \leq \theta(2k + 2l)$  avec  $\theta$  la fonction définie par  $\theta(k) = \min\{s : \binom{s}{\lceil \frac{s}{2} \rceil} \geq k\}$ . Nous étudions ensuite plus en détails des propriétés de  $\Phi$  et  $\Phi^\vee$ . Enfin, nous donnons les valeurs exactes de  $\Phi(k)$  et  $\Phi^\vee(k, l)$  pour  $l \leq k \leq 3$ .

**Mots-clés :** arc-coloration, nombre arc-chromatique, théorie des fonctions

## 1 Introduction

A *directed graph* or *digraph*  $D$  is a pair  $(V(D), E(D))$  of disjoint sets (of *vertices* and *arcs*) together with two maps  $tail : E(D) \rightarrow V(D)$  and  $head : E(D) \rightarrow V(D)$  assigning to every arc  $e$  a *tail*,  $tail(e)$ , and a *head*,  $head(e)$ . The tail and the head of an arc are its *ends*. An arc with tail  $u$  and head  $v$  is denoted by  $uv$ ; we say that  $u$  *dominates*  $v$  and write  $u \rightarrow v$ . We also say that  $u$  and  $v$  are adjacent. The *order* of a digraph is its number of vertices. In this paper, all the digraph we consider are *loopless*, that is that every arc has its tail distinct from its head.

Let  $D$  be a digraph. The *line-digraph* of  $D$  is the digraph  $L(D)$  such that  $V(L(D)) = E(D)$  and an arc  $a \in E(D)$  dominates an arc  $b \in E(D)$  in  $L(D)$  if and only if  $head(a) = tail(b)$ .

A *vertex-colouring* or *colouring* of  $D$  is an application  $c$  from the vertex-set  $V(D)$  into a set of colours  $S$  such that for any arc  $uv$ ,  $c(u) \neq c(v)$ . The *chromatic number* of  $D$ , denoted  $\chi(D)$ , is the minimum number of colours of a colouring of  $D$ .

An *arc-colouring* of  $D$  is an application  $c$  from the arc-set  $E(D)$  into a set of colours  $S$  such that if the tail of an arc  $e$  is the head of an arc  $e'$  then  $c(e) \neq c(e')$ . Trivially, there is a one-to-one correspondence between arc-colourings of  $D$  and colourings of  $L(D)$ . The *arc-chromatic number* of  $D$ , denoted  $\chi_a(D)$ , is the minimum number of colours of an arc-colouring of  $D$ . Clearly  $\chi_a(D) = \chi(L(D))$ .

A *k-digraph* is a digraph in which every vertex has outdegree at most  $k$ . A *(k ∨ l)-digraph* is a digraph in which a vertex has either outdegree at most  $k$  or indegree at most  $l$ .

For any digraph  $D$  and set of vertices  $V' \subset V(D)$ , we denote by  $D[V']$ , the subdigraph of  $D$  induced by the vertices of  $V'$ . For any subdigraph  $F$  of  $D$ , we denote by  $D - F$  the digraph  $(V(D) \setminus V(F), E(D) \setminus E(F))$ . For any arc-set  $E' \subset E$ , we denote by  $D - E'$  the digraph  $(V(D), E(D) \setminus E')$  and for any vertex  $x \in V(D)$ , we denote by  $D - x$  the digraph induced by  $V(D) \setminus \{x\}$ .

Let  $D$  be a  $(k \vee l)$ -digraph. We denote by  $V^+(D)$ , or  $V^+$  if  $D$  is clearly understood, the subset of the vertices of  $D$  with outdegree at most  $k$ , and by  $V^-(D)$  or  $V^-$  the complement of  $V^+(D)$  in  $V(D)$ . Also  $D^+$  (resp.  $D^-$ ) denotes  $D[V^+]$  (resp.  $D[V^-]$ ).

In this paper, we study the arc-chromatic number of  $k$ -digraphs and  $(k \vee l)$ -digraphs. This is motivated by the following interpretation in function theory as shown by El-Sahili in [3].

Let  $f$  and  $g$  be two maps from a finite set  $A$  into a set  $B$ . Suppose that  $f$  and  $g$  are *nowhere coinciding*, that is for all  $a \in A$ ,  $f(a) \neq g(a)$ . A subset  $A'$  of  $A$  is *(f, g)-independant* if  $f(A') \cap g(A') = \emptyset$ . We are interested by finding the largest  $(f, g)$ -independant subset of  $A$  and the minimum number of  $(f, g)$ -independant subsets to partition  $A$ . As shown by El-Sahili [3], this can be translated into an arc-colouring problem.

Let  $D_{f,g}$  and  $H_{f,g}$  be the digraphs defined as follows :

- $V(D_{f,g}) = B$  and  $(b, b') \in E(D_{f,g})$  if there exists an element  $a$  in  $A$  such that  $g(a) = b$  and  $f(a) = b'$ . Note that if for all  $a$ ,  $f(a) \neq g(a)$ , then  $D_{f,g}$  has no loop.

- $V(H_{f,g}) = A$  and  $(a, a') \in E(H_{f,g})$  if  $f(a) = g(a')$ .

We associate to each arc  $(b, b')$  in  $D_{f,g}$  the vertex  $a$  of  $A$  such that  $g(a) = b$  and  $f(a) = b'$ . Then  $(a, a')$  is an arc in  $H_{f,g}$  if, and only if,  $head(a) = tail(a')$  (as arcs in  $D_{f,g}$ ). Thus  $H_{f,g} = L(D_{f,g})$ . Note that for every digraph  $D$ , there exists maps  $f$  and  $g$  such that  $D = D_{f,g}$ .

It is easy to see that an  $(f, g)$ -independant subset of  $A$  is an independant set in  $H_{f,g}$ . In [2] El-Sahili proved the following :

**Theorem 1 (El-Sahili [2])** *Let  $f$  and  $g$  be two nowhere coinciding maps from a finite set  $A$  into a set  $B$ . Then there exists an  $(f, g)$ -independant subset  $A'$  of cardinality at least  $|A|/4$ .*

Let  $f$  and  $g$  be two nowhere coinciding maps from a finite set  $A$  into  $B$ . We define  $\phi(f, g)$  as the minimum number of  $(f, g)$ -independant sets to partition  $A$ . Then  $\phi(f, g) = \chi(H_{f,g}) = \chi_a(D_{f,g})$ .

Let  $\Phi(k)$  (resp.  $\Phi^\vee(k, l)$ ) be the maximum value of  $\phi(f, g)$  for two nowhere coinciding maps  $f$  and  $g$  from  $A$  into  $B$  such that for every  $z$  in  $B$ ,  $g^{-1}(z) \leq k$  (resp. either  $g^{-1}(z) \leq k$  or  $f^{-1}(z) \leq l$ ). The condition  $f^{-1}(z)$  (resp.  $g^{-1}(z)$ ) has at most  $k$  elements means that each vertex has indegree (resp. outdegree) at most  $k$  in  $D_{f,g}$ . Hence  $\Phi(k)$  (resp.  $\Phi^\vee(k, l)$ ) is the maximum value of  $\chi_a(D)$  for  $D$  a  $k$ -digraph (resp.  $(k \vee l)$ -digraph).

**Remark 2** Let  $f$  and  $g$  be two nowhere coinciding maps from  $A$  into  $B$ . Then  $A$  may be partitionned into  $\Phi(|A| - 1)$   $(f, g)$ -independant sets.

The functions  $\Phi^\vee$  and  $\Phi$  are very close to each other:

**Proposition 3**

$$\Phi(k) \leq \Phi^\vee(k, 0) \leq \dots \leq \Phi^\vee(k, k) \leq \Phi(k) + 2$$

**Proof.** The sole inequality that does not immediatly follow the definitions is  $\Phi^\vee(k, k) \leq \Phi(k) + 2$ . Let us prove it.

Let  $D$  be a  $(k \vee k)$ -digraph. One can colour the arcs in  $D^+ \cup D^-$  with  $\Phi(k)$  colours. It remains to colour the arcs with tail in  $V^-$  and head in  $V^+$  with one new colour and the arcs with tail in  $V^+$  and head in  $V^-$  with a second new colour.  $\square$

Moreover, we conjecture that  $\Phi^\vee(k, k)$  is never equal to  $\Phi(k) + 2$ .

**Conjecture 4**

$$\Phi^\vee(k, k) \leq \Phi(k) + 1$$

In [3], El-Sahili gave the following upper bound on  $\Phi^\vee(k, k)$ :

**Theorem 5 (El-Sahili [3])**  $\Phi^\vee(k, k) \leq 2k + 1$

In this paper, we first give simple proofs of Theorems 1 and 5. Then, in Section 3, we improve the upper bounds on  $\Phi(k)$  and  $\Phi^\vee(k, l)$ . We show (Theorem 18) that  $\Phi(k) \leq \theta(2k)$  if  $k \geq 2$ , and  $\Phi^\vee(k, l) \leq \theta(2k + 2l)$  if  $k + l \geq 3$ , where  $\theta$  is the function defined by  $\theta(k) = \min\{s : \binom{s}{\lfloor \frac{s}{2} \rfloor} \geq k\}$ . We also establish (Corollary 21) that  $\Phi^\vee(k, l) \leq \theta(2k)$  if  $\theta(2k) \geq 2l + 1$ .

