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INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

## Random thresholds for linear model selection

Marc Lavielle - Carenne Ludeña

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#### Abstract

A method is introduced to estimate the number of significant coefficients in non ordered model selection problems. The method is based on a convenient random centering of the partial sums of the ordered observations. Based on $L$-statistics methods we show consistency of the proposed estimator. An extension to unknown parametric distributions is considered. The method is then applied to a regression model and interpreted as a random threshold procedure. Simulated examples are included to show the accuracy of the estimator.


Key-words: adaptive estimation, linear model selection, hard thresholding, random thresholding, $L$ statistics.

[^0][^1]
## Sélection de modèles linéaires par seuillage aléatoire

Résumé : Une nouvelle méthode est proposée pour estimer le nombre de coefficients significatifs dans un problème de sélection de modèles. Cette méthode utilise un centrage aléatoire bien choisi des sommes cumulées partielles des observations ordonnées. En utilisant des propriétés des $L$-statistiques, nous montrons la consistance de l'estimateur proposé. Une extension à des distributions paramétriques inconnues est considérée. La procédure est ensuite appliquée à un modèle de régression et est interprétée comme une procédure de seuillage aléatoire. Des exemples numériques illustrent l'intérêt pratique de la méthode.

Mots-clés : estimation adaptative, sélection de modèle linéaire, seuillage dur, seuillage aléatoire, $L$-statistique.

## 1 Introduction

Consider the following model

$$
y_{i}=\mu_{i}+\varepsilon_{i}, i=1, \ldots, n
$$

where $\left(\mu_{i}\right)$ is an unknown sequence of constants some of which are zero and $\left(\varepsilon_{i}\right)$ are independent random variables with common cumulative distribution $F_{\varepsilon}$. The problem we study in this article is choosing the significant, non zero, coefficients based on the observations $\left(y_{i}\right)$. Obviously, significant coefficients will be those which are greater than a certain threshold, i.e. choose index $i$ if

$$
\begin{equation*}
\left|y_{i}\right|>\tau \tag{1}
\end{equation*}
$$

The choice of $\tau$ in (1) will depend on the distribution of the sequence $\left(\varepsilon_{i}\right)$. So in practice $\tau$ must be calibrated in terms of the data. A usual technique is to consider a sequence of thresholds $\left(\tau_{j}\right)$ for values ranging from very small (many significant coefficients) to very big (few significant coefficients) and study the point where a substantial decrease in the number of significant coefficients occurs. Of course choosing the "right" $\tau$ is equivalent to choosing the "right" number $k$ of significant coefficients, with the advantage that this can be done independently of the choice of $\left(\tau_{j}\right)$ by looking at the relative size of the observations. Indeed, a jump in the relative size of the observations should indicate the existence of significant (not noise) coefficients.

This has been considered by a number of authors (see, for example, [6, 7, 10]) and many adaptive procedures aimed at studying the correct "jump point" have been developed.

A natural approach seems to consider the partial sums of the absolute (or squared) observations ordered decreasingly, and study the fluctuations of these partial sums around some kind of centering factor.

In this article we tackle this problem based on the use of order statistics by considering a convenient random centering: the conditional expectation, with respect to the total sum, of these partial sums. Even when this conditional expectation cannot be computed in a closed form, an exponential change of variable makes this centering possible. We then construct an $L$-statistic and study its weak convergence. Empirical probability tables or simulated ones based on the limiting process can then be constructed to accept or reject the null hypothesis of all coefficients being equal to zero. If we reject the null hypothesis, further inspection of the test statistic yields the subset of significant coefficients. Indeed, we construct a test statistic based on the minimization of a certain functional of the conveniently centered partial sums and show consistency of the estimated number of significant coefficients under mild assumptions over the gap between significant and non significant coefficients.

The above method requires previous knowledge of $F_{\varepsilon}$. In a parametric setting, $F_{\varepsilon}=F_{\varepsilon}\left(\cdot ; \theta^{\star}\right)$, we show that the proposed method can also address the case $\theta^{\star}$ unknown, assuming the existence of a consistent estimator of $\theta^{\star}$. When $\theta^{\star}$ is a scale parameter, an appropriate modification of the estimating procedure yields a scale free statistic, which is also shown to be consistent.

We then apply our method to the problem of estimating the number of significant coefficients for the regression problem

$$
y_{i}=f\left(x_{i}\right)+\eta_{i}, i=1, \ldots, n
$$

where $f$ is an unknown function in some function space $S$ and $\eta_{i}$ are independent random variables with variance $\sigma^{2}$. A usual estimation procedure is to consider $f \in L^{2}(\mu)$ and a finite orthonormal system $\left\{\phi_{\lambda}\right\}_{\Lambda}$, with $|\Lambda|=M_{n}$. Denoting by $\left\langle y, \phi_{j}\right\rangle_{n}=1 / n \sum_{i=1}^{n} y_{i} \phi_{\lambda}\left(x_{i}\right)$ the empirical coefficients, Donoho and Johnstone [5] in their seminal article proposed choosing only those coefficients whose absolute value exceeded a certain threshold $u=\sqrt{\frac{\tau \sigma^{2} \log (n)}{n}}$. This procedure has since been refereed to as hard thresholding.

In a very interesting reinterpretation, Barron, Birge and Massart [2] study the problem of hard thresholding in the context of non ordered model selection based on the addition of a penalization term. Their arguments are combinatorial based on the complexity of the underlying linear spaces: the size of the set of all possible models of size $k$ out of $K$ is bounded by $(e K / k)^{k}$ and a logarithmic factor depending on $K$ must be introduced in order to bound the probabilities. In terms of (1) our observations would now be the empirical coefficients $\left\langle y, \phi_{j}\right\rangle_{n}$. Of course, except for the case $\eta_{i} \sim N\left(0, \sigma^{2}\right)$, the empirical coefficients will not be necessarily independent, although uncorrelated, so that the problem does not comply to our assumptions. However, in practice the method works well. As discussed in section 4.3, our method can be interpreted as a random threshold procedure .

The article is organized as follows: in section 2 we introduce the problem and basic notation as well as the proposed test procedure. In section 3 we state and prove theoretical results that justify our procedure, namely consistency of the selected subset of significant coefficients. In section 4.3 we consider certain extensions which include the parametric distribution case, an application to the problem of non ordered linear model selection for the regression setting and interpret out testing scheme in terms of a random penalization procedure. In section 5 we present simulated examples.

## 2 Describing the procedure

### 2.1 A first hypothesis testing procedure

Assume we observe $y_{i}=\mu_{i}+\varepsilon_{i}$. Variables $\varepsilon_{i}$ are assumed to be independent and identically distributed with common cumulative distribution $F_{\varepsilon}$. We begin by assuming that the cumulative distribution function $F_{|\varepsilon|}$ of the $\left|\varepsilon_{i}\right|$ 's is known. In section 4 we will deal with the unknown $F_{|\varepsilon|}$ case.

Given the collection $\left(y_{i} ; 1 \leq i \leq n\right)$, we are interested in this section in testing if all the $\mu_{i}$ 's are null or not. Thus, we introduce the following hypothesis:

Null hypothesis:
$\boldsymbol{H}_{\mathbf{0}}: \quad \mu_{i} \equiv 0$ for $i=1, \ldots, n$.
Alternative hypothesis:
$\boldsymbol{H}_{1}$ : there exists a non empty subset $I$ of $\{1,2, \ldots, n\}$ such that $\mu_{i} \neq 0$ for $i \in I$.
Then, the test procedure is defined as follows:
i) Order the observations $\left|y_{(1)}\right| \geq\left|y_{(2)}\right| \geq \ldots \geq\left|y_{(n)}\right|$
ii) For $i=1, \ldots, n$, let $X_{(i)}=-\log \left(1-F_{|\varepsilon|}\left(\left|y_{(i)}\right|\right)\right)$,
iii) Let $T_{j}=\sum_{i=1}^{j} X_{(i)}$ and $Q_{j}=\mathbb{E}_{H_{0}}\left(T_{j} \mid T_{n}\right)$.
iv) Define the test statistic $D_{n}=\max _{j}\left|T_{j}-Q_{j}\right| / \sqrt{n}$. We will reject the null hypothesis if $D_{n}>d_{\alpha}$, where $d_{\alpha}$ is defined in Section 3.

