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Abstract: We are interested in the relation between the pathwidth of a biconnected outerplanar graph and the pathwidth of its (geometric) dual. Bodlaender and Fomin [2], after having proved that the pathwidth of every biconnected outerplanar graph is always at most twice the pathwidth of its (geometric) dual plus two, conjectured that there exists a constant c such that the pathwidth of every biconnected outerplanar graph is at most c plus the pathwidth of its dual. They also conjectured that this was actually true with c being 1 for every biconnected planar graph. Fomin [7] proved that the second conjecture is true for all planar triangulations, and made a stronger conjecture about the linear width of planar graphs. First, we construct for each $p \geq 1$ a biconnected outerplanar graph of pathwidth $2p+1$ whose (geometric) dual has pathwidth $p+1$, thereby disproving all three conjectures. Then we prove, in an algorithmic way, that the pathwidth of every biconnected outerplanar graph is at most twice the pathwidth of its (geometric) dual minus 1. A tight interval for the studied relation is therefore obtained, and we show that all the gaps within the interval actually happen.

Key-words: pathwidth, vertex separation, outerplanar graph, biconnected, linear width

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Sommet-séparation des graphes planaires extérieurs

Résumé : Nous étudions la relation entre la sommet-séparation d'un graphe planaire extérieur 2-connexe G et celle de son dual. Bodlaender et Fomin [2], après avoir prouvé que la sommet-séparation d'un tel graphe G est au plus deux fois celle de son dual plus deux, ont conjecturé que la sommet-séparation d'un tel graphe G est à une constante c de celle de son dual. Ils ont également conjecturé que ceci est vrai avec $c = 1$ pour tout graphe planaire 2-connexe. Fomin [7] a montré que cette seconde conjecture est vraie si G est une triangulation du plan, et a fait une conjecture plus forte à propos de la largeur linéaire des graphes planaires. En premier lieu, nous construisons pour tout $p \geq 1$ un graphe planaire extérieur 2-connexe de sommet-séparation $2p+1$ tel que la sommet séparation de son dual soit $p+1$, ce qui établit que les trois conjectures précédentes sont fausses. Ensuite nous prouvons, de façon algorithmique, que la sommet-séparation d'un graphe planaire extérieur 2-connexe est au plus 2 fois celle de son dual moins 1. Un intervalle serré pour la relation étudiée est ainsi obtenu, et nous montrons que tous les écarts de l'intervalle sont atteints.

Mots-clés : sommet-séparation, graphe planaire extérieur, largeur linéaire, biconnexe

1 Introduction

A *planar graph* is a graph that can be embedded in the plane without crossing edges. It is said to be *outerplanar* if it can be embedded in the plane without crossing edges and such that all its vertices are incident to the unbounded face. For any graph G, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. The *dual* of the planar graph G, denoted by G^*, is the graph obtained by putting one vertex for each face, and joining two vertices if and only if the corresponding faces are adjacent. The *weak dual* of G, denoted by G^{**}, is the induced subgraph of G^* obtained by removing the vertex corresponding to the unbounded face. As is well known, the weak dual of an outerplanar graph is a forest, and the weak dual of a biconnected outerplanar graph is a tree. Furthermore, linear-time algorithms to recognise and embed outerplanar graphs are known (see for instance [12, 19]). Note that the dual of a planar graph can also be computed in linear-time.

The notion of pathwidth was introduced by Robertson and Seymour [14]. A *path decomposition* of a graph $G=(V,E)$ is a set system $(X_1,\dots,X_r)$ of V such that

(i) $\bigcup_{i=1}^r X_i = V$;

(ii) $\forall xy \in E, \exists i \in \{1,2,\dots,r\} : \{x,y\} \subset X_i$;

(iii) $\forall (i_0,i_1,i_2) \in \{1,2,\dots,r\}^3, i_0 < i_1 < i_2 \Rightarrow X_{i_0} \cap X_{i_2} \subseteq X_{i_1}$.

The *width* of the path decomposition $(X_1,\dots,X_r)$ is $\max_{1\le i\le r}|X_i|-1$. The *pathwidth* of G, denoted by $\mathrm{pw}(G)$, is the minimum width over its path decompositions.

The pathwidth of a graph was shown to be equal to its vertex separation [9]: a *layout* (or *vertex-ordering*) L of a graph $G=(V,E)$ is a one-to-one correspondence between V and $\{1,\dots,|V|\}$. The *vertex separation* of (G,L) is $\max_{1\le i\le |V|}|M(i)|$ where

$$M(i) := \{v \in V : L(v) > i \text{ and } \exists u \in N(v) : L(u) \le i\}.$$

The *vertex separation* of G, denoted by $\mathrm{vs}(G)$, is the minimum of the vertex separation of (G,L) taken over all vertex-orderings L.