In Section 4, we study in more details the relations between  $\Phi^\vee(k, l)$  and  $\Phi(k)$ . We conjecture that if  $k$  is very large compared to  $l$  then  $\Phi^\vee(k, l) = \Phi(k)$ . We prove that  $\Phi^\vee(k, 0) = \Phi(k)$  and conjecture that  $\Phi^\vee(k, 1) = \Phi(k)$  if  $k \geq 1$ . We prove that for a fixed  $k$  either this latter conjecture holds or Conjecture 4 holds. This implies that  $\Phi^\vee(k, 1) \leq \Phi(k) + 1$ .

Finally, in Section 5, we give the exact values of  $\Phi(k)$  and  $\Phi^\vee(k, l)$  for  $l \leq k \leq 3$ . They are summarized in the following table :

|                       |                                 |                                 |                                 |
|-----------------------|---------------------------------|---------------------------------|---------------------------------|
| $\Phi^\vee(0, 0) = 1$ | $\Phi^\vee(1, 0) = \Phi(1) = 3$ | $\Phi^\vee(2, 0) = \Phi(2) = 4$ | $\Phi^\vee(3, 0) = \Phi(3) = 4$ |
|                       | $\Phi^\vee(1, 1) = 3$           | $\Phi^\vee(2, 1) = 4$           | $\Phi^\vee(3, 1) = 4$           |
|                       |                                 | $\Phi^\vee(2, 2) = 4$           | $\Phi^\vee(3, 2) = 5$           |
|                       |                                 |                                 | $\Phi^\vee(3, 3) = 5$           |

## 2 Simple proofs of Theorems 1 and 5

**Proof of Theorem 1.** Let  $D = D_{f,g}$ . Let  $(V_1, V_2)$  be a partition of  $V(D)$  that maximizes the number of arcs  $a$  with one end in  $V_1$  and one end in  $V_2$ . It is well-known that  $a \geq |E(D)|/2$ . Now let  $A_1$  be the set of arcs with head in  $V_1$  and tail in  $V_2$  and  $A_2$  be the set of arcs with head in  $V_2$  and tail in  $V_1$ . Then  $A_1$  and  $A_2$  corresponds to independent sets of  $L(D)$  and  $|A_1| + |A_2| = a$ . Hence one of the  $A_i$  has cardinality at least  $a/2 = \frac{|E(D)|}{4}$ .  $\square$

Before giving a short proof of Theorem 5, we precise few standard definitions.

**Definition 6** A *path* is a non-empty digraph  $P$  of the form

$$V(P) = \{v_0, v_1, \dots, v_k\} \quad E(P) = \{v_0v_1, v_1v_2, \dots, v_{k-1}v_k\},$$

where the  $v_i$  are all distinct. The vertices  $v_0$  and  $v_k$  are respectively called the *origin* and *terminus* of  $P$ .

A *circuit* is a non-empty digraph  $C$  of the form

$$V(C) = \{v_0, v_1, \dots, v_k\} \quad E(C) = \{v_0v_1, v_1v_2, \dots, v_{k-1}v_k, v_kv_0\},$$

where the  $v_i$  are all distinct.

A digraph is *strongly connected* or *strong* if for every two vertices  $u$  and  $v$  there is a path with origin  $u$  and terminus  $v$ . A maximal strong subdigraph of a digraph  $D$  is called a *strong component* of  $D$ . A component  $I$  of  $D$  is *initial* if there is no arc with tail in  $V(D) \setminus V(I)$



and head in  $V(I)$ . A component  $I$  of  $D$  is *terminal* if there is no arc with tail in  $V(I)$  and head in  $V(D) \setminus V(I)$ . A digraph is *connected* if its underlying graph is connected.

A digraph  $D$  is  *$l$ -degenerate* if every subdigraph  $H$  has a vertex of degree at most  $l$ .

The following lemma corresponding to the greedy colouring algorithm is a piece of folklore.

**Lemma 7** *Every  $l$ -degenerate digraph is  $(l + 1)$ -colourable.*

**Proof of Theorem 5.** Let  $D$  be a  $(k \vee k)$ -digraph. According to Lemma 7, it suffices to prove that  $L(D)$  is  $2k$ -degenerate.

In every initial strong component  $C$  there is a vertex with indegree at most  $k$ . Indeed if there is no such vertex then  $(k + 1)|C| \leq \sum_{v \in C} d^-(v) \leq \sum_{v \in C} d^+(v) \leq k|C|$ . Analogously, in every terminal strong component there is a vertex with outdegree at most  $k$ .

Now, there is a path originating in a minimal component and terminating in a terminal one. Hence there is a path whose origin has indegree at most  $k$  and whose terminus has outdegree at most  $k$ . Hence there is an arc  $e$  whose tail has indegree at most  $k$  and whose head has outdegree at most  $k$ . Thus  $e$  has degree at most  $2k$  in  $L(D)$ .  $\square$

### 3 Lower and upper bounds for $\Phi$ and $\Phi^\vee$

We will now search for bounds on  $\Phi$  since they also give bounds on  $\Phi^\vee$ .

Theorem 1 and an easy induction yields  $\chi_a(D) \leq \log_{4/3} |D|$ . However there exists better upper bounds provided by Poljak and Rödl [5]. For sake of completeness and in order to introduce useful tools, we provide a proof of Theorem 11.

**Definition 8** We denote by  $\overline{H}_k$  the complementary of the hypercube of dimension  $k$ , that is the digraph with vertex-set all the subsets of  $\{1, \dots, k\}$  and with arc-set  $\{xy : x \not\subseteq y\}$ .

A *homomorphism*  $h : D \rightarrow D'$  is a mapping  $h : V(D) \rightarrow V(D')$  such that for every arc  $xy$  of  $D$ ,  $h(x)h(y)$  is an arc of  $D'$ .

Let  $c$  be an arc-colouring of a digraph  $D$  into a set of colours  $S$ . For any vertex  $x$  of  $D$ , we denote by  $Col_c^+(x)$  or simply  $Col^+(x)$  the set of colours assigned to the arcs with tail  $x$ . We define  $Col^-(x) = S \setminus Col^+(x)$ . Note that  $Col^-(x)$  contains (but may be bigger than) the set of colours assigned to the arcs with head  $x$ . The cardinality of  $Col^+(x)$  (resp.  $Col^-(x)$ ) is denoted by  $col^+(x)$  (resp.  $col^-(x)$ ).

**Theorem 9** *For every digraph  $D$ ,  $\chi_a(D) = \min\{k : D \rightarrow \overline{H}_k\}$ .*

**Proof.** Assume that  $D$  admits an arc-colouring with  $\{1, \dots, k\}$ . It is easy to check that  $Col^+$  is a homomorphism from  $D$  to  $\overline{H}_k$ .

Conversely, suppose that there exists a homomorphism  $h$  from  $D$  to  $\overline{H}_k$ . Assign to each arc  $xy$  an element of  $h(y) \setminus h(x)$ , which is not empty. This provides an arc-colouring of  $D$ .  $\square$

**Definition 10** The *complete digraph of order  $n$* , denoted  $\vec{K}_n$ , is the digraph with vertex-set  $\{v_1, v_2, \dots, v_n\}$  and arc-set  $\{v_i v_j : i \neq j\}$ .

The *transitive tournament of order  $n$* , denoted  $TT_n$ , is the digraph with vertex-set  $\{v_1, v_2, \dots, v_n\}$  and arc-set  $\{v_i v_j : i < j\}$ .

The following corollary of Theorem 9 provides bounds on the arc-chromatic number of a digraph according to its chromatic number.

**Theorem 11 (Poljak and Rödl [5])** For every digraph  $D$ ,

$$\lceil \log(\chi(D)) \rceil \leq \chi_a(D) \leq \theta(\chi(D)).$$

**Proof.** By definition of the chromatic number,  $D \rightarrow \vec{K}_{\chi(D)}$ . As the subsets of  $\{1, \dots, k\}$  with cardinality  $\lceil \frac{k}{2} \rceil$  induce a complete digraph on  $\binom{k}{\lceil \frac{k}{2} \rceil}$  vertices in  $\vec{H}_k$ , we obtain a homomorphism from  $D$  to  $\vec{H}_{\theta(\chi(D))}$ . So  $\chi_a(D) \leq \theta(\chi(D))$ .

By Theorem 9, we have  $D \rightarrow \vec{H}_{\chi_a(D)}$ . As  $\chi(\vec{H}_{\chi_a(D)}) = 2^{\chi_a(D)}$ , we obtain  $D \rightarrow \vec{K}_{2^{\chi_a(D)}}$ .  $\square$

These bounds are tight since the lower one is achieved by transitive tournaments and the upper one by complete digraphs by Sperner's Lemma (see [6]). However, the lower bound may be increased if the digraph has no sink (vertex with outdegree 0) or/and no source (vertex with indegree 0).

**Theorem 12** Let  $D$  be a digraph.

(i) If  $D$  has no sink then  $\log(\chi(D) + 1) \leq \chi_a(D)$ .

(ii) If  $D$  has no source and no sink then  $\log(\chi(D) + 2) \leq \chi_a(D)$ .