Remark 1: Under the null hypothesis, the sequence $\left(X_{(i)}\right)$ is a decreasing sequence of exponential random variables with parameter 1 . Then, the conditional expectation $\mathbb{E}_{H_{0}}\left(T_{j} \mid T_{n}\right)$ can easily be computed using the following proposition (the proof is given in the appendix):

Proposition 2.1 Assume $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is an ordered sequence of Exp(1) random variables, with $X_{(1)} \geq X_{(2)} \geq \ldots X_{(n)}$. For any $1 \leq j \leq n$, let $T_{j}=\sum_{i=1}^{j} X_{(i)}$. Then, for any $j \leq K \leq n$,

$$
\begin{align*}
& \mathbb{E}\left(X_{(i)}\right)=\sum_{\ell=1}^{n} \frac{1}{\ell}  \tag{2}\\
& \mathbb{E}\left(T_{j}\right)=j+j \sum_{i=j+1}^{n} \frac{1}{i}  \tag{3}\\
& \mathbb{E}\left(T_{j} \mid T_{K}\right)=\mathbb{E}\left(T_{j}\right)  \tag{4}\\
& \mathbb{E}\left(T_{K}\right) \\
& T_{K}
\end{align*}
$$

Remark 2: The distribution of the test statistic $D_{n}$ cannot be computed in a closed form. Nevertheless, the following standard result will allow us to construct probability tables (the proof is given in the Appendix):

Theorem 2.1 Assume $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ is an ordered sequence of $\operatorname{Exp}(1)$ random variables, with $X_{(1)} \geq X_{(2)} \geq \ldots X_{(n)}$. For any $1 \leq j \leq n$, let $T_{j}=\sum_{i=1}^{j} X_{(i)}$. Introduce for $t \in[0,1]$ the random process $d_{n}(t)=T_{[n t]}-\mathbb{E}\left(T_{[n t]} \mid T_{n}\right)$. Then, $\frac{1}{\sqrt{n}} d_{n}(t)$, as a stochastic process indexed on $t \in[0,1]$, converges in distribution to a zero mean Gaussian process $\Delta$ with covariance function defined by

$$
\begin{aligned}
\mathbb{E}(\Delta(t) \Delta(s))=\int_{0}^{1} & \int_{0}^{1}[(1-u) \wedge(1-v)-(1-u)(1-v)]\left[\mathbb{1}_{[0, t]}(u)-t+t \log (t)\right] \\
& \times\left[\mathbb{I}_{[0, s]}(v)-s+s \log (s)\right] d G^{-1}(u) d G^{-1}(v)
\end{aligned}
$$

where $G(x)$ is the distribution function of an exponential r.v.

Using Theorem 2.1, we can conclude that statistic $D_{n}$ defined in the test procedure converges weakly to $\Delta_{\infty}=\sup _{t} \Delta(t)$. Then, $d_{\alpha}$ is defined as the $\alpha$ quantile of $\Delta_{\infty}$.

Remark 3: Instead of assuming that the distribution of the $\left|\varepsilon_{i}\right|$ is known, we can assume that there exists an increasing continuous function $h: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$such that the cumulative distribution function $F_{h}$ of $h(|\varepsilon|)$ is known. Then, $X_{(i)}$ is defined as $-\log \left(1-F_{h}\left(h\left(\left|y_{(i)}\right|\right)\right)\right)$. Without any loss of generality, we will consider the case $h=i d$ in the following.

Remark 4: A uniform change of variable can also be used, by setting $X_{(i)}=F_{|\varepsilon|}\left(\left|y_{(i)}\right|\right)$. Indeed, the conditional expectation of $T_{j}$ can also be computed here:

$$
\mathbb{E}\left(T_{j} \mid T_{K}\right)=\frac{j(K-j)}{K+1}+\frac{j(j+1)}{K(K+1)} T_{K}
$$

### 2.2 Choosing the right coefficients

If we reject the null hypothesis, the next step is to select the significant coefficients.
Let $s$ be the one to one mapping from $\{1,2, \ldots, n\}$ to $\{1,2, \ldots, n\}$ defined by $Y_{s(i)}=Y_{(i)}$ (recall that $\left(Y_{(i)}\right)$ is a decreasing sequence).

Then, define the set of alternative hypotheses:
Alternative hypothesis:
$\boldsymbol{H}_{\mathbf{1}}(\boldsymbol{k})$ : there exists a subset $I_{k} \subset\{1,2, \ldots, n\}$ such that,

- for any $i \in I_{k}, s(i) \leq k$ and $\mathbb{E}_{H_{1}(k)}\left(Y_{i}\right) \neq 0$,
- for any $i \notin I_{k}, s(i) \geq k+1$ and $\mathbb{E}_{H_{1}(k)}\left(Y_{i}\right)=0$,

Under $\boldsymbol{H}_{\mathbf{1}}(\boldsymbol{k})$, there are $k$ significant coefficients and $\left|y_{(k+1)}\right|, \ldots,\left|y_{(n)}\right|$ have distribution $F_{|\varepsilon|}$. Then, we define the following test procedure:
i) For $i=1, \ldots, n$, let $X_{(i)}=-\log \left(1-F_{|\varepsilon|}\left(\left|y_{(i)}\right|\right)\right)$,
ii) Let $K_{n}$ be some positive integer. For $1 \leq k \leq n-K_{n}$ and $1 \leq j \leq K_{n}$, compute

$$
\begin{align*}
T_{k, j} & =\sum_{i=k+1}^{k+j} X_{(i)},  \tag{5}\\
Q_{k, j} & =\mathbb{E}_{H_{1}(k)}\left(T_{k, j} \mid T_{k, K_{n}}\right),  \tag{6}\\
\eta_{k} & =\max _{1 \leq j \leq K_{n}} \frac{\left|T_{k, j}-Q_{k, j}\right|}{\sqrt{n}} . \tag{7}
\end{align*}
$$

iii) Let

$$
\hat{k}=\operatorname{Arg} \min _{1 \leq k \leq n-K_{n}} \eta_{k}
$$

Remark 1: The $\ell_{1}$ or the $\ell_{2}$ norms can be used instead of the $\ell_{\infty}$ norm to define $\eta$ by setting

$$
\eta_{k}=n^{-\frac{3}{2}} \sum_{j=1}^{K_{n}}\left|T_{k, j}-Q_{k, j}\right|
$$

or

$$
\eta_{k}=n^{-2} \sum_{j=1}^{K_{n}}\left(T_{k, j}-Q_{k, j}\right)^{2}
$$

Remark 2: $Q_{k, j}$ can easily be computed using the results of the previous section. Indeed, Let

$$
\begin{equation*}
B_{k, j, n}=\frac{\mathbb{E}_{k}\left(T_{k, j}\right)}{\mathbb{E}_{k}\left(T_{k, K_{n}}\right)}=\frac{j\left(1+\sum_{i=j+1}^{n-k} 1 / i\right)}{K_{n}\left(1+\sum_{i=K_{n}+1}^{n-k} 1 / i\right)} \tag{8}
\end{equation*}
$$

Then, Proposition 2.1 yields $Q_{k, j}=B_{k, j, n} T_{k, K_{n}}$.
In order to state our main consistency result, we consider the following asymptotic framework:

AF1 There exists $t^{\star} \in(0,1)$ and a subset $I_{k_{n}^{\star}}$ of $\{1,2, \ldots, n\}$ with $k_{n}^{\star}=\left[t^{\star} n\right]$, such that $\mu_{i} \neq 0$ if $i \in I_{k_{n}^{\star}}$. For all other index, $\mu_{i}=0$.

AF2 For any $i \in I_{k_{n}^{\star}},\left|\mu_{i}\right| \geq \alpha_{n}$, where $\alpha_{n} \rightarrow \infty$ according to the distribution of the $\left(\varepsilon_{i}\right)$. Let $\Phi_{(1)}$ be the distribution of $\max _{1 \leq i \leq n}\left|\varepsilon_{i}\right|$ and $\left(a_{n}, b_{n}\right)$ such that $\Phi_{(1)}\left(a_{n}+b_{n} x\right) \rightarrow W(x)$ for some fixed distribution $W$. Then $\left(\alpha_{n}\right)$ satisfies

$$
\begin{equation*}
\frac{\alpha_{n}-2 a_{n}}{b_{n}} \rightarrow \infty \tag{9}
\end{equation*}
$$

AF3 $K_{n} / n \rightarrow c$ such that $0<c<1-t^{\star}$.

