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► **To cite this version:**

Jean-Gabriel Houot, Alexandre Munnier. On the motion and collisions of rigid bodies in an ideal fluid. 2006. inria-00001149v1

**HAL Id: inria-00001149**

**<https://inria.hal.science/inria-00001149v1>**

Submitted on 13 Mar 2006 (v1), last revised 6 Sep 2006 (v3)

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# On the motion and collisions of rigid bodies in an ideal fluid

Jean-Gabriel Houot\*      Alexandre Munnier†

December 15, 2005

## Abstract

In this paper we study a coupled system of partial differential equations and ordinary differential equations. This system is a model for the 3d interactive free motion of rigid bodies immersed in an ideal fluid. Applying the least action principle of Lagrangian mechanics, we prove that the solids degrees of freedom, solve a second order system of non-linear ordinary differential equations. Under suitable smoothness assumption on the solids and on the fluid domain boundaries, we prove the existence and  $C^\infty$  regularity, up to a collision between solids or between a solid with the boundary of the fluid domain, of the solids motion. Studying the particular case of an infinite cylinder surrounded by a fluid occupying an half space, we prove that collisions with non zero relative velocity can occur.

**Keywords and Phrases:** Fluid-solid interaction, ideal fluid, Lagrangian mechanics, collisions, PDE-ODE coupled system.

**AMS Subject Classification:** 35Q35, 35R35, 34A34, 34A12, 76B99.

## 1 Introduction and main results

### 1.1 Modelling

This paper deals with the free motion of rigid bodies in an incompressible, inviscid and irrotational fluid flow in  $\mathbb{R}^3$ . We denote by  $S(t) = \cup_{i=1}^n S^i(t)$  the domain occupied by a collection of  $n$  solids at the instant  $t > 0$ , by  $\Omega(t)$  the domain of the surrounding fluid with  $\Omega(0) = \Omega$  and  $S^i(0) = S^i$ . Thus, the boundary of  $\Omega(t)$  decomposes into  $\Gamma_1 = \partial\Omega(t) \setminus \partial S(t)$  and  $\Gamma_2(t) = \cup_{i=1}^n \Gamma_2^i(t)$  with  $\Gamma_2^i(t) = \partial S^i(t)$  (see figure 1). The center of mass of the  $i$ -th solid occupies the position  $\mathbf{h}^i(t)$  and we denote  $\mathbf{h}^i(0) = \mathbf{h}_0^i$ . By definition of a rigid motion, for any  $i = 1, \dots, n$  and for any  $t > 0$ , there exists an orthogonal matrix  $[R^i(t)] \in SO(3)$  such that the position  $\mathbf{x}^i(t)$  of a point occupying the position  $\mathbf{x}_0^i \in S^i$  at the instant  $t = 0$  is:

$$\mathbf{x}^i(t) = [R^i(t)](\mathbf{x}_0^i - \mathbf{h}_0^i) + \mathbf{h}^i(t). \quad (1.1)$$

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\*Insitut Elie Cartan UMR 7502, Université Henri Poincaré, Nancy 1, B.P. 239, F-54506 Vandoeuvre-lès-Nancy Cedex, France, [Jean-Gabriel.houot@iecn.u-nancy.fr](mailto:Jean-Gabriel.houot@iecn.u-nancy.fr)

†Insitut Elie Cartan UMR 7502, Université Henri Poincaré, Nancy 1, B.P. 239, F-54506 Vandoeuvre-lès-Nancy Cedex, France, [Alexandre.munnier@iecn.u-nancy.fr](mailto:Alexandre.munnier@iecn.u-nancy.fr)

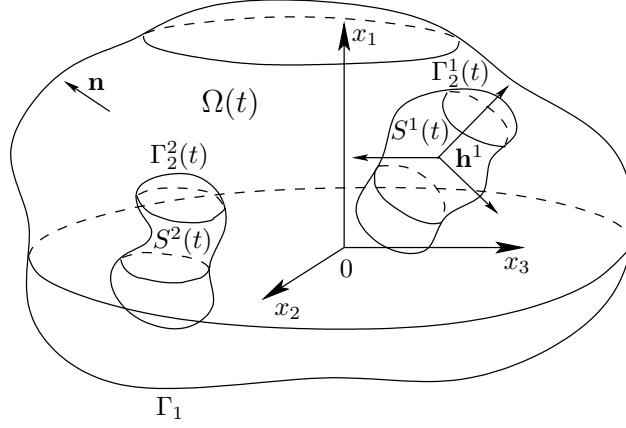


Figure 1: The rigid bodies  $S^i(t)$  and the fluid domain  $\Omega(t)$ .

### Admissible positions of the solids, global and local coordinates

Thus, at any time, the position of the  $i$ -th solid is described by a couple of  $SO(3) \times \mathbb{R}^3$ . In a general way, for any  $\mathbf{P}^i = ([R^i], \mathbf{h}^i) \in SO(3) \times \mathbb{R}^3$ , we denote  $S^i(\mathbf{P}^i) = [R^i]S^i + \mathbf{h}^i$  and  $\Gamma_2^i(\mathbf{P}^i) = \partial S^i(\mathbf{P}^i)$ . We shall call  $\mathbf{P} = (\mathbf{P}^1, \dots, \mathbf{P}^n) \in (SO(3) \times \mathbb{R}^3)^n$  the global coordinates of the solids and we introduce

$$\mathcal{P} = \{\mathbf{P} \in (SO(3) \times \mathbb{R}^3)^n \mid S^i(\mathbf{P}^i) \cap S^j(\mathbf{P}^j) = S^i(\mathbf{P}^i) \cap \Gamma_1 = \emptyset, \forall 1 \leq i, j \leq n, i \neq j\},$$

the set of the admissible positions of the solids. It is an indefinitely differentiable,  $6n$ -dimensional sub-manifold of  $(\mathcal{M}(3) \times \mathbb{R}^3)^n$ ,  $\mathcal{M}(3)$  standing for the vectorial space of the square  $3 \times 3$  matrices. In order to build a local coordinate chart of  $\mathcal{P}$ , let us recall how  $SO(3)$  is classically parameterized in a neighborhood of a given rotation  $[\tilde{R}]$ . One introduces the tensors:

$$[A_1] = \mathbf{e}_3 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_3, \quad [A_2] = \mathbf{e}_1 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_1, \quad [A_3] = \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2,$$

where  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  is the canonical basis of  $\mathbb{R}^3$  and the mapping:

$$\mathcal{R}([\tilde{R}], \cdot) : \mathbb{R}^3 \rightarrow SO(3), \quad (1.2a)$$

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)^\top \mapsto \mathcal{R}([\tilde{R}], \boldsymbol{\theta}) = \prod_{k=1}^3 \exp(\theta_k [A_k]) [\tilde{R}]. \quad (1.2b)$$

Denoting  $\Theta = ]-\pi, \pi[ \times ]-\pi/2, \pi/2[ \times ]-\pi, \pi[$ , the couple  $(\Theta, \mathcal{R}([\tilde{R}], \cdot))$  is a local  $C^\infty$  chart from a neighborhood of  $\mathbf{0}_3 \in \mathbb{R}^3$  onto a neighborhood of  $[\tilde{R}] \in SO(3)$ . The coordinates  $\theta_k$  are the so-called Euler angles (with the “ $xyz$ ” (pitch-roll-yaw) convention, see [8] page 603).

Then, for any given  $\tilde{\mathbf{P}} = (([\tilde{R}^1], \tilde{\mathbf{h}}^1), \dots, ([\tilde{R}^n], \tilde{\mathbf{h}}^n)) \in \mathcal{P}$  and any  $\mathbf{Q} = ((\boldsymbol{\theta}^1, \mathbf{h}^1), \dots, (\boldsymbol{\theta}^n, \mathbf{h}^n)) \in (\mathbb{R}^3 \times \mathbb{R}^3)^n$ , we define the function

$$\Lambda(\tilde{\mathbf{P}}, \mathbf{Q}) = ((\mathcal{R}([\tilde{R}^1], \boldsymbol{\theta}^1), \mathbf{h}^1), \dots, (\mathcal{R}([\tilde{R}^n], \boldsymbol{\theta}^n), \mathbf{h}^n)) \in (SO(3) \times \mathbb{R}^3)^n, \quad (1.3)$$

and the set  $\mathcal{Q}(\tilde{\mathbf{P}}) = (\Theta \times \mathbb{R}^3)^n \cap \Lambda^{-1}(\tilde{\mathbf{P}}, \mathcal{P})$ . One can easily check that  $(\mathcal{Q}(\tilde{\mathbf{P}}), \Lambda(\tilde{\mathbf{P}}, \cdot))$  is a local chart of  $\mathcal{P}$  from a neighborhood of  $\tilde{\mathbf{Q}} = ((\mathbf{0}_3, \tilde{\mathbf{h}}^1), \dots, (\mathbf{0}_3, \tilde{\mathbf{h}}^n))$  onto a neighborhood of  $\tilde{\mathbf{P}}$ , satisfying  $\Lambda(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}}) = \tilde{\mathbf{P}}$ . Therefore:

$$\mathcal{A} = \{(\mathcal{Q}(\mathbf{P}), \Lambda(\mathbf{P}, \cdot)), \mathbf{P} \in \mathcal{P}\},$$

is an atlas of  $\mathcal{P}$ . The tangent space to  $\mathcal{P}$  at  $\tilde{\mathbf{P}}$  is denoted  $T\mathcal{P}(\tilde{\mathbf{P}})$  and  $\partial\Lambda/\partial\mathbf{Q}(\tilde{\mathbf{P}}, \tilde{\mathbf{Q}})$  is an isomorphism from  $(\mathbb{R}^3 \times \mathbb{R}^3)^n$  onto  $T\mathcal{P}(\tilde{\mathbf{P}})$ .

For any  $\mathbf{Q} \in \mathcal{Q}(\tilde{\mathbf{P}})$  and any  $i = 1, \dots, n$ , we denote  $\mathbf{Q}^i = (\boldsymbol{\theta}^i, \mathbf{h}^i)$  and  $\Lambda^i(\tilde{\mathbf{P}}, \mathbf{Q}^i) = (\mathcal{R}([\tilde{R}^i], \boldsymbol{\theta}^i), \mathbf{h}^i)$ , so that  $\mathbf{Q} = (\mathbf{Q}^1, \dots, \mathbf{Q}^n)$  and  $\Lambda(\tilde{\mathbf{P}}, \mathbf{Q}) = (\Lambda^1(\tilde{\mathbf{P}}, \mathbf{Q}^1), \dots, \Lambda^n(\tilde{\mathbf{P}}, \mathbf{Q}^n))$ .

When there is no confusion possible, we shall dropped  $\tilde{\mathbf{P}}$  in the notations and denote merely  $\mathcal{Q}$  instead of  $\mathcal{Q}(\tilde{\mathbf{P}})$  and  $\mathcal{R}(\boldsymbol{\theta})$  instead of  $\mathcal{R}([\tilde{R}], \boldsymbol{\theta})$ . In the same way, in local coordinates,  $S^i(\mathbf{Q}^i)$  will stand for  $S^i(\Lambda^i(\tilde{\mathbf{P}}, \mathbf{Q}^i))$ ,  $\Gamma_2^i(\mathbf{Q}^i)$  for  $\Gamma_2^i(\Lambda^i(\tilde{\mathbf{P}}, \mathbf{Q}^i))$ ,  $\Gamma_2(\mathbf{Q}) = \cup_{i=1}^n \Gamma_2^i(\mathbf{Q}^i)$  and  $\Omega(\mathbf{Q})$  for  $\Omega(\Lambda(\tilde{\mathbf{P}}, \mathbf{Q}))$ .

### Euler-Lagrange equations of the fluid-solids system in local coordinates

The time derivative of (1.1) yields:

$$\dot{\mathbf{x}}(t) = \sum_{k=1}^3 \dot{\theta}_k^i(t) \frac{\partial \mathcal{R}}{\partial \theta_k^i}(\boldsymbol{\theta}^i(t))(\mathbf{x}_0 - \mathbf{h}_0^i) + \dot{\mathbf{h}}^i(t),$$

where  $\boldsymbol{\theta}^i(t) = (\theta_1^i(t), \theta_2^i(t), \theta_3^i(t))^\top = \mathcal{R}^{-1}([R^i(t)])$ . Defining for all  $\boldsymbol{\theta} \in \Theta$  the vectors:

$$\boldsymbol{\omega}_1(\boldsymbol{\theta}) = \mathbf{e}_1, \quad \boldsymbol{\omega}_2(\boldsymbol{\theta}) = e^{\theta_1[A_1]}\mathbf{e}_2, \quad \boldsymbol{\omega}_3(\boldsymbol{\theta}) = e^{\theta_1[A_1]}e^{\theta_2[A_2]}\mathbf{e}_3,$$

and the  $3 \times 3$  square matrix  $[\boldsymbol{\omega}(\boldsymbol{\theta})]$  as the horizontal concatenation of the column vectors  $\boldsymbol{\omega}_k(\boldsymbol{\theta})$  for  $k = 1, 2, 3$ , the following formula holds:

$$\frac{\partial \mathcal{R}}{\partial \boldsymbol{\theta}_k}(\boldsymbol{\theta})\mathcal{R}(\boldsymbol{\theta})^\top \mathbf{x} = \boldsymbol{\omega}_k(\boldsymbol{\theta}) \wedge \mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^3, \forall \boldsymbol{\theta} \in \Theta, \forall k = 1, 2, 3, \quad (1.4)$$

and the Eulerian velocity  $\mathbf{v}^i(t, \mathbf{x})$  at a point  $\mathbf{x}$  of  $S^i(t)$  is given by:

$$\mathbf{v}^i(t, \mathbf{x}) = [\boldsymbol{\omega}(\boldsymbol{\theta}^i)]\dot{\boldsymbol{\theta}}^i \wedge (\mathbf{x} - \mathbf{h}^i) + \dot{\mathbf{h}}^i. \quad (1.5)$$

To simplify notations, we set  $\boldsymbol{\omega}_k^i = \boldsymbol{\omega}_k(\boldsymbol{\theta}^i)$ ,  $k = 1, 2, 3$  and  $[\boldsymbol{\omega}^i] = [\boldsymbol{\omega}(\boldsymbol{\theta}^i)]$ .

For  $i = 1, \dots, n$ ,  $\rho^i$  stands for the density of the solid  $S^i$ ,  $m^i > 0$  is its mass and  $[J^i]$  is its inertia tensor defined by:

$$[J^i] = \int_{S^i} \rho^i (|\mathbf{x}_0 - \mathbf{h}_0^i|^2 [I_3] - (\mathbf{x}_0 - \mathbf{h}_0^i) \otimes (\mathbf{x}_0 - \mathbf{h}_0^i)) \, d\mathbf{x}_0.$$

The kinetic energy of the solid  $S^i$  reads  $\mathcal{K}^i(\mathbf{Q}^i, \dot{\mathbf{Q}}^i) = (1/2) \int_{S^i(\mathbf{Q}^i)} \rho^i |\mathbf{v}^i(t, \mathbf{x})|^2 \, d\mathbf{x}$ .