Computing the pathwidth of graphs is an active research area, in which a lot of work has been done (survey papers are for instance [5, 1, 13]). It was shown [3] that the pathwidth of graphs with bounded treewidth can be computed in polynomial time. As outerplanar graphs have treewidth 2, the pathwidth of an outerplanar graph is polynomially computable. However, the exponent in the running time of the algorithm is rather large, so the algorithm is not useful in practice. This is why Govindan et al. [8] gave an $O(n\log(n))$ time algorithm for approximating the pathwidth of outerplanar graphs with a multiplicative factor of 3. For biconnected outerplanar graphs, Bodlaender and Fomin [2] improved upon this result by giving a linear-time algorithm which approximates the pathwidth of biconnected outerplanar graphs with a multiplicative factor 2 (and a corresponding path decomposition is obtained in time $O(n\log(n))$). To do so, they exhibited a relationship between the pathwidth of an outerplanar graph and the pathwidth of its dual. More precisely, the following holds.

Theorem 1 (Bodlaender and Fomin [2]) *Let G be a biconnected outerplanar graph without loops and multiple edges. Then* $\mathrm{pw}(G^*) \le \mathrm{pw}(G) \le 2\mathrm{pw}(G^*)+2$.

Since the weak dual of an outerplanar graph (which can be computed in linear-time) is a tree and there exist linear-time algorithms to compute the pathwidth of a tree [16, 17, 6], this yields the desired approximation (obtaining a corresponding path decomposition needs more work).

Bodlaender and Fomin [2] suggested that a stronger relationship holds between the pathwidth of a planar graph and the pathwidth of its dual.

Conjecture 1 (Bodlaender and Fomin [2]) *There is a constant c such that for every biconnected outerplanar graph G without loops and multiple edges* $\mathrm{pw}(G) \le \mathrm{pw}(G^*)+c$.

Conjecture 2 (Bodlaender and Fomin [2]) *For every biconnected planar graph G without loops and multiple edges,* $\mathrm{pw}(G) \le \mathrm{pw}(G^*)+1$.

Fomin [7] proved that if G is any biconnected planar graph of maximum degree at most 3, then $\mathrm{pw}(G) \geq \mathrm{pw}(G^*) - 1$. This implies that Conjecture 2 is true for every planar triangulation (since any planar triangulation is the dual of a biconnected planar graph of maximum degree 3). Actually, Fomin made an even stronger conjecture. We need two new definitions to state it. Given an edge-ordering σ of $G = (V, E)$, let $\delta(i)$ be the number of vertices incident to at least two edges e, e' such that $\sigma(e) \leq i$ and $\sigma(e') > i$. The *linear width* of (G, σ) is the maximum of $\delta(i), i \in \{1, 2, \ldots, |E|\}$. The *linear width* of G is the minimum of the linear width of (G, σ) taken over all the edge-orderings σ. Notice that if G has minimum degree at least 2, then $\mathrm{pw}(G) \leq \mathrm{lw}(G) \leq \mathrm{pw}(G) + 1$. For a planar graph G, a *split* H of G is a graph obtained by a sequence of the following operations: take a vertex v, partition its neighbourhood in two sets M and N, replace v by two new vertices x, y. Link x to $M \cup \{y\}$ and y to N.

Conjecture 3 (Fomin [7]) *For every planar graph G, there exists a planar split H of maximum degree* 3 *such that* $\mathrm{lw}(H) = \mathrm{lw}(G)$.

According to [7], this conjecture implies the preceding ones. It is worth noting that these conjectures are motivated by the following result about the treewidth, conjectured by Robertson and Seymour [15] and proved by Lapoire [10] using algebraic methods (notice that Bouchitté, Mazoit and Todinca [4] gave a shorter and combinatorial proof of this result).

Theorem 2 (Lapoire [10]) *For every planar graph G,* $\mathrm{tw}(G) \leq \mathrm{tw}(G^*)$.

In Section 2, we exhibit a family $(G_p)_{p \geq 1}$ of biconnected outerplanar graphs with maximum degree 4 such that $\mathrm{pw}(G_p) = 2p + 1$ and $\mathrm{pw}(G_p^*) = p + 1$, thereby disproving all three conjectures. To construct these graphs, we introduce a general construction which actually allows us to prove the following result.