We have the following result

Theorem 2.2 Let $\left(u_{n}\right)$ be any positive and decreasing sequence such that $\sqrt{n} u_{n} \rightarrow \infty$. Then, under the asymptotic framework defined by AF1, AF2,AF3,

$$
\begin{equation*}
P\left(\left|\frac{\hat{k}}{n}-t^{\star}\right|>u_{n}\right) \rightarrow 0 \tag{10}
\end{equation*}
$$

Moreover, for $a>0$ there exist constants $c_{1}, c_{2}$ which depend on a such that if

$$
u_{n}=\frac{c_{1} \alpha_{n} \sqrt{\log n}}{2 \sqrt{n}}+\frac{c_{2} \alpha_{n} \log (n)}{2 n}
$$

then

$$
\begin{equation*}
\mathbb{P}_{H_{1}\left(k_{n}^{\star}\right)}\left(\left|\frac{\hat{k}}{n}-t^{\star}\right|>u_{n}\right) \leq 2 e^{-a \log (n)}+2 \mathbb{P}\left(\max _{1 \leq i \leq n}\left|\varepsilon_{i}\right|>\alpha_{n}\right) \tag{11}
\end{equation*}
$$

The proof of Theorem 2.2 is given in Section 3.

## 3 Proof of Theorem 2.2

Our procedure is based on two facts: a)under mild assumptions over the error distribution, if the null hypothesis is rejected, that is, if there is a group of significant coefficients and one of non significant coefficients, both groups of observations will be stochastically in order with high probability and b) for
two separate groups, separated at index $k_{n}^{\star}=\left[t^{\star} n\right], T_{k_{n}, j}-Q_{k_{n}, j}$ will only converge at rate $\sqrt{n}$ for index $k_{n}$ such that $\left|k_{n}-k_{n}^{\star}\right|=o(\sqrt{n})$.

Set $u_{i}=y_{i}$ for $i \in I_{k_{n}^{\star}}$ and $v_{i}=y_{i}$ for $i \notin I_{k_{n}^{\star}}$. Thus $\left(v_{i}\right)$ is an i.i.d. sequence with distribution $F_{\varepsilon}$.
We have the following lemma that assures that both collections are stochastically in order with high probability:

Lemma 3.1 Let $\left(u_{(i)}\right)$ and $\left(v_{(i)}\right)$ be the sequences $\left(\left|u_{i}\right|\right)$ and $\left(\left|v_{i}\right|\right)$ in a decreasing order. Then

$$
\mathbb{P}\left(v_{(1)}>u_{\left(k_{n}^{\star}\right)}\right) \rightarrow 0
$$

and

$$
\mathbb{P}\left(v_{(1)}>\frac{\alpha_{n}}{2}\right) \rightarrow 0
$$

Proof: By assumption $\left(v_{(1)}-a_{n}\right) / b_{n} \xrightarrow{D} W$. On the other hand, let $\left(\tilde{v}_{i}\right)$ be a sequence of i.i.d. r.v. with distribution $F_{\varepsilon}$. Then,

$$
\begin{aligned}
\mathbb{P}\left(v_{(1)}>u_{\left(k_{n}^{*}\right)}\right) & \leq \mathbb{P}\left(v_{(1)}+\tilde{v}_{(1)}>\alpha_{n}\right) \\
& \leq \mathbb{P}\left(v_{(1)}>\alpha_{n} / 2\right)+\mathbb{P}\left(\tilde{v}_{(1)}>\alpha_{n} / 2\right) \\
& \leq 2 \mathbb{P}\left(v_{(1)}>\alpha_{n} / 2\right) \rightarrow 2 W\left(\frac{\alpha_{n} / 2-a_{n}}{b_{n}}\right) \rightarrow 0
\end{aligned}
$$

Lemma 3.1 yields the $\left(u_{(i)}\right)$ and $\left(v_{(i)}\right)$ are stochastically in order with high probability. Let $\Omega_{n}$ be the subset of $\Omega$ where $v_{(1)}<\alpha_{n} / 2$ and $u_{\left(k_{n}^{\star}\right)}>\alpha_{n} / 2$. Clearly $P\left(\Omega_{n}\right) \rightarrow 1$. In what follows we will restrict our proof to $\Omega_{n}$.

We will denote $\mathbb{E}_{k}()$ the expectation under $\boldsymbol{H}_{\mathbf{1}}(\boldsymbol{k})$ (instead of $\left.\mathbb{E}_{H_{1}(k)}()\right)$. On the other hand, let $a_{i}=\mathbb{E}_{0}\left(X_{(i)}\right)=\sum_{\ell=1}^{n} 1 / \ell$.

1) Consider first the case $k>k_{n}^{\star}$. On $\Omega_{n}$,

$$
\begin{aligned}
T_{k, j}-Q_{k, j}= & T_{k, j}-B_{k, j, n} T_{k, K_{n}} \\
= & \left(T_{k, j}-\mathbb{E}_{k_{n}^{\star}}\left(T_{k, j}\right)\right)-B_{k, j, n}\left(T_{\left(k, K_{n}\right.}-\mathbb{E}_{k_{n}^{\star}}\left(T_{k, K_{n}}\right)\right) \\
& +\mathbb{E}_{k_{n}^{\star}}\left(T_{k, j}\right)-B_{k, j} \mathbb{E}_{k_{n}^{\star}}\left(T_{k, K_{n}}\right) \\
= & R_{k, j}+S_{k, j}
\end{aligned}
$$

We have decomposed the statistics $T_{k, j}-Q_{k, j}$ into a random part $R_{k, j}$ and a deterministic part $S_{k, j}$. First, let $k=[t n]$ and $j=[s n]$ for $t \leq s$. As in Theorem 2.1, $R_{k, j} \mathbb{I}_{\Omega_{n}}$ (normalized by $\sqrt{n}$ ) as a process indexed by $(t, s) \in(0,1)^{2}$ converges in distribution to a zero-mean Gaussian process $\Gamma_{t, s}=$ $\left(1-t^{\star}\right)\left[\Delta\left(s-t^{\star}\right)-\Delta\left(t-t^{\star}\right)\right]$.

On the other hand,

$$
\begin{aligned}
S_{k, j} & =\mathbb{E}_{k_{n}^{\star}}\left(T_{k, j}\right)-\frac{\mathbb{E}_{k}\left(T_{k, j}\right)}{\mathbb{E}_{k}\left(T_{k, K_{n}}\right)} \mathbb{E}_{k_{n}^{\star}}\left(T_{k, K_{n}}\right) \\
& =\mathbb{E}_{k_{n}^{\star}}\left(T_{k, j}\right)-\mathbb{E}_{k}\left(T_{k, j}\right)-\frac{\mathbb{E}_{k}\left(T_{k, j}\right)}{\mathbb{E}_{k}\left(T_{k, K_{n}}\right)}\left(\mathbb{E}_{k_{n}^{\star}}\left(T_{k, K_{n}}\right)-\mathbb{E}_{k}\left(T_{k, K_{n}}\right)\right) \\
& =\sum_{i=j+1}^{j+k-k_{n}^{\star}} a_{i}-\sum_{i=1}^{k-k_{n}^{\star}} a_{i}+B_{k, j, n}\left(\sum_{i=K_{n}+1}^{K_{n}+k-k_{n}^{\star}} a_{i}-\sum_{i=1}^{k-k_{n}^{\star}} a_{i}\right) \\
& =\sum_{i=1}^{k-k_{n}^{\star}}\left(a_{i+j}-a_{i}+B_{k, j, n}\left(a_{i+K_{n}}-a_{i}\right)\right)
\end{aligned}
$$