Denoting  $[M^i]$  the  $3 \times 3$  diagonal matrix  $\text{diag}(m^i, m^i, m^i)$ , and  $[\mathcal{J}^i(\mathbf{Q}^i)] = [\boldsymbol{\omega}^i]^\top [J^i] [\boldsymbol{\omega}^i]$  this expression reads:

$$\mathcal{K}^i(\mathbf{Q}^i, \dot{\mathbf{Q}}^i) = \frac{1}{2} \left( \dot{\boldsymbol{\theta}}^i \cdot [\mathcal{J}^i(\mathbf{Q}^i)] \dot{\boldsymbol{\theta}}^i + \dot{\mathbf{h}}^i \cdot [M^i] \dot{\mathbf{h}}^i \right).$$

Setting then the  $(6 \times 6)^n$  bloc-diagonal matrix

$$[K_S(\mathbf{Q})] = \text{diag}([\mathcal{J}^1(\mathbf{Q}^1)], [M^1], [\mathcal{J}^2(\mathbf{Q}^2)], [M^2], \dots, [\mathcal{J}^n(\mathbf{Q}^n)], [M^n]), \quad (1.6)$$

the total kinetic energy of the  $n$  solids reads:

$$\mathcal{K}_S(\mathbf{Q}, \dot{\mathbf{Q}}) = \sum_{i=1}^n \mathcal{K}^i(\mathbf{Q}^i, \dot{\mathbf{Q}}^i) = \frac{1}{2} \dot{\mathbf{Q}} \cdot [K_S(\mathbf{Q})] \dot{\mathbf{Q}}.$$

The Eulerian velocity  $\mathbf{u}(t, \cdot)$  of the fluid in  $\Omega(\mathbf{Q}(t))$  is given by:

$$\mathbf{u}(t, \cdot) = \nabla \varphi(t, \cdot),$$

where  $\varphi(t, \cdot)$  is the harmonic potential. It decomposes into  $\varphi(t, \cdot) = \sum_{i=1}^n \varphi^i(t, \cdot)$ , each function  $\varphi^i(t, \cdot)$  being harmonic in  $\Omega(\mathbf{Q}(t))$  and satisfying the Neumann boundaries conditions:

$$\begin{aligned} \frac{\partial \varphi^i}{\partial \mathbf{n}}(t, \cdot) &= \mathbf{v}^i(t) \cdot \mathbf{n}^i(\mathbf{Q}^i(t)) && \text{on } \Gamma_2^i(\mathbf{Q}^i(t)), \\ \frac{\partial \varphi^i}{\partial \mathbf{n}}(t, \cdot) &= 0 && \text{on } \Gamma_1 \cup_{j \neq i} \Gamma_2^j(\mathbf{Q}^j(t)), \end{aligned}$$

where the velocity  $\mathbf{v}^i(t)$  is given by (1.5) and the vector  $\mathbf{n}^i(\mathbf{Q}^i(t))$  stands for the unit normal to  $\Gamma_2^i(\mathbf{Q}^i(t))$  directed towards the exterior of the fluid. According to (1.5),  $\varphi^i(t, \cdot)$  is a linear combination of the functions  $\varphi_k^i(\mathbf{Q}(t), \cdot)$ ,  $i = 1, \dots, n$ ,  $k = 1, \dots, 6$  defined, for any  $\mathbf{Q} \in \mathcal{Q}$  as the solution (in a sense that will be made precise later in) of:

$$-\Delta \varphi_k^i(\mathbf{Q}, \cdot) = 0 \quad \text{in } \Omega(\mathbf{Q}), \quad (1.7a)$$

$$\frac{\partial \varphi_k^i}{\partial \mathbf{n}}(\mathbf{Q}, \cdot) = g_k^i(\mathbf{Q}^i) \quad \text{on } \Gamma_2^i(\mathbf{Q}^i), \quad (1.7b)$$

$$\frac{\partial \varphi_k^i}{\partial \mathbf{n}}(\mathbf{Q}, \cdot) = 0 \quad \text{on } \Gamma_1 \cup_{j \neq i} \Gamma_2^j(\mathbf{Q}^j), \quad (1.7c)$$

where  $g_k^i(\mathbf{Q}^i, \cdot) = (\boldsymbol{\omega}_k^i \wedge (\mathbf{x} - \mathbf{h}^i)) \cdot \mathbf{n}^i(\mathbf{Q}^i, \cdot)$  for  $k = 1, 2, 3$  and  $g_{3+k}^i(\mathbf{Q}^i, \cdot) = \mathbf{e}_k \cdot \mathbf{n}^i(\mathbf{Q}^i, \cdot)$  for  $k = 1, 2, 3$ .

Thus, denoting  $\boldsymbol{\varphi}^i(\mathbf{Q}, \cdot) = (\varphi_1^i(\mathbf{Q}, \cdot), \dots, \varphi_6^i(\mathbf{Q}, \cdot))^\top$ , and  $\boldsymbol{\varphi}(\mathbf{Q}, \cdot)$  the vertical concatenation of  $\boldsymbol{\varphi}^i(\mathbf{Q}, \cdot)$ ,  $i = 1, \dots, n$ , we obtain the formula:

$$\varphi^i(t, \cdot) = \dot{\mathbf{Q}}^i(t) \cdot \boldsymbol{\varphi}^i(\mathbf{Q}(t), \cdot), \quad \varphi(t, \cdot) = \dot{\mathbf{Q}}(t) \cdot \boldsymbol{\varphi}(\mathbf{Q}(t), \cdot). \quad (1.8)$$

The kinetic energy of the fluid which is assumed to be of constant density  $\rho_F$  is  $\mathcal{K}_F(\mathbf{Q}, \dot{\mathbf{Q}}) = (1/2) \int_{\Omega(\mathbf{Q})} |\mathbf{u}(t)|^2 d\mathbf{x} = (1/2) \int_{\Omega(\mathbf{Q})} |\nabla \varphi(t, \mathbf{x})|^2 d\mathbf{x}$ . Taking into account (1.8), we define for all  $\mathbf{Q} \in \mathcal{Q}$  the  $6^n \times 6^n$  square matrix

$$[K_F(\mathbf{Q})] = \rho_F \int_{\Omega(\mathbf{Q})} [D\boldsymbol{\varphi}(\mathbf{Q}, \mathbf{x})][D\boldsymbol{\varphi}(\mathbf{Q}, \mathbf{x})]^\top d\mathbf{x},$$

and rewrite that:

$$\mathcal{K}_F(\mathbf{Q}, \dot{\mathbf{Q}}) = \frac{1}{2} \dot{\mathbf{Q}} \cdot [K_F(\mathbf{Q})] \dot{\mathbf{Q}}.$$

Following Lamb [11], we will derive the governing equations for the solids motion by using Lagrangian mechanics. In the absence of body forces, the Lagrangian of the system is the sum of the kinetic energy of the fluid and the kinetic energy of the solids:

$$\mathcal{L}(\mathbf{Q}, \dot{\mathbf{Q}}) = \frac{1}{2} \dot{\mathbf{Q}} \cdot ([K_S(\mathbf{Q})] + [K_F(\mathbf{Q})]) \dot{\mathbf{Q}}, \quad (1.9)$$

and (see [1] page 53) the equations of the motion are:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{Q}}}(\dot{\mathbf{Q}}, \mathbf{Q}) - \frac{\partial \mathcal{L}}{\partial \mathbf{Q}}(\dot{\mathbf{Q}}, \mathbf{Q}) = \mathbf{0}_{6^n}. \quad (1.10)$$

Various fluid-solids interaction models were studied during the last years. We refer for example to [12] for a precise bibliography on this topic. The originalities of our approach is to use Euler-Lagrange equations and tools of shape sensitivity analysis deriving from shape optimization theory. These technics allow us to deal with a 3d model involving any number of solids of general shapes immersed in either bounded or unbounded fluid. The firsts who worked on solids dynamics in a frictionless fluid were Thomson, Tait and Kirchhoff. They were also the firsts who introduced the generalized coordinates and the Euler-Lagrange equations to study this kind of problems. Lamb, in [11], used their works to study the motion of solids in unbounded irrotational fluid. For example he dealt with the motion of a cylinder, an ellipsoid or two spheres. However the interesting case of shocks was not studied.

## 1.2 Notations

Throughout this article, we shall use bold print notations for vectors like  $\mathbf{x}$ ,  $\mathbf{h}$ ,  $\mathbf{Q}$  whereas we keep the usual characters for their coordinates  $x_j$ ,  $h_j$ ,  $q_j$  and in a general way for any real valued functions like  $\phi$ ,  $\varphi$ . The canonical basis of the physical space  $\mathbb{R}^3$  is  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  and the canonical basis of the  $6n$  dimensional space of the local coordinates  $\mathbf{Q}$  is  $\{\mathbf{e}_1^1, \dots, \mathbf{e}_6^1, \mathbf{e}_1^2, \dots, \mathbf{e}_6^2, \dots, \mathbf{e}_1^n, \dots, \mathbf{e}_6^n\}$ .

A matrix is denoted in square brackets  $[M]$ , its entries are  $M_{j_1, j_2}$  and  $[M]^\top$  is the transposed matrix. We use double square brackets for three-rank tensors like  $\llbracket T \rrbracket$ . Its entries are  $T_{j_1, j_2, j_3}$ .

In section 4, we will identify  $\mathbb{R}^2$  with the complex plane  $\mathbb{C}$  and any vector  $\mathbf{x} = (x_1, x_2)^\top$  with the complex number  $x_1 + ix_2$  where  $i^2 = -1$ . Due to this identification and according to the rules of notations above, we shall use bold print notations for complex numbers  $\mathbf{z} = x_1 + ix_2$ . In the same way, we will make no difference between a complex valued function  $\mathbf{f}(x_1 + ix_2) = f_1(x_1 + ix_2) + if_2(x_1 + ix_2)$  and the vector valued one  $\mathbf{f}(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))^\top$ . At last, for an holomorphic function, the complex derivative  $\mathbf{f}'(x_1 + ix_2)$  will coincide with the Jacobian matrix of  $\mathbf{f}$  and sometimes be denoted  $[\mathbf{f}']$ .

## 1.3 Main results

The first Theorem concerns the existence, uniqueness and regularity of local solutions for the Euler-Lagrange system of ODE (1.10):

**Theorem 1.1** *Assume that  $\partial\Omega$  is Lipschitz continuous. Then, for any time  $\tilde{t} \geq 0$ , any admissible position of the solids  $\tilde{\mathbf{P}} \in \mathcal{P}$  and any initial data  $(\tilde{\mathbf{Q}}, \tilde{\dot{\mathbf{Q}}}) \in \mathcal{Q}(\tilde{\mathbf{P}}) \times (\mathbb{R}^3 \times \mathbb{R}^3)^n$ , there exists an interval  $(\tilde{t}_1, \tilde{t}_2) \subset \mathbb{R}$  containing  $\tilde{t}$  and a unique solution  $t \in (\tilde{t}_1, \tilde{t}_2) \mapsto \mathbf{Q}^*(t, \tilde{t}, \tilde{\mathbf{Q}}, \tilde{\dot{\mathbf{Q}}}) \in \mathcal{Q}(\tilde{\mathbf{P}})$  to the ODE (1.10) satisfying  $\mathbf{Q}^*(\tilde{t}, \tilde{t}, \tilde{\mathbf{Q}}, \tilde{\dot{\mathbf{Q}}}) = \tilde{\mathbf{Q}}$  and  $\dot{\mathbf{Q}}^*(\tilde{t}, \tilde{t}, \tilde{\mathbf{Q}}, \tilde{\dot{\mathbf{Q}}}) = \tilde{\dot{\mathbf{Q}}}$ . Moreover, the function  $\mathbf{Q}^*(\cdot, \tilde{t}, \tilde{\mathbf{Q}}, \tilde{\dot{\mathbf{Q}}})$  is indefinitely differentiable.*

Combining this Theorem with a priori estimates, it shall allow us to prove:

**Theorem 1.2** *Assume that  $\partial\Omega$  is Lipschitz continuous. Then, for any initial data  $(([I_3], [\dot{R}_0^1]), \dots, ([I_3], [\dot{R}_0^n])) \in (TSO(3))^n$  and  $((\mathbf{h}_0^1, \dot{\mathbf{h}}_0^1), \dots, (\mathbf{h}_0^n, \dot{\mathbf{h}}_0^n)) \in (\mathbb{R}^3 \times \mathbb{R}^3)^n$  (initial positions and initial velocities of the solids), there exists a times  $T^* > 0$  and a unique indefinitely differentiable function:*

$$\begin{aligned} \mathbf{P}^* : [0, T^*) &\rightarrow \mathcal{P} \\ t &\mapsto (([R^1(t)], \mathbf{h}^1(t)), \dots, ([R^n(t)], \mathbf{h}^n(t))), \end{aligned}$$

for which  $\mathbf{P}^*(0) = (([I_3], \mathbf{h}_0^1), \dots, ([I_3], \mathbf{h}_0^n))$  and  $\dot{\mathbf{P}}^*(0) = (([\dot{R}_0^1], \dot{\mathbf{h}}_0^1), \dots, ([\dot{R}_0^n], \dot{\mathbf{h}}_0^n))$  and such that the trajectory of a point  $\mathbf{x}^i(t)$ , occupying the position  $\mathbf{x}_0^i \in S^i$  at the instant  $t = 0$  be given by (1.1).

The time  $T^*$  corresponds to the time of the first collision between two solids or between a solid with the fluid domain boundary. If there is no collision, then  $T^* = \infty$ .

Assuming additional regularity for  $\partial\Omega$ , it is possible to make explicit the Euler-Lagrange equations (1.10). The expression shall involve  $\varphi(\mathbf{Q}, \cdot)$  and  $[D\varphi(\mathbf{Q}, \cdot)]$  on  $\Gamma_2(\mathbf{Q})$  and geometric data of the solids. Thus, for any  $\mathbf{x} \in \Gamma_2^i(\mathbf{Q}^i)$ ,  $\tau_1^i(\mathbf{Q}^i, \mathbf{x})$  and  $\tau_2^i(\mathbf{Q}^i, \mathbf{x})$  stand for the principal directions on  $\Gamma_2^i(\mathbf{Q})$  at the point  $\mathbf{x}$  and  $\kappa_1^i(\mathbf{Q}^i, \mathbf{x})$  and  $\kappa_2^i(\mathbf{Q}^i, \mathbf{x})$  are the principal curvatures. The set  $\{\tau_1^i(\mathbf{Q}^i, \mathbf{x}), \tau_2^i(\mathbf{Q}^i, \mathbf{x}), \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x})\}$  is a direct orthonormal basis of  $\mathbb{R}^3$  and  $\Pi^i(\mathbf{Q}^i, \mathbf{x}) = \kappa_1^i(\mathbf{Q}^i, \mathbf{x})\tau_1^i(\mathbf{Q}^i, \mathbf{x}) \otimes \tau_1^i(\mathbf{Q}^i, \mathbf{x}) + \kappa_2^i(\mathbf{Q}^i, \mathbf{x})\tau_2^i(\mathbf{Q}^i, \mathbf{x}) \otimes \tau_2^i(\mathbf{Q}^i, \mathbf{x})$  is the second fundamental form on  $\Gamma_2^i(\mathbf{Q})$  at the point

$\mathbf{x}$ . We refer to [4] for details and precisions on this topic. We introduce also the 6–length column vectors defined on  $\Gamma_2^i(\mathbf{Q})$  by:

$$\mathbf{W}_{\mathbf{n}}^i(\mathbf{Q}^i, \mathbf{x}) = \begin{bmatrix} [\omega^i]((\mathbf{x} - \mathbf{h}^i) \wedge \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x})) \\ \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}) \end{bmatrix}, \quad (1.11a)$$

$$\mathbf{W}_{\tau_k}^i(\mathbf{Q}^i, \mathbf{x}) = \begin{bmatrix} [\omega^i]((\mathbf{x} - \mathbf{h}^i) \wedge \tau_k^i(\mathbf{Q}^i, \mathbf{x})) \\ \tau_k^i(\mathbf{Q}^i, \mathbf{x}) \end{bmatrix}, \quad k = 1, 2, \forall \mathbf{x} \in \Gamma_2(\mathbf{Q}^i), \quad (1.11b)$$

and  $\mathbf{W}_{\mathbf{n}}^i(\mathbf{Q}^i, \mathbf{x}) = \mathbf{W}_{\tau_k}^i(\mathbf{Q}^i, \mathbf{x}) = \mathbf{0}_6$  if  $\mathbf{x} \in \Gamma_2^j(\mathbf{Q}^j)$ ,  $j \neq i$ . We define also the  $6n$ –length column vectors  $\mathbf{W}_{\mathbf{n}}(\mathbf{Q}, \mathbf{x})$  and  $\mathbf{W}_{\tau_k}(\mathbf{Q}, \mathbf{x})$  obtain by concatenating respectively the  $n$  vectors  $\mathbf{W}_{\tau_k}^i(\mathbf{Q}^i, \mathbf{x})$  and the  $n$  vectors  $\mathbf{W}_{\mathbf{n}}^i(\mathbf{Q}^i, \mathbf{x})$ . For all  $\mathbf{x} \in \Gamma_2(\mathbf{Q})$  and  $k = 1, 2$ ,  $\kappa_k(\mathbf{Q}, \mathbf{x}) = \kappa_k^i(\mathbf{Q}^i, \mathbf{x})$  if  $\mathbf{x} \in \Gamma_2^i(\mathbf{Q}^i)$ . We set then the following three-rank  $6n \times 6n \times 6n$  tensors:

$$\begin{aligned} \llbracket T_{F_n}(\mathbf{Q}) \rrbracket &= -\rho_F \int_{\Gamma_2(\mathbf{Q})} [D\varphi(\mathbf{Q})][D\varphi(\mathbf{Q})]^\top \otimes \mathbf{W}_{\mathbf{n}}(\mathbf{Q}) \, d\sigma_x \\ &\quad + \rho_F \frac{1}{2} \int_{\Gamma_2(\mathbf{Q})} \mathbf{W}_{\mathbf{n}}(\mathbf{Q}) \otimes [D\varphi(\mathbf{Q})][D\varphi(\mathbf{Q})]^\top \, d\sigma_x \\ &\quad - \rho_F \int_{\Gamma_2(\mathbf{Q})} (\kappa_1(\mathbf{Q}) + \kappa_2(\mathbf{Q})) \varphi(\mathbf{Q}) \otimes \mathbf{W}_{\mathbf{n}}(\mathbf{Q}) \otimes \mathbf{W}_{\mathbf{n}}(\mathbf{Q}) \, d\sigma_x, \end{aligned} \quad (1.12)$$

and for  $k = 1, 2$ :

$$\llbracket T_{F_k}(\mathbf{Q}) \rrbracket = \rho_F \int_{\Gamma_2(\mathbf{Q})} \kappa_k(\mathbf{Q}) \varphi(\mathbf{Q}) \otimes \mathbf{W}_{\tau_k}(\mathbf{Q}) \otimes \mathbf{W}_{\tau_k}(\mathbf{Q}) \, d\sigma_x. \quad (1.13)$$