Theorem 3 *For every integer $p \geq 1$ and every integer $k \in \{1, 2, \ldots, p+1\}$, there exists a biconnected outerplanar graph of pathwidth $p + k$ whose weak dual has pathwidth p.*

In Section 3, we prove the following result which improves the upper bound given by Theorem 1.

Theorem 4 *Let G be a biconnected outerplanar graph without loops and multiple edges. Then* $\mathrm{pw}(G) \leq 2\mathrm{pw}(G^*) - 1$.

As a consequence, the previous approximation for the pathwidth of biconnected outerplanar graphs is also improved. We give an algorithmic proof which allows to obtain a layout of the outerplanar graph G considered (and whose vertex separation is hence at most $2\mathrm{pw}(G) - 1$).

Furthermore, Theorem 3 shows that this bound is best possible in general.

2 Counter-examples

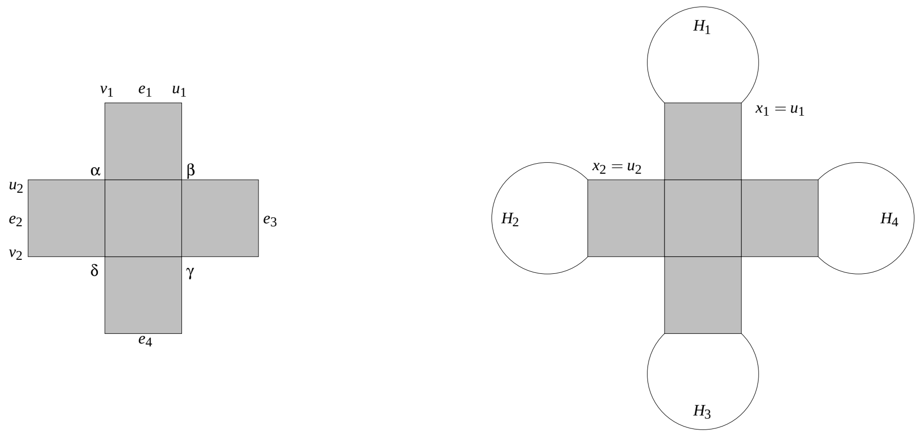
In this section, we establish Theorem 3 and deduce the following corollary which disproves Conjectures 1, 2 and 3.

Corollary 1 *For every integer $p \geq 1$, there exists a triangle-free biconnected outerplanar graph G_p of maximum degree* 4 *whose pathwidth is* $2p + 1$ *such that the pathwidth of its dual is* $p + 1$.

For each $i \in \{1, 2, 3, 4\}$, let H_i be a biconnected outerplanar graph of pathwidth p whose weak dual has pathwidth p'. We shall describe a construction which yields a biconnected outerplanar graph $\mathcal{C}(H_1, H_2, H_3, H_4)$ of pathwidth $p + 2$ whose weak dual has pathwidth $p' + 1$. This construction will be illustrated by examples yielding the graphs G_p of Corollary 1.

A 4-cycle is called a *square*. Two squares are *adjacent* if they share exactly one edge. The *degree* of the square S is the number of squares adjacent to S. Let the *cross* K be the biconnected outerplanar graph consisting of 4 squares of degree 1 and 1 square of degree 4 (see Figure 1(a)).

For each $i \in \{1, 2, 3, 4\}$, let $x_i y_i$ be an edge of H_i incident to the unbounded face in an outerplanar embedding of H_i. For $i \in \{1, 2\}$, we denote by L_i an optimal layout of H_i, i.e. a layout with vertex separation p, and without loss of generality we assume that $L_i(x_i) < L_i(y_i)$.



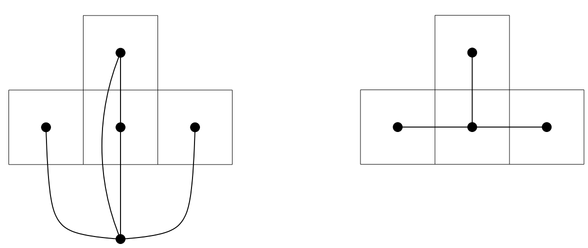
(a) The cross K.

(b) Gluing 4 graphs H_1, H_2, H_3, H_4 on the cross K

Figure 1: When identifying the edges, we ensure that x_1 is identified with u_1 and x_2 with u_2.

Consider the cross K of Figure 1(a). For each $i \in \{1,2,3,4\}$, the edge e_i of K is identified with the edge $x_i y_i$. We assume moreover that the vertices x_1 and x_2 are identified with the vertices u_1 and u_2 respectively (see Figure 1(b)). Notice that there is generally not a unique way to achieve this construction, but we shall denote by $\mathcal{C}(H_1, H_2, H_3, H_4)$ any graph obtained from H_1, H_2, H_3, H_4 in this way.