Thus, there exists a constant, $\gamma>0$, which depends on $c$ in [AF3], such that $\sup _{j}\left|S_{k, j}\right| \geq \gamma\left(k-k_{n}^{\star}\right)$ and

$$
\begin{align*}
& \mathbb{P}_{k_{n}^{\star}}\left(k_{n}^{\star}-\widehat{k}>n u_{n}\right) \leq \mathbb{P}\left(\eta_{k_{n}^{\star}}>\sup \eta_{k},\left(k-k_{n}^{\star}\right)>n u_{n}\right)  \tag{12}\\
\leq & \mathbb{P}\left(2 \sup _{k} \sup _{j} R_{k, j}>\inf _{k} \sup _{j}\left|S_{k, j}\right|,\left(k-k_{n}^{\star}\right)>n u_{n}\right)+\mathbb{P}\left(\Omega_{n}^{c}\right) \\
\leq & \mathbb{P}\left(2 \sup _{k} \sup _{j} R_{k, j}>\gamma n u_{n}\right)+\mathbb{P}\left(\Omega_{n}^{c}\right) .
\end{align*}
$$

Because of the weak convergence of $R_{k, j} \mathbb{I}_{\Omega_{n}}$ the above probability tends to zero when $n$ goes to infinity.
2) Consider now the case $k<k_{n}^{\star}$. On $\Omega_{n}$,

$$
\begin{aligned}
T_{k, j}-Q_{k, j}= & T_{k, j}-B_{k, j, n} T_{k, K_{n}} \\
= & \left(1-B_{k, j, n}\right) T_{k, k_{n}^{\star}-k}+\left(T_{k_{n}^{\star}, j}-\mathbb{E}\left(T_{k_{n}^{\star}, j}\right)\right)-B_{k, j, n}\left(T_{k_{n}^{\star}, K_{n}}-\mathbb{E}\left(T_{k_{n}^{\star}, K_{n}}\right)\right) \\
& +\mathbb{E}\left(T_{k_{n}^{\star}, j}\right)-B_{k, j, n} \mathbb{E}\left(T_{k_{n}^{\star}, K_{n}}\right) \\
= & A_{k, j}+R_{k_{n}^{\star}, j}+U_{k, j}
\end{aligned}
$$

where $A_{k, j}=\left(1-B_{k, j, n}\right) T_{k, k_{n}^{\star}-k}$. Remark that over $\Omega_{n},\left|y_{(i)}\right|>\alpha_{n} / 2$. Then, there exists $c\left(\alpha_{n}\right)>0$ such that $T_{k, k_{n}^{\star}-k}>c\left(\alpha_{n}\right)\left(k_{n}^{\star}-k\right)$, thus $A_{k, j}=\mathcal{O}\left(k_{n}^{\star}-k\right)$. On the other hand, $R_{k_{n}^{\star}, j} \mathbb{I}_{\Omega_{n}}$ converges in distribution to a zero-mean Gaussian process $\Gamma_{t^{\star}, s}$. Consider now the bias term $U_{k, j}$ :

$$
\begin{aligned}
U_{k, j}= & \mathbb{E}_{k_{n}^{\star}}\left(T_{k_{n}^{\star}, j}\right)-\frac{\mathbb{E}_{k}\left(T_{k, j}\right)}{\mathbb{E}_{k}\left(T_{k, K_{n}}\right)} \mathbb{E}_{k_{n}^{\star}}\left(T_{k_{n}^{\star}, K_{n}}\right) \\
= & \mathbb{E}_{k_{n}^{\star}}\left(T_{k_{n}^{\star}, j}\right)-\mathbb{E}_{k}\left(T_{k_{n}^{\star}, j}\right)-\frac{\mathbb{E}_{k}\left(T_{k_{n}^{\star}, j}\right)}{\mathbb{E}_{k}\left(T_{k, K_{n}}\right)}\left(\mathbb{E}_{k_{n}^{\star}}\left(T_{k_{n}^{\star}, K_{n}}\right)-\mathbb{E}_{k}\left(T_{k, K_{n}}\right)\right) \\
& +\frac{\mathbb{E}_{k_{n}^{\star}}\left(T_{k_{n}^{\star}, K_{n}}\right)}{\mathbb{E}_{k}\left(T_{k, K_{n}}\right)} \mathbb{E}_{k}\left(T_{k, k_{n}^{\star}}\right) \\
= & \sum_{i=j+k-k_{n}^{\star}+1}^{j-k} a_{i}-\sum_{i=1}^{k_{n}^{\star}-k} a_{i}+\frac{\mathbb{E}_{k}\left(T_{k_{n}^{\star}, j}\right)}{\mathbb{E}_{k}\left(T_{k, K_{n}}\right)} \sum_{i=K_{n}+k-k_{n}^{\star}+1}^{K_{n}} a_{i}-\frac{\mathbb{E}_{k_{n}^{\star}}\left(T_{k_{n}^{\star}, K_{n}}\right)}{\mathbb{E}_{k}\left(T_{k, K_{n}}\right)} \sum_{i=1}^{k_{n}^{\star}-k} a_{i} .
\end{aligned}
$$

Thus, there exists a constant $\delta>0$, which depends on $c$ in [AF3], such that $\sup _{j}\left|U_{k, j}\right| \geq \delta\left(k-k_{n}^{\star}\right)$ and $\mathbb{P}_{k_{n}^{\star}}\left(\widehat{k}-k_{n}^{\star}>n u_{n}\right) \rightarrow 0$ when $n \rightarrow \infty$. The latter together with (12) shows (10).

In order to show (11), sharper bounds on $\mathbb{P}_{k_{n}^{\star}}\left(\sup _{k} \sup _{j} R_{k, j}>C n u_{n}\right)$ are required for any given constant $C$. As above, we will restrict our attention to the set $\Omega_{n}$ and drop this fact from the notation. Consider first as above the case $k>k_{n}^{\star}$. Write, over $\Omega_{n}$,

$$
\begin{aligned}
R_{k, j} & =\left(T_{k, j}-\mathbb{E}_{k_{n}^{\star}}\left(T_{k, j}\right)\right)-B_{k, j, n}\left(T_{k, K_{n}}-\mathbb{E}_{k_{n}^{\star}}\left(T_{k, K_{n}}\right)\right) \\
& =R_{k, j}^{(1)}+R_{k, j}^{(2)} .
\end{aligned}
$$

Remark $\sup _{j} B_{k, j, n}=1$. So that $\sup _{j}\left|R_{k, j}^{(2)}\right|=\left|T_{k, K_{n}}-\mathbb{E}_{k_{n}^{\star}}\left(T_{k, K_{n}}\right)\right|$.
Let $G$ denote the common distribution function of the collection $\left(X_{i}\right)$. We can rewrite

$$
T_{k, K_{n}}-\mathbb{E}_{k_{n}^{\star}}\left(T_{k, K_{n}}\right)=\sum_{i}\left[X_{i}-\mathbb{E}_{k_{n}^{\star}}\left(X_{i}\right)\right] \mathbb{I}_{\left\{G^{-1}\left(1-K_{n} / n\right)<X_{i}\right\}}
$$

Thus,

$$
\frac{T_{k, K_{n}}-\mathbb{E}_{k_{n}^{\star}}\left(T_{k, K_{n}}\right)}{\alpha_{n} / 2}
$$

is the sum of independent bounded r.v. with variance bounded by 1 , so that by Bennet's inequality

$$
\begin{aligned}
\mathbb{P}\left(\frac{\sup _{j}\left|R_{k, j}^{(2)}\right|}{\alpha_{n} / 2}>\frac{\gamma / 2 n u_{n}}{\alpha_{n} / 2}\right) & \leq \mathbb{P}\left(\frac{\left|T_{k, K_{n}}-\mathbb{E}_{k_{n}^{\star}}\left(T_{k, K_{n}}\right)\right|}{\alpha_{n} / 2}>\frac{c_{1} \gamma}{2 \sqrt{2}} \sqrt{2 n \log n}+\frac{3 c_{2} \gamma}{2} \frac{\log n}{3}\right) \\
& \leq e^{-(a+1) \log (n)}
\end{aligned}
$$

choosing $c_{2} \geq \frac{2(a+1)}{3 \gamma}$ and $c_{1} \geq 2 \frac{\sqrt{2} \sqrt{a+1}}{\gamma}$.
Hence, summing in $k$

$$
\mathbb{P}\left(\sup _{k} \sup _{j<k+K_{n}} R_{k, j}^{(2)}>\frac{\gamma}{2 n u_{n}}\right) \leq e^{-a \log n}
$$