In the other hand, for any  $i = 1, \dots, n$ , we introduce the three-rank  $6 \times 6 \times 6$  tensor  $\llbracket T^i(\boldsymbol{\theta}^i) \rrbracket$ :

$$\begin{aligned} T_{k,j_1,j_2}^i(\boldsymbol{\theta}^i) &= (\omega_{\min\{j_1,j_2\}}^i \wedge \omega_{\max\{j_1,j_2\}}^i) [J^i] \omega_k^i + (\omega_{j_1}^i \wedge \omega_k^i) [J^i] \omega_{j_2}^i \\ &\quad + (\omega_{j_2}^i \wedge \omega_k^i) [J^i] \omega_{j_1}^i, \end{aligned} \quad (1.14)$$

if  $1 \leq k, j_1, j_2 \leq 3$  and  $T_{k,j_1,j_2}^i(\boldsymbol{\theta}^i) = 0$  if  $4 \leq k, j_1, j_2 \leq 6$ . Thus,  $\langle \llbracket T^i(\boldsymbol{\theta}^i) \rrbracket, \dot{\mathbf{Q}}^i, \dot{\mathbf{Q}}^i \rangle$  yields a 6–length column vector. With the tensors  $\llbracket T^i(\boldsymbol{\theta}^i) \rrbracket$ , we build the three-rank  $6n \times 6n \times 6n$  tensor defined for all  $(\mathbf{Q}, \dot{\mathbf{Q}}) \in \mathcal{Q} \times (\mathbb{R}^6)^n$  by:

$$\langle \llbracket T_S(\mathbf{Q}) \rrbracket, \dot{\mathbf{Q}}, \dot{\mathbf{Q}} \rangle = \begin{pmatrix} \langle \llbracket T^1(\mathbf{Q}^1) \rrbracket, \dot{\mathbf{Q}}^1, \dot{\mathbf{Q}}^1 \rangle \\ \vdots \\ \langle \llbracket T^n(\mathbf{Q}^n) \rrbracket, \dot{\mathbf{Q}}^n, \dot{\mathbf{Q}}^n \rangle \end{pmatrix}. \quad (1.15)$$

**Theorem 1.3** *Assume that  $\partial\Omega$  is of class  $C^2$ . Then, the Euler-Lagrange system of ODE (1.10) reads:*

$$\begin{aligned} &([K_S(\mathbf{Q})] + [K_F(\mathbf{Q})])\ddot{\mathbf{Q}} + \langle \llbracket T_S(\mathbf{Q}) \rrbracket, \dot{\mathbf{Q}}, \dot{\mathbf{Q}} \rangle \\ &\quad + \langle \llbracket T_{F_n}(\mathbf{Q}) \rrbracket, \dot{\mathbf{Q}}, \dot{\mathbf{Q}} \rangle + \langle \llbracket T_{F_1}(\mathbf{Q}) \rrbracket, \dot{\mathbf{Q}}, \dot{\mathbf{Q}} \rangle + \langle \llbracket T_{F_2}(\mathbf{Q}) \rrbracket, \dot{\mathbf{Q}}, \dot{\mathbf{Q}} \rangle = \mathbf{0}_{6n}. \end{aligned}$$

Furthermore, all the matrices and tensors involved in this equation are well defined and indefinitely differentiable with respect to  $\mathbf{Q}$  for all  $\mathbf{Q} \in \mathcal{Q}$ .

At last, assuming there is only one solid in the fluid, we shall prove:

**Theorem 1.4** *If we specify the solid to be an infinite cylinder and the domain of the fluid to be an half space, then there exist initial conditions such that the cylinder collides with the boundary in finite time with non zero relative velocity.*

## 1.4 Plan of the paper

This paper is organized as follows : in the following section, we deal with a generic problem of shape sensitivity analysis. The results are used in section 3 for the proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3. Then, in section 4, specifying the solid to be an infinite cylinder and the domain occupied by the fluid to be an half space, we shall prove that finite time collisions happen. In the last section, we will study the behavior of the solid just before the collision time.

## 2 Shape sensitivity analysis

### 2.1 The context

Let  $\Omega$  be an open connected subset of  $\mathbb{R}^3$  satisfying one of the following assertions:

$$\Omega \text{ bounded,} \quad (2.1a)$$

$$\Omega^c \text{ bounded,} \quad (2.1b)$$

$$\partial\Omega \text{ is an hyperplan outside a bounded set.} \quad (2.1c)$$

We denote  $\Gamma = \partial\Omega$  and we assume furthermore that

$$\Gamma = \Gamma_1 \cup \Gamma_2, \quad \Gamma_1 \cap \Gamma_2 = \emptyset \text{ and } \Gamma_2 \text{ is bounded.} \quad (H_1)$$

Following A. Henrot and M. Pierre in [10], we introduce for all  $m \geq 1$  the Banach spaces  $C^{m,\infty}(\mathbb{R}^3, \mathbb{R}^3) = C^m(\mathbb{R}^3, \mathbb{R}^3) \cap W^{m,\infty}(\mathbb{R}^3, \mathbb{R}^3)$  and  $C_0^{m,\infty}(\mathbb{R}^3, \mathbb{R}^3)$  the subset of the functions compactly supported in  $C^{m,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ . The following properties, which proof can be found in [10] page 184, will be useful in the sequel:

**Lemma 2.1** *There exists a neighborhood  $\mathcal{U}$  of  $[I_3]$  in  $C^{1,\infty}(\mathbb{R}^3, \mathbb{R}^3)$  such that the following mappings are well defined and indefinitely differentiable:*

$$\Theta \in \mathcal{U} \mapsto [D\Theta]^{-1} \in L^\infty(\mathbb{R}^3, \mathcal{M}(3)), \quad \Theta \in \mathcal{U} \mapsto |\det[D\Theta]| \in L^\infty(\mathbb{R}^3). \quad (2.2a)$$

Let now  $\mathcal{Q}$  be an open subset of  $\mathbb{R}^p$  ( $p \geq 1$ ) containing  $\mathbf{0}_p$  and consider a  $C^\alpha$  mapping ( $\alpha \geq 1$ ):

$$\mathbf{Q} = (q_1, \dots, q_p)^\top \in \mathcal{Q} \mapsto \phi(\mathbf{Q}, \cdot) \in C^{m,\infty}(\mathbb{R}^3, \mathbb{R}^3), \quad (2.3)$$

such that

- For all  $\mathbf{Q} \in \mathcal{Q}$ ,  $\phi(\mathbf{Q}, \cdot) \in \mathcal{U}$ ,  $\phi(\mathbf{Q}, \cdot) - [I_3] \in C_0^{m,\infty}(\mathbb{R}^3, \mathbb{R}^3)$  and  $\phi(\mathbf{0}_p, \cdot) = [I_3]$ . (H<sub>2</sub>)
- For all  $\mathbf{Q} \in \mathcal{Q}$ ,  $\phi(\mathbf{Q}, \Gamma_1) = \Gamma_1$  and  $\phi(\mathbf{Q}, \cdot)$  is a rigid motion in a neighborhood of  $\Gamma_2$ . (H<sub>3</sub>)

We denote  $\Omega(\mathbf{Q}) = \phi(\mathbf{Q}, \Omega)$ ,  $\Gamma(\mathbf{Q}) = \phi(\mathbf{Q}, \Gamma)$  and  $\Gamma_2(\mathbf{Q}) = \phi(\mathbf{Q}, \Gamma_2)$  and we define for all  $\mathbf{Q} \in \mathcal{Q}$ :

$$[A(\mathbf{Q}, \cdot)] = [D\phi(\mathbf{Q}, \cdot)]^{-1} [D\phi(\mathbf{Q}, \cdot)]^{-1\top} |\det[D\phi(\mathbf{Q}, \cdot)]| \in L^\infty(\mathbb{R}^3, \mathcal{M}(3)).$$

The hypothesis (H<sub>2</sub>) leads to  $[A(\mathbf{0}_p, \cdot)] = [I_3]$  and  $[A(\mathbf{Q}, \cdot)] - [I_3] \in C_0^{m,\infty}(\mathbb{R}^3, \mathbb{R}^3)$ . Let  $0 < \delta < 1$ . Lemma 2.1 allows us to choose  $\mathcal{Q}$  small enough such that:

$$\|[A(\mathbf{Q}, \cdot)] - [I_3]\|_{L^\infty(\mathbb{R}^3, \mathcal{M}(3))} \leq \delta, \quad \forall \mathbf{Q} \in \mathcal{Q}. \quad (H_4)$$



For any  $\mathbf{Q} \in \mathcal{Q}$ , we consider the solutions (in a sense that will be made precise later in)  $u_1(\mathbf{Q}, \cdot)$  and  $u_2(\mathbf{Q}, \cdot)$ , of the following Neumann problems:

$$-\Delta u_k(\mathbf{Q}, \cdot) = 0 \quad \text{in } \Omega(\mathbf{Q}), \quad (2.4a)$$

$$\frac{\partial u_k}{\partial \mathbf{n}}(\mathbf{Q}, \cdot) = g_k(\mathbf{Q}, \cdot) \text{ on } \Gamma(\mathbf{Q}), \quad k = 1, 2, \quad (2.4b)$$

where  $g_1(\mathbf{Q}, \cdot)$  and  $g_2(\mathbf{Q}, \cdot)$  are given functions on  $\Gamma(\mathbf{Q})$ . We are interested in studying the sensitivity with respect to  $\mathbf{Q}$  of the functional:

$$\Upsilon(\mathbf{Q}) = \int_{\Omega(\mathbf{Q})} \nabla u_1(\mathbf{Q}, \mathbf{x}) \cdot \nabla u_2(\mathbf{Q}, \mathbf{x}) \, d\mathbf{x}. \quad (2.5)$$

## 2.2 The functional spaces

We denote:

$$\tilde{L}^2(\Gamma) = \left\{ G \in L^2(\Gamma) \mid \int_{\Gamma} G \, d\sigma_x = 0 \text{ if (2.1a) holds and } G|_{\Gamma_1} \equiv 0 \text{ if (2.1c) holds} \right\},$$

$$\tilde{H}^{1/2}(\Gamma) = H^{1/2}(\Gamma) \cap \tilde{L}^2(\Gamma),$$

and the weighted Lebesgue spaces:

$$\begin{aligned} \mathcal{L}^2(\Omega) &= \{u \in \mathcal{D}'(\Omega) \mid (1 + |\mathbf{x}|^2)^{3/4} u \in L^2(\Omega)\}, & \text{if } \Omega \text{ is not bounded,} \\ \mathcal{L}^2(\Omega) &= L^2(\Omega), & \text{if } \Omega \text{ is bounded.} \end{aligned}$$

We define also:

$$\tilde{\mathcal{V}}^1 = \left\{ u \in \mathcal{D}'(\mathbb{R}^3) \mid u/(1 + |\mathbf{x}|^2)^{3/4} \in L^2(\mathbb{R}^3), \nabla u \in L^2(\mathbb{R}^3, \mathbb{R}^3) \right\},$$

and, if  $\Omega$  is bounded, we set for all  $\mathbf{Q} \in \mathcal{Q}$  the quotient space:

$$\mathcal{V}^1(\mathbf{Q}) = \{u \in \mathcal{D}'(\Omega(\mathbf{Q})) \mid u \in L^2(\Omega(\mathbf{Q})), \nabla u \in L^2(\Omega(\mathbf{Q}), \mathbb{R}^3)\} / \mathbb{R}.$$

If  $\Omega$  is not bounded, the definition above turns into:

$$\mathcal{V}^1(\mathbf{Q}) = \left\{ u \in \mathcal{D}'(\Omega(\mathbf{Q})) \mid u/(1 + |\mathbf{x}|^2)^{3/4} \in L^2(\Omega(\mathbf{Q})), \nabla u \in L^2(\Omega(\mathbf{Q}), \mathbb{R}^3) \right\}.$$

At last, we will need:

$$\tilde{\mathcal{V}}^2 = \left\{ u \in \tilde{\mathcal{V}}^1(\mathbf{Q}) \mid (1 + |\mathbf{x}|^2)^{3/4} [D^2 u] \in L^2(\mathbb{R}^3, \mathcal{M}(3)) \right\},$$

$$\mathcal{V}^2(\mathbf{Q}) = \left\{ u \in \mathcal{V}^1(\mathbf{Q}) \mid (1 + |\mathbf{x}|^2)^{3/4} [D^2 u] \in L^2(\Omega(\mathbf{Q}), \mathcal{M}(3)) \right\}, \quad \text{if } \Omega \text{ is not bounded,}$$

$$\mathcal{V}^2(\mathbf{Q}) = \left\{ u \in \mathcal{V}^1(\mathbf{Q}) \mid [D^2 u] \in L^2(\Omega(\mathbf{Q}), \mathcal{M}(3)) \right\}, \quad \text{if } \Omega \text{ is bounded,}$$

where  $[D^2 u]$  stands for the Hessian matrix of  $u$ . We denote merely  $\mathcal{V}^1 = \mathcal{V}^1(\mathbf{0}_p)$  and  $\mathcal{V}^2 = \mathcal{V}^2(\mathbf{0}_p)$ . The spaces  $\mathcal{V}^1(\mathbf{Q})$  and  $\mathcal{V}^2(\mathbf{Q})$  are endowed respectively with the scalar products:

$$(u, v)_{\mathcal{V}^1(\mathbf{Q})} = \int_{\Omega(\mathbf{Q})} \nabla u \cdot \nabla v \, d\mathbf{x},$$

$$(u, v)_{\mathcal{V}^2(\mathbf{Q})} = (u, v)_{\mathcal{V}^1(\mathbf{Q})} + \int_{\Omega(\mathbf{Q})} [D^2 u] : [D^2 v] (1 + |\mathbf{x}|^2)^{3/2} \, d\mathbf{x}.$$

We refer to the PhD thesis [6] for details about these spaces. Dealing with the existence of solutions for (2.4), we claim that:

**Lemma 2.2** *Assume that  $\Gamma$  is lipschitz continuous and that  $m = 1$  in (2.3). Then for any  $g_k(\mathbf{Q}, \cdot)$  ( $k = 1, 2$ ) on  $\Gamma(\mathbf{Q})$  such that  $g_k(\mathbf{Q}, \phi(\mathbf{Q}, \cdot)) \in \tilde{L}^2(\Gamma)$ , the system (2.4) admits in  $\mathcal{V}^1(\mathbf{Q})$  a unique weak solution  $u_k(\mathbf{Q}, \cdot)$  defined by:*

$$\int_{\Omega(\mathbf{Q})} \nabla u_k(\mathbf{Q}, \mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) \, d\mathbf{x} = \int_{\Gamma(\mathbf{Q})} g_k(\mathbf{Q}, \mathbf{x}) \varphi(\mathbf{x}) \, d\sigma_x, \quad \forall \varphi \in \mathcal{V}^1(\mathbf{Q}). \quad (2.6)$$

*If  $\Gamma$  is of class  $C^2$  and  $g_k(\mathbf{Q}, \phi(\mathbf{Q}, \cdot)) \in \tilde{H}^{1/2}(\Gamma)$  the solution  $u_k(\mathbf{Q}, \cdot)$  is in  $\mathcal{V}^2(\mathbf{Q})$  ( $k = 1, 2$ ).*

The proof is classical if  $\Omega$  is bounded and we refer to [7] for the case  $\Omega^c$  bounded or the case (2.1c), which is similar.

### 2.3 Regularity results

From now on we shall denote  $G_k(\mathbf{Q}, \cdot) = g_k(\mathbf{Q}, \phi(\mathbf{Q}, \cdot))$  and  $U_k(\mathbf{Q}, \cdot) = u_k(\mathbf{Q}, \phi(\mathbf{Q}, \cdot))$  for all  $\mathbf{Q} \in \mathcal{Q}$ ,  $k = 1, 2$ .