**Proof.** The proof is identical to the proof of Theorem 11. But if a digraph has no source (resp. no sink) then for every  $v$ ,  $Col^+(v) \neq S$  (resp.  $Col^+(v) \neq \emptyset$ ).  $\square$

Again these two lower bounds are also tight. Let  $Q_n$  (resp.  $W_n$ ) be the tournament of order  $n$  obtained from  $TT_n$  by reversing the arc  $v_1 v_n$  (resp.  $v_2 v_n$ ). One can easily check that  $\chi_a(W_n) = \lceil \log(n + 1) \rceil = \lceil \log(\chi(W_n) + 1) \rceil$  and  $\chi_a(Q_n) = \lceil \log(n + 2) \rceil = \lceil \log(\chi(Q_n) + 2) \rceil$ .

**Proposition 13** Every  $k$ -digraph is  $2k$ -degenerate.

**Proof.** Every subdigraph of a  $k$ -digraph is also a  $k$ -digraph. Hence it suffices to prove that every  $k$ -digraph has a vertex with degree at most  $2k$ . Since the sum of outdegrees equals the sum of indegrees, there is a vertex with indegree at most  $k$  and thus with degree at most  $2k$ .  $\square$

**Corollary 14**

$$\max\{\log(2k+3), \theta(k+1)\} \leq \Phi(k) \leq \theta(2k+1)$$

**Proof.** The upper bound follows from Theorem 11, Proposition 13 and Lemma 7. The lower bound comes from a regular tournament on  $2k+1$  vertices  $T_{2k+1}$  and the complete digraph on  $k+1$  vertices  $\vec{K}_{k+1}$ . Indeed  $\chi(T_{2k+1}) = 2k+1$ , so  $\chi_a(T_{2k+1}) \geq \log(2k+3)$  by Theorem 12 and  $\chi_a(\vec{K}_{k+1}) = \theta(k+1)$ .  $\square$

**Corollary 15**

$$\max\{\log(2k+2l+4), \theta(k+1), \theta(l+1)\} \leq \Phi^\vee(k, l) \leq \theta(2k+2l+2)$$

**Proof.** The upper bound follows Theorem 11 and Proposition degen since every  $(k \vee l)$ -digraph  $D$  is  $2k+2l+2$ -colourable ( $D^+$  is  $2k$ -degenerate and so  $(2k+1)$ -colourable and  $D^-$  is  $2l$ -degenerate and so  $(2l+1)$ -colourable). The lower bound comes from  $\vec{K}_{k+1}$ ,  $\vec{K}_{l+1}$  and a tournament  $T$  composed of a regular tournament on  $2l+1$  vertices dominating a regular tournament on  $2k+1$  vertices. Indeed,  $\chi_a(\vec{K}_{k+1}) = \theta(k+1)$ ,  $\chi_a(\vec{K}_{l+1}) = \theta(l+1)$  and  $T$  has no source, no sink and chromatic number  $2k+2l+2$ , by Theorem 12,  $\chi_a(T) \geq \log(2k+2l+4)$ .  $\square$

We can obtain a slightly better upper bound on  $\Phi$ . Bounds on  $\Phi^\vee$  follow.

**Definition 16** For any integer  $k \geq 1$ , let  $T_k^+$  ( $k \geq 1$ ) be the complete digraph on  $t_1^+, \dots, t_{2k+1}^+$  minus the arcs  $\{t_1^+ t_2^+, t_1^+ t_3^+, \dots, t_1^+ t_{k+1}^+\}$ .

**Lemma 17** Let  $k \geq 1$  be an integer. If  $D$  is a  $k$ -digraph, then there exists a homomorphism  $h^+$  from  $D$  to  $T_k^+$  such that if  $h^+(x) = t_1^+$  then  $d^+(x) = k$ .

**Proof.** For a fixed  $k$ , we prove it by induction on  $|V(D)|$ .

First, suppose that there exists in  $D$  a vertex  $x$  with  $d^-(x) < k$ . Then,  $d^+(x) + d^-(x) < 2k$ . By induction on  $D - x$ , there is a homomorphism  $h$  from  $D - x$  to  $T_k^+$  such that if  $h^+(v) = t_1^+$  then  $d_{D-x}^+(v) = k$ . Hence,  $h^+(y) \neq t_1^+$  for every inneighbour  $y$  of  $x$ , because  $d_{D-x}^+(y) < k$ . As  $x$  has at most  $2k-1$  neighbours, we find  $i \in \{2, \dots, 2k+1\}$  such that no neighbour  $y$  of  $x$  satisfies  $h^+(y) = t_i^+$ . So,  $h^+(x) = t_i^+$  extends  $h^+$  to a homomorphism from  $D$  to  $T_k^+$ .

Suppose now that every vertex  $v$  of  $D$  has indegree at least  $k$ . Since  $\sum_{v \in V(D)} d^-(v) = \sum_{v \in V(D)} d^+(v) \leq k|V(D)|$ , every vertex has indegree and outdegree  $k$ . Hence, by Brooks Theorem (see [1]) either  $D$  is  $2k$ -colourable and  $D \rightarrow T_k^+[\{t_2^+, \dots, t_{2k+1}^+\}]$ , or  $D$  is a regular tournament on  $2k+1$  vertices. In this latter case, label the vertices of  $D$  with  $v_1, v_2, \dots, v_{2k+1}$  such that  $N^-(v_1) = \{v_2, \dots, v_{k+1}\}$ . Then  $h^+$  defined by  $h^+(v_i) = t_i^+$  is the desired homomorphism.  $\square$

**Theorem 18** *Let  $k$  and  $l$  be two positive integers.*

(i) *If  $k \geq 2$ , then  $\Phi(k) \leq \theta(2k)$ .*

(ii) *If  $k + l \geq 3$ , then  $\Phi^\vee(k, l) \leq \theta(2k + 2l)$ .*

**Proof.** (i) If  $k = 2$ , the result follows Corollary 14 since  $\theta(4) = \theta(5) = 4$ . Suppose now that  $k \geq 3$ . Let  $D$  be a  $k$ -digraph. By Lemma 17 there is a homomorphism from  $D$  to  $T_k^+$ . We will provide a homomorphism  $g$  from  $T_k^+$  to  $\overline{H}_{\theta(2k)}$ . Fix  $S_1, \dots, S_{2k}$ ,  $2k$  subsets of  $\{1, \dots, \theta(2k)\}$  with cardinality  $\lfloor \theta(2k)/2 \rfloor$  and  $S$  a subset of  $\{1, \dots, 2k\}$  with cardinality  $\lfloor \theta(2k)/2 \rfloor - 1$ . Without loss of generality, the  $S_i$  containing  $S$  are  $S_1, \dots, S_l$  with  $l \leq \theta(2k)/2 + 1 \leq k$ . (One can easily check that  $\theta(2k)/2 + 1 \leq k$  provided that  $k \geq 3$ .) Now, set  $g(t_1^+) = S$  and  $g(t_{i+1}^+) = S_i$  for  $1 \leq i \leq 2k$ . It is straightforward to check that  $g$  is a homomorphism.

(ii) Let  $D$  be a  $(k \vee l)$ -digraph. By Lemma 17, there exists a homomorphism  $h^+$  from  $D^+$  to  $T_k^+$  such that if  $h^+(x) = t_1^+$  then  $d^+(x) = k$  and, by symmetry, a homomorphism  $h^-$  from  $D^-$  to  $T_l^-$ , the complete digraph on  $\{t_1^-, \dots, t_{2l+1}^-\}$  minus the arcs  $\{t_1^- t_2^-, t_1^- t_3^-, \dots, t_1^- t_{l+1}^-\}$ , such that if  $h^-(x) = t_1^-$  then  $d^-(x) = l$ . We now provide a homomorphism from  $D$  to  $\overline{H}_{\theta(2k+2l)}$ .

Fix  $S^+$  and  $S^-$ , two subsets of  $\{1, \dots, \theta(2k+2l)\}$  with cardinality  $\lfloor \theta(2k+2l)/2 \rfloor - 1$  for  $S^+$  and  $\lfloor \theta(2k+2l)/2 \rfloor + 1$  for  $S^-$  such that  $S^+ \not\subseteq S^-$ . (This is possible since  $\theta(2k+2l) \geq 4$ , because  $k+l \geq 2$ .) Set  $\mathcal{N} = \{U \subseteq \{1, \dots, \theta(2k+2l)\} : |U| = \lfloor \theta(2k+2l)/2 \rfloor\}$ . We want a partition of  $\mathcal{N}$  into two parts  $\mathcal{A}$  and  $\mathcal{B}$  with  $|\mathcal{A}| \geq 2k$  and  $|\mathcal{B}| \geq 2l$ , such that  $S^+$  is included in at most  $k$  sets of  $\mathcal{A}$  and  $S^-$  contains at most  $l$  sets of  $\mathcal{B}$ . Let  $\mathcal{N}_{S^+}$  (resp.  $\mathcal{N}_{S^-}$ ) be the set of elements of  $\mathcal{N}$  containing  $S^+$  (resp. contained in  $S^-$ ). We have  $|\mathcal{N}_{S^+}| = \lceil \theta(2k+2l)/2 \rceil + 1$  and  $|\mathcal{N}_{S^-}| = \lfloor \theta(2k+2l)/2 \rfloor + 1$ ; because  $k+l \geq 3$ , it follows  $|\mathcal{N}_{S^+}| \leq k+l$  and  $|\mathcal{N}_{S^-}| \leq k+l$ . Moreover, the sets  $\mathcal{N}_{S^+}$  and  $\mathcal{N}_{S^-}$  are disjoint and  $|\mathcal{N}_{S^-}| \leq k+l$ . Let us sort the elements of  $\mathcal{N}$  beginning with those of  $\mathcal{N}_{S^-}$  and ending with those of  $\mathcal{N}_{S^+}$ . Let  $\mathcal{A}$  be the  $2k$  first sets in this sorting and  $\mathcal{B}$  what remains ( $|\mathcal{B}| \geq 2l$ ). We claim that  $\mathcal{A}$  contains at most  $k$  elements of  $\mathcal{N}_{S^+}$ . If not, then  $|\mathcal{A}| > \binom{\theta(2k+2l)}{\lfloor \theta(2k+2l)/2 \rfloor} - |\mathcal{N}_{S^+}| + k$ . We obtain  $2k > 2k + 2l - |\mathcal{N}_{S^+}| + k$  which contradicts  $|\mathcal{N}_{S^+}| \leq k+l$ . With same argument,  $\mathcal{B}$  contains at most  $l$  elements of  $\mathcal{N}_{S^-}$ .