For $R_{k, j}^{(1)}$ we have

$$
\frac{T_{k, j}-\mathbb{E}_{k_{n}^{\star}}\left(T_{k, j}\right)}{\alpha_{n} / 2}=\sum_{i}\left[X_{i}-\mathbb{E}_{k_{n}^{\star}}\left(X_{i}\right)\right] \mathbb{I}_{\left\{G^{-1}(1-j / n)<X_{i}\right\}}
$$

Hence in this case we must use a functional version of Bennet's inequality (Theorem 7.3 in [4]) which yields

$$
\mathbb{P}\left(\frac{\sup _{j}\left|R_{k, j}^{(2)}\right|}{\alpha_{n} / 2}>\mathbb{E}\left(\frac{\sup _{j}\left|R_{k, j}^{(2)}\right|}{\alpha_{n} / 2}\right)+\sqrt{2 x v}+\frac{x}{3}\right) \leq e^{-x},
$$

for $v \geq n+2 \mathbb{E}\left(\frac{\sup _{j}\left|R_{k, j}^{(2)}\right|}{\alpha_{n} / 2}\right)$. Thus it remains to bound $A=\mathbb{E}\left(\frac{\sup _{j}\left|R_{k, j}^{(2)}\right|}{\alpha_{n} / 2}\right)$. This can be done using standard symmetrization and entropy techniques to obtain, $A \leq 4 \sqrt{n \log n}$, as the random entropy of the class $\mathcal{A}=\left\{\mathbb{I}_{G^{-1}(1-t), t \in[0,1]}\right\}$ (as it is a collection of increasing functions) is bounded by $2 \log n$.

As above,

$$
\begin{aligned}
\mathbb{P}\left(\frac{\sup _{j}\left|R_{k, j}^{(1)}\right|}{\alpha_{n} / 2}>\frac{\gamma / 2 n u_{n}}{\alpha_{n} / 2}\right) \leq & \mathbb{P}\left(\frac{2\left|T_{k, j}-\mathbb{E}_{k_{n}^{\star}}\left(T_{k, j}\right)\right|}{\alpha_{n}}>4 \sqrt{n \log n}\right. \\
& \left.+\sqrt{2(a+1) \log n(n+4 \sqrt{n \log n})}+(a+1) \frac{\log n}{3}\right) \\
\leq & e^{-(a+1) \log (n)}
\end{aligned}
$$

choosing $c_{i}, i=1,2$ appropriately.
The case $k<k_{n}^{\star}$ follows analogously.

## 4 Some extensions

### 4.1 Unknown distribution

Assume now that the distribution $F_{\varepsilon}$ of the $\varepsilon_{i}$ 's is a parametric distribution $F_{\varepsilon}\left(\cdot ; \theta^{\star}\right)$, but where the parameter $\theta^{\star}$ is unknown. For any $0 \leq k \leq n-1$, let $\widehat{\theta}_{k}=\widehat{\theta}\left(y_{(k+1)}, y_{(k+2)}, \ldots, y_{(n)}\right)$ be an estimator of $\theta$. Let $F_{|\varepsilon|}\left(\cdot ; \theta^{\star}\right)$ be the distribution of the $\left|\varepsilon_{i}\right|$ 's. We will consider the following assumptions:

F1 The cumulative distribution function $F_{|\varepsilon|}$ is two times differentiable as a function of $\theta$ with a.e. strictly positive derivative at $\theta=\theta^{\star}$.

F2 $\theta^{\star}$ belongs to some compact set $\Theta$ and there exists, under $H_{k_{n}^{\star}}$, a consistent estimator $\widehat{\theta}_{k_{n}^{\star}}=$ $\widehat{\theta}\left(y_{\left(k_{n}^{\star}+1\right)}, y_{\left(k_{n}^{\star}+2\right)}, \ldots, y_{(n)}\right)$ of $\theta^{\star}$.

F3 There exists $(a, b)$ such that $0<a<t^{\star}<b<1$ and a Lipschitz continuous function $\tilde{\theta}$ defined on [ $a, b]$ such that, under $H_{k_{n}^{\star}},\left(\widehat{\theta}_{[t n]}\right)$ converges uniformly on $[a, b]$ in probability to $(\tilde{\theta}(t))$.

Remark 1: under hypothesis F2 and F3, $\widehat{\theta}_{k_{n}^{\star}}$ is a consistent estimator of $\tilde{\theta}\left(t^{\star}\right)=\theta^{\star}$.
Remark 2: When $t<t^{\star}$, convergence of $\widehat{\theta}_{[t n]}$ can be difficult to check with any estimator, since $\widehat{\theta}_{[t n]}$ depends on some $y_{i}$ 's that are not distributed under distribution $F_{\varepsilon}$. Nevertheless, it is possible to use an estimator based on some empirical quantiles and that only depends on the smallest observations, that is, that depends only on the observations distributed under $F_{\varepsilon}$.

For any $\theta \in \Theta$, let $X_{i}(\theta)=-\log \left(1-F_{|\varepsilon|}\left(\left|y_{i}\right|, \theta\right)\right)$, and $T_{k, j}(\theta)=\sum_{i=k+1}^{k+j} X_{(i)}(\theta)$.
Then, we define the following procedure:
i) Let $K_{n} \leq[(1-b) n]$ be some positive integer. For $[a n] \leq k \leq n-K_{n}$,

1. let $\widehat{\theta}_{k}=\widehat{\theta}\left(y_{(k+1)}, y_{(k+2)}, \ldots, y_{(n)}\right)$,
2. for $i=1, \ldots, n$, let $X_{(i)}\left(\widehat{\theta}_{k}\right)=-\log \left(1-F_{|\varepsilon|}\left(\left|y_{(i)}\right| ; \widehat{\theta}_{k}\right)\right)$,
3. for $1 \leq j \leq K_{n}$, compute

$$
\begin{aligned}
T_{k, j}\left(\widehat{\theta}_{k}\right) & =\sum_{i=k+1}^{k+j} X_{(i)}\left(\widehat{\theta}_{k}\right), \\
Q_{k, j}\left(\widehat{\theta}_{k}\right) & =B_{k, j, n} T_{k, K_{n}}\left(\widehat{\theta}_{k}\right) \\
\eta_{k}\left(\widehat{\theta}_{k}\right) & =\max _{k+1 \leq j \leq n} \frac{\left|T_{k, j}\left(\widehat{\theta}_{k}\right)-Q_{k, j}\left(\widehat{\theta}_{k}\right)\right|}{\sqrt{n}} .
\end{aligned}
$$

iii) Let

$$
\hat{k}=\operatorname{Arg} \min _{a n \leq k \leq b n} \eta_{k}\left(\widehat{\theta}_{k}\right)
$$

Remark: Here, $Q_{k, j}\left(\widehat{\theta}_{k}\right)=B_{k, j, n} T_{k, K_{n}}\left(\widehat{\theta}_{k}\right)$ is the conditional expectation of $T_{k, j}$, conditionally to $T_{k, K_{n}}$, assuming that $k_{n}^{\star}=k$ and that $\theta^{\star}=\widehat{\theta}_{k}$.

Then, we have the following result,

Theorem 4.1 Assume F1, F2, F3.
i) Introduce for $t \in[0,1]$ and $s \in[a, b]$ the random process

$$
\hat{d}_{n}(t, s)=T_{k_{n}^{\star},\left[K_{n} t\right]}\left(\widehat{\theta}_{[n s]}\right)-\mathbb{E}_{H_{k_{n}^{\star}}}\left(T_{k_{n}^{\star},\left[K_{n} t\right]}\left(\widehat{\theta}_{[n s]}\right) \mid T_{k_{n}^{\star}, K_{n}}\left(\widehat{\theta}_{[n s]}\right)\right) .
$$

Then, $\hat{d}_{n}(t, s) / \sqrt{n}$, as a stochastic process indexed on $[0,1] \times[a, b]$, converges in distribution, under $\boldsymbol{H}_{\boldsymbol{k}_{n}^{\star}}$, to a zero mean Gaussian process $(\Lambda(t, s))$.
ii) Let $\left(u_{n}\right)$ be any positive and decreasing sequence such that $\sqrt{n} u_{n} \rightarrow \infty$. Then, under the asymptotic framework defined by AF1, AF2, AF3,

$$
\begin{equation*}
\mathbb{P}_{H_{1}\left(k_{n}^{\star}\right)}\left(\left|\frac{\hat{k}}{n}-t^{\star}\right|>u_{n}\right) \rightarrow 0 \tag{13}
\end{equation*}
$$