It is clear by the construction that any such graph $\mathcal{C}(H_1, H_2, H_3, H_4)$ is a biconnected outerplanar graph. As an example, let G_1 be the biconnected outerplanar graph consisting of 3 squares of degree 1 and 1 square of degree 3 (see Figure 2). For any integer $p \geq 2$, let G_p be the graph $\mathcal{C}(G_{p-1}, G_{p-1}, G_{p-1}, G_{p-1})$, obtained as indicated in Figures 3 and 4. Remark that the condition on the vertices x_1 and x_2 is clearly fulfilled in this case thanks to the symmetry of the graphs G_p, and that the maximum degree of G_p is 4.



(a) G_1 and G_1^*

(b) G_1 and G_1^{**}

Figure 2: G_1, graph consisting of 1 square of degree 3 and 3 squares of degree 1, the dual G_1^* and the weak dual G_1^{**}, a star.

In the following three lemmata, we prove the announced properties of the construction. The central square of the cross is denoted by S, and the corresponding vertex of the dual is s.

Lemma 1 *For each $i \in \{1,2,3,4\}$, let H_i be a biconnected outerplanar graph whose weak dual T_i has pathwidth $p \geq 1$. The pathwidth of the weak dual of graph $\mathcal{C}(H_1, H_2, H_3, H_4)$ is $p+1$.*

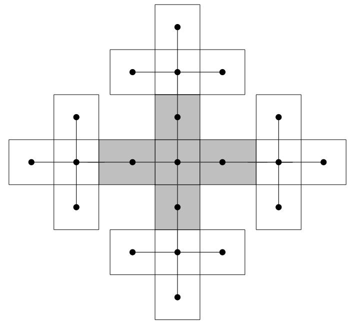


Figure 3: G_2, 4 disjoint copies of G_1 glued with a grey cross K, and its weak dual G_2^{**}.