Finally, set  $\mathcal{A} = \{A_1, \dots, A_{2k}\}$  such that none of  $A_{k+1}, \dots, A_{2k}$  contains  $S^+$  and  $\{B_1, \dots, B_{2l}\}$   $2l$  sets of  $\mathcal{B}$  such that none of  $B_{l+1}, \dots, B_{2l}$  is contained in  $S^-$ .

Let us define  $h : D \rightarrow \overline{H}_{2k+2l}$ . If  $x \in V^+$  and  $h^+(x) = t_i^+$  then  $h(x) = S^+$  if  $i = 1$  and  $h(x) = A_{i-1}$  otherwise. If  $x \in V^-$  and  $h^-(x) = t_i^-$  then  $h(x) = S^-$  if  $i = 1$  and  $h(x) = B_{i-1}$  otherwise. Let us check that  $h$  is a homomorphism. Let  $xy$  be an arc of  $D$ .  $T_k^+$  is a subdigraph of  $\overline{H}_{2k+2l}[\{A_1, \dots, A_{2k}, S^+\}]$  and  $T_l^-$  is a subdigraph of  $\overline{H}_{2k+2l}[\{B_1, \dots, B_{2l}, S^-\}]$ . So,  $h(x)h(y)$  is an arc of  $\overline{H}_{2k+2l}$  if  $x$  and  $y$  are both in  $V^+$  or both in  $V^-$ . Suppose now that  $x \in V^+$  and  $y \in V^-$ , then  $h^+(x) \neq t_1^+$  because  $d_{D^+}(x) < k$  and  $h(x) \neq S^+$ . Similarly,  $h^-(y) \neq S^-$ . Thus  $h(x)$  and  $h(y)$  are elements of  $\mathcal{N}$ , so  $h(x)h(y) \in E(\overline{H}_{2k+2l})$ . Finally, suppose that  $x \in V^-$  and  $y \in V^+$ . Then  $h(x)h(y) \in E(\overline{H}_{2k+2l})$  because no element of  $\{B_1, \dots, B_{2l}, S^-\}$  is a subset of an element of  $\{A_1, \dots, A_{2k}, S^+\}$ .  $\square$

**Remark 19** Note that the homomorphism provided in (i) has for image subsets of  $\{1, \dots, \theta(2k)\}$  with cardinality at most  $\lfloor \theta(2k)/2 \rfloor$ . So, using the method developed in Theorem 9, we provide an arc-colouring of a  $k$ -digraph  $D$  with  $\theta(2k)$  colours which satisfies  $col^+(x) \leq \lfloor \theta(2k)/2 \rfloor$ , so  $col^-(x) \geq \lceil \theta(2k)/2 \rceil$ , for every vertex  $x$  of  $D$ .

We will now improve the bound (ii) of Theorem 18 when  $l$  is very small compared to  $k$ .

**Lemma 20** *Let  $D$  be a  $(k \vee l)$ -digraph and  $D^1$  the subdigraph of  $D$  induced by the arcs with tail in  $V^+$ . If there exists an arc-colouring of  $D^1$  with  $m \geq 2l + 1$  colours such that for every  $x$  in  $V^+$ ,  $col^-(x) \geq l + 1$  then  $\chi_\alpha(D) = m$ .*

**Proof.** We will extend the colouring as stated into an arc-colouring of  $D$ .

Let us now first extend this colouring to the arcs of  $D^-$ . Since  $\sum_{v \in V^-} d_{D^-}^+(v) = \sum_{v \in V^-} d_{D^-}^-(v) \leq l|V^-|$ , there is a vertex  $v_1 \in V^-$  such that  $d_{D^-}^+(v_1) \leq l$ . And so on, by induction, there is an ordering  $(v_1, v_2, \dots, v_p)$  of the vertices of  $D^-$  such that, for every  $1 \leq i \leq p$ ,  $v_i$  dominates at most  $l$  vertices in  $\{v_j, j > i\}$ . Let us colour the arcs of  $D^-$  in decreasing order of their head; that is first colour the arcs with head  $v_p$  then those with head  $v_{p-1}$ , and so on. This is possible since at each stage  $i$ , an arc  $uv_i$  has at most  $2l < m$  forbidden colours ( $l$  ingoing  $u$  and  $l$  outgoing  $v_i$  to a vertex in  $\{v_j : j > i\}$ ).

It remains to colour the arcs with tail in  $V^-$  and head in  $V^+$ . Let  $v^-v^+$  be such an arc. Since  $col^-(v^+) \geq l + 1$  and  $d^-(v^-) \leq l$ , there is a colour  $\alpha$  in  $Col^-(v^+)$  that is assigned to no arc ingoing  $v^-$ . Hence, assigning  $\alpha$  to  $v^-v^+$ , we extend the arc-colouring to  $v^-v^+$ .  $\square$

**Corollary 21** *If  $\theta(2k) \geq 2l + 1$ , then  $\phi^\vee(k, l) \leq \theta(2k)$ .*

**Proof.** The digraph  $D^1$ , as defined in Lemma 20, is a  $k$ -digraph. The result follows directly from Remark 19 and Lemma 20.  $\square$

## 4 Relations between $\Phi(k)$ and $\Phi^\vee(k, l)$

**Conjecture 22** *Let  $l$  be a positive integer. There exists an integer  $k_l$  such that if  $k \geq k_l$  then  $\Phi^\vee(k, l) = \Phi(k)$ .*

We now prove Conjecture 22, for  $l = 0$ , showing that  $k_0 = 1$ .

**Theorem 23** *If  $k \geq 1$ ,*

$$\Phi^\vee(k, 0) = \Phi(k).$$

**Proof.** Let  $D = (V, A)$  be a  $(k \vee 0)$  digraph. Let  $V_0$  be the set of vertices with indegree 0. Let  $D'$  be the digraph obtained from  $D$  by splitting each vertex  $v$  of  $V_0$  into  $d^+(v)$  vertices with outdegree 1. Formally, for each vertex  $v \in V_0$  incident to the arcs  $vw_1, \dots, vw_{d^+(v)}$ ,

replace  $v$  by  $\{v_1, v_2, \dots, v_{d^+(v)}\}$  and  $vw_i$  by  $v_iw_i$ ,  $1 \leq i \leq d^+(v)$ . By construction,  $D'$  is a  $k$ -digraph and  $L(D) = L(D')$ . So  $\chi_a(D) = \chi_a(D') \leq \Phi(k)$ .  $\square$

We conjecture that if  $l = 1$ , Conjecture 22 holds with  $k_1 = 1$ .

**Conjecture 24** *If  $k \geq 1$ ,*

$$\Phi^\vee(k, 1) = \Phi(k)$$

**Theorem 25** *If  $\Phi(k) = \Phi(k - 1)$  or  $\Phi(k) = \Phi(k + 1)$  then  $\Phi^\vee(k, 1) = \Phi(k)$ .*

**Proof.** By Lemma 20, it suffices to prove that if  $\Phi(k) = \Phi(k - 1)$  or  $\Phi(k) = \Phi(k + 1)$  every  $k$ -digraph admits an arc-colouring with  $\Phi(k)$  colours such that for every vertex  $x$ ,  $col^-(x) \geq 2$ .

Suppose that  $\Phi(k) = \Phi(k - 1)$ . Let  $D$  be a  $k$ -digraph and  $D'$  be a  $(k - 1)$ -digraph such that  $\chi_a(D') = \Phi(k)$ . Let  $C$  be the digraph constructed as follows: for every vertex  $x \in V(D)$  add a copy  $D'(x)$  of  $D'$  such that every vertex of  $D'(x)$  dominates  $x$ . Then  $C$  is a  $k$ -digraph, so it admits an arc-colouring  $c$  with  $\Phi(k)$  colours. Note that  $c$  is also an arc-colouring of  $D$  which is a subdigraph of  $C$ . Let us prove that for every vertex  $x \in V(D)$ ,  $col^-(x) \geq 2$ .

Suppose, reductio ad absurdum, that there is a vertex  $x \in V(D)$  such that  $col^-(x) \leq 1$ . Since there are arcs ingoing  $x$  in  $C$  (those from  $V(D'(x))$ ), then  $Col^-(x)$  is a singleton  $\{\alpha\}$ . Now every arc  $vx$  with  $v \in D'(x)$  is coloured  $\alpha$  so any arc  $uv \in E(D'(x))$  is not coloured  $\alpha$ . Hence  $c$  is an arc-colouring with  $\Phi(k) - 1$  colours which is a contradiction.

