



HAL
open science

A note on maximally repeated sub-patterns of a point set

Véronique Cortier, Xavier Goaoc, Mira Lee, Hyeon-Suk Na

► **To cite this version:**

Véronique Cortier, Xavier Goaoc, Mira Lee, Hyeon-Suk Na. A note on maximally repeated sub-patterns of a point set. [Research Report] RR-5773, INRIA. 2005, pp.5. inria-00070247

HAL Id: inria-00070247

<https://inria.hal.science/inria-00070247>

Submitted on 19 May 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

*A note on maximally repeated sub-patterns
of a point set*

Véronique Cortier — Xavier Goaoc — Mira Lee — Hyeon-Suk Na

N° 5773

Decembre 2005

Thème SYM

A large blue rectangle occupies the lower half of the page. Overlaid on it is a large, light grey stylized 'R' logo. To the right of the 'R', the words 'Rapport de recherche' are written in a white serif font. A horizontal grey brushstroke is positioned below the text.

*Rapport
de recherche*



A note on maximally repeated sub-patterns of a point set

Véronique Cortier^{*}, Xavier Goaoc[†], Mira Lee[‡], Hyeon-Suk Na[§]

Thème SYM — Systèmes symboliques
Projets Cassis et Vegas

Rapport de recherche n° 5773 — Decembre 2005 — 5 pages

Abstract: We answer a question raised by P. Brass on the number of maximally repeated sub-patterns in a set of n points in \mathbb{R}^d . We show that this number, which was conjectured to be polynomial, is in fact $\Theta(2^{n/2})$ in the worst case, regardless of the dimension d .

Key-words: Discrete geometry, point sets, repeated configurations.

^{*} Projet Cassis, LORIA - CNRS, Nancy, France. Email: cortier@loria.fr.

[†] Projet Vegas, LORIA - INRIA Lorraine, Nancy, France. Email: goaoc@loria.fr.

[‡] Division of Computer Science, Korea Advanced Institute of Science and Technology (KAIST), Republic of Korea. Email: mira@kaist.ac.kr

[§] School of Computing, Soongsil University, Seoul, Republic of Korea. Email: hsnaa@computing.ssu.ac.kr

Une note sur les sous-motifs maximallement répétés d'un nuage de points

Résumé : Nous répondons à une question de P. Brass sur le nombre de sous-motifs maximallement répétés d'un ensemble de n points de \mathbb{R}^d . Nous montrons que ce nombre, conjecturé polynomial, s'avère être $\Theta(2^{n/2})$ dans le cas le pire, et ce en toute dimension d .

Mots-clés : Géométrie discrète, nuages de points, sous-motifs répétés.

1 Introduction

Let \mathcal{S} be a set of n points in \mathbb{R}^d . A *sub-pattern*, i.e. a subset, of \mathcal{S} is repeated if it can be translated to another subset of \mathcal{S} . A sub-pattern $P \subseteq \mathcal{S}$ is *maximally repeated* if for any subset Q such that $P \subsetneq Q \subseteq \mathcal{S}$ there exists a translation that maps P to a subset of \mathcal{S} without mapping Q to a subset of \mathcal{S} . In other words, a pattern is maximally repeated if it cannot be extended without losing at least one of its occurrences. Maximally repeated sub-patterns (MRSP for short) originated from the field of pattern matching to solve the following problem: given two point sets X and Y , can Y be translated to a subset of X ? P.Brass [1, Theorem3] gave an algorithm that answers such queries in time $O(|Y| \log |X|)$ whose preprocessing time depends on the number of distinct MRSP of X , where two MRSP are *distinct* if they are not equal up to a translation. A natural question is thus to give a theoretical bound on this number of MRSP in order to provide an upper bound on the time requirement of that algorithm. This number was conjectured [1] [2, p.267] to be $O(n^d)$ where d is the dimension in which the point set is embedded.

In this note we show that the number of MRSP of a set of n points is actually $\Theta(2^{n/2})$ in the worst case, which shows that finding sub-patterns via this approach may lead to exponential worst-case running time. Our proof is based on combinatorial rather than geometrical properties of the point set, which explains that the bound is independent of the dimension d in which the points are considered.