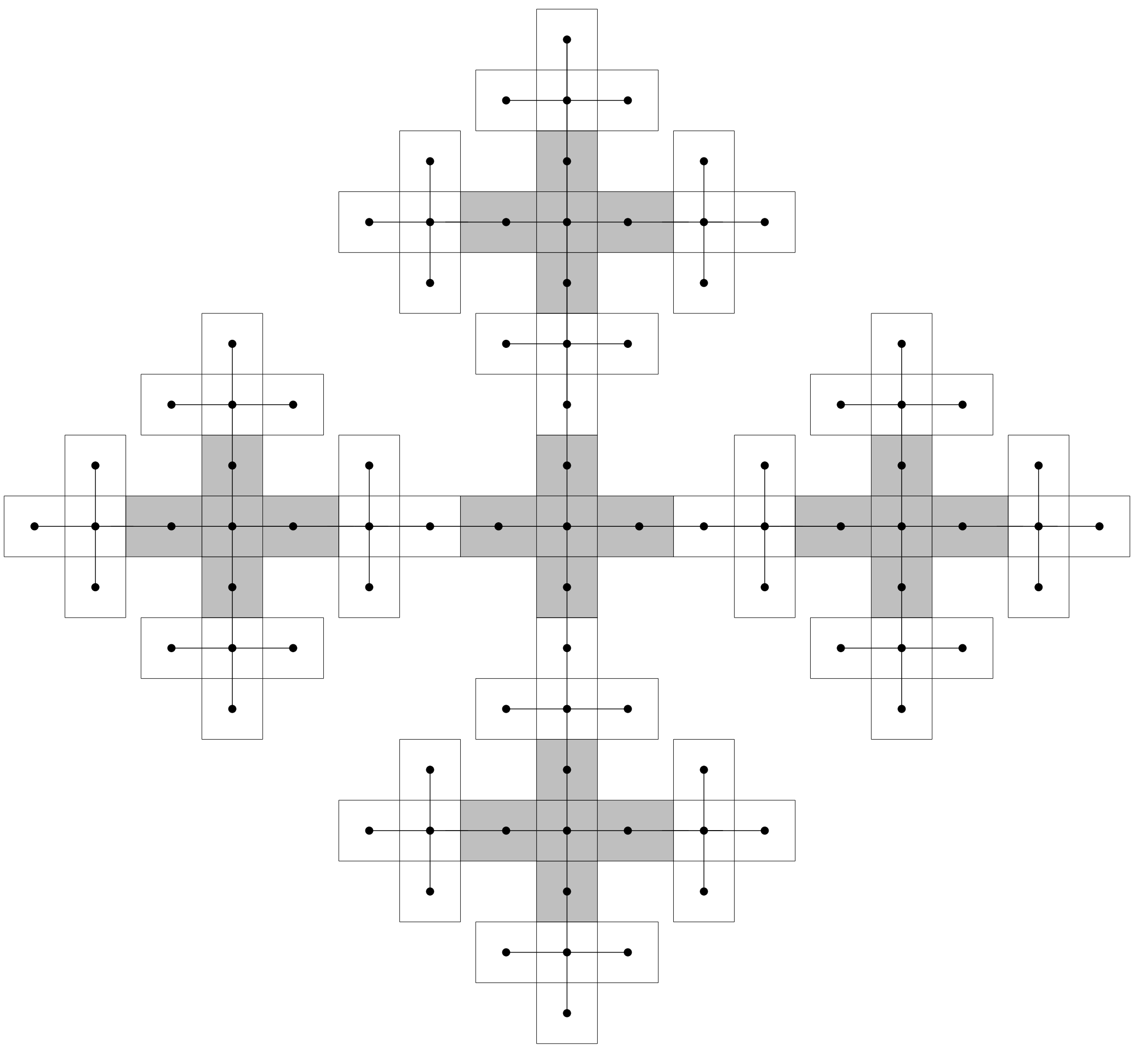


Figure 4: G_3, 4 disjoint copies of G_2 glued with a grey cross K, and its weak dual G_3^{**}.

We introduce the following definition: for every vertex v of a tree T, a *branch at* v is any maximal subtree which contains a neighbour of v without containing v. The following result will be useful to prove Lemma 1.

Theorem 5 (Scheffler [16]) *For every integer* $p \geq 1$ *and every tree* T, $\mathrm{pw}(T) \geq p+1$ *if and only if there exists a vertex* t *of* T *with at least three branches of pathwidth at least* p.

Proof of Lemma 1. As the pathwidth of each tree T_i is p, it is clear how to construct a path decomposition of the weak dual of $\mathcal{C}(H_1,H_2,H_3,H_4)$ of width $p+1$. Moreover, the pathwidth of the weak dual of $\mathcal{C}(H_1,H_2,H_3,H_4)$ is more than p by Theorem 5 since the vertex s has 4 branches with pathwidth p. □

Lemma 2 *For each* $i \in \{1,2,3,4\}$, *let* H_i *be a biconnected outerplanar graph of pathwidth* $p \geq 1$. *The vertex separation of the graph* $\mathcal{C}(H_1,H_2,H_3,H_4)$ *is at least* $p+2$.

Proof. Consider any layout L of $H := \mathcal{C}(H_1,H_2,H_3,H_4)$. We shall prove that the vertex separation of (H,L) is at least $p+2$. The subgraph of H induced by removing the vertices of the square S is the disjoint union of the 4 graphs H_1,H_2,H_3 and H_4, each of them having pathwidth p. We moreover assume that the vertex a such that $L(a)=1$ and the vertex b such that $L(b)=|V(H)|$ are in $V(H_1)\cup V(S)$ and $V(H_1)\cup V(H_2)\cup V(S)$ respectively. By hypothesis, there exists $i \in L(V(H_4))$ such that there are p vertices x of H_4 with $L(x)>i$, each having a neighbour y in H_4 with $L(y)\leq i$. As a similar integer exists for H_3, we suppose without loss of generality that there exists a vertex $v \in V(H_3)$ with $L(v)>i$. Let $X := \cup_{j=1}^{3} V(H_j)\cup V(S)$. Label every vertex $x \in X$ with m if $L(x)<i$ and M if $L(x)>i$. Note that a is labelled m (we say it is an *m-vertex*) and b is labelled M (we say it is an *M-vertex*). An edge is *bad* if it links an m-vertex to an M-vertex. A *bad pair* is a pair of bad edges that are either disjoint, or incident to the same m-vertex. Note that the existence of a bad pair in the complement of H_4 implies that $\mathrm{vs}(H,L)\geq \mathrm{vs}(H_4)+2 = p+2$.

(i) There are at least 2 vertices labelled m in $V(H_1)\cup V(S)$.

Otherwise, there is only a labelled m in $V(H_1)\cup V(S)$, and as the subgraph of H induced by the vertices of $V(H_1)\cup V(S)$ has minimum degree 2, there is a bad pair in H_1, so we get the desired result.

(ii) There is exactly one M-vertex in X.