The proof is analogous if  $\Phi(k) = \Phi(k + 1)$  with  $D'$  a  $k$ -digraph such that  $\chi_a(D') = \Phi(k)$ . Then  $C$  is a  $(k + 1)$ -digraph and we get the result in the same way.  $\square$

The next theorem shows that for a fixed integer  $k$ , one of the Conjectures 24 and 4 holds.

**Theorem 26** *Let  $k$  be an integer. Then  $\Phi^\vee(k, 1) = \Phi(k)$  or  $\Phi^\vee(k, k) \leq \Phi(k) + 1$ .*

**Proof.** Suppose that  $\Phi^\vee(k, 1) \neq \Phi(k)$ . Let  $C$  be a  $(k \vee 1)$ -digraph such that  $\chi_a(C) = \Phi(k) + 1$  and  $C^1$  the subdigraph of  $C$  induced by the arcs with tail in  $V^+(C)$ . By Lemma 20, for every arc-colouring of  $C^1$  with  $\Phi(k)$  colours there exists a vertex  $x$  of  $C^+$  with  $col^-(x) \leq 1$ .

Let  $D$  be a  $(k \vee k)$ -digraph. Let  $D^1$  (resp.  $D^2$ ) be the subdigraph of  $D$  induced by the arcs with tail in  $V^+(D)$  (resp. head in  $V^-(D)$ ). Let  $E'$  be the set of arcs of  $D$  with tail in  $V^-$  and head in  $V^+$ . Let  $F^1$  be the digraph constructed from  $C^1$  as follows: for every vertex  $x \in V^+(C)$ , add a copy  $D^+(x)$  of  $D^+$  and the arcs  $\{u(x)x, uv \in E(D), u \in V^+(D), \text{ and } v \in V^-(D)\}$ . Then  $F^1$  is a  $k$ -digraph so it admits an arc-colouring  $c_1$  with  $\{1, \dots, \Phi(k)\}$ . Now there is a vertex  $x \in V^+(C)$  such that  $col^-(x) \leq 1$ . So all the arcs from  $D^+(x)$  to  $x$  are coloured the same. Free to permute the labels, we may assume they are coloured 1. Since  $F^1[V^+(D) \cup x]$  has the same line-digraph than  $D^1$ , the arc-colourings of  $F^1[V^+(D) \cup x]$  is in one-to-one correspondence with the arc-colourings of  $D^1$ . So  $D^1$  admits an arc-colouring  $c^1$  with  $\{1, \dots, \Phi(k)\}$  such that every arc with head in  $V^-$  is coloured 1.

Analogously,  $D^2$  admits an arc-colouring  $c^2$  with  $\{1, \dots, \Phi(k)\}$  such that every arc with head in  $V^-$  is coloured 1. The union of  $c_1$  and  $c_2$  is an arc-colouring of  $D - E'$  with

$\{1, \dots, \Phi(k)\}$ . Hence assigning  $\Phi(k) + 1$  to every arc of  $E'$ , we obtain an arc-colouring of  $D$  with  $\Phi(k) + 1$  colours.  $\square$

**Corollary 27**  $\Phi^\vee(k, 1) \leq \Phi(k) + 1$ .

Note that since  $\Phi(k)$  is bounded by  $\theta(2k)$ , the condition  $\Phi^\vee(k, 1) = \Phi(k)$  or  $\Phi^\vee(k, k) \leq \Phi(k) + 1$  is very often true. Indeed, we conjecture that it is always true and that  $\Phi$  behaves “smoothly”.

**Conjecture 28** (i) If  $k \geq 1$ ,  $\Phi(k + 1) \leq \Phi(k) + 1$ .

(ii) If  $k \geq 1$ ,  $\Phi(k + 2) \leq \Phi(k) + 1$ .

(iii)  $\Phi(k_1 k_2) \leq \Phi(k_1) + \Phi(k_2)$ .

Note that (ii) implies (i) and Conjecture 24.

The arc-set of a  $(k_1 + k_2)$ -digraph  $D$  may trivially be partitioned into two sets  $E_1$  and  $E_2$  such that  $(V(D), E_1)$  is a  $k_1$ -digraph and  $(V(D), E_2)$  is a  $k_2$ -digraph. So  $\Phi(k_1 + k_2) \leq \Phi(k_1) + \Phi(k_2)$ . In particular,  $\Phi(k + 1) \leq \Phi(k) + \Phi(1) = \Phi(k) + 3$ . Despite we were not able to prove Conjecture 28-(i), we now improve the above trivial result.

**Theorem 29** If  $k \geq 1$  then,  $\Phi(k + 1) \leq \Phi(k) + 2$ .

**Proof.** Let  $D$  be a  $(k + 1)$ -digraph. Free to add arcs, we may assume that  $d^+(v) = k + 1$  for every  $v \in V(D)$ . Let  $T_1, \dots, T_p$  be the terminal components of  $D$ . Each  $T_i$  contains a circuit  $C_i$  which has a chord. Indeed consider a maximal path  $P$  in  $T_i$  and two arcs with tail its terminus and head in  $P$ , by maximality. One can extend  $\bigcup C_i$  into a subdigraph  $F$  spanning  $D$  such that  $d_F^+(v) \geq 1$  for every  $v \in V(D)$  and the sole circuits are the  $C_i$ ,  $1 \leq i \leq p$ .  $F$  is the union of  $p$  connected components  $F_1, \dots, F_p$ , each  $F_i$  being the union of  $C_i$  and inarborescences  $A_i^1, \dots, A_i^{q_i}$  with roots  $r_i^1, \dots, r_i^{q_i}$  in  $C_i$  such that  $(V(C_i), V(A_i^1) \setminus \{r_i^1\}, \dots, V(A_i^{q_i}) \setminus \{r_i^{q_i}\})$  is a partition of  $V(D)$ .

Now  $D - F$  is a  $k$ -digraph. So we colour the arcs of  $D - F$  with  $\Phi(k)$  colours. Let  $\alpha$  and  $\beta$  be two new colours. Let us colour the arcs of  $F$ . Let  $1 \leq i \leq p$ . If  $C_i$  is an even circuit then  $F_i$  is bipartite and its arcs may be coloured by  $\alpha$  and  $\beta$ . If  $C_i$  is an odd circuit, consider its chord  $xy$  in  $E(D - F)$ . In the colouring of  $D - F$ ,  $Col^+(x) \not\subseteq Col^+(y)$  thus there is an arc  $x'y'$  of  $E(C_i)$  such that  $Col^+(x') \not\subseteq Col^+(y')$ . Hence we may assign to  $x'y'$  a colour of  $Col^+(x') \setminus Col^+(y')$ . Now  $F_i - x'y'$  is bipartite and its arcs may be coloured by  $\alpha$  and  $\beta$ .  $\square$

## 5 $\Phi$ and $\Phi^\vee$ for small value of $k$ or $l$ .

### 5.1 $\Phi(1)$ , $\Phi^\vee(1, 0)$ and $\Phi^\vee(1, 1)$ .

**Theorem 30**

$$\Phi^\vee(1, 1) = \Phi^\vee(1, 0) = \Phi(1) = 3$$

**Proof.** By Theorem 5,  $\Phi^\vee(1, 1) \leq 3$ . The 3-circuit is its own line-digraph and is not 2-colourable.  $\square$

### 5.2 $\Phi(2)$ and $\Phi^\vee(2, l)$ , for $l \leq 2$ .

The aim of this subsection is to prove Theorem 35, that is  $\Phi(2) = \Phi^\vee(2, 0) = \Phi^\vee(2, 1) = \Phi^\vee(2, 2) = 4$ . Therefore, we first exhibit a 2-digraph which is not 3-arc-colourable. Then we show that  $\Phi^\vee(2, 2) \leq 4$ .

**Definition 31** For any integer  $k \geq 1$ , the *rotative tournament* on  $2k + 1$  vertices, denoted  $R_{2k+1}$ , is the tournament with vertex-set  $\{v_1, \dots, v_{2k+1}\}$  and arc-set  $\{v_i v_j, j - i \pmod{2k+1} \in \{1, \dots, k\}\}$ .

**Proposition 32** *The tournament  $R_5$  is not 3-arc-colourable. So  $\Phi(2) \geq 4$ .*

**Proof.** Suppose that  $R_5$  admits a 3-arc-colouring  $c$  in  $\{1, 2, 3\}$ . Then, for any two vertices  $x$  and  $y$ ,  $Col^+(x) \neq Col^+(y)$  and  $1 \leq col^+(x) \leq 2$ . Hence there is a vertex, say  $v_1$ , such that  $col^+(v_1) = 1$ , say  $Col^+(v_1) = \{1\}$ . Then  $Col^+(v_2)$  and  $Col^+(v_3)$  are subsets of  $\{2, 3\}$  and  $Col^+(v_2) \not\subseteq Col^+(v_3)$ . It follows that  $col^+(v_3) = 1$ . Repeating the argument for  $v_3$ , we obtain  $col^+(v_5) = 1$  and then  $col^+(v_i) = 1$ , for every  $1 \leq i \leq 5$ , which is a contradiction.  $\square$

In order to prove that  $\Phi^\vee(2, 2) \leq 4$ , we need to show that every  $(2 \vee 2)$ -digraph admits homomorphism  $h$  into  $\vec{H}_4$ . In order to exhibit such a homomorphism, we first show that there is a homomorphism  $h^+$  from  $D^+$  into a subdigraph  $S_2^+$  of  $\vec{H}_4$  and a homomorphism  $h^-$  from  $D^-$  into a subdigraph  $S_2^-$  of  $\vec{H}_4$  with specific properties allowing us to extend  $h^+$  and  $h^-$  into a homomorphism  $h$  from  $D$  into  $\vec{H}_4$ .