**Proposition 2.1** *Assume that  $\Gamma$  is Lipschitz continuous,  $m = 1$  in (2.3) and that  $\mathbf{Q} \in \mathcal{Q} \mapsto G_k(\mathbf{Q}, \cdot) \in \tilde{L}^2(\Gamma)$  is of class  $C^\beta$ ,  $\beta \geq 1$ . Then, the mappings:*

$$\mathbf{Q} \in \mathcal{Q} \mapsto U_k(\mathbf{Q}, \cdot) \in \mathcal{V}^1, \quad \mathbf{Q} \in \mathcal{Q} \mapsto \Upsilon(\mathbf{Q}) \in \mathbb{R},$$

*are of class  $C^{\min\{\alpha, \beta\}}$  in a neighborhood of  $\mathbf{Q} = \mathbf{0}_p$ .*

*If  $\Gamma$  is of class  $C^2$ ,  $m = 2$  in (2.3) and  $\mathbf{Q} \in \mathcal{Q} \mapsto G_k(\mathbf{Q}, \cdot) \in \tilde{H}^{1/2}(\Gamma)$  is of class  $C^1$ , then:*

$$\mathbf{Q} \in \mathcal{Q} \mapsto U_k(\mathbf{Q}, \cdot) \in \mathcal{V}^2,$$

*is of class  $C^1$  in a neighborhood of  $\mathbf{Q} = \mathbf{0}_p$ . Furthermore, for all compact set  $K \subset \Omega$ , there exists a neighborhood  $\mathcal{Q}_K$  of  $\mathbf{Q} = \mathbf{0}_p$  in  $\mathcal{Q}$  such that  $\phi(\mathbf{Q}, K) \subset \Omega$ , for all  $\mathbf{Q} \in \mathcal{Q}_K$  and the mappings:*

$$\mathbf{Q} \in \mathcal{Q}_K \mapsto u_k(\mathbf{Q}, \cdot)|_K \in L^2(K), \quad \mathbf{Q} \in \mathcal{Q}_K \mapsto \nabla u_k(\mathbf{Q}, \cdot)|_K \in L^2(K, \mathbb{R}^3),$$

*are also of class  $C^1$ . That allows us to define  $\mathbf{x}$ -a.e. in  $\Omega$  the functions  $\partial u_k / \partial q_i(\mathbf{0}_p, \cdot)$  and  $\partial(\nabla u_k) / \partial q_i(\mathbf{0}_p, \cdot)$  for all  $i = 1, \dots, p$ . The first one lies in  $\mathcal{V}^1$  and the second one in  $L^2(\Omega, \mathbb{R}^3)$ .*

The local results of this Proposition which are available only in a neighborhood of  $\mathbf{Q} = \mathbf{0}_p$  can easily be extended to the whole set  $\mathcal{Q}$ . Indeed, denoting  $\tilde{\Omega} = \phi(\tilde{\mathbf{Q}}, \Omega)$  for any  $\tilde{\mathbf{Q}} \in \mathcal{Q}$  and defining, for all  $\mathbf{Q}$  in a neighborhood of  $\mathbf{Q} = \mathbf{0}_p$  the function

$$\tilde{\phi}(\mathbf{Q}, \cdot) = \phi(\tilde{\mathbf{Q}} + \mathbf{Q}, \phi^{-1}(\tilde{\mathbf{Q}}, \cdot)),$$

there is only to apply the Proposition with the function  $\tilde{\phi}$ . Therefore, it makes sense to define, for all  $\mathbf{Q} = (q_1, \dots, q_n)^\top \in \mathcal{Q}$ :

$$\frac{\partial u_k}{\partial q_i}(\mathbf{Q}, \cdot) \in \mathcal{V}^1(\mathbf{Q}) \quad \text{and} \quad \frac{\partial(\nabla u_k)}{\partial q_i}(\mathbf{Q}, \cdot) \in L^2(\Omega(\mathbf{Q}), \mathbb{R}^3), \quad i = 1, \dots, p.$$

Assuming additional regularity for  $\Gamma$ , it is possible to compute explicitly the expressions of the partial derivatives of  $\Upsilon$  with respect to  $q_i$ :

**Proposition 2.2** Assume that  $\Gamma$  is of class  $C^2$ ,  $m = 2$  in (2.3) and  $\mathbf{Q} \in \mathcal{Q} \mapsto G_k(\mathbf{Q}, \cdot) \in \tilde{H}^{1/2}(\Gamma)$  is of class  $C^1$ . Then, we have:

$$\begin{aligned}
\frac{\partial \Upsilon}{\partial q_i}(\mathbf{Q}) = & - \int_{\Gamma_2(\mathbf{Q})} \nabla u_1(\mathbf{Q}) \cdot \nabla u_2(\mathbf{Q})(\mathbf{V}^i(\mathbf{Q}) \cdot \mathbf{n}) \, d\sigma_x \\
& + \int_{\Gamma_2(\mathbf{Q})} \left( \frac{\partial G_1}{\partial q_i}(\phi^{-1}(\mathbf{Q})) - \nabla_\tau g_1(\mathbf{Q}) \cdot \mathbf{V}^i(\mathbf{Q}) \right) u_2(\mathbf{Q}) \, d\sigma_x \\
& + \int_{\Gamma_2(\mathbf{Q})} \left( \frac{\partial G_2}{\partial q_i}(\phi^{-1}(\mathbf{Q})) - \nabla_\tau g_2(\mathbf{Q}) \cdot \mathbf{V}^i(\mathbf{Q}) \right) u_1(\mathbf{Q}) \, d\sigma_x \\
& + 2 \int_{\Gamma_2(\mathbf{Q})} g_1(\mathbf{Q})g_2(\mathbf{Q})(\mathbf{V}^i(\mathbf{Q}) \cdot \mathbf{n}) \, d\sigma_x \\
& - \int_{\Gamma_2(\mathbf{Q})} (\kappa_1(\mathbf{Q}) + \kappa_2(\mathbf{Q}))(g_1(\mathbf{Q})u_2(\mathbf{Q}) + g_2(\mathbf{Q})u_1(\mathbf{Q}))(\mathbf{V}^i(\mathbf{Q}) \cdot \mathbf{n}) \, d\sigma_x \\
& + \int_{\Gamma_1} \frac{\partial g_1}{\partial q_i}(\mathbf{Q})u_2(\mathbf{Q}) + \frac{\partial g_2}{\partial q_i}(\mathbf{Q})u_1(\mathbf{Q}) \, d\sigma_x, \quad (2.7)
\end{aligned}$$

where  $\kappa_1(\mathbf{Q})$  and  $\kappa_2(\mathbf{Q})$  stand for the principal curvatures of the surface  $\Gamma_2(\mathbf{Q})$  (we refer to [4], page 129 for a definition),  $\nabla_\tau$  is the tangential gradient and

$$\mathbf{V}^i(\mathbf{Q}, \cdot) = \frac{\partial \phi}{\partial q_i}(\mathbf{Q}, \phi^{-1}(\mathbf{Q}, \cdot)), \quad i = 1, \dots, p.$$

## 2.4 Proofs of the Propositions

**Proof of Proposition 2.1:** Let  $\varphi$  be a function of  $\mathcal{V}^1$  and define  $\tilde{\varphi}(\mathbf{Q}, \cdot) = \varphi(\phi^{-1}(\mathbf{Q}, \cdot))$ . Simple computations yield:

$$\begin{aligned}
\nabla \tilde{\varphi}(\mathbf{Q}, \cdot) &= \nabla \varphi(\phi^{-1}(\mathbf{Q}, \cdot)) [D\phi(\mathbf{Q}, \phi^{-1}(\mathbf{Q}, \cdot))]^{-1}, \\
\int_{\Omega(\mathbf{Q})} |\nabla \tilde{\varphi}(\mathbf{Q}, \mathbf{x})|^2 \, d\mathbf{x} &= \int_{\Omega} \nabla \varphi(\mathbf{x}) [A(\mathbf{Q}, \mathbf{x})] \nabla \varphi^\top(\mathbf{x}) \, d\mathbf{x},
\end{aligned}$$

and according to the hypothesis (H<sub>4</sub>),  $\varphi \circ \phi^{-1}(\mathbf{Q}, \cdot)$  lies in  $\mathcal{V}^1(\mathbf{Q})$ . The variational formulation (2.6) can be rewritten with the test function  $\varphi \circ \phi^{-1}(\mathbf{Q}, \cdot)$  and yields, upon a change of variables:

$$\int_{\Omega} \nabla U_k(\mathbf{Q}, \mathbf{x}) [A(\mathbf{Q}, \mathbf{x})] \nabla \varphi^\top(\mathbf{x}) \, d\mathbf{x} = \int_{\Gamma} G(\mathbf{Q}, \mathbf{x}) \varphi(\mathbf{x}) \, d\sigma_x. \quad (2.8)$$

Notice that the expression of the integral on  $\Gamma$  results from the hypothesis (H<sub>3</sub>). We define then the mapping:

$$\begin{aligned}
F_k^1 : \mathcal{Q} \times \mathcal{V}^1 &\rightarrow (\mathcal{V}^1)' \\
(\mathbf{Q}, U) &\mapsto \mathcal{A}(\mathbf{Q})U - \mathcal{B}_k(\mathbf{Q}), \quad k = 1, 2,
\end{aligned}$$

where  $\mathcal{A}(\mathbf{Q}) : \mathcal{V}^1 \rightarrow (\mathcal{V}^1)'$  and  $\mathcal{B}_k(\mathbf{Q}) \in (\mathcal{V}^1)'$  are defined by

$$\begin{aligned}
\langle \mathcal{A}(\mathbf{Q})U, \varphi \rangle_{(\mathcal{V}^1)' \times \mathcal{V}^1} &= \int_{\Omega} \nabla U [A(\mathbf{Q})] \nabla \varphi^\top \, d\mathbf{x}, \quad \forall U, \varphi \in \mathcal{V}^1, \\
\langle \mathcal{B}_k(\mathbf{Q}), \varphi \rangle_{(\mathcal{V}^1)' \times \mathcal{V}^1} &= \int_{\Gamma} G_k(\mathbf{Q}) \varphi \, d\sigma_x, \quad \forall \varphi \in \mathcal{V}^1.
\end{aligned}$$

The mapping  $\mathbf{Q} \in \mathcal{Q} \mapsto \mathcal{B}_k(\mathbf{Q}) \in (\mathcal{V}^1)'$  is of class  $C^\beta$  because  $\mathbf{Q} \in \mathcal{Q} \mapsto G_k(\mathbf{Q}, \cdot) \in L^2(\Gamma)$  is. According to the Lemma 2.1, one deduces as well that  $\mathbf{Q} \in \mathcal{Q} \mapsto [\mathcal{A}(\mathbf{Q})] \in$

$L^\infty(\mathbb{R}^3, \mathcal{M}(3))$  is of class  $C^\alpha$ . At last, the mapping

$$\begin{aligned} L^\infty(\mathbb{R}^3, \mathcal{M}(3)) \times \mathcal{V}^1 &\rightarrow (\mathcal{V}^1)' \\ ([A], U) &\mapsto \mathcal{A}U, \end{aligned}$$

defined by

$$\langle \mathcal{A}U, \varphi \rangle_{(\mathcal{V}^1)' \times \mathcal{V}^1} = \int_{\Omega} \nabla U [A] \nabla \varphi^\top \, d\mathbf{x}, \quad \forall U, \varphi \in \mathcal{V}^1,$$

is of class  $C^\infty$ . Therefore  $(\mathbf{Q}, U) \in \mathcal{Q} \times \mathcal{V}^1 \mapsto \mathcal{A}(\mathbf{Q})U \in (\mathcal{V}^1)'$  is of class  $C^\alpha$  and  $F_k^1$  is of class  $C^{\min\{\alpha, \beta\}}$  (remark that  $F_k^1$  is of class  $C^\infty$  with respect to  $U$  because it is linear and continuous). Moreover,  $F_k^1(\mathbf{0}_p, u_k(\mathbf{0}_p, \cdot)) = 0$  and  $\partial F_k^1 / \partial U(\mathbf{0}_p, u_k(\mathbf{0}_p, \cdot))$  is an isomorphism from  $\mathcal{V}^1$  onto  $(\mathcal{V}^1)'$ , according to the Riesz representation Theorem. Indeed, for all  $U, \varphi \in \mathcal{V}^1$  we have:

$$\left\langle \frac{\partial F_k^1}{\partial U}(\mathbf{0}_p, u_k(\mathbf{0}_p, \cdot))U, \varphi \right\rangle_{(\mathcal{V}^1)' \times \mathcal{V}^1} = \int_{\Omega} \nabla U \cdot \nabla \varphi \, d\mathbf{x}.$$

Applying then the implicit function Theorem, we conclude that there exists a neighborhood  $\mathcal{Q}_0$  of  $\mathbf{Q} = \mathbf{0}_p$  in  $\mathcal{Q}$  and a function  $\mathbf{Q} \in \mathcal{Q}_0 \mapsto U_k(\mathbf{Q}, \cdot) \in \mathcal{V}^1$  of class  $C^{\min\{\alpha, \beta\}}$  such that  $F_k^1(\mathbf{Q}, U_k(\mathbf{Q}, \cdot)) = 0$  for all  $\mathbf{Q} \in \mathcal{Q}_0$ . By uniqueness of the solution of (2.6) in  $\mathcal{V}^1(\mathbf{Q})$ , we conclude that  $U_k(\mathbf{Q}, \cdot) = u_k(\mathbf{Q}, \phi(\mathbf{Q}, \cdot))$ .

Proceeding as in (2.8), we rewrite  $\Upsilon(\mathbf{Q})$ :

$$\Upsilon(\mathbf{Q}) = \int_{\Omega} \nabla U_1(\mathbf{Q}, \mathbf{x}) [A(\mathbf{Q}, \mathbf{x})] \nabla U_2(\mathbf{Q}, \mathbf{x})^\top \, d\mathbf{x}.$$

Since  $\mathbf{Q} \in \mathcal{Q}_0 \mapsto \nabla U_k(\mathbf{Q}, \cdot) \in L^2(\Omega, \mathbb{R}^3)$  is of class  $C^{\min\{\alpha, \beta\}}$  and  $\mathbf{Q} \in \mathcal{Q} \mapsto [A(\mathbf{Q}, \cdot)] \in L^\infty(\mathbb{R}^3, \mathcal{M}(3))$  is of class  $C^\infty$ , we obtain the expected regularity result for  $\mathbf{Q} \in \mathcal{Q}_0 \mapsto \Upsilon(\mathbf{Q})$ , namely  $C^{\min\{\alpha, \beta\}}$ .

Assume now that  $\Gamma$  is of class  $C^2$  and that  $G_k(\mathbf{Q}, \cdot)$  is like in the hypothesis of the second part of the Proposition. We define

$$\begin{aligned} F_k^2 : \mathcal{Q} \times \mathcal{V}^2 &\rightarrow \mathcal{L}^2(\Omega) \times \tilde{H}^{1/2}(\Gamma) \\ (\mathbf{Q}, U) &\mapsto \begin{pmatrix} -\operatorname{div}([A(\mathbf{Q})]\nabla U) \\ \nabla U|_{\Gamma} \cdot \mathbf{n} - G_k(\mathbf{Q}, \cdot) \end{pmatrix}. \end{aligned}$$

Taking into account (H<sub>2</sub>),  $[A(\mathbf{Q})] - [I_3]$  is compactly supported and  $\operatorname{div}([A(\mathbf{Q})]\nabla U) = \operatorname{div}([A(\mathbf{Q})]) \cdot \nabla U + [A(\mathbf{Q})] : [D^2U]$  lies in  $\mathcal{L}^2(\Omega)$ . Proceeding as for  $F_k^1$ , we prove that  $F_k^2$  is continuously derivable. Moreover  $F_k^2(\mathbf{0}_p, u_k(\mathbf{0}_p, \cdot)) = 0$  and:

$$\frac{\partial F_k^2}{\partial U}(\mathbf{0}_p, u_k(\mathbf{0}_p, \cdot))U = \begin{pmatrix} -\Delta U \\ \nabla U|_{\Gamma} \cdot \mathbf{n} \end{pmatrix}.$$

Classical results for elliptic equations [2], [7] ensure that  $\partial F_k^2 / \partial U(\mathbf{0}_p, u_k(\mathbf{0}_p, \cdot))$  is an isomorphism from  $\mathcal{V}^2$  onto  $\mathcal{L}^2(\Omega)$ . We use then the implicit function Theorem and conclude as for  $F_1^k$ .