There is at least one such vertex, namely b. So, as the subgraph of H induced by the vertices of X is connected, there exists a bad edge $e=xy$, x being an m-vertex. Observe then that all the neighbours of x except y must be m-vertices (otherwise there would be a bad pair). This implies that all the neighbours of y are m-vertices. And then by connectivity all the vertices of $X\setminus\{y\}$ are m-vertices.

Item **(ii)** contradicts the existence of the vertex $v \in V(H_3)$.

□

Lemma 3 *For each* $i \in \{1,2,3,4\}$, *let* H_i *be a biconnected outerplanar graph of pathwidth* $p \geq 1$. *The pathwidth of* $\mathcal{C}(H_1,H_2,H_3,H_4)$ *is at most* $p+2$.

Proof. We shall construct a layout of $\mathcal{C}(H_1,H_2,H_3,H_4)$ from optimal layouts L_i of H_i, $i\in\{1,2,3,4\}$. Start by labelling all the vertices of H_4 according to L_4. The vertex separation never exceeds $p+2$, since the only unlabelled vertices not in H_4 that might have labelled neighbours are β and γ. By the construction, the optimal layout L_1 of H_1 can be chosen such that $L_1(x_1)<L_1(y_1)$. Label the vertices of H_1 until the vertex x_1 is labelled. As previously, the vertex separation does not exceed $p+2$ when doing so. Now, label the vertex β, which does not change the vertex separation, as β has exactly one unlabelled vertex, α. Now go on labelling the vertices of H_1 according to L_1. The vertex-separation still does not exceed $p+2$, the only unlabelled vertices not in H_1 with labelled neighbours being α and γ. By the construction again, the layout L_2 of H_2 can be chosen such that $L_2(x_2)<L_2(y_2)$. Therefore we can apply the same procedure to label the vertices of H_2: first label them until x_2 is labelled, then label the vertex α and finish labelling the vertices

of H_2. At last, label the vertices of H_3 (the vertex separation does not exceed $p+2$ when doing so, since the only unlabelled vertices with labelled neighbours not in H_3 are δ and γ), and then label the vertices δ and γ. The obtained layout has vertex separation at most $p+2$. □

Proof of Theorem 3. The proof is by induction on $p \geq 1$. If $p = 1$, a square and G_1 give the desired result when $k = 1$ and $k = 2$ respectively.

Suppose that the result is true for $p-1 \geq 1$, and let $k \in \{1,2,\dots,p+1\}$. If $k = 1$, then let T_p be the complete binary tree of height $2p-1$. As shown in [16], T_p has pathwidth p, and it is not hard to see that the triangulated outerplanar graph whose weak dual is T_p has pathwidth $p+1$. If $k \in \{2,3,\dots,p+1\}$ then $k-1 \in \{1,2,\dots,p\}$ so by the induction hypothesis there exists a biconnected outerplanar H of pathwidth $(p-1)+(k-1)$ whose weak dual has pathwidth $p-1$. Then $\mathcal{C}(H,H,H,H)$ has pathwidth $p+k$ and its weak dual has pathwidth p, as desired. □

3 Upper bound

We shall present in this section an algorithm which, given a biconnected outerplanar graph G, computes a layout of G with vertex separation at most $2\mathrm{pw}(G^{**})+1$. As $\mathrm{pw}(G^{**}) = \mathrm{pw}(G^{*})-1$ for any biconnected outerplanar G (see [2]), this establishes Theorem 4.

First, recall that a *caterpillar* is a tree in which a single path, the *spine*, is incident to (or contains) every edge. The caterpillars are the only trees of pathwidth 1: every caterpillar has surely pathwidth 1, and if a tree T is not a caterpillar, then it contains a spider with three legs of length 2 (see Figure 5). But such a tree has pathwidth at least 2 by Theorem 5.

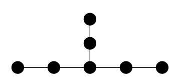


Figure 5: A spider with three legs of length 2.

Proposition 1 *Let G be a biconnected outerplanar graph whose weak dual is a caterpillar. Then G has pathwidth at most* 3*.*

Proof. Here is a layout of the vertices of G with vertex separation at most 3. Let $P := v_1 v_2 \dots v_k$ be a longest path of G^{**}. Denote by F_i the face of G corresponding to the vertex v_i, $i \in \{1,2,\dots,k\}$. Label by 1 a vertex v of F_1 of degree 2 (such a vertex exists as G is outerplanar and v_1 is a leaf of T). Then recursively label every vertex of F_1 of degree 2 which is adjacent to a labelled vertex.