**Definition 33** Let  $S_2^+$  be the digraph with vertex-set  $\{s_1^+, \dots, s_6^+\}$  with arc-set  $\{s_i^+ s_j^+ : i \neq j\} \setminus \{s_2^+ s_1^+, s_4^+ s_3^+, s_6^+ s_5^+\}$ .

**Lemma 34** *Let  $D$  be a 2-digraph. There exists a homomorphism  $h^+$  from  $D$  to  $S_2^+$  such that the vertices  $x$  with  $h^+(x) \in \{s_2^+, s_4^+, s_6^+\}$  have outdegree 2.*

**Proof.** Let us prove it by induction on  $|V(D)|$ . If  $d^+(x) \leq 1$  for every vertex  $x$  of  $D$  then  $D$  is 3-colourable and  $D \rightarrow S_2^+[\{s_1^+, s_3^+, s_5^+\}]$ . So, we assume that there exists a vertex  $x$  with outdegree 2. By induction hypothesis, there is a homomorphism  $h^+ : D - x \rightarrow S_2^+$  with



the required condition. Note that every inneighbour of  $x$  has outdegree at most 1 in  $D - x$  and thus can not have image  $s_2^+$ ,  $s_4^+$  or  $s_6^+$  by  $h^+$ . Denote by  $y$  and  $z$  the outneighbours of  $x$ . The set  $\{h^+(y), h^+(z)\}$  does not intersect one set of  $\{s_1^+, s_2^+\}$ ,  $\{s_3^+, s_4^+\}$  and  $\{s_5^+, s_6^+\}$ , say  $\{s_1^+, s_2^+\}$ . Then, setting  $h^+(x) = s_2^+$ , we extend  $h^+$  into a homomorphism from  $D$  to  $S_2^+$  with the required condition.  $\square$

**Theorem 35**

$$\Phi(2) = \Phi^\vee(2, 0) = \Phi^\vee(2, 1) = \Phi^\vee(2, 2) = 4$$

**Proof.**

By Proposition 32,  $4 \leq \Phi(2) \leq \Phi^\vee(2, 0) \leq \Phi^\vee(2, 1) \leq \Phi^\vee(2, 2)$ .

Let us prove that  $\Phi^\vee(2, 2) \leq 4$ . Let  $D$  be a  $(2 \vee 2)$ -digraph. We will provide a homomorphism from  $D$  to  $\overline{H}_4$ .

Let  $S_2^-$  be the dual of  $S_2^+$ , that is the digraph on  $\{s_1^-, \dots, s_6^-\}$  with arc-set  $\{s_i^- s_j^- : i \neq j\} \setminus \{s_1^- s_2^-, s_3^- s_4^-, s_5^- s_6^-\}$ . By Lemma 34, there is a homomorphism  $h^+ : D^+ \rightarrow S_2^+$  such that if  $h^+(x) \in \{s_2^+, s_4^+, s_6^+\}$  then  $d_{D^+}^+(x) = 2$ . Symmetrically, there exists a homomorphism  $h^- : D^- \rightarrow S_2^-$  such that if  $h^-(x) \in \{s_2^-, s_4^-, s_6^-\}$  then  $d_{D^-}^-(x) = 2$ .

Let  $S_2$  be the digraph obtained from the disjoint union of  $S_2^+$  and  $S_2^-$  by adding the arcs of  $\{s_i^- s_j^+ : 1 \leq i \leq 6, 1 \leq j \leq 6\} \cup \{s_i^+ s_j^- : i = 1, 3, 5, j = 1, 3, 5\}$ . The mapping  $h : D \rightarrow S_2$  defined by  $h(x) = h^+(x)$  if  $x \in V^+$  and  $h(x) = h^-(x)$  if  $x \in V^-$  is a homomorphism. Indeed if  $xy$  is an arc of  $D$  with  $x \in V^+$  and  $y \in V^-$ , conditions on  $h^+$  and  $h^-$  imply that  $h(x) = h^+(x) \in \{s_1^+, s_3^+, s_5^+\}$  and  $h(y) = h^-(y) \in \{s_1^-, s_3^-, s_5^-\}$ . To conclude, Figure 1 provides a homomorphism  $g$  from  $S_2$  to  $\overline{H}_4$ . The non-oriented arcs on the figure corresponds to circuits of length 2 and all the arcs from  $S_2^-$  to  $S_2^+$  are not represented.  $\square$

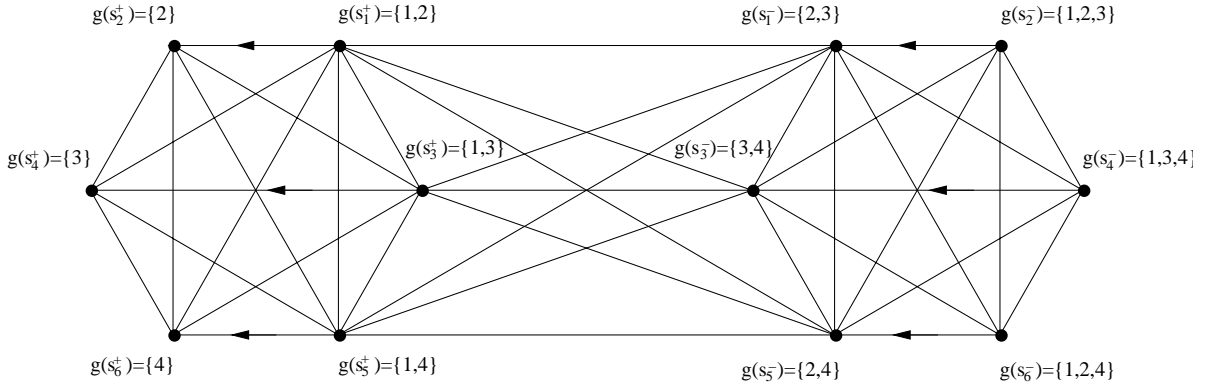


Figure 1: The homomorphism  $g$  from  $S_2$  to  $\overline{H}_4$ .

### 5.3 $\Phi(3)$ and $\Phi^\vee(3, l)$ , for $l \leq 3$ .

#### Theorem 36

$$\Phi(3) = \Phi^\vee(3, 0) = \Phi^\vee(3, 1) = 4$$

**Proof.**  $4 \leq \Phi(2) \leq \Phi(3) \leq \Phi^\vee(3, 0) \leq \Phi^\vee(3, 1) \leq \theta(6) = 4$  by Corollary 21.  $\square$

In the remaining of this subsection, we shall prove Theorem 35, that is  $\Phi^\vee(3, 2) = \Phi^\vee(3, 3) = 5$ . Therefore, we first exhibit a  $(3 \vee 2)$ -digraph which is not 4-arc-colourable. Then we show that  $\Phi^\vee(3, 3) \leq 5$ .

**Definition 37** Let  $G^-$  be the digraph obtained from the rotative tournament on five vertices  $R_5$ , with vertex set  $\{v_1^-, \dots, v_5^-\}$  and arc-set  $\{v_i^- v_j^- : j - i \pmod{5} \in \{1, 2\}\}$  and five copies of the 3-circuits  $R_3^1, \dots, R_3^5$  by adding, for  $1 \leq i \leq 5$ , the arcs  $vv_i^-$ , for  $v \in R_3^i$ .

Let  $G^+$  be the digraph obtained from the rotative tournament on seven vertices  $R_7$ , with vertex set  $\{v_1^+, \dots, v_7^+\}$  and arc-set  $\{v_i^+ v_j^+ : j - i \pmod{7} \in \{1, 2, 3\}\}$  and seven copies of the rotative tournament of,  $R_5^1, \dots, R_5^7$  by adding, for  $1 \leq i \leq 7$ , the arcs  $vv_i^+$ , for  $v \in R_5^i$ .

Finally, let  $G$  be the  $(3 \vee 2)$ -digraph obtained from the disjoint union of  $G^-$  and  $G^+$  by adding all the arcs of the form  $v^- v^+$  with  $v^- \in V(G^-)$  and  $v^+ \in V(G^+)$ . See Figure 2.