Let  $K$  be a compact subset in  $\Omega$  and consider a function  $\zeta_K \in C_0^\infty(\Omega)$  such that  $\zeta_K \equiv 1$  in a neighborhood of  $K$ . The function  $\mathbf{Q} \mapsto \tilde{U}_k(\mathbf{Q}, \cdot) = \zeta_K(\cdot)U_k(\mathbf{Q}, \cdot) \in H^1(\mathbb{R}^3)$  is of class  $C^1$  and according to the Lemma 5.3.3 page 181 of [10] we deduce that  $\mathbf{Q} \mapsto \tilde{u}_k(\mathbf{Q}, \cdot) = \zeta_K(\phi^{-1}(\mathbf{Q}, \cdot))u_k(\mathbf{Q}, \cdot) \in L^2(\mathbb{R}^3)$  is differentiable. Furthermore  $\tilde{U}_k$  and  $\tilde{u}_k$  are linked through the formula:

$$\frac{\partial \tilde{u}_k}{\partial q_i}(\mathbf{Q}, \cdot) = \frac{\partial \tilde{U}_k}{\partial q_i}(\mathbf{Q}, \phi^{-1}(\mathbf{Q}, \cdot)) - \nabla \tilde{U}_k(\mathbf{Q}, \phi^{-1}(\mathbf{Q}, \cdot)) \cdot \mathbf{V}^i(\mathbf{Q}, \cdot), \quad i = 1, \dots, p.$$

We can define the function  $\partial u_k/\partial q_i(\mathbf{0}_p, \cdot)$  in the whole set  $\Omega$  through its restriction to any compact subset of  $\Omega$  :

$$\left. \frac{\partial u_k}{\partial q_i}(\mathbf{0}_p, \cdot) \right|_K = \left. \frac{\partial \tilde{u}_k}{\partial q_i}(\mathbf{0}_p, \cdot) \right|_K, \quad \forall K \subset \Omega.$$

We obtain therefore that:

$$\frac{\partial u_k}{\partial q_i}(\mathbf{0}_p, \cdot) = \frac{\partial U_k}{\partial q_i}(\mathbf{0}_p, \cdot) - \nabla U_k(\mathbf{0}_p, \cdot) \cdot \mathbf{V}_i(\mathbf{0}_p, \cdot) \in \mathcal{V}^1,$$

since  $\partial U_k/\partial q_i(\mathbf{0}_p, \cdot) \in \mathcal{V}^1$ ,  $\nabla U_k(\mathbf{0}_p, \cdot) \in L^2(\Omega, \mathbb{R}^3)$  and  $V^i(\mathbf{0}_p, \cdot) \equiv 0$  on  $\Gamma_1$ . We proceed as well to define  $\partial \nabla u_k/\partial q_i(\mathbf{0}_p, \cdot) \in L^2(\Omega, \mathbb{R}^3)$ , taking into account that  $U_k(\mathbf{0}_p, \cdot) \in \mathcal{V}^2$ . ■

**Proof of Proposition 2.2:** The domain  $\Omega$  being of class  $C^2$ , we can extend any function  $\varphi \in \mathcal{V}^2$  to be a function of  $\tilde{\mathcal{V}}^2$  and for all  $\mathbf{Q} \in \mathcal{Q}$ ,  $\varphi|_{\Omega(\mathbf{Q})}$  lies in  $\mathcal{V}^1(\mathbf{Q})$ . Choosing this test function in (2.6), we compute the derivative of the variational formulation:

$$\int_{\Omega(\mathbf{Q})} \nabla u_k(\mathbf{Q}, \mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) \, d\mathbf{x} = \int_{\Gamma} G_k(\mathbf{Q}, \mathbf{x}) \varphi(\phi(\mathbf{Q}, \mathbf{x})) \, d\sigma_x,$$

at the point  $\mathbf{Q} = \mathbf{0}_p$  with respect to  $q_i$ . Applying the Corollary 5.2.5 of [10] page 173, we obtain that:

$$\begin{aligned} & \int_{\Omega(\mathbf{Q})} \frac{\partial \nabla u_k}{\partial q_i}(\mathbf{0}_p, \mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) \, d\mathbf{x} + \int_{\Gamma_2} \nabla u_k(\mathbf{0}_p, \mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) (\mathbf{V}^i \cdot \mathbf{n}) \, d\sigma_x \\ &= \int_{\Gamma_1} \frac{\partial G_k}{\partial q_i}(\mathbf{0}_p, \mathbf{x}) \varphi \, d\sigma_x + \frac{d}{dq_i} \left( \int_{\Gamma_2} G_k(\mathbf{Q}, \mathbf{x}) \varphi(\phi(\mathbf{Q}, \mathbf{x})) \, d\sigma_x \right). \end{aligned} \quad (2.9)$$

The Lemma 5.3.3 page 181 of [10] ensures that  $\mathbf{Q} \in \mathcal{Q} \mapsto \nabla \varphi(\phi(\mathbf{Q}, \cdot)) \in \tilde{\mathcal{V}}^1$  is differentiable and that

$$\frac{\partial \nabla \varphi}{\partial q_i}(\mathbf{Q}, \cdot) = \nabla \varphi(\phi(\mathbf{Q}, \cdot)) \cdot \mathbf{V}^i(\phi(\mathbf{Q}, \cdot)) \in \tilde{\mathcal{V}}^1.$$

In particular  $\partial \nabla \varphi/\partial q_i(\mathbf{Q}, \cdot)|_{\Gamma_2} \in H^{1/2}(\Gamma_2)$  and we have :

$$\begin{aligned} & \frac{d}{dq_i} \left( \int_{\Gamma_2} G_k(\mathbf{Q}, \mathbf{x}) \varphi(\phi(\mathbf{Q}, \mathbf{x})) \, d\sigma_x \right) = \\ & \int_{\Gamma_2} \frac{\partial G_k}{\partial q_i}(\mathbf{Q}, \mathbf{x}) \varphi(\phi(\mathbf{Q}, \mathbf{x})) + G_k(\mathbf{Q}, \mathbf{x}) \nabla \varphi(\phi(\mathbf{Q}, \mathbf{x})) \cdot \mathbf{V}^i(\phi(\mathbf{Q}, \mathbf{x})) \, d\sigma_x. \end{aligned}$$

Plugging this expression into (2.9), specifying  $\mathbf{Q} = \mathbf{0}_p$  and denoting merely  $u_k$ ,  $G_k$  and  $\mathbf{V}^i$  for  $u_k(\mathbf{0}_p, \cdot)$ ,  $G_k(\mathbf{0}_p, \cdot)$  and  $\mathbf{V}^i(\mathbf{0}_p, \cdot)$ , we obtain in one hand the variational formulation for  $\nabla u_k$ :

$$\begin{aligned} & \int_{\Omega(\mathbf{Q})} \frac{\partial \nabla u_k}{\partial q_i} \cdot \nabla \varphi \, d\mathbf{x} + \int_{\Gamma_2} \nabla u_k \cdot \nabla \varphi (\mathbf{V}^i \cdot \mathbf{n}) \, d\sigma_x = \int_{\Gamma_1} \frac{\partial G_k}{\partial q_i} \varphi \, d\sigma_x \\ & \quad + \int_{\Gamma_2} \frac{\partial G_k}{\partial q_i} \varphi + G_k \nabla \varphi \cdot \mathbf{V}^i \, d\sigma_x, \quad \forall \varphi \in \tilde{\mathcal{V}}^2. \end{aligned} \quad (2.10)$$

In the other hand, computing the derivative at the point  $\mathbf{Q} = \mathbf{0}_p$  of the expression of  $\Upsilon(\mathbf{Q})$  and applying the Corollary 5.2.5 of [10] page 173, we get:

$$\begin{aligned} \frac{d}{dq_i} \left( \int_{\Omega(\mathbf{Q})} \nabla u_1(\mathbf{Q}, \mathbf{x}) \cdot \nabla u_2(\mathbf{Q}, \mathbf{x}) \, d\mathbf{x} \right) \Big|_{\mathbf{Q}=\mathbf{0}_p} &= \int_{\Gamma} \frac{\partial \nabla u_1}{\partial q_i} \cdot \nabla u_2 \, d\sigma_x \\ &+ \int_{\Gamma} \frac{\partial \nabla u_2}{\partial q_i} \cdot \nabla u_1 \, d\sigma_x + \int_{\Gamma_2} \nabla u_1 \cdot \nabla u_2(\mathbf{V}^i \cdot \mathbf{n}) \, d\sigma_x. \end{aligned}$$

Since  $u_1$  and  $u_2$  lie in  $\mathcal{V}^2$ , we can use the identity (2.10) in order to transform both right hand side terms. We get therefore:

$$\begin{aligned} \frac{d}{dq_i} \left( \int_{\Omega(\mathbf{Q})} \nabla u_1(\mathbf{Q}, \mathbf{x}) \cdot \nabla u_2(\mathbf{Q}, \mathbf{x}) \, d\mathbf{x} \right) \Big|_{\mathbf{Q}=\mathbf{0}_p} &= - \int_{\Gamma_2} \nabla u_1 \cdot \nabla u_2(\mathbf{V}^i \cdot \mathbf{n}) \, d\sigma_x \\ &+ \int_{\Gamma_1} \frac{\partial G_1}{\partial q_i} u_2 + \frac{\partial G_2}{\partial q_i} u_1 \, d\sigma_x + \int_{\Gamma_2} G_1 \nabla u_2 \cdot \mathbf{V}^i + G_2 \nabla u_1 \cdot \mathbf{V}^i \, d\sigma_x. \quad (2.11) \end{aligned}$$

Then, decomposing  $\mathbf{V}^i$  into its normal and tangential part on  $\Gamma_2$  :  $\mathbf{V}^i = (\mathbf{V}^i \cdot \mathbf{n})\mathbf{n} + \mathbf{V}_\tau^i$ , one obtains that:

$$\nabla u_k \cdot \mathbf{V}^i = \frac{\partial u_k}{\partial \mathbf{n}}(\mathbf{V}^i \cdot \mathbf{n}) + \nabla_\tau u_k \cdot \mathbf{V}_\tau^i, \quad (2.12)$$

where  $\nabla_\tau$  stands for the tangential gradient on  $\Gamma_2$ . Using some results of differential geometry (see [10] pages 192-197), we have in one hand:

$$G_1 \nabla_\tau u_2 \cdot \mathbf{V}_\tau^i = \operatorname{div}_\tau(G_1 u_2 \mathbf{V}_\tau^i) - \nabla_\tau G_1 \cdot \mathbf{V}_\tau^i u_2 - G_1 u_2 \operatorname{div}_\tau(\mathbf{V}_\tau^i),$$

and in the other hand:

$$\operatorname{div}_\tau(\mathbf{V}_\tau^i) = \operatorname{div}_\tau(\mathbf{V}^i) + (\kappa_1 + \kappa_2)(\mathbf{V}^i \cdot \mathbf{n}), \quad \operatorname{div}_\tau(\mathbf{V}^i) = \operatorname{div}(\mathbf{V}^i) - \mathbf{n}^\top [D\mathbf{V}^i]\mathbf{n},$$

where  $\kappa_1 + \kappa_2 = -\operatorname{div}_\tau(\mathbf{n})$ . Taking into account (H<sub>3</sub>), we deduce that  $\operatorname{div}(\mathbf{V}^i) \equiv 0$  and  $\mathbf{n}^\top [D\mathbf{V}^i]\mathbf{n} \equiv 0$  since  $[D\mathbf{V}^i]$  is a skew symmetric matrix. We remark also, applying Lemma 5.4.10 page 194 of [10], that:

$$\int_{\Gamma_2} \operatorname{div}_\tau(G_1 u_2 \mathbf{V}_\tau^i) \, d\sigma_x = \int_{\Gamma} \operatorname{div}_\tau(G_1 u_2 \mathbf{V}_\tau^i) \, d\sigma_x = 0,$$

the first equality resulting from the hypothesis (H<sub>3</sub>) which means in particular that  $\mathbf{V}^i \equiv \mathbf{0}$  on  $\Gamma_1$ . Finally we get:

$$\int_{\Gamma_2} G_1 \nabla_\tau u_2 \cdot \mathbf{V}_\tau^i \, d\sigma_x = - \int_{\Gamma_2} (\kappa_1 + \kappa_2) G_1 u_2 (\mathbf{V}^i \cdot \mathbf{n}) + u_2 \nabla_\tau G_1 \cdot \mathbf{V}_\tau^i \, d\sigma_x. \quad (2.13)$$

Summarizing (2.12) and (2.13) and plugging the result into (2.11), we obtain the expression (2.7) of the Proposition. ■

## 3 Proof of Theorems

### 3.1 Proof of Theorem 1.1

Going back to the expression (1.9) of the Lagrangian and taking into account the symmetry of the matrices, we deduce easily that  $\mathcal{L}$  is analytic with respect to  $\dot{\mathbf{Q}}$  and that

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{Q}}}(\mathbf{Q}) = ([K_S(\mathbf{Q})] + [K_F(\mathbf{Q})])\dot{\mathbf{Q}}.$$

Assuming for a while the matrices  $[K(\mathbf{Q})] = [K_S(\mathbf{Q})] + [K_F(\mathbf{Q})]$  to be derivable with respect to  $\mathbf{Q}$  and applying the chain rule

$$\frac{d}{dt}[K(\mathbf{Q})] = \left\langle \frac{\partial[K(\mathbf{Q})]}{\partial \mathbf{Q}}, \dot{\mathbf{Q}} \right\rangle,$$

we obtain that:

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{Q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{Q}} = [K(\mathbf{Q})]\ddot{\mathbf{Q}} + \left\langle \frac{\partial[K(\mathbf{Q})]}{\partial \mathbf{Q}}, \dot{\mathbf{Q}} \right\rangle \dot{\mathbf{Q}} - \frac{1}{2} \dot{\mathbf{Q}}^\top \left\langle \frac{\partial[K(\mathbf{Q})]}{\partial \mathbf{Q}}, \cdot \right\rangle \dot{\mathbf{Q}}. \quad (3.1)$$

The Euler-Lagrange system of ODE (1.10) can formally be rewritten under the normal form:

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} \dot{\mathbf{Q}} \\ \mathbf{Q} \end{bmatrix} = \\ \begin{bmatrix} [K(\mathbf{Q})]^{-1} & [0_3] \\ [0_3] & [I_3] \end{bmatrix} \begin{bmatrix} - \left\langle \frac{\partial[K(\mathbf{Q})]}{\partial \mathbf{Q}}, \dot{\mathbf{Q}} \right\rangle \dot{\mathbf{Q}} + \frac{1}{2} \dot{\mathbf{Q}}^\top \left\langle \frac{\partial[K(\mathbf{Q})]}{\partial \mathbf{Q}}, \cdot \right\rangle \dot{\mathbf{Q}} \\ \dot{\mathbf{Q}} \end{bmatrix}. \end{aligned} \quad (3.2)$$

To study this system, one begins with remarking that:

**Lemma 3.1** Denote  $\lambda_i > 0$  the smallest eigenvalue of the inertia tensor  $[J^i]$  (see [8] page 195) and

$$\alpha = \min\{\lambda_i, m_i, i = 1, \dots, n\},$$

where we recall that  $m_i$  is the mass of the  $i$ -th solid. Then  $\alpha > 0$  and

$$\mathbf{X}^\top [K(\mathbf{Q})] \mathbf{X} = |[K(\mathbf{Q})] \mathbf{X}|^2 \geq \alpha |\mathbf{X}|^2, \quad \forall \mathbf{X} \in (\mathbb{R}^6)^n, \forall \mathbf{Q} \in \mathcal{Q}. \quad (3.3)$$

In particular, the matrix  $[K(\mathbf{Q})]$  is invertible for all  $\mathbf{Q} \in \mathcal{Q}$ .

**Proof :** The matrix  $[K(\mathbf{Q})]$  is the sum of  $[K_F(\mathbf{Q})]$  which is positive and  $[K_S(\mathbf{Q})]$  which is definite positive for all  $\mathbf{Q} \in \mathcal{Q}$ . Indeed, remind that  $[K_S(\mathbf{Q})]$  is a bloc diagonal matrix. Each bloc is either a  $3 \times 3$  diagonal matrix  $\text{diag}(m^i, m^i, m^i)$  with  $m^i > 0$  for all  $i = 1, \dots, n$ , either a matrix  $[\omega^i]^\top [J^i] [\omega^i]$ . The matrix  $[J^i]$  is definite positive and simple computations yield  $\det[\omega^i] = \cos(\theta_2^i) \neq 0$  because  $\theta^i = (\theta_1^i, \theta_2^i, \theta_3^i)^\top \in \Theta = ]-\pi, \pi[ \times ]-\pi/2, \pi/2[ \times ]-\pi, \pi[$ . ■

Therefore, in order to apply the Cauchy-Lipschitz Theorem to the ODE (3.2), the key point consists in proving that  $\partial[K(\mathbf{Q})]/\partial \mathbf{Q}$  is well defined, at last Lipschitz continuous and to compute it when it is possible. This task will be achieved using arguments of shape sensitivity analysis developed in section 2.