Proof: We first show $i$ ).
With the above notation, for any $\theta \in \Theta$, let

$$
\begin{aligned}
\Psi_{j}(\theta) & =T_{k_{n}^{\star}, j}(\theta)-Q_{k_{n}^{\star}, j}(\theta) \\
\Psi_{j}^{\prime}(\theta) & =\frac{\partial \Psi_{j}}{\partial \theta}(\theta)
\end{aligned}
$$

For any $t \in[0,1]$ and $s \in[a, b]$, let

$$
\begin{aligned}
\hat{d}_{n}(t, s) & =\Psi_{[n t]}\left(\widehat{\theta}_{[n s]}\right) \\
& =\Psi_{[n t]}(\tilde{\theta}(s))+\left(\widehat{\theta}_{[n s]}-\tilde{\theta}(s)\right) \Psi_{[n t]}^{\prime}(\tilde{\theta}(s))++\mathcal{O}\left(\left(\tilde{\theta}(s)-\widehat{\theta}_{[n s]}\right)^{2}\right) .
\end{aligned}
$$

Using the same proof used for the convergence of $\left(\Psi_{[n t]}\left(\theta^{\star}\right)\right) / \sqrt{n}$ (see the Appendix), we show that, for any $s \in[a, b],\left(\Psi_{[n t]}(\tilde{\theta}(s))\right) / \sqrt{n}$ and $\left(\Psi_{[n t]}^{\prime}(\tilde{\theta}(s))\right) / \sqrt{n}$ also converge to two zero-mean Gaussian processes. Then, using hypothesis $\mathbf{F 3}$, $\widehat{\theta}_{[n s]} \rightarrow \tilde{\theta}(s)$ uniformly over $[a, b]$, and then, $\left(\hat{d}_{n}(t, s)\right)$ converges to a zero mean Gaussian process $(\Lambda(t, s))$.

We show now $i i)$. For any $\theta \in \Theta$, let $a_{i}(\theta)=\mathbb{E}_{H_{0}}\left(X_{(i)}(\theta)\right)$. Following the proof of Theorem 2.2, consider first the case $k>k_{n}^{\star}$. On $\Omega_{n}$, for any $\theta \in \Theta$,

$$
\begin{aligned}
T_{k, j}(\theta)-Q_{k, j}(\theta)= & \left(T_{k, j}(\theta)-\mathbb{E}_{k_{n}^{*}}\left(T_{k, j}(\theta)\right)\right)-B_{k, j, n}\left(T_{k, K_{n}}(\theta)-\mathbb{E}_{k_{n}^{*}}\left(T_{k, K_{n}}(\theta)\right)\right) \\
& +\mathbb{E}_{k_{n}^{*}}\left(T_{k, j}(\theta)\right)-B_{k, j} \mathbb{E}_{k_{n}^{*}}\left(T_{k, K_{n}}(\theta)\right) \\
= & R_{k, j}(\theta)+S_{k, j}(\theta)
\end{aligned}
$$

As in Theorem 2.2, $R_{k, j}$ (normalized by $\sqrt{n}$ ) as a process indexed by $(t, s) \in(0,1)^{2}$ converges in distribution on $\Omega_{n}$ to a zero-mean Gaussian process $\Gamma_{t, s}(\theta)$.

On the other hand,

$$
S_{k, j}(\theta)=\sum_{i=1}^{k-k_{n}^{\star}}\left(a_{i+j}(\theta)-a_{i}(\theta)+B_{k, j, n}\left(a_{i+K_{n}}(\theta)-a_{i}(\theta)\right)\right)
$$

Thus, there exists a constant, $\gamma>0$, which depends on $a, b$ in [F3], such that $\sup _{j}\left|S_{k, j}\left(\widehat{\theta}_{k}\right)\right| \geq\left(k-k_{n}^{\star}\right) \gamma$. We conclude that $\mathbb{P}_{k_{n}^{\star}}\left(k_{n}^{\star}-\widehat{k}>n u_{n}\right) \rightarrow 0$ using the arguments used for Theorem 2.2. The case $k<k_{n}^{\star}$ is identical.

### 4.2 The unknown variance case

When $\theta^{\star}$ is a scale parameter, i.e. $F_{\varepsilon}\left(y ; \theta^{\star}\right)=F_{\varepsilon}\left(y / \theta^{\star} ; 1\right)$, we introduce the following procedure which is scale invariant:
i) For $i=1, \ldots, n$, let $X_{(i)}=\left|y_{(i)}\right|$,
ii) Let $K_{n}$ be some positive integer. For $1 \leq k \leq n-K_{n}$ and $1 \leq j \leq K_{n}$, compute

$$
\begin{align*}
T_{k, j} & =\sum_{i=1}^{k+j} X_{(i)},  \tag{14}\\
Q_{k, j} & =\frac{\mathbb{E}_{H_{1}(k)}\left(\sum_{i=k}^{k+j} X_{(i)}\right)}{\mathbb{E}_{H_{1}(k)}\left(\sum_{i=k}^{k+K_{n}} X_{(i)}\right)} T_{k, K_{n}},  \tag{15}\\
\eta_{k} & =\max _{1 \leq j \leq K_{n}} \frac{\left|T_{k, j}-Q_{k, j}\right|}{\sqrt{n}} . \tag{16}
\end{align*}
$$

iii) Let

$$
\hat{k}_{u}=\operatorname{Arg} \min _{1 \leq k \leq n-K_{n}} \eta_{k}
$$

Remark: Notice, the minimization problem at hand is not changed if we consider $\frac{\left|T_{k, j}-Q_{k, j}\right|}{\sigma \sqrt{n}}$, so the procedure is indeed scale invariant. We have the following result, whose proof is omitted as it resembles quite closely that of Theorem 2.2.

Theorem 4.2 Let $\left(u_{n}\right)$ be any positive and decreasing sequence such that $\sqrt{n} u_{n} \rightarrow \infty$. Then, under the asymptotic framework defined by AF1, AF2,AF3,

$$
P\left(\left|\frac{\hat{k}_{u}}{n}-t^{\star}\right|>u_{n}\right) \rightarrow 0
$$

Moreover, for $a>0$ there exist constants $c_{1}, c_{2}$ which depend on a such that if $u_{n}=\frac{c_{1} \alpha_{n} \sqrt{\log n}}{2 \sqrt{n}}+\frac{c_{2} \alpha_{n} \log (n)}{2 n}$ then

$$
P\left(\left|\frac{\hat{k}_{u}}{n}-t^{\star}\right|>u_{n}\right) \leq 2 e^{-a \log n}+2 P\left(\max _{1 \leq i \leq n} \frac{\left|\varepsilon_{i}\right|}{\sigma}>\alpha_{n}\right)
$$

### 4.3 Application to a regression problem

Consider as discussed in section 1 the following setting.

1. Assume we observe $y_{i}=f\left(x_{i}\right)+\varepsilon_{i}$ for a fixed collection $x_{i}$. Variables $\left(\varepsilon_{i}\right)$ are assumed to be independent and identically distributed with variance $\operatorname{Var}(\varepsilon)=\sigma^{2}$.
2. Associated to the collection $\left(x_{i}\right)$, we introduce the empirical inner product $\langle t, s\rangle_{n}=\frac{1}{n} \sum_{i} t\left(x_{i}\right) s\left(x_{i}\right)$ and its associated empirical norm $\|\cdot\|_{n}$.
3. We are interested in approximating $f$ in terms of a certain orthormal basis $\left\{\phi_{\lambda}\right\}_{\lambda}$. We assume that the basis is such that it is also orthonormal in the empirical norm $\langle,\rangle_{n}$.
4. Given the basis define the absolute empirical coefficients $\widehat{\beta}_{j}=\left|\left\langle y, \phi_{j}\right\rangle_{n}\right| \sqrt{n}$. More generally, we could consider the collection of the transformed coefficients $\gamma_{j}(h)=h\left(\widehat{\beta}_{j} / \sigma\right)$ for any given strictly increasing function $h$ such that there exists $\beta$ satisfying $h(a x)=a^{\beta} h(x)$ for any positive constant $a$.