Now, apply the following procedure in which we suppose that $V(F_{i-1}) \cap V(F_i) = \{x_i, y_i\}$ and $V(F_i) \cap V(F_{i+1}) = \{x_{i+1}, y_{i+1}\}$, see Figure 6.

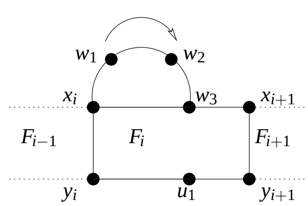


Figure 6: Vertices for step i.

for $i = 2$ to $k-1$ **do**
  let $P := x_i w_1 \dots w_j x_{i+1}$ be the path of G from x_i to x_{i+1} consisting of edges incident to the unbounded face. Label the vertices of P from x_i to w_k

let $P' := y_i u_1 \dots u_t y_{i+1}$ be the path of G from y_i to y_{i+1} consisting of edges incident to the unbounded face. Label the vertices of P' from y_i to u_t

end for

Last, label the vertices clockwise from x_k to y_k.

The obtained layout surely has vertex separation at most 3. □

The procedure given in the preceding proof actually achieves an optimal layout of the corresponding graph. Indeed, such a graph has pathwidth 2 if its weak dual is path, and pathwidth 3 otherwise. Note also that the time complexity of the procedure is linear.

We will use the following result about the pathwidth of trees.

Theorem 6 (Ellis, Sudborough and Turner [6]) *For any tree T, and any integer $p \geq 2$, $\mathrm{pw}(T) \leq p$ if and only if there is a path P such that every connected component of the forest induced by the vertices of $V(T) \setminus V(P)$ has pathwidth at most $p-1$.*

We consider the recursive procedure given by Algorithm 1. It computes a layout of G stored in the list l, which is initialised by $l(v) := \infty$ for every vertex $v \in V(G)$ (this means that all vertices are unlabelled at the beginning).

Notice that what is done in lines 14–15 and 25–26 is equivalent to label all the vertices of H except y (or y' respectively), and to keep s updated.

The following lemma suffices to establish Theorem 4.

Lemma 4 *For any biconnected outerplanar graph G whose weak dual T has pathwidth p, the procedure* `Layout` *of Algorithm 1 returns a layout with vertex separation at most $2p+1$.*

Proof. Algorithm 1 clearly assigns a unique label to every vertex of G.

For the vertex separation of the obtained layout, the proof is by induction on the pathwidth p of T. If p is 1, then T is a caterpillar and Proposition 1 gives the conclusion.

Suppose now that for every biconnected outerplanar graph whose weak dual has pathwidth at most $p-1 \geq 1$, the procedure `Layout` of Algorithm 1 returns a layout with vertex separation at most $2p-1$. Let us prove that the obtained layout for G has pathwidth at most $2p+1$.

Stop the labelling of G at any moment and denote by F the set of unlabelled vertices with a labelled neighbour. If no subgraph H has been labelled yet, then the set F consists of x and x', so its size is at most $2p+1$. If a subgraph H has just been labelled, then F consists of two vertices, namely x' and y or x and y'.

Suppose now that a subgraph H is being labelled. Without loss of generality, say that its intersection with the current face F_i is $\{x,y\}$. There is only one vertex of F not in H, namely x'. Therefore, if $|F \cap V(H)| \leq 2p$ we have $|F| \leq 2p+1$ as wanted. As the vertex separation of the layout used to label H is at most $2p-1$, the only problem that might occur is if $|F \cap (V(H) \setminus \{x,y\})| = 2p-1$, and x, x' and y also belong to F. This implies that y was requested to be labelled in the original layout l used for H, but kept unlabelled as indicated in the algorithm. But in this case, in the labelling l of H, the vertex x is unlabelled, and has at least a labelled neighbour, y. So the number of unlabelled vertices of H with a labelled neighbour in H is $|F \cap (V(H) \setminus \{x,y\})| + 1 = 2p$, a contradiction. □

As one can see in the preceding proof, the subgraphs H, labelled in lines 13 and 24, can actually be labelled by any layout with vertex separation at most $2p-1$.

Corollary 2 *For any biconnected outerplanar graph G, $\mathrm{pw}(G^{**}) + 1 \leq \mathrm{pw}(G) \leq 2\mathrm{pw}(G^{**}) + 1$. Furthermore the bounds are tight.*

As proved in [16], the pathwidth of a tree with f vertices is less than $\log_3(2f+1)$. Thus we have the following corollary.