**Proposition 38** *The digraph  $G$  is not 4-arc-colourable. So  $\Phi(3, 2) \geq 5$ .*

**Proof.** Suppose for a contradiction that  $G$  admits an arc-colouring  $c$  in  $\{1, 2, 3, 4\}$ .

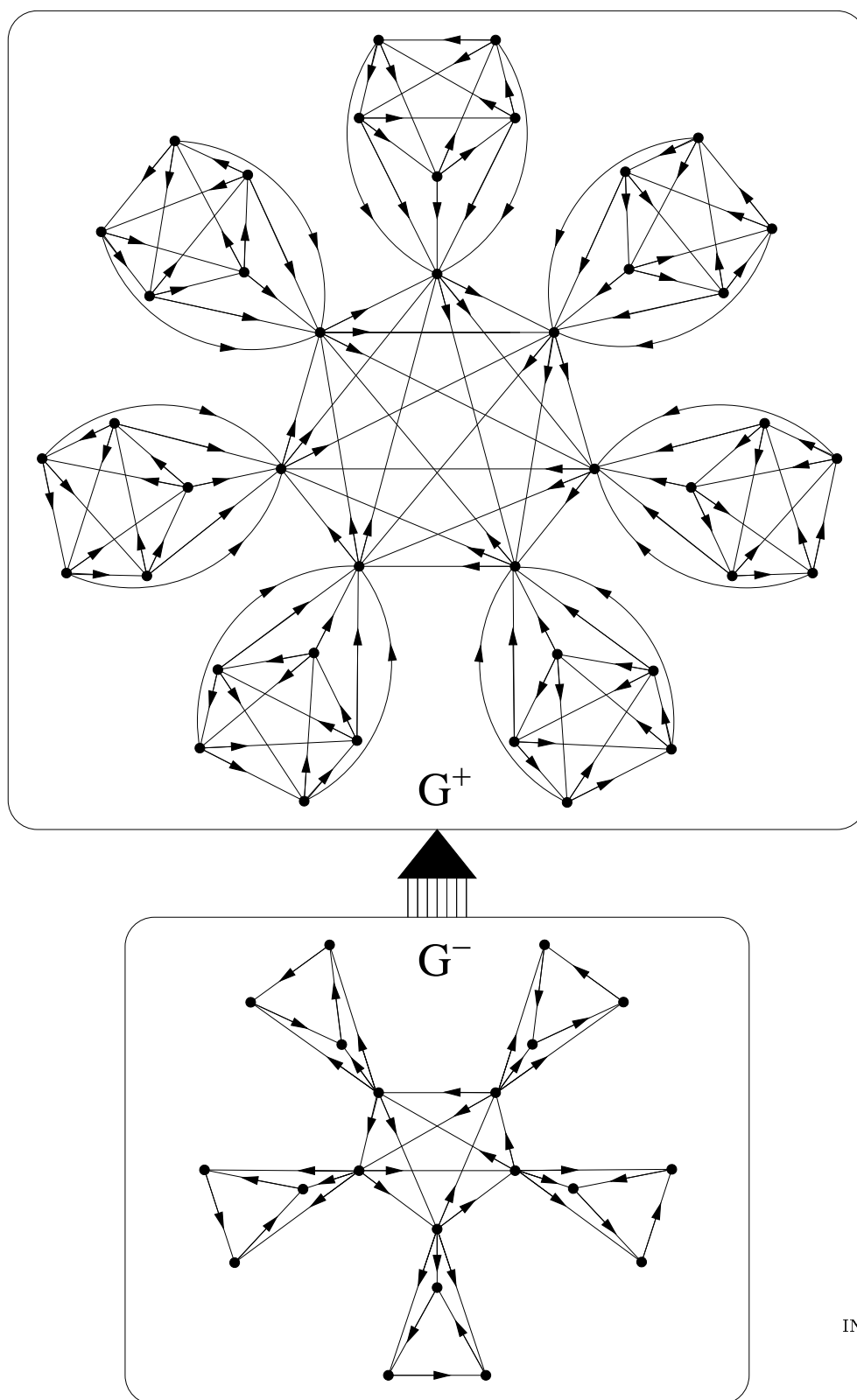
Let  $v^+$  be a vertex of  $G^+$  and  $v^-$  a vertex of  $G^-$ . Then since  $v^- v^+$  is an arc,  $Col^+(v^+) \neq Col^+(v^-)$ . We will show:

- (1) there are two 2-subsets  $S$  of  $\{1, 2, 3, 4\}$  such that a vertex  $v^- \in G^-$  satisfies  $Col^+(v^-) = S$ ;
- (2) there are five 2-subsets  $S$  of  $\{1, 2, 3, 4\}$  such that a vertex  $v^+ \in G^+$  satisfies  $Col^+(v^+) = S$ .

This gives a contradiction since there are only six 2-subsets in  $\{1, 2, 3, 4\}$ .

Let us first show (1). Every vertex of  $G^-$  satisfies  $col^+ \geq 2$  otherwise all the arcs of  $R_7$  in  $G^+$  must be coloured with three colours, a contradiction to Theorem 12. Hence, since in  $R_5$  all the  $Col^+$  are distinct and not  $\{1, 2, 3, 4\}$ , a vertex of  $R_5$ , say  $v_1^-$ , has  $col^+ = 2$ , say  $Col^+(v_1^-) = \{1, 2\}$ . Consider now the vertices of  $R_3^1$ . None of them has  $Col^+ = \{1, 2, 3\}$  nor  $Col^+ = \{1, 2, 4\}$  since they are dominated by  $v_1^-$ . Moreover they all have different  $Col^+$  since  $R_3^1$  is a tournament. Hence one of them, say  $v$ , satisfies  $col^+(v) = 2$ . Now  $Col^+(v) \neq Col^+(v_1^-)$  since  $v_1^- \rightarrow v$ .

Let us now prove (2). Let  $\mathcal{S} = \{2\text{-subsets } S \text{ such that } \exists v^+ \in G^+, Col^+(v^+) = S\}$  and suppose that  $|\mathcal{S}| \leq 4$ . Every vertex of  $G^+$  has  $col^+ \leq 2$  otherwise all the arcs of  $R_5$  in  $G^-$  must be coloured with three colours, a contradiction to Proposition 32. Now, all the vertices



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Figure 2: The non 4-arc-colourable  $(3 \vee 2)$ -digraph  $G$ .

of  $R_7$  have distinct and non-empty  $Col^+$ . So at least three vertices of  $R_7$  have  $col^+ = 2$  and  $|\mathcal{S}| \geq 3$ . Thus, without loss of generality, we are in one these three following cases:

- (a)  $\mathcal{S} \subset \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$  and  $Col^+(v_1^+) = \{1, 2\}$ ;
- (b)  $\mathcal{S} \subset \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 4\}\}$  and  $Col^+(v_1^+) = \{2, 3\}$ ;
- (c)  $\mathcal{S} \subset \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 4\}\}$  and  $Col^+(v_1^+) = \{1, 4\}$ .

Let  $x_1, \dots, x_5$  be the vertices of  $R_5^1$  such that  $x_i x_j$  is an arc if and only if  $j = i + 1 \pmod 5$  or  $j = i + 2 \pmod 5$  and  $\mathcal{F} = \{Col^+(x_i) : 1 \leq i \leq 5\}$ . Recall that  $|\mathcal{F}| = 5$  since  $R_5^1$  is a tournament and that every element  $S$  of  $\mathcal{F}$  is not included in  $Col^+(v_1^+)$  since  $x_i \rightarrow v_1^+$  for every  $1 \leq i \leq 5$ .

Case (a): We have  $\mathcal{F} = \{\{3\}, \{4\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\}$ . So we may assume that  $Col^+(x_1) = \{3\}$ . Now because  $x_1 \rightarrow x_2$ ,  $x_1 \rightarrow x_3$  and  $x_2 \rightarrow x_3$ ,  $Col^+(x_1) \not\subset Col^+(x_2)$ ,  $Col^+(x_1) \not\subset Col^+(x_3)$  and  $Col^+(x_2) \not\subset Col^+(x_3)$ . It follows that  $Col^+(x_2) = \{1, 4\}$  and  $Col^+(x_3) = \{4\}$ . Hence, none of  $Col^+(x_4)$  and  $Col^+(x_5)$  is  $\{3, 4\}$  since  $x_3 \rightarrow x_4$  and  $x_3 \rightarrow x_5$ , a contradiction.

Case (b): We have  $\mathcal{F} = \{\{1\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}\}$ . So we may assume that  $Col^+(x_1) = \{1\}$ . Since  $x_1 \rightarrow x_2$ ,  $Col^+(x_1) \not\subset Col^+(x_2)$ , so  $Col^+(x_2) = \{4\}$ . Similarly,  $Col^+(x_3) = \{4\}$  which is a contradiction.

Case (c): We have  $\mathcal{F} = \{\{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}$ . So we may assume that  $Col^+(x_1) = \{2\}$ . It follows that  $Col^+(x_2) = \{1, 3\}$  and  $Col^+(x_3) = \{3\}$ . Hence, none of  $Col^+(x_4)$  and  $Col^+(x_5)$  is  $\{2, 3\}$  since  $x_3 \rightarrow x_4$  and  $x_3 \rightarrow x_5$ , a contradiction.  $\square$

We will now prove that  $\Phi^\vee(3, 3) \leq 5$ . As in the proof of Theorem 35, in order to exhibit a homomorphism from a  $(3 \vee 3)$ -digraph  $D$  to  $\vec{H}_5$ , we first show that there are two homomorphisms,  $h^+$  from  $D^+$  into a subdigraph  $S_3^+$  of  $\vec{H}_5$  and  $h^-$  from  $D^-$  into another subdigraph  $S_3^-$  of  $\vec{H}_5$ , with specific properties.