According to the definition (1.6) of  $[K_S(\mathbf{Q})]$ , it is clear that the mapping:

$$\mathbf{Q} \in \mathcal{Q} \mapsto [K_S(\mathbf{Q})] \in \mathcal{M}(6n),$$

is analytic. Moreover, the formula easy to prove,

$$\frac{\partial \omega_k}{\partial \theta_j}(\theta) = \begin{cases} \omega_j(\theta) \wedge \omega_k(\theta), & \text{if } 1 \leq j < k \leq 3, \\ \mathbf{0}_3, & \text{if } 1 \leq k \leq j \leq 3, \end{cases} \quad (3.4)$$

leads to:

$$\frac{\partial [\omega^i]}{\partial \theta_1^i} = \mathbf{e}_2 \otimes (\omega_1^i \wedge \omega_3^i) + \mathbf{e}_3 \otimes (\omega_1^i \wedge \omega_2^i), \quad \frac{\partial [\omega^i]}{\partial \theta_2^i} = \mathbf{e}_3 \otimes (\omega_2^i \wedge \omega_3^i), \quad \forall i = 1, \dots, n.$$

The differential of the matrix  $[\omega^i]$  with respect to  $\mathbf{Q}$  is therefore the three-rank tensor:

$$\frac{\partial[\omega^i]}{\partial \mathbf{Q}} = \mathbf{e}_2 \otimes (\omega_1^i \wedge \omega_2^i) \otimes \mathbf{e}_1 + \mathbf{e}_3 \otimes (\omega_1^i \wedge \omega_3^i) \otimes \mathbf{e}_1 + \mathbf{e}_3 \otimes (\omega_2^i \wedge \omega_3^i) \otimes \mathbf{e}_2.$$

After some algebra, one obtains that:

$$\left\langle \frac{\partial[\mathcal{J}^i(\mathbf{Q}^i)]}{\partial \mathbf{Q}^i}, \dot{\mathbf{Q}}^i \right\rangle \dot{\mathbf{Q}}^i - \frac{1}{2} \dot{\mathbf{Q}}^{i\top} \left\langle \frac{\partial[\mathcal{J}^i(\mathbf{Q}^i)]}{\partial \mathbf{Q}^i}, \cdot \right\rangle \dot{\mathbf{Q}}^i = \langle \llbracket T^i(\mathbf{Q}^i) \rrbracket, \dot{\mathbf{Q}}^i, \dot{\mathbf{Q}}^i \rangle, \quad i = 1, \dots, n,$$

where the three-rank tensor  $\llbracket T^i(\mathbf{Q}^i) \rrbracket$  arising in the right hand side has been defined by (1.14). We get finally the equality:

$$\left\langle \frac{\partial[K_S(\mathbf{Q})]}{\partial \mathbf{Q}}, \dot{\mathbf{Q}} \right\rangle \dot{\mathbf{Q}} - \frac{1}{2} \dot{\mathbf{Q}}^\top \left\langle \frac{\partial[K_S(\mathbf{Q})]}{\partial \mathbf{Q}}, \cdot \right\rangle \dot{\mathbf{Q}} = \langle \llbracket T_S(\mathbf{Q}) \rrbracket, \dot{\mathbf{Q}}, \dot{\mathbf{Q}} \rangle, \quad (3.5)$$

where  $\llbracket T_S(\mathbf{Q}) \rrbracket$  is defined by (1.15). It is clear that  $\llbracket T_S(\mathbf{Q}) \rrbracket$  is analytic with respect to  $\mathbf{Q}$ .

The use of the Proposition 2.1 will allow us to prove that  $[K_F(\mathbf{Q})]$  is continuously differentiable with respect to  $\mathbf{Q}$  if  $\partial\Omega$  is Lipschitz continuous. Nevertheless, the computation of  $\partial[K_F(\mathbf{Q})]/\partial \mathbf{Q}$  is only possible when  $\partial\Omega$  is of class  $C^2$ . Remind that:

$$[K_F(\mathbf{Q})] = \rho_F \int_{\Omega(\mathbf{Q})} \nabla \varphi_{k_1}^{i_1}(\mathbf{Q}, \mathbf{x}) \cdot \nabla \varphi_{k_2}^{i_2}(\mathbf{Q}, \mathbf{x}) \, d\mathbf{x}, \quad 1 \leq i_1, i_2 \leq n, \quad 1 \leq k_1, k_2 \leq 3,$$

where  $\varphi_k^i$  solves the Neuman problem (1.7). Hence, each entries of the matrix  $[K_F(\mathbf{Q})]$  has the form of the functional  $\Upsilon$  defined by (2.5) in section 2. We define the function  $\phi$  arising in (2.3) as follows: let  $\mathcal{O}^0$  be an open set such that  $\Gamma_1 \subset \mathcal{O}^0$ . For each  $S^i$  it is possible to find an open set  $\mathcal{O}^i$  such that  $\overline{S^i} \subset \mathcal{O}^i$ ,  $\overline{\mathcal{O}^i}$  compact,  $\overline{\mathcal{O}^i} \cap \overline{\mathcal{O}^j} = \emptyset$  for all  $i = 1, \dots, n$  and  $\overline{\mathcal{O}^i} \cap \overline{\mathcal{O}^j} = \emptyset$  if  $i \neq j$ . We consider then a  $C^\infty$  partition of unity  $\{\zeta^0, \zeta^i, i = 1, \dots, n\}$  such that  $\zeta^i \equiv 1$  in  $\mathcal{O}^i$ ,  $i = 0, 1, \dots, n$ . For any  $\mathbf{Q}$  in a neighborhood of  $((\mathbf{0}_3, \mathbf{h}_0^1), \dots, (\mathbf{0}_3, \mathbf{h}_0^n))$  in  $\mathcal{Q}$ , the function

$$\phi(\mathbf{Q}, \mathbf{x}) = \zeta^0(\mathbf{x})\mathbf{x} + \sum_{i=1}^n \zeta^i(\mathbf{x})(\mathcal{R}(\theta^i)(\mathbf{x} - \mathbf{h}_0^i) + \mathbf{h}^i),$$

lies in  $C^{\infty, \infty}(\mathbb{R}^3, \mathbb{R}^3)$  and satisfies the hypothesis (H<sub>2</sub>) and (H<sub>3</sub>) of section 2. Then we introduce, for all  $i = 1, \dots, n$ , the following vectorial fields on  $\Gamma_2(\mathbf{Q})$ :

$$\begin{aligned} \mathbf{V}_k^i(\mathbf{Q}^i, \mathbf{x}) &= \omega_k^i \wedge (\mathbf{x} - \mathbf{h}^i), & k = 1, 2, 3 \quad \text{on } \Gamma_2^i(\mathbf{Q}^i), \\ \mathbf{V}_{k+3}^i(\mathbf{Q}^i, \mathbf{x}) &= \mathbf{e}_k^i, & k = 1, 2, 3 \quad \text{on } \Gamma_2^i(\mathbf{Q}^i), \\ \mathbf{V}_k^i(\mathbf{Q}^i, \mathbf{x}) &= \mathbf{0}_3, & k = 1, \dots, 6 \quad \text{on } \Gamma(\mathbf{Q}) \setminus \Gamma_2^i(\mathbf{Q}^i). \end{aligned}$$

The boundaries values of  $\partial\varphi_k^i/\partial \mathbf{n}$  in the system (1.7) read for all  $i = 1, \dots, n$  and  $k = 1, \dots, 6$ :

$$\begin{aligned} g_k^i(\mathbf{Q}^i, \mathbf{x}) &= \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}) \cdot \mathbf{V}_k^i(\mathbf{Q}^i, \mathbf{x}), & \text{on } \Gamma_2^i(\mathbf{Q}^i), \\ g_k^i(\mathbf{Q}^i, \mathbf{x}) &= 0, & \text{on } \Gamma(\mathbf{Q}) \setminus \Gamma_2^i(\mathbf{Q}^i), \end{aligned}$$

and the functions  $G_k^i(\mathbf{Q}^i, \cdot) = g_k^i(\mathbf{Q}^i, \phi(\mathbf{Q}, \cdot))$  are:

$$\begin{aligned} G_k^i(\mathbf{Q}^i, \mathbf{x}) &= \mathbf{n}^i \cdot ((\mathcal{R}(\theta^i)^\top \omega_k^i) \wedge (\mathbf{x} - \mathbf{h}_0^i)), \\ G_{k+3}^i(\mathbf{Q}^i, \mathbf{x}) &= \mathbf{n}^i \cdot \mathcal{R}(\theta^i)^\top \mathbf{e}_k, & \text{on } \Gamma_2^i, \quad 1 \leq k \leq 3, \\ G_k^i(\mathbf{Q}^i, \mathbf{x}) &= 0, & \text{on } \Gamma \setminus \Gamma_2^i, \quad 1 \leq k \leq 6. \end{aligned}$$



Since  $G_k^i(\mathbf{Q}^i, \cdot) \in \tilde{L}^2(\Gamma_2)$  for all  $\mathbf{Q} \in \mathcal{Q}$ , Lemma 2.2 applies. It ensures the existence and uniqueness of the functions  $\varphi_k^i(\mathbf{Q}, \cdot)$  in suitable weighted Sobolev spaces. The matrix  $[K_F(\mathbf{Q})]$  is hence well defined for all  $\mathbf{Q} \in \mathcal{Q}$ . Moreover, it is clear that the functions  $\mathbf{Q} \mapsto G_k^i(\mathbf{Q}^i, \cdot) \in \tilde{L}^2(\Gamma)$  are indefinitely differentiable. According to the Proposition 2.1, the mapping  $\mathbf{Q} \mapsto [K_F(\mathbf{Q})]$  is also of class  $C^\infty$  in a neighborhood of  $((\mathbf{0}_3, \mathbf{h}_0^1), \dots, (\mathbf{0}_3, \mathbf{h}_0^n))$ . The proof of Theorem 1.1 is then completed.

### 3.2 Proof of Theorem 1.2

Let  $(\mathbf{P}_0, \dot{\mathbf{P}}_0) \in \mathcal{P} \times T\mathcal{P}(\mathbf{P}_0)$  (namely, initial positions and initial velocities of the solids) and define  $\mathbf{Q}_0 = ((\mathbf{0}_3, \mathbf{h}_0^1), \dots, (\mathbf{0}_3, \mathbf{h}_0^n)) \in \mathcal{Q}(\mathbf{P}_0)$  and  $\dot{\mathbf{Q}}_0 \in (\mathbb{R}^3 \times \mathbb{R}^3)^n$  such that

$$\left\langle \frac{\partial \Lambda}{\partial \mathbf{Q}}(\mathbf{P}_0, \mathbf{Q}_0), \dot{\mathbf{Q}}_0 \right\rangle = \dot{\mathbf{P}}_0.$$

We can apply Theorem 1.1: there exists a real interval  $[0, T)$  and a unique  $C^\infty$  function  $\mathbf{Q}^*(\cdot, 0, \mathbf{Q}_0, \dot{\mathbf{Q}}_0)$  that solves (1.10) on  $[0, T)$ . We set then

$$\mathbf{P}^*(t) = \Lambda(\mathbf{P}_0, \mathbf{Q}^*(t, 0, \mathbf{Q}_0, \dot{\mathbf{Q}}_0)), \quad \forall t \in [0, T).$$

The principle of conservation of energy ensures that

$$\mathcal{L}(\mathbf{Q}^*(t, 0, \mathbf{Q}_0, \dot{\mathbf{Q}}_0), \dot{\mathbf{Q}}^*(t, 0, \mathbf{Q}_0, \dot{\mathbf{Q}}_0)) = \mathcal{L}(\mathbf{Q}_0, \dot{\mathbf{Q}}_0), \quad \forall t \in [0, T).$$

The definition (1.9) of the Lagrangian and the formula (3.3) yield:

$$|\dot{\mathbf{Q}}^*(t, 0, \mathbf{Q}_0, \dot{\mathbf{Q}}_0)|^2 \leq \frac{1}{\alpha} \mathcal{L}(\mathbf{Q}_0, \dot{\mathbf{Q}}_0), \quad \forall t \in [0, T). \quad (3.6)$$

Classical results on the ODE (see for example [3] page 13) allow to make precise the behavior of the point  $(t, \mathbf{Q}^*(t, 0, \mathbf{Q}_0, \dot{\mathbf{Q}}_0), \dot{\mathbf{Q}}^*(t, 0, \mathbf{Q}_0, \dot{\mathbf{Q}}_0)) \in \mathbb{R} \times \mathcal{Q}(\mathbf{P}_0) \times (\mathbb{R}^6)^n$  as  $t \rightarrow T$ . One of the following alternatives holds:

- either  $T = \infty$ ,
- either  $|\dot{\mathbf{Q}}^*(t, 0, \mathbf{Q}_0, \dot{\mathbf{Q}}_0)| \rightarrow \infty$  as  $t \rightarrow T$  but this case is excluded by the estimate (3.6).
- either  $\mathbf{Q}^*(t, 0, \mathbf{Q}_0, \dot{\mathbf{Q}}_0) \rightarrow \tilde{\mathbf{Q}}_1 \in \partial\mathcal{Q}(\mathbf{P}_0)$  and  $\dot{\mathbf{Q}}^*(t, 0, \mathbf{Q}_0, \dot{\mathbf{Q}}_0) \rightarrow \check{\dot{\mathbf{Q}}}_1 \in (\mathbb{R}^6)^n$  as  $t \rightarrow T$ .

In the last case, if the point  $\mathbf{Q}_1$  does not correspond to a collision, we can set  $\mathbf{P}_1 = \Lambda(\mathbf{P}_0, \tilde{\mathbf{Q}}_1)$ ,  $\dot{\mathbf{P}}_1 = \left\langle \frac{\partial \Lambda}{\partial \mathbf{Q}}(\mathbf{P}_0, \tilde{\mathbf{Q}}_1), \check{\dot{\mathbf{Q}}}_1 \right\rangle$ , and  $\mathbf{Q}^1 = \Lambda^{-1}(\mathbf{P}_1, \tilde{\mathbf{Q}}_1)$ ,

$$\dot{\mathbf{Q}}_1 = \left\langle \frac{\partial \Lambda^{-1}}{\partial \mathbf{Q}}(\mathbf{P}_1, \tilde{\mathbf{Q}}_1), \check{\dot{\mathbf{Q}}}_1 \right\rangle.$$

We can then build a new solution with initial data  $(\mathbf{Q}_1, \dot{\mathbf{Q}}_1)$ . This solution does continue  $\mathbf{P}^*$  to the right of  $T$ .

### 3.3 Proof of Theorem 1.3

We assume now that  $\partial\Omega$  is of class  $C^2$  and we shall compute the expression of  $\partial[K_F(\mathbf{Q})]/\partial\mathbf{Q}$ . Applying the formula (1.4) and (3.4), we get:

$$\begin{aligned} \frac{\partial}{\partial \theta_l^i} (\mathcal{R}(\theta^i)^\top \omega_k^i) &= -\mathcal{R}(\theta^i)^\top (\omega_l^i \wedge \omega_k^i) + \mathcal{R}(\theta^i)^\top (\omega_l^i \wedge \omega_k^i) = \mathbf{0}_3, \quad 1 \leq l < k \leq 3, \\ \frac{\partial}{\partial \theta_l^i} (\mathcal{R}(\theta^i)^\top \omega_k^i) &= -\mathcal{R}(\theta^i)^\top (\omega_l^i \wedge \omega_k^i), \quad 1 \leq k \leq l \leq 3. \end{aligned}$$

From these identities, we deduce that, on  $\Gamma_2^i$  and for all  $i = 1, \dots, n$ :

$$\begin{aligned}\frac{\partial G_k^i}{\partial \theta_l^i}(\mathbf{Q}^i, \mathbf{x}) &= 0, & 1 \leq l < k \leq 3, \\ \frac{\partial G_k^i}{\partial \theta_l^i}(\mathbf{Q}^i, \mathbf{x}) &= \mathbf{n}^i \cdot ((-\mathcal{R}(\boldsymbol{\theta}^i)^\top (\boldsymbol{\omega}_l^i \wedge \boldsymbol{\omega}_k^i)) \wedge (\mathbf{x} - \mathbf{h}_0^i)), & 1 \leq k \leq l \leq 3, \\ \frac{\partial G_{k+3}^i}{\partial \theta_l^i}(\mathbf{Q}^i, \mathbf{x}) &= -\mathcal{R}(\boldsymbol{\theta}^i)^\top \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}) \cdot (\boldsymbol{\omega}_l^i \wedge \mathbf{e}_k), & 1 \leq k, l \leq 3\end{aligned}$$

and

$$\frac{\partial G_k^i}{\partial h_l^i}(\mathbf{Q}^i, \mathbf{x}) = 0, \quad 1 \leq l \leq 3, 1 \leq k \leq 6.$$

We deduce that, on  $\Gamma_2^i$ :

$$\begin{aligned}\frac{\partial G_k^i}{\partial \theta_l^i}(\mathbf{Q}^i, \phi^{-1}(\mathbf{Q}, \mathbf{x})) &= 0, & 1 \leq l < k \leq 3, \\ \frac{\partial G_k^i}{\partial \theta_l^i}(\mathbf{Q}^i, \phi^{-1}(\mathbf{Q}, \mathbf{x})) &= (\boldsymbol{\omega}_l^i \wedge \mathbf{n}^i(\mathbf{Q}^i)) \cdot \mathbf{V}_k^i(\mathbf{Q}^i) \\ &\quad - (\boldsymbol{\omega}_k^i \wedge \mathbf{n}^i(\mathbf{Q}^i)) \cdot \mathbf{V}_l^i(\mathbf{Q}^i), & 1 \leq k \leq l \leq 3, \\ \frac{\partial G_{k+3}^i}{\partial \theta_l^i}(\mathbf{Q}^i, \phi^{-1}(\mathbf{Q}, \mathbf{x})) &= (\boldsymbol{\omega}_l^i \wedge \mathbf{n}^i(\mathbf{Q}^i)) \cdot \mathbf{e}_k, & 1 \leq l \leq 3, \quad 1 \leq k \leq 3, \\ \frac{\partial G_k^i}{\partial h_l^i}(\mathbf{Q}^i, \phi^{-1}(\mathbf{Q}, \mathbf{x})) &= 0, & 1 \leq l \leq 3, \quad 1 \leq k \leq 6.\end{aligned}$$