Corollary 3 *The pathwidth of any biconnected outerplanar graph G with f inner faces is less than $2\log_3(2f+1)+1$.*

Proposition 2 *The time complexity of Algorithm 1 is $O(n\log(n))$.*

Algorithm 1 Procedure `Layout`

Require: a biconnected outerplanar graph G, a list l and an integer s.

Ensure: returns the integer j, which is one more than the biggest label used. Every vertex v of G is given a unique label, stored in $l(v)$.

if the weak dual T of G is a caterpillar **then**
  label it according to Proposition 1 and starting with the label s.
  return $s+|V(G)|$.
end if
Compute a path $P := v_1 v_2 \dots v_k$ of T fulfilling the property of Theorem 6, with the additional property that its endvertices are leaves. Denote by F_i the face of G corresponding to the vertex v_i of P, $i \in \{1,2,\dots k\}$.
Let v be a vertex of degree 2 of the face F_1, and denote by x and x' its clockwise and counter-clockwise neighbours respectively {note that such a vertex always exists}
$l(v) := s$
$s := s+1$
for $i = 1$ to k **do** {throughout the following, y and y' respectively denote the clockwise neighbour of x and the counter-clockwise neighbour of x' on F_i}
  while $x \notin V(F_{i+1})$ **do**
    if x has at most one unlabelled neighbour different from x' **then**
      $l(x) := s$
      $s := s+1$
    else
      let H be the maximal biconnected subgraph of G whose intersection with F_i is $\{x,y\}$.
      $s :=$ `Layout`(H,l,s)
      $l(y) := \infty$
    end if
    if $y == x'$ **then**
      $l(x') := s$
      return $s+1$
    else
      $x := y$
    end if
  end while
  while $x' \notin V(F_{i+1})$ **do**
    if x' has at most one unlabelled neighbour (different from x) **then**
      $l(x') := s$
      $s := s+1$
    else
      let H be the maximal biconnected subgraph of G whose intersection with F_i is $\{x,y\}$.
      $s :=$ `Layout`(H,l,s)
      $l(y') := \infty$
    end if
    $x' := y'$
  end while
end for

Proof. It is easy to see that the time complexity of Algorithm 1 depends mainly on the recursive calls and on the time complexity of line 5 (since computing the weak dual of a biconnected outerplanar graph and determining whether a tree is a caterpillar is linear in time, as is the procedure of Proposition 1). We first show that the time complexity of Algorithm 1 without line 5 is linear.

For that, remark that a node x of face F_i is labelled directly during the processing of face F_i if it has at most one unlabelled neighbour different from x', otherwise during the recursive call, or it will be considered again during the processing of face F_{i+1}. So a node x is considered once in each inner face to which it belongs, that is its degree minus one. So altogether we have $2(|E|-|V|)$ steps, which is equal to $2(f-1)$ using Euler's formula for planar graphs, where f is the total number of inner faces. Since the number of faces of a biconnected outerplanar graph is smaller that its number of vertices, the time complexity of Algorithm 1 without line 5 is linear.

The computation of a path P fulfilling the property of Theorem 6, with the additional property that its endvertices are leaves, is similar in style to the techniques used in [6, 20, 11] on trees to compute vertex separation, cutwidth and search number. Thus it can be done in linear-time. Furthermore, the pathwidth of a tree with f vertices being less than $\log_3(2f+1)$ [16], the computation of all paths takes time $O(f\log_3(2f+1))$, that is $O(n\log(n))$. □

Theorem 4 clearly provides a linear-time algorithm to approximate the pathwidth of a biconnected outerplanar graph G since computing the dual tree of G and its pathwidth can both be done in linear-time. A corresponding layout is given by Algorithm 1, whose time complexity is $O(n\log(n))$. As noted in [2], there exist trees and outerplanar graphs for which a straight representation of a layout needs $\Omega(n\log(n))$ in time just to be written. Skodinis [18] developed a representation so that path decompositions (and layouts) can be written in linear-time. We did not try to use it for Algorithm 1 but we suspect that it can be used to precompute all paths in linear-time and thus reduce the complexity to $O(n)$.

Corollary 4 *For any biconnected outerplanar graph G, Algorithm 1 provides in time* $O(n\log(n))$ *a layout of G with vertex separation at most* $2\text{pw}(G)-1$.

4 Conclusion

We strengthened the previously known relation between the pathwidth of a biconnected outerplanar graph and the pathwidth of its dual. We did so in an algorithmic way and thus obtained a new approximation algorithm. We established the tightness of our bound, thereby disproving a series of conjectures of Bodlaender and Fomin [2, 7] and moreover we showed that all cases in the interval happen.

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