2 Lower and upper bounds

Let us first introduce some terminology. Given a set of points $P \subseteq \mathbb{R}^d$ and a translation $t \in \mathbb{R}^d$, $P+t := \{x+t \mid x \in P\}$ is the set of translated points of P by t . A subset $P \subseteq \mathcal{S}$ is a repeated sub-pattern if there exists a translation $t \neq \mathbf{0}$ such that $P+t \subseteq \mathcal{S}$. P is a *maximally repeated sub-pattern* (MRSP) if, in addition, for any subset Q such that $P \subsetneq Q \subseteq \mathcal{S}$ there exists a translation t such that $P+t \subseteq \mathcal{S}$ and $Q+t \not\subseteq \mathcal{S}$. Two MRSP are *distinct* if they are not equal up to a translation.

In the sequel, we present a set of n points in \mathbb{R} having at least $2^{\lfloor n/2 \rfloor - 1}$ distinct MRSP (Section 2.1) and then prove that any set of n points in \mathbb{R}^d can have at most $16 \cdot 2^{\lfloor n/2 \rfloor}$ distinct MRSP (Section 2.2).

2.1 Lower bound

We build our example on a 1-dimensional grid which can, of course, be considered as embedded in \mathbb{R}^d for any $d \geq 1$. Let k be an integer, G_k denotes the set of integers $\{1, \dots, k\}$ and $\mathcal{S}_k = G_k \cup (G_k + (k+1))$, that is, two copies of G_k separated by a gap of one point at $k+1$.

Proposition 1 *The set \mathcal{S}_k has at least 2^{k-1} distinct MRSP.*

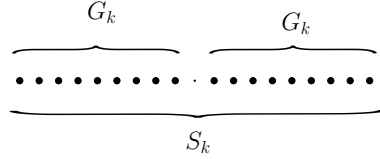


Figure 1: S_k is a set of $2k$ points on a 1-dimensional grid having at least 2^{k-1} distinct MRSP.

We show that any subset $P \subseteq G_k$ is a MRSP by arguing that for any $p^* \in S_k \setminus P$, one of the translations that keeps P in S_k sends p^* either to $\{k+1\}$ or outside of S_k . Indeed, let $Q \subseteq S_k$ be a proper super-set of P and $p^* \in Q \setminus P$. If $p^* \geq k+2$ then $P + (k+1) \subseteq S_k$ and $Q + (k+1) \not\subseteq S_k$. If $p^* \leq k$ then $P + (k+1-p^*) \subseteq S_k$ and $Q + (k+1-p^*) \not\subseteq S_k$. This proves that any subset $P \subseteq G_k$ is a MRSP of S_k . No translation can map a subset of G_k that contains 1 to another subset of G_k that contains 1, so all the subsets of G_k containing 1 are distinct. Therefore, at least 2^{k-1} of the subsets of S_k are distinct MRSP.

2.2 Upper bound

Let $\mathcal{S} = \{a_1, \dots, a_n\} \subseteq \mathbb{R}^d$ be a set of n points and $\mathcal{T} \subseteq \mathbb{R}^d$ the *set of translations* defined by

$$\mathcal{T} := \mathcal{S} - \mathcal{S} = \{x - y \mid (x, y) \in \mathcal{S}^2\}.$$

Both the points in \mathcal{S} and the translations in \mathcal{T} are *ordered lexicographically* as vectors of d real numbers, in the sense that if $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, then $x < y$ if $x_1 < y_1$ or $x_1 = y_1, \dots, x_r = y_r$ and $x_{r+1} < y_{r+1}$ for some $r = 1, \dots, d-1$. Let \mathcal{A} denote the family of all *first* occurrences of subsets of \mathcal{S} that are MRSP. By “first” we mean that a MRSP P is in \mathcal{A} if and only if no translation $t < \mathbf{0}$ satisfies $P + t \subseteq \mathcal{S}$. We choose one representative of each equivalence class of MRSP under translation, so the number of distinct MRSP of \mathcal{S} is $|\mathcal{A}|$. The following function maps each pattern to its set of translations:

$$\phi: \begin{cases} 2^{\mathcal{S}} & \rightarrow & 2^{\mathcal{T}} \\ P & \mapsto & \{t \in \mathcal{T} \mid P + t \subseteq \mathcal{S}\} \end{cases}$$

For any repeated sub-pattern P , $|\phi(P)| \geq 2$ and if $P \in \mathcal{A}$ then $t \geq \mathbf{0}$ for every $t \in \phi(P)$.