In order to apply formula (2.7), we need also to compute the quantities  $\nabla_\tau g_k^i(\mathbf{Q}^i, \mathbf{x}) \cdot \mathbf{V}_l^i(\mathbf{Q}^i, \mathbf{x})$ . We will denote merely  $\mathbf{V}_l^i$  instead of  $\mathbf{V}_l^i(\mathbf{Q}^i, \mathbf{x})$  in the following. We have,  $\forall i = 1, \dots, n, \forall k = 1, 2, 3$ :

$$\begin{aligned}\nabla_\tau(\mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}) \cdot (\boldsymbol{\omega}_k^i \wedge (\mathbf{x} - \mathbf{h}^i))) \cdot \mathbf{V}_l^i &= (\nabla_\tau \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}) \cdot \mathbf{V}_l^i) \cdot (\boldsymbol{\omega}_k^i \wedge (\mathbf{x} - \mathbf{h}^i)) \\ &\quad + \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}) \cdot (\boldsymbol{\omega}_k^i \wedge (\nabla_\tau(\mathbf{x} - \mathbf{h}^i) \cdot \mathbf{V}_l^i)).\end{aligned}$$

According to the definition of the tangential gradient in [10] page 192, we get:

$$\nabla_\tau(\mathbf{x} - \mathbf{h}^i) \cdot \mathbf{V}_l^i = \mathbf{V}_l^i - (\mathbf{V}_l^i \cdot \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}))\mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}),$$

and therefore:

$$\nabla_\tau g_k^i(\mathbf{Q}^i, \mathbf{x}) \cdot \mathbf{V}_l^i = (\nabla_\tau \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}) \cdot \mathbf{V}_l^i) \cdot \mathbf{V}_k^i - (\boldsymbol{\omega}_k^i \wedge \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x})) \cdot \mathbf{V}_l^i \quad 1 \leq k \leq 3. \quad (3.7a)$$

Proceeding as well, we get:

$$\nabla_\tau g_k^i(\mathbf{Q}^i, \mathbf{x}) \cdot \mathbf{V}_l^i = (\nabla_\tau \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}) \cdot \mathbf{V}_l^i) \cdot \mathbf{V}_k^i, \quad 4 \leq k \leq 6. \quad (3.7b)$$

Denoting then  $T\Gamma_2^i(\mathbf{Q}^i)$  the tangent bundle of  $\Gamma_2^i(\mathbf{Q}^i)$ ,  $TS^2$  the tangent bundle of the 2-dimensional sphere  $S^2$  and  $\Pi^i : T\Gamma_2^i(\mathbf{Q}^i) \times T\Gamma_2^i(\mathbf{Q}^i) \rightarrow TS^2$  the second fundamental form on  $\Gamma_2^i(\mathbf{Q}^i)$  (see [4] for a definition of these notions), we rewrite (3.7):

$$\begin{aligned}\nabla_\tau g_k^i(\mathbf{Q}^i, \mathbf{x}) \cdot \mathbf{V}_l^i &= -\langle \Pi^i(\mathbf{Q}^i, \mathbf{x}), \mathbf{V}_l^i, \mathbf{V}_k^i \rangle - (\boldsymbol{\omega}_k^i \wedge \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x})) \cdot \mathbf{V}_l^i, & 1 \leq k \leq 3, \\ \nabla_\tau g_k^i(\mathbf{Q}^i, \mathbf{x}) \cdot \mathbf{V}_l^i &= -\langle \Pi^i(\mathbf{Q}^i, \mathbf{x}), \mathbf{V}_l^i, \mathbf{V}_k^i \rangle, & 4 \leq k \leq 6.\end{aligned}$$

According to the notations introduced in subsection 1.3, we get the relation:

$$\Pi^i(\mathbf{Q}^i, \mathbf{x}) = \kappa_1^i(\mathbf{Q}^i, \mathbf{x})\boldsymbol{\tau}_1^i(\mathbf{Q}^i, \mathbf{x}) \otimes \boldsymbol{\tau}_1^i(\mathbf{Q}^i, \mathbf{x}) + \kappa_2^i(\mathbf{Q}^i, \mathbf{x})\boldsymbol{\tau}_2^i(\mathbf{Q}^i, \mathbf{x}) \otimes \boldsymbol{\tau}_2^i(\mathbf{Q}^i, \mathbf{x}).$$

If  $1 \leq l, k \leq 3$ , we can rewrite, according to the definition of the vectors  $\mathbf{V}_l^i$  and the properties of the wedge product:

$$\begin{aligned} (\boldsymbol{\omega}_l^i \wedge \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x})) \cdot \mathbf{V}_k^i &= (\boldsymbol{\omega}_l^i \wedge \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x})) \cdot (\boldsymbol{\omega}_k^i \wedge (\mathbf{x} - \mathbf{h}^i)) \\ &= \boldsymbol{\omega}_k^{i\top} [((\mathbf{x} - \mathbf{h}^i) \cdot \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}))[\mathbf{I}_3] - \mathbf{n}^i(\mathbf{Q}^i, \mathbf{x}) \otimes (\mathbf{x} - \mathbf{h}^i)]\boldsymbol{\omega}_l^i. \end{aligned}$$

Denoting  $\mathbf{g}^i(\mathbf{Q}^i, \cdot) = (g_1^i(\mathbf{Q}^i, \cdot), \dots, g_6^i(\mathbf{Q}^i, \cdot))^\top$  and  $\mathbf{G}^i(\mathbf{Q}^i, \cdot) = (G_1^i(\mathbf{Q}^i, \cdot), \dots, G_6^i(\mathbf{Q}^i, \cdot))^\top$ ,  $[\partial \mathbf{G}^i / \partial \mathbf{Q}^i(\mathbf{Q}^i, \phi^{-1}(\mathbf{Q}, \mathbf{x}))]$  is a  $6 \times 6$  matrix whose column vectors are  $\partial \mathbf{G}^i / \partial \theta_l^i$ ,  $\partial \mathbf{G}^i / \partial h_l^i$ ,  $l = 1, 2, 3$ . We define as well that matrix  $[\nabla_\tau \mathbf{g}^i(\mathbf{Q}^i, \mathbf{x})]$  whose row vectors are  $\nabla_\tau g_k^i(\mathbf{Q}^i, \mathbf{x})$  for  $k = 1, \dots, 6$  and we denote  $[\mathbf{V}^i]$  the matrix built by the horizontal concatenation of the column vectors  $\mathbf{V}_k^i$ ,  $k = 1, \dots, 6$ . We can summarize the results still obtain by:

$$\begin{aligned} & \left[ \frac{\partial \mathbf{G}^i}{\partial \mathbf{Q}^i}(\mathbf{Q}^i, \phi^{-1}(\mathbf{Q}, \mathbf{x})) \right] - [\nabla_\tau \mathbf{g}^i(\mathbf{Q}^i, \mathbf{x})] [\mathbf{V}^i] - (\kappa_1^i(\mathbf{Q}^i) + \kappa_2^i(\mathbf{Q}^i)) \mathbf{g}^i(\mathbf{Q}^i, \mathbf{x}) [\mathbf{V}^i]^\top \mathbf{n}^i \\ &= \begin{bmatrix} [\boldsymbol{\omega}^i]^\top & [\mathbf{I}_3] \end{bmatrix} \left( \begin{bmatrix} [A_{RR}^i(\mathbf{Q}^i)] & [A_{RT}^i(\mathbf{Q}^i)]^\top \\ [A_{RT}^i(\mathbf{Q}^i)] & [A_{TT}^i(\mathbf{Q}^i)] \end{bmatrix} \right. \\ & \quad \left. + \begin{bmatrix} [B_{RR}^i(\mathbf{Q}^i)] & [B_{RT}^i(\mathbf{Q}^i)]^\top \\ [B_{RT}^i(\mathbf{Q}^i)] & [0_{3 \times 3}] \end{bmatrix} \right) \begin{bmatrix} [\boldsymbol{\omega}^i] \\ [\mathbf{I}_3] \end{bmatrix}, \end{aligned}$$

where (we dropped  $\mathbf{Q}^i$  and  $\mathbf{x}$  in the following to simplify the notations):

$$\begin{aligned} [A_{RR}^i] &= \sum_{k=1}^2 \kappa_k^i ((\mathbf{x} - \mathbf{h}^i) \wedge \boldsymbol{\tau}_k^i \otimes (\mathbf{x} - \mathbf{h}^i) \wedge \boldsymbol{\tau}_k^i - (\mathbf{x} - \mathbf{h}^i) \wedge \mathbf{n}^i \otimes (\mathbf{x} - \mathbf{h}^i) \wedge \mathbf{n}^i), \\ [B_{RR}^i] &= ((\mathbf{x} - \mathbf{h}^i) \cdot \mathbf{n}^i)[\mathbf{I}_3] - \mathbf{n}^i \otimes (\mathbf{x} - \mathbf{h}^i), \\ [A_{RT}^i] &= \sum_{k=1}^2 \kappa_k^i (\boldsymbol{\tau}_k^i \otimes (\mathbf{x} - \mathbf{h}^i) \wedge \boldsymbol{\tau}_k^i - \mathbf{n}^i \otimes (\mathbf{x} - \mathbf{h}^i) \wedge \mathbf{n}^i), \\ [B_{RT}^i] &= \boldsymbol{\tau}_1^i \otimes \boldsymbol{\tau}_2^i - \boldsymbol{\tau}_2^i \otimes \boldsymbol{\tau}_1^i, \\ [A_{TT}^i] &= \sum_{k=1}^2 \kappa_k^i (\boldsymbol{\tau}_k^i \otimes \boldsymbol{\tau}_k^i - \mathbf{n}^i \otimes \mathbf{n}^i). \end{aligned}$$

Setting then the  $6n \times 6n$  bloc-diagonal symmetric matrix:

$$[B] = \text{diag}([B^1], [B^2], \dots, [B^n]),$$

and applying formula (2.7), we get:

$$\begin{aligned} & \left\langle \frac{\partial [K_F(\mathbf{Q})]}{\partial \mathbf{Q}}, \dot{\mathbf{Q}} \right\rangle \dot{\mathbf{Q}} = -\rho_F \int_{\Gamma_2(\mathbf{Q})} [D\varphi][D\varphi]^\top \dot{\mathbf{Q}}(\mathbf{W}_n \cdot \dot{\mathbf{Q}}) d\sigma_x \\ & + \sum_{k=1}^2 \rho_F \int_{\Gamma_2(\mathbf{Q})} [\kappa_k (\varphi(\mathbf{W}_{\tau_k} \cdot \dot{\mathbf{Q}}) + \mathbf{W}_{\tau_k}(\varphi \cdot \dot{\mathbf{Q}}))(\mathbf{W}_{\tau_k} \cdot \dot{\mathbf{Q}}) + (\varphi(\dot{\mathbf{Q}}^\top [B] \dot{\mathbf{Q}}) + ([B] \dot{\mathbf{Q}})(\varphi \cdot \dot{\mathbf{Q}}))] d\sigma_x \\ & \quad - \rho_F \int_{\Gamma_2(\mathbf{Q})} (\kappa_1 + \kappa_2) (\varphi(\mathbf{W}_n \cdot \dot{\mathbf{Q}}) + \mathbf{W}_n(\varphi \cdot \dot{\mathbf{Q}}))(\mathbf{W}_n \cdot \dot{\mathbf{Q}}) d\sigma_x, \end{aligned}$$

where the vectors  $\mathbf{W}_{\tau_1}(\mathbf{Q}, \cdot)$ ,  $\mathbf{W}_{\tau_2}(\mathbf{Q}, \cdot)$  and  $\mathbf{W}_n(\mathbf{Q}, \cdot)$  are defined on  $\Gamma_2(\mathbf{Q})$  by (1.11a). Proceeding as well we get:

$$\begin{aligned} \dot{\mathbf{Q}}^\top \left\langle \frac{\partial[K_F(\mathbf{Q})]}{\partial \mathbf{Q}}, \cdot \right\rangle \dot{\mathbf{Q}} &= -\rho_F \int_{\Gamma_2(\mathbf{Q})} (\dot{\mathbf{Q}}^\top [D\varphi][D\varphi]^\top \dot{\mathbf{Q}}) \mathbf{W}_n \, d\sigma_x \\ &+ \sum_{k=1}^2 \rho_F \int_{\Gamma_2(\mathbf{Q})} [2\kappa_k(\varphi \cdot \dot{\mathbf{Q}})(\mathbf{W}_{\tau_k} \cdot \dot{\mathbf{Q}}) \mathbf{W}_{\tau_k} + 2(\varphi \cdot \dot{\mathbf{Q}})[B]\dot{\mathbf{Q}}] \, d\sigma_x \\ &- \rho_F \int_{\Gamma_2(\mathbf{Q})} 2(\kappa_1 + \kappa_2)(\varphi \cdot \dot{\mathbf{Q}})(\mathbf{W}_n \cdot \dot{\mathbf{Q}}) \, d\sigma_x. \end{aligned}$$

After some simplifications, we deduce that:

$$\begin{aligned} \left\langle \frac{\partial[K_F(\mathbf{Q})]}{\partial \mathbf{Q}}, \dot{\mathbf{Q}} \right\rangle \dot{\mathbf{Q}} - \frac{1}{2} \dot{\mathbf{Q}}^\top \left\langle \frac{\partial[K_F(\mathbf{Q})]}{\partial \mathbf{Q}}, \cdot \right\rangle \dot{\mathbf{Q}} &= \\ -\rho_F \int_{\Gamma_2(\mathbf{Q})} [D\varphi][D\varphi]^\top \dot{\mathbf{Q}} (\mathbf{W}_n \cdot \dot{\mathbf{Q}}) \, d\sigma_x + \frac{1}{2} \rho_F \int_{\Gamma_2(\mathbf{Q})} |[D\varphi]^\top \dot{\mathbf{Q}}|^2 \mathbf{W}_n \, d\sigma_x \\ + \sum_{k=1}^2 \rho_F \int_{\Gamma_2(\mathbf{Q})} \kappa_k \varphi [(\mathbf{W}_{\tau_k} \cdot \dot{\mathbf{Q}})^2 - (\mathbf{W}_n \cdot \dot{\mathbf{Q}})^2] \, d\sigma_x. \quad (3.8) \end{aligned}$$

Introducing the tensors  $\llbracket T_{F_n}(\mathbf{Q}) \rrbracket$  and  $\llbracket T_{F_k}(\mathbf{Q}) \rrbracket$  defined respectively by (1.12) and (1.13) and plugging the expressions (3.5) and (3.8) into (3.1), we obtain the formula of Theorem 1.3.

## 4 Motion of an infinite cylinder

In the remaining part of this paper we study the motion of an infinite cylinder in an half space. For symmetry reasons, the problem reduces to a plane configuration and the cylinder to a disk. Therefore, let us denote  $\Omega$  the upper half-plane  $\Omega = \{(x_1, x_2)^\top \in \mathbb{R}^2, \text{ s.t. } x_2 > 0\}$ . The solid  $S(0)$  is supposed to be a disk of center  $\mathbf{h}_0 = (0, h_0)^\top$  and of radius  $r = 1$ . Since we shall consider only initial velocities of the form  $\dot{\mathbf{h}}_0 = (0, \dot{h}_0)^\top$ , the center of the disk remains in the axis  $x_1 = 0$  for all  $t > 0$  and then  $\mathbf{h}(t) = (0, h(t))^\top$ . The velocity is therefore  $\dot{\mathbf{h}}(t) = (0, \dot{h}(t))^\top$ . We assume also that the initial angular velocity is zero and hence that the disk will not rotate in his motion. In this case, the potential of the fluid is linear with respect to  $\dot{h}$  and reads merely  $\dot{h}\varphi$  where the function  $\varphi$  solves:

$$-\Delta\varphi(t, \cdot) = 0 \quad \text{in } \Omega \setminus S(t), \quad (4.1a)$$

$$\frac{\partial\varphi}{\partial \mathbf{n}}(t, \cdot) = h - x_2 \quad \text{on } \partial S(t), \quad (4.1b)$$

$$\frac{\partial\varphi}{\partial \mathbf{n}}(t, \cdot) = 0 \quad \text{on } \partial\Omega. \quad (4.1c)$$

We shall compute explicitly the solution of (4.1), using a conformal mapping method. Then we will determine the kinetic energy of the fluid in terms of  $h$  and  $\dot{h}$  and apply the Lagrangian formulation (1.10) in order to show that the disk hits the boundary with non-zero velocity.