In this section we are interested in the partial sums of the ordered variables $\widehat{\beta}$. If $\varepsilon_{i}$ follows a Gaussian distribution then $y_{j}=\left\langle y, \phi_{j}\right\rangle_{n} \sqrt{n} / \sigma$ are independent normal variables and it is straightforward to show that the procedure considered in section 4.2 can be applied. More precisely assume the following assumptions are satisfied

R1 $\varepsilon_{i}$ are an i.i.d. collection of centered normal r.v. with variance $\sigma^{2}$.
$\mathbf{R 2}\left\{\phi_{1}, \ldots, \phi_{n}\right\}$ is orthonormal w.r.t. the empirical norm $\langle,\rangle_{n}$.
R3 For any $i \in I_{k_{n}^{\star}},\left|\left\langle f, \phi_{j}\right\rangle_{n}\right|>a \sigma \sqrt{\log 2 n} / \sqrt{n}$, with $a \geq 2 \sqrt{2}$.

We have the following result,

Lemma 4.1 Assume AF1, AF3, R1, R2 and R3 hold true. Let $\hat{k}_{u}$, be the estimator defined in section 4.2. Then, for $b>0$ there exist constants $c_{1}, c_{2}$ which depend on $a$ and $b$ such that if $u_{n}=$ $\frac{c_{1} \log (n)}{2 \sqrt{n}}+\frac{c_{2} \log ^{2}(n)}{2 n}$ then

$$
P\left(\left|\frac{\hat{k}_{u}}{n}-t^{\star}\right|>u_{n}\right) \leq 2 e^{-b \log n}+2 e^{-(a / 2-\sqrt{2}) \log (n)} .
$$

Proof: It follows directly from Theorem 4.2 by checking that if $\epsilon_{i}, i=1, \ldots, n$ are independent standard normal random variables, then

$$
P\left(\max _{1 \leq i \leq n}\left|\epsilon_{i}\right|>a \sqrt{2 n}\right) \leq e^{-(a / 2-\sqrt{2}) \log (n)}
$$

### 4.4 Random thresholding

It is interesting we can link this procedure to a random thresholding one, or as in [2, 3] in terms of penalized estimation. This link clearly appears when we use the $\ell_{2}$-norm to define $\eta_{k}$ :

$$
\eta_{k}=n^{-2} \sum_{j=k+1}^{k+K_{n}}\left(T_{k, j}-Q_{k, j}\right)^{2}
$$

Lemma 2.2 ensures that good choice for the cutpoint between significant and non significant coefficients is $\widehat{k}=\arg \min \eta_{k}$. Thus, it is reasonable to assume we are looking from left to right to the first $k$ such that $\eta_{k}>\eta_{k-1}$. We will assume coefficients are significant while $\eta_{k}<\eta_{k-1}$. In order to develop this idea we must understand how $\eta_{k}-\eta_{k-1}$ looks like. We have

$$
\begin{aligned}
n^{2}\left(\eta_{k}-\eta_{k-1}\right)= & \left.\sum_{j=1}^{K_{n}}\left(T_{k, j}-B_{k, j} T_{k, K_{n}}\right)^{2}-\sum_{j=1}^{K_{n}-1} T_{k-1, j}-B_{k-1, j} T_{k-1, n}\right)^{2} \\
= & \left.\left(X_{k}-B_{k-1, k} T_{k-1, k-1+K_{n}}\right)^{2}+\sum_{j=1}^{K_{n}-1}\left(T_{k, j}-B_{k, j} T_{k, K_{n}}\right)\right)^{2} \\
& -\sum_{j=1}^{K_{n}-1}\left(X_{k}+T_{k, j}-B_{k, j}\left(X_{k}+T_{k, K_{n}}\right)(1+o(1))^{2}\right. \\
\approx & \left(X_{k}-B_{k-1, k} T_{k-1, k-1+K_{n}}\right)^{2} \\
& +\sum_{j=1}^{K_{n}-1} X_{k}^{2}\left(1-B_{k, j, n}\right)^{2}+2 X_{k}\left(1-B_{k, j, n}\right)\left(T_{k, j}-B_{k, j, n} T_{k, k+K_{n}}\right)
\end{aligned}
$$

Hence coefficients will be significant approximatively until the first $k$ such that

$$
X_{k} \leq \tau_{k, n}:=\frac{\sum_{j=1}^{K_{n}-1}\left(T_{k, j}-B_{k, j} T_{k, K_{n}}\right)\left(1-B_{k, j, n}\right)}{\sum_{j=1}^{K_{n}-1}\left(1-B_{k, j, n}\right)^{2}}
$$

Remark: If using the estimator $\widehat{k}_{u}$, this would yield a scale free random estimator $\tau_{k, n}$ of the threshold $\tau$. In the regression case, we obtain the traditional hard threshold scheme

$$
\left|\left\langle y, \phi_{j}\right\rangle_{n}\right|>\frac{\tau_{k, n}}{\sqrt{n}}
$$

## 5 Numerical experiments

We consider here the model

$$
\begin{equation*}
y_{i}=\mu_{i}+\varepsilon_{i} \tag{17}
\end{equation*}
$$

where $\left(\varepsilon_{i}\right)$ is a collection of i.i.d. r.v.

### 5.1 Testing the null hypothesis $H_{0}$

The distribution of $D_{n}=\max _{j}\left|T_{j}-\widehat{T}_{j}\right| / \sqrt{n}$ under $H_{0}$ is estimated by Monte-Carlo (using 5000 simulated samples). Here, the $\left(y_{i} ; 1 \leq i \leq n\right)$ are i.i.d. $\mathcal{N}(0,1)$ r.v. We set $X_{(i)}=-\log \left(1-F\left(y_{(i)}^{2}\right)\right)$ where $F$ is the cumulative distribution of a $\chi^{2}(1)$ distribution. Then, $\left(T_{j}\right),\left(\widehat{T}_{j}\right)$ and $D_{n}$ are computed as described in Section 2.1.

Table 5.1 displays the estimated percentiles of order $0.50,0.90,0.95$ and 0.99 obtained with different values of $n$. We see in this table that the distribution of $D_{n}$ (except the tail) does not depend on $n$ for $n \geq 20$. In particular, $\mathbb{P}_{H_{0}}\left(D_{n}>0.65\right) \approx 0.05$ for any $n \geq 20$.

| $n \backslash \alpha$ | 0.50 | 0.90 | 0.95 | 0.99 |
| :---: | :---: | :---: | :---: | :---: |
| 20 | 0.27 | 0.55 | 0.67 | 0.93 |
| 50 | 0.29 | 0.55 | 0.65 | 0.82 |
| 500 | 0.29 | 0.56 | 0.65 | 0.83 |
| 5000 | 0.30 | 0.55 | 0.64 | 0.79 |

Table 1: Estimated percentiles of $D_{n}$ under $H_{0}$ obtained with different values of $n$
Using a level $\alpha=5 \%$, the test consist in rejecting the null hypothesis $H_{0}$ if $D_{n}>0.65$. We estimated the power of this test, by simulating data under $H_{1}$. Here, the $\left(y_{i} ; 1 \leq i \leq n / 5\right)$ are i.i.d. $\mathcal{N}(\mu, 1)$ r.v. Figure 5.1 displays the estimated probability to reject the null hypothesis $H_{0}$ for different values of $\mu$ and $n$.

### 5.2 Estimating the number of significant coefficients

### 5.2.1 A Gaussian example

In the following experiment, we have simulated 500 Gaussian random variables, with $\mu_{i}=4$ for $1 \leq i \leq$ 100 , and $\mu_{i}=0$ for $101 \leq i \leq 500$. $\left(\varepsilon_{i}\right)$ is a collection of $\mathcal{N}(0,1)$ i.i.d. r.v.


Figure 1: The estimated power of the $5 \%$ level test, for different values of $\mu$ and $n$.

Assuming that the variance of the $\varepsilon_{i}$ 's is known, we set

$$
X_{i}=-\log \left(1-F\left(y_{i}^{2}\right)\right)
$$

where $F$ is the cumulative distribution function of a $\chi^{2}$ r.v.
Then, we used the procedure described in Section 2.1. Figure 5.2.1 displays the two sequences $\left(T_{k}\right)$ and $\left(Q_{k}\right)$. We find $D_{n}=14.95$ and reject the null hypothesis.