For $1 \leq i \leq j \leq n$ let $\mathcal{A}_{ij} = \{P \in \mathcal{A} \mid \{a_i, a_j\} \subseteq P \subseteq \{a_i, \dots, a_j\}\}$ be the set of all occurrences of MRSP spanning the range $\{a_i, \dots, a_j\}$ and $\mathcal{T}_{ij} = \{t \in \mathcal{T} \mid t \geq \mathbf{0} \text{ and } \{a_i, a_j\} \subseteq \mathcal{S} \cap (\mathcal{S} - t)\}$ be the set of all non-negative translations compatible with a_i and a_j . Note that $\{\mathcal{A}_{ij}\}$ is a partition of \mathcal{A} , $\mathcal{A}_{11} = \{a_1\}$ and \mathcal{A}_{ii} is empty for $i \geq 2$. So we have

$$|\mathcal{A}| = 1 + \sum_{1 \leq i < j \leq n} |\mathcal{A}_{ij}|. \quad (1)$$

We can now prove our upper bound.

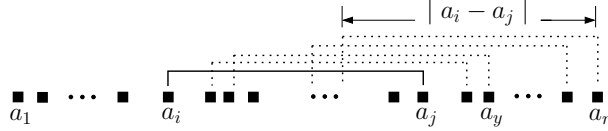


Figure 2: Bounding $|\mathcal{T}_{ij}|$ in 1-dimensional case; the same reasoning holds in \mathbb{R}^d thanks to the total ordering.

Proposition 2 *A set of n points has at most $16 \cdot 2^{\lceil n/2 \rceil}$ distinct MRSP.*

Let P_1 and P_2 be two MRSP such that $\phi(P_1) = \phi(P_2)$. Then $\phi(P_1 \cup P_2) = \phi(P_1) = \phi(P_2)$ which leads to $P_1 \cup P_2 = P_1$, since P_1 is a MRSP, and $P_1 \cup P_2 = P_2$, as P_2 is also a MRSP. Thus, ϕ defines an injection from \mathcal{A} on the subsets of \mathcal{T} . If $P \in \mathcal{A}_{ij}$ then $\phi(P) \subseteq \mathcal{T}_{ij}$ and ϕ induces an injection from \mathcal{A}_{ij} on the subsets of \mathcal{T}_{ij} . Hence,

$$|\mathcal{A}_{ij}| \leq 2^{|\mathcal{T}_{ij}|}.$$

For each $t \in \mathcal{T}_{ij} \setminus \{\mathbf{0}\}$, $t > \mathbf{0}$ and there exists unique $y > j$ such that $a_j + t = a_y$. Hence, $|\mathcal{T}_{ij} \setminus \{\mathbf{0}\}| \leq n - j$. Because $\phi(P \in \mathcal{A}_{ij})$ includes $\mathbf{0}$ and at least one translation $t \in \mathcal{T}_{ij} \setminus \{\mathbf{0}\}$, it follows that

$$|\mathcal{A}_{ij}| \leq 2^{n-j} - 1.$$

As any MRSP in \mathcal{A}_{ij} corresponds to a subset of $\{a_{i+1}, \dots, a_{j-1}\}$ we also have that

$$|\mathcal{A}_{ij}| \leq 2^{j-i-1}.$$

Applying these to equation (1), we get

$$|\mathcal{A}| \leq 1 + \sum_{1 \leq i < j \leq n} 2^{\min(n-j, j-i-1)}.$$

Splitting the sum at $j = \lceil \frac{n+i}{2} \rceil + 1$, we have

$$|\mathcal{A}| \leq 1 + 2 \sum_{i=1}^n \sum_{j=i+1}^{\lceil \frac{n+i}{2} \rceil + 1} 2^{j-i-1} \leq 1 + 2 \sum_{i=1}^n 2^{\lceil \frac{n-i}{2} \rceil + 1} \leq 1 + 8 \sum_{\ell=1}^{\lceil \frac{n}{2} \rceil} 2^\ell$$

and finally $|\mathcal{A}| \leq 16 \cdot 2^{\lceil n/2 \rceil}$.

References

- [1] P. Brass. Combinatorial geometry problems in pattern recognition. *Discrete and Computational Geometry*, 28:495–510, 2002.
- [2] P. Brass, W. Moser, and J. Pach. *Research Problems in Discrete Geometry*. Springer-Verlag, 2005.



Unité de recherche INRIA Lorraine
LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)

Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)

Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)

Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Unité de recherche INRIA Sophia Antipolis : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
<http://www.inria.fr>
ISSN 0249-6399