**Definition 39** Let  $S_3^+$  be the complete digraph with vertex-set  $\{s_1^+, \dots, s_7^+\}$ . Let  $S_3^-$  be the digraph with vertex-set  $\{s_1^-, \dots, s_9^-\}$  and arc-set  $\{s_i^- s_j^- : i \neq j\} \setminus \{s_2^- s_1^-, s_3^- s_1^-\}$ .

**Lemma 40** Let  $D$  be a 3-digraph. There exists a homomorphism  $h^+$  from  $D$  to  $S_3^+$  such that the vertices  $x$  with  $h^+(x) \in \{s_6^+, s_7^+\}$  have outdegree 3.

**Proof.** Let us prove it by induction on  $n = |V(D)|$ . If there exists a vertex  $x$  with  $d^+(x) + d^-(x) \leq 4$  then we obtain the desired homomorphism  $h^+$  from  $D - x$  to  $S_3^+$  and extend it with a suitable choice of  $h^+(x)$  in  $\{s_1^+, \dots, s_5^+\}$ .

Assume now that  $d^+(x) + d^-(x) \geq 5$  for every  $x$ . Let  $n_i$  be the number of vertices with outdegree  $i$ . Clearly,  $n = n_0 + n_1 + n_2 + n_3$ . Moreover, we have:

$$3n \geq \sum_{x \in V} d^+(x) = \sum_{x \in V} d^-(x) = \sum_{d^+(x)=0} d^-(x) + \sum_{d^+(x)=1} d^-(x) + \sum_{d^+(x)=2} d^-(x) + \sum_{d^+(x)=3} d^-(x)$$

Then, by assumption:

$$3n \geq 5n_0 + 4n_1 + 3n_2 + \sum_{d^+(x)=3} d^-(x)$$

If there is no vertex with outdegree 3, then  $D$  is 5-colourable and there is an homomorphism  $h^+$  from  $D$  to  $S_3^+[\{s_1^+, \dots, s_5^+\}]$ . Suppose now that there exists a vertex with outdegree 3. Then, there exists a vertex with outdegree 3 and indegree at most 3. If not,  $d^-(x) \geq 4$  for every  $x$  with  $d^+(x) = 3$  and the previous inequality implies  $3n \geq 5n_0 + 4n_1 + 3n_2 + 4n_3$  with  $n_3 \neq 0$ , what contradicts  $n = n_0 + n_1 + n_2 + n_3$ .

Finally, let  $x$  be a vertex with outdegree 3 and indegree at most 3. By induction hypothesis, there is a homomorphism  $h^+$  from  $D - x$  to  $S_3^+$  with the required property. As  $x$  has at most 6 neighbours, we extend  $h^+$  with a suitable choice for  $h^+(x)$  in  $\{s_1^+, \dots, s_7^+\}$ .  $\square$

**Lemma 41** *Let  $D$  be a digraph with maximal indegree at most 3. There exists a homomorphism  $h^-$  from  $D$  to  $S_3^-$  such that the vertices  $x$  with  $h^-(x) \in \{s_6^-, \dots, s_9^-\}$  have indegree 3.*

**Proof.** We prove the result by induction on  $|V(D)|$ .

If every vertex have indegree at most 2 then, by the dual form of the Lemma 17, there exists a homomorphism from  $D$  to  $S_3^-[\{s_1^-, \dots, s_5^-\}]$ .

Now, let  $x$  be a vertex with indegree 3. Let  $y_1, y_2$  and  $y_3$  be the outneighbours of  $x$ . By induction, there is a homomorphism  $h^-$  from  $D - x$  to  $S_3^-$  with the required property. In particular, as the vertex  $y_i, 1 \leq i \leq 3$ , has indegree at most 2 in  $D - x$ , we have  $h^-(y_i) \in \{s_1^-, \dots, s_5^-\}$ . So, as  $x$  has 3 inneighbours, we can extend  $h^-$  with a suitable choice for  $h^-(x)$  in  $\{s_6^-, \dots, s_9^-\}$ .  $\square$

**Theorem 42**

$$\Phi^\vee(3, 2) = \Phi^\vee(3, 3) = 5$$

**Proof.** By Proposition 38,  $5 \leq \Phi^\vee(3, 2) \leq \Phi^\vee(3, 3)$ . We will prove that  $\Phi^\vee(3, 3) \leq 5$ . Let  $D$  be a  $(3 \vee 3)$ -digraph, we will provide a homomorphism from  $D$  to  $\overline{H}_5$ .

By Lemma 40, there is a homomorphism  $h^+ : D^+ \rightarrow S_3^+$  such that if  $h^+(x) \in \{s_6^+, s_7^+\}$  then  $d_{D^+}^+(x) = 3$ . Moreover by Lemma 41, there is a homomorphism  $h^- : D^- \rightarrow S_3^-$  such that if  $h^-(x) \in \{s_6^-, \dots, s_9^-\}$ , then  $d_{D^-}^-(x) = 3$ .

Let  $S_3$  be the digraph obtained from the disjoint union of  $S_3^+$  and  $S_3^-$  by adding the arcs of  $\{s_i^- s_j^+ : 1 \leq i \leq 9, 1 \leq j \leq 7\} \cup \{s_i^+ s_j^- : i = 1, \dots, 5, j = 1, \dots, 5\}$ . The mapping  $h : D \rightarrow S_3$  defined by  $h(x) = h^+(x)$  if  $x \in V^+$  and  $h(x) = h^-(x)$  if  $x \in V^-$  is a homomorphism. Indeed if  $xy$  is an arc of  $D$  with  $x \in V^+$  and  $y \in V^-$ , conditions on  $h^+$  and  $h^-$  imply that  $h(x) = h^+(x) \in \{s_1^+, \dots, s_5^+\}$  and  $h(y) = h^-(y) \in \{s_1^-, \dots, s_5^-\}$ . To conclude, Figure 3 provides a homomorphism  $g$  from  $S_3$  to  $\overline{H}_5$ . Inside  $S_3^-$  and  $S_3^+$ , only the arcs which are not in a circuit of length 2 are represented, every pair of not adjacent vertices are, in fact, linked by two arcs, one in each way.  $\square$

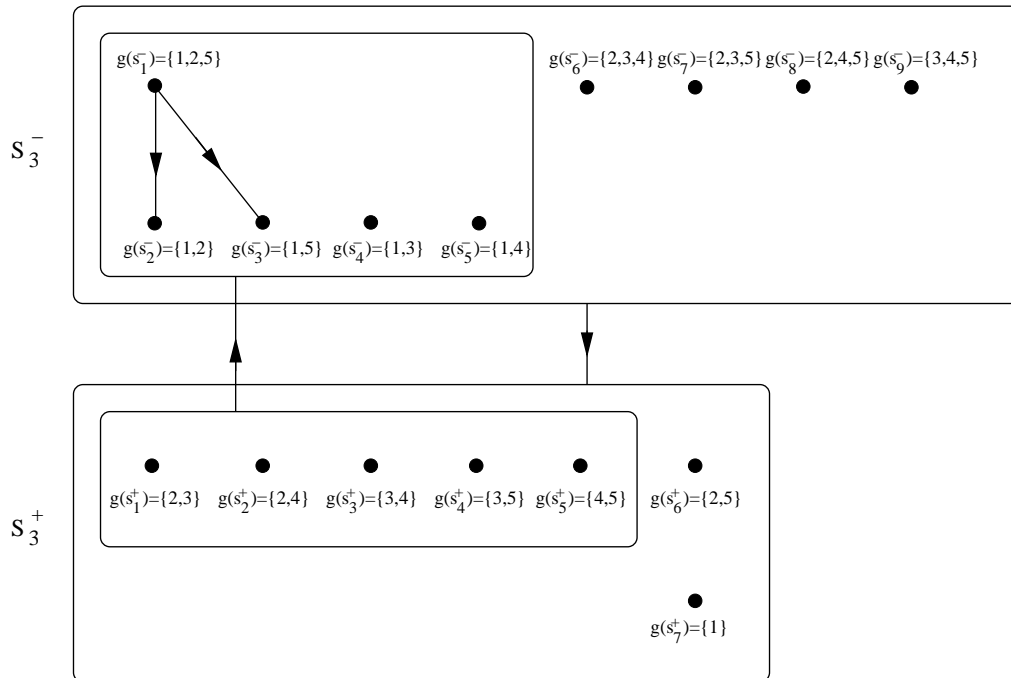


Figure 3: The homomorphism  $g$  from  $S_3$  to  $\overline{H}_5$ .

## References

- [1] R. L. Brooks, On colouring the nodes of a network. *Proc Cambridge Phil. Soc.* **37** (1941), 194–197.
- [2] A. El-Sahili, Fonctions de graphes disjoints, *C. R. Acad. Sci. Paris* **319** Série I (1994), 519–521.
- [3] A. El-Sahili, Functions and Line Digraphs, *J. of Graph Theory* **44** (2003), no. 4, 296–303.
- [4] P. Erdős, A. L. Rubin, H. Taylor, Choosability in graphs, *Proc. West-Coast Conference on Combinatorics, Graph Theory and Computing*, Arcata, California, *Congressus Numerantium* **XXVI** (1979), 125–157.
- [5] S. Poljak and V. Rödl, On the arc-chromatic number of a digraph, *J. Combin. Theory Ser. B* **31** (1981), no. 2, 190–198.
- [6] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, *Math. Zeit.* **27** (1928).



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