Figure 2: Example 1. The two sequences $\left(T_{k}\right)$ and $\left(Q_{k}\right)$
After rejecting the null hypothesis, we will estimate the number of significant coefficients, following the procedure described in Section 2.2. We use here $K=200$. Then, for $k=1,2, \ldots, 300$, we computed the sequences $\left(T_{k, j}, 1 \leq j \leq 200\right)$ and $\left(Q_{k, j}, 1 \leq j \leq 200\right)$. Figure 5.2.1 displays these two sequences for
$k=70, k=100$ and $k=130$. We see that $\left(T_{k, j}\right)$ concentrates around its conditional expected value $\left(\mathbb{E}_{H_{1}(k)}\left(T_{k, j} \mid T_{k, 200}\right)\right)$ only for $k=100$. A bias is clearly present for $k=70$ and $k=130$. The sequence $\left(\eta_{k}\right)$ defined by $\eta_{k}=\sum_{j=1}^{200}\left(T_{k, j}-Q_{k, j}\right)^{2} / \sqrt{n-k}$ is displayed Figure 5.2.1. A minimum at $\hat{k}=97$ is obvious.




Figure 3: Example 1. The two sequences $\left(T_{k, j}, 1 \leq j \leq 200\right)$ and $\left(Q_{k, j}, 1 \leq j \leq 200\right)$ with (a) $k=70$, (b) $k=100$, (c) $k=130$.


Figure 4: Example 1. The sequence $\left(\eta_{k}\right)$ (in a semilog scale)
Repeating the same procedure with 100 simulated sequences, we obtained 100 values of $\hat{k}$. The mean value of $\hat{k}$ is 97.6 and the standard deviation is 4.8 .

If we consider now that the variance is unknown, we use the procedure described Section 4.1, estimating the variance under $\mathbf{H}_{1}(k)$ by

$$
\widehat{\theta_{k}}=\frac{1}{n-k} \sum_{i=k+1}^{n} y_{(i)}^{2}
$$

The results obtained when the variance is unknown are very similar than those obtained when the variance is known. The mean value of $\hat{k}$ is 97.3 and the standard deviation is 5.1.

### 5.2.2 A Exponential example

In this second example, $n=500$ again, but $\left(\varepsilon_{i}\right)$ is a collection of $\operatorname{Expo}(1)$ i.i.d. r.v. Here, $\mu_{i}$ is uniformly distributed in $[3,6]$ for $1 \leq i \leq 100$, and $\mu_{i}=0$ for $101 \leq i \leq 500$.

When the parameter of the exponential distribution is known, we use the procedure described Section 2.1, setting $X_{i}=y_{i}$. Figure 5.2.2 displays the two sequences $\left(T_{k}\right)$ and $\left(Q_{k}\right)$. We find $D_{n}=3.72$ and reject the null hypothesis.


Figure 5: Example 2. The two sequences $\left(T_{k}\right)$ and $\left(Q_{k}\right)$

The number of significant coefficients is estimated as before. Figure 5.2.2 displays these two sequences $\left(T_{k, j}, 1 \leq j \leq 200\right)$ and $\left(Q_{k, j}, 1 \leq j \leq 200\right)$ for $k=70, k=100$ and $k=130$. In this example, the sequence $\left(\eta_{k}\right)$ displayed Figure 5.2 .2 is defined by $\eta_{k}=\sum_{j=1}^{200}\left|T_{k, j}-Q_{k, j}\right| / \sqrt{(n-k)^{3}}$. A minimum at $\hat{k}=99$ is obvious.

Repeating the same procedure with 100 simulated sequences, we obtained 100 values of $\hat{k}$. The mean value of $\hat{k}$ is 103.8 and the standard deviation is 6.2 .

The results obtained using the procedure described Section 4.1 when $\theta^{\star}$ is unknown are very similar: the mean value of $\hat{k}$ is 102.9 and the standard deviation is 5.6 .

## 6 Appendix

## Proof of proposition 2.1





Figure 6: Example 2. The two sequences $\left(T_{k, j}, 1 \leq j \leq 200\right)$ and $\left(Q_{k, j}, 1 \leq j \leq 200\right)$ with (a) $k=70$, (b) $k=100$, (c) $k=130$.


Figure 7: Example 2. The sequence $\left(\eta_{k}\right)$ (in a semilog scale)

For any $1 \leq i \leq n$, let

$$
D_{i}=X_{i}-X_{i+1}
$$

with $X_{n+1}=0$. Thus, $X_{i}=\sum_{j=i}^{n} D_{j}$. Next, let $Z_{j}=j D_{j}$. As is well known, $\left(Z_{j} ; 1 \leq j \leq n\right)$ is a sequence of i.i.d random variables $(\operatorname{Exp}(1))$, so that, for any $1 \leq k \leq K \leq n$,

$$
\begin{aligned}
\mathbb{E}\left(T_{k} \mid T_{K}\right) & =\sum_{i=1}^{k} \sum_{j=i}^{n} \mathbb{E}\left(D_{j} \mid T_{K}\right) \\
& =\left(\sum_{i=1}^{k} \sum_{j=i}^{n} \frac{1}{j}\right) \mathbb{E}\left(Z_{1} \mid T_{K}\right)
\end{aligned}
$$

Since

$$
\begin{aligned}
\mathbb{E}\left(T_{K} \mid T_{K}\right) & =\left(\sum_{i=1}^{K} \sum_{j=i}^{n} \frac{1}{j}\right) \mathbb{E}\left(Z_{1} \mid T_{K}\right) \\
& =T_{K}
\end{aligned}
$$

and

$$
\sum_{i=1}^{k} \sum_{j=i}^{n} \frac{1}{j}=k+k \sum_{j=k+1}^{n} \frac{1}{j}
$$

we obtain

$$
\begin{equation*}
\mathbb{E}\left(T_{k} \mid T_{K}\right)=\frac{k+k \sum_{j=k+1}^{n} 1 / j}{K+K \sum_{j=K+1}^{n} 1 / j} T_{K}=\frac{B_{k, n}}{B_{K, n}} T_{K} \tag{18}
\end{equation*}
$$

## Proof of Theorem 2.1

Let $c_{n t, i}=\mathbb{1}_{[0,[n t]]}(i)$. By definition $\mathbb{E}\left(T_{[n t]} \mid T_{n}\right)=B_{[n t]} T_{n}$, so that $\mathbb{E}\left(T_{[n t]}\right)=\sum_{i}^{n} c_{n t, i} \mathbb{E}\left(X_{(i)}\right)=$ $n B_{[n t]}$. Thus,

$$
d_{n}(t)=\sum_{i}^{n} c_{n t, i}\left[X_{(i)}-\mathbb{E}\left(X_{(i)}\right)\right]-\frac{\sum_{i}^{n} c_{n t, i} \mathbb{E}\left(X_{(i)}\right)}{n}\left(T_{n}-n\right)=I_{n}(t)-I I_{n}(t)
$$

Let $G_{n}=\sum_{i=1}^{n} \zeta_{n-i}$ stand for the empirical sum of uniform r.v. $\zeta_{i}$. Then, as in [9] it can be seen that

$$
\frac{1}{\sqrt{n}} I_{n}(t)=-\frac{1}{\sqrt{n}} \int_{0}^{1}\left[G_{n}-I\right](s) \mathbb{I}_{[0, t]}(s) d F^{-1}(s)+o_{p}(1)
$$

and

$$
\frac{1}{\sqrt{n}} I I_{n}(t)=-(t-t \log (t)) \frac{1}{\sqrt{n}} \int_{0}^{1}\left[G_{n}-I\right](s) d F^{-1}(s)+o_{p}(1)
$$

where $o_{p}(1)$ is uniform for all $t \in[0,1)$. So that,

$$
\frac{1}{\sqrt{n}} d_{n}(t)=\int_{0}^{1}\left[R^{t}(u)-(t-t \log (t)) F^{-1}(u)\right] d\left[G_{n}(s)-(1-s)\right]+o_{p}(1)
$$

with $R^{t}(u)=\int_{0}^{u} d F^{-1}(s) \mathbb{1}_{[0, t]}(s) d s$. The result now follows because

$$
\mathcal{G}=\left\{R^{t}-(t-t \log (t)) F^{-1}, t \in[0,1]\right\}
$$

is a Donsker class.
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