**Lemma 4.1** *For all  $t > 0$ , the conformal mapping  $\mathbf{G}$  defined by:*

$$\mathbf{G}(\mathbf{z}) = \frac{\mathbf{z} - ia}{\mathbf{z} + ia},$$

where  $a = \sqrt{h^2 - 1}$ , maps  $S(t)$  onto the disk  $\mathcal{D}_\sigma$  of center  $\mathbf{0}$  and radius  $\sigma = 1/(h + \sqrt{h^2 - 1})$  and  $\Omega \setminus S(t)$  onto  $\mathcal{A}_\sigma = \mathcal{A}(0, \sigma, 1)$ , the annulus of center  $\mathbf{0}$  and radii  $\sigma$  and 1. For all  $\mathbf{w} \in \mathcal{A}_\sigma$ , the inverse function  $\mathbf{g} = \mathbf{G}^{-1}$  is given by:

$$\mathbf{g}(\mathbf{w}) = ia \frac{1 + \mathbf{w}}{1 - \mathbf{w}},$$

and we have  $h = (\sigma^{-1} + \sigma)/2$ ,  $a = (\sigma^{-1} - \sigma)/2$ .

**Proof :** A proof can be found in [9]. ■

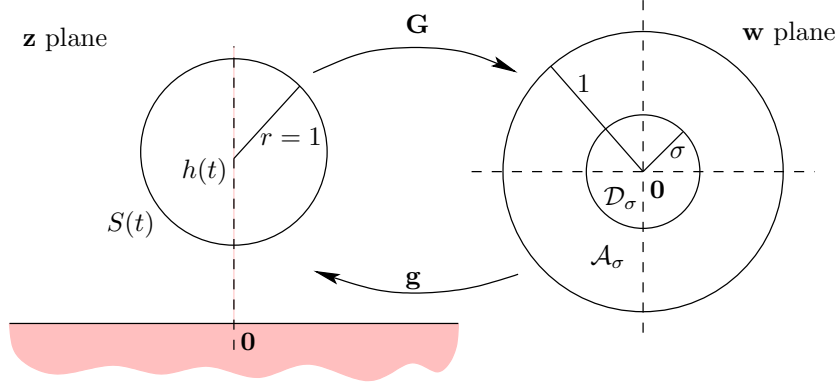


Figure 2: The conformal mappings  $\mathbf{G}$  and  $\mathbf{g}$ .

Let us recall some useful classical properties of an holomorphic change of variables, the proof being let to the reader:

**Lemma 4.2** Let  $\mathcal{O}$  and  $\tilde{\mathcal{O}}$  be two open subsets of  $\mathbb{R}^2$  and  $\mathbf{P} = P_1 + iP_2 : \mathcal{O} \rightarrow \tilde{\mathcal{O}}$  be a conformal mapping.

1. Considering a function  $f \in C^2(\tilde{\mathcal{O}})$ , we denote  $F = f \circ \mathbf{P}$ . Then  $F \in C^2(\mathcal{O})$  and we have the relation:

$$\Delta F(u_1, u_2) = \left[ \left( \frac{\partial P_1}{\partial u_1} \right)^2 + \left( \frac{\partial P_2}{\partial u_1} \right)^2 \right] \Delta f(\mathbf{P}(u_1, u_2)),$$

for all  $(u_1, u_2)^\top \in \mathcal{O}$ .

2. Let  $\Gamma$  be a  $C^1$  curve,  $\Gamma \subset \mathcal{O}$  and  $\mathbf{n}$  be the unit normal to  $\Gamma$ . Denote also  $\tilde{\Gamma} = \mathbf{P}(\Gamma) \subset \tilde{\mathcal{O}}$  and  $\tilde{\mathbf{n}}$  the unit normal to  $\tilde{\Gamma}$ . Then we have:

$$\frac{\partial F}{\partial \mathbf{n}}(u_1, u_2) = \left| \frac{\partial \mathbf{P}}{\partial u_1}(u_1, u_2) \right| \frac{\partial f}{\partial \tilde{\mathbf{n}}}(\mathbf{P}(u_1, u_2)). \quad (4.2)$$

Applying Lemma 4.2, the new function  $\phi(u_1, u_2) = \varphi(\mathbf{g}(u_1, u_2))$  solves the Neumann problem:

$$\Delta \phi(u_1, u_2) = 0 \quad \text{for } (u_1, u_2) \in \mathcal{A}_\sigma, \quad (4.3a)$$

$$\frac{\partial \phi}{\partial \mathbf{n}}(u_1, u_2) = 0 \quad \text{for } u_1^2 + u_2^2 = 1, \quad (4.3b)$$

$$\frac{\partial \phi}{\partial \mathbf{n}}(u_1, u_2) = \frac{a[2\sigma^2 - (1 + \sigma^2)u_1]}{\sigma(1 + \sigma^2 - 2u_1)^2} \quad \text{for } u_1^2 + u_2^2 = \sigma^2. \quad (4.3c)$$

For all  $t > 0$ , according to [5, Theorem 3.1], the harmonicity of  $\phi$  ensures the existence of an harmonic conjugate function  $\psi$  and of an holomorphic function  $\mathbf{F}$  defined in  $\Omega \setminus S(t)$  by  $\mathbf{F} = \phi + i\psi$ . Remark that, since  $\Omega \setminus S(t)$  is not a simple connected set, this existence result is not obvious. The function  $\mathbf{F}$  is defined up to an additive complex constant because  $\phi$  is the solution of a Neumann problem. From the boundaries conditions (4.3b) and (4.3c) for  $\phi$ , we deduce the boundaries conditions for  $\mathbf{F}$ . On the boundary  $|\mathbf{w}| = 1$  of  $\mathcal{A}_\sigma$ , the exterior unit normal is  $\mathbf{n} = \mathbf{w}$ , then  $\partial\mathbf{F}/\partial\mathbf{n}(\mathbf{w}) = \mathbf{F}'(\mathbf{w})\mathbf{w}$ . On the boundary  $|\mathbf{w}| = \sigma$ , the normal vector is  $\mathbf{n} = -\mathbf{w}/\sigma$ , therefore  $-\sigma\partial\mathbf{F}/\partial\mathbf{n}(\mathbf{w}) = \mathbf{F}'(\mathbf{w})\mathbf{w}$ . Finally, with  $\mathbf{w} = u_1 + iu_2$ , (4.3b) and (4.3c) yield for  $\mathbf{F}$  the conditions:

$$\mathcal{R}(\mathbf{F}'(\mathbf{w})\mathbf{w}) = 0 \quad \text{for } |\mathbf{w}| = 1, \quad (4.4a)$$

$$\mathcal{R}(\mathbf{F}'(\mathbf{w})\mathbf{w}) = \frac{-a[2\sigma^2 - (1 + \sigma^2)u_1]}{(1 + \sigma^2 - 2u_1)^2} \quad \text{for } |\mathbf{w}| = \sigma. \quad (4.4b)$$

The function  $\mathbf{F}$  being holomorphic in the annulus  $\mathcal{A}_\sigma$ , it admits an expansion in the form of a Laurent series.

**Lemma 4.3** *The Laurent series of the function  $\mathbf{F}$ , holomorphic in  $\mathcal{A}_\sigma$  and satisfying (4.4) is:*

$$\mathbf{F}(\mathbf{w}) = \gamma - a \sum_{n \geq 1} \frac{\sigma^{2n}}{1 - \sigma^{2n}} (\mathbf{w}^n + \mathbf{w}^{-n}), \quad \forall \mathbf{w} \in \mathcal{A}_\sigma, \quad (4.5)$$

where  $\gamma \in \mathbb{C}$  is an arbitrary constant.

**Remark 4.1** *The above series converges for  $\sigma^2 < |\mathbf{w}| < \sigma^{-2}$ , what is a larger domain than  $\mathcal{A}_\sigma$ .*

**Proof :** Seeking the coefficients of the Laurent series (4.5), we write formally that:

$$\mathbf{F}(\mathbf{w}) = \gamma + \sum_{n \geq 1} (a_n + ib_n)\mathbf{w}^n + (c_n + id_n)\mathbf{w}^{-n} \quad \text{for all } \mathbf{w} \in \mathcal{A}_\sigma, \quad (4.6a)$$

and then

$$\begin{aligned} \mathcal{R}(\mathbf{F}'(\mathbf{w})\mathbf{w}) &= \sum_{n \geq 1} n[r^n a_n - r^{-n} c_n] \cos(n\theta) \\ &\quad - n[r^n b_n + r^{-n} d_n] \sin(n\theta) \quad \text{for all } \mathbf{w} = re^{i\theta}. \end{aligned} \quad (4.6b)$$

For  $r = 1$ , according to (4.4a), we obtain that  $a_n = c_n$  and  $b_n = -d_n$ . For  $r = \sigma$ , the identity (4.4b) allows us to deduce the coefficients  $a_n$  and  $b_n$ . Thus, we get:

$$\begin{aligned} a_n + ib_n &= \frac{-\sigma^n}{n(1 - \sigma^{2n})} \frac{1}{\pi} \int_0^{2\pi} \alpha(\theta) e^{-in\theta} d\theta = \frac{-\sigma^n}{n(1 - \sigma^{2n})} \hat{\alpha}(n), \\ c_n + id_n &= \frac{-\sigma^n}{n(1 - \sigma^{2n})} \frac{1}{\pi} \int_0^{2\pi} \alpha(\theta) e^{in\theta} d\theta = \frac{-\sigma^{-n}}{-n(1 - \sigma^{-2n})} \hat{\alpha}(-n), \end{aligned}$$

where  $\hat{\alpha}(n)$  is the  $n$ -th Fourier coefficient of  $\alpha$ , the function defined for all  $\theta \in \mathbb{R}$  by:

$$\alpha(\theta) = \frac{-a\sigma[2\sigma - (1 + \sigma^2) \cos(\theta)]}{[1 + \sigma^2 - 2\sigma \cos(\theta)]^2}.$$

Plugging the expressions of  $a_n$ ,  $b_n$ ,  $c_n$  and  $d_n$  in the formal expansion (4.6a), we get:

$$\mathbf{F}(\mathbf{w}) = \gamma + \sum_{n \in \mathbb{Z}^*} \frac{-\sigma^n}{n(1 - \sigma^{2n})} \hat{\alpha}(n) \mathbf{w}^n, \quad (4.7)$$

where  $\gamma \in \mathbb{C}$  is an arbitrary constant. We next compute explicitly the Fourier coefficients of  $\alpha$ . Set  $\mathbf{z} = e^{i\theta}$ , it follows that:

$$\widehat{\alpha}(n) = \frac{1}{2i\pi} \int_{\partial\mathcal{D}} \frac{-a\sigma[4\sigma\mathbf{z} - (1 + \sigma^2)(\mathbf{z}^2 + 1)] d\mathbf{z}}{[(1 + \sigma^2)\mathbf{z} - \sigma(\mathbf{z}^2 + 1)]^2 \mathbf{z}^n}. \quad (4.8)$$

Since:

$$\frac{-a\sigma[4\sigma\mathbf{z} - (1 + \sigma^2)(\mathbf{z}^2 + 1)]}{[(1 + \sigma^2)\mathbf{z} - \sigma(\mathbf{z}^2 + 1)]^2} = a\sigma \left[ \frac{1}{(\mathbf{z} - \sigma)^2} + \frac{1}{\sigma^2} \frac{1}{(\mathbf{z} - \sigma^{-1})^2} \right],$$

the relation (4.8), combined with (4.7) gives, according to the residue Theorem:

$$\begin{aligned} \widehat{\alpha}(n) &= na\sigma^n \quad \text{for all } n \geq 1, \\ \widehat{\alpha}(n) &= -na\sigma^{-n} \quad \text{for all } n \leq -1. \end{aligned}$$

Plugging these identities into (4.7), we get (4.5) and the proof is then completed. ■

We shall compute now the kinetic energy of the system fluid-solid for all  $0 < \sigma < 1$  (and hence for all  $h > 1$ , according to the relation  $h = (\sigma^{-1} - \sigma)/2$ ). Then, using the Lagrangian principle (1.10), we will be able to determine the velocity of the disk and prove that for an adequate initial velocity the disk collides the fluid boundary with non zero velocity. The following Lemma displays some properties of the energy of the fluid:

**Lemma 4.4** *For all  $t > 0$ , the energy of the fluid reads  $E_F(t) = 1/2|\dot{h}|^2\rho_F\mathcal{E}(\sigma)$  where  $\mathcal{E}(\sigma)$  is given by the following series:*

$$\mathcal{E}(\sigma) = \pi a^2 \sum_{n \geq 1} n\sigma^{2n} \frac{(1 + \sigma^{2n})}{(1 - \sigma^{2n})}.$$

Moreover, the function  $\mathcal{E}(\sigma)$  is continuous with respect to  $\sigma$ , for all  $\sigma \in ]0, 1[$  and verifies:

$$\lim_{\sigma \rightarrow 1^-} \mathcal{E}(\sigma) = \frac{\pi^3}{12} - \frac{\pi}{4}, \quad \lim_{\sigma \rightarrow 0^+} \mathcal{E}(\sigma) = \frac{\pi}{4}. \quad (4.9)$$

**Proof :** The kinetic energy of the fluid is:

$$E_F(t) = \frac{1}{2}|\dot{h}|^2\rho_F \int_{\Omega \setminus S(t)} |\nabla\varphi|^2 d\mathbf{x},$$

where  $\varphi$  is the solution of the system (4.1). The functions  $\mathbf{G}$  and  $\mathbf{g}$  are holomorphic. Hence, taking into account the Cauchy Riemann conditions, their Jacobian matrices can be written as:

$$\begin{aligned} [J_G](x_1, x_2) &= \left| \frac{\partial \mathbf{G}}{\partial x_1}(x_1, x_2) \right| [Q_G](x_1, x_2) \quad \text{for all } (x_1, x_2) \in \Omega \setminus S(t), \\ [J_g](u_1, u_2) &= \left| \frac{\partial \mathbf{g}}{\partial u_1}(u_1, u_2) \right| [Q_g](u_1, u_2) \quad \text{for all } (u_1, u_2) \in \mathcal{A}_\sigma, \end{aligned}$$

where  $[Q_G]$  and  $[Q_g]$  are orthogonal matrices. We deduce the expression of  $\nabla\varphi$  in terms of  $\nabla\phi$ :

$$\begin{aligned} \nabla\varphi(x_1, x_2) &= \left| \frac{\partial \mathbf{G}}{\partial x_1}(x_1, x_2) \right| [Q_G](x_1, x_2) \nabla\phi(\mathbf{G}(x_1, x_2)) \\ &= \left| \frac{\partial \mathbf{g}}{\partial u_1}(\mathbf{G}(x_1, x_2)) \right|^{-1} [Q_G](x_1, x_2) \nabla\phi(\mathbf{G}(x_1, x_2)). \end{aligned}$$

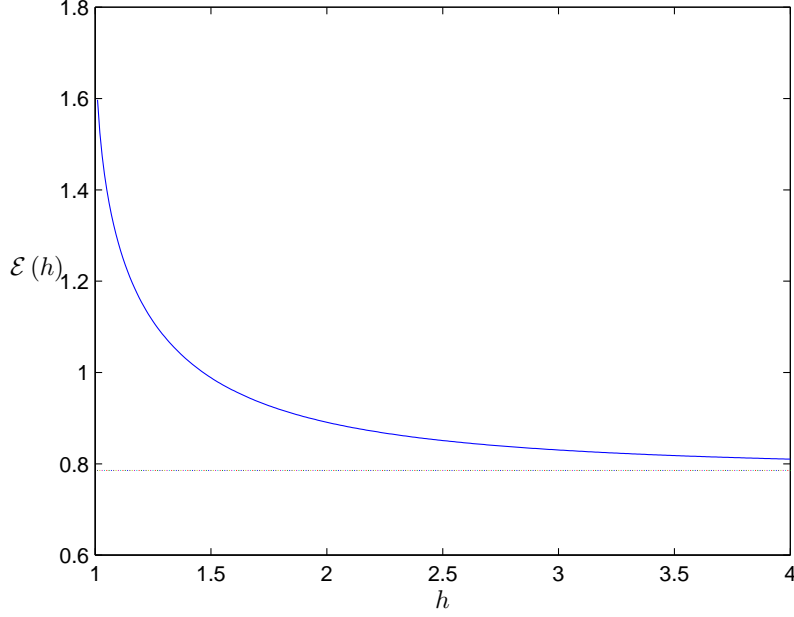


Figure 3: Graph of the function  $\mathcal{E}$  with respect to  $h$

Finally the energy of the fluid can be expressed as follows:

$$E_F(t) = \frac{1}{2} \rho_F |\dot{h}|^2 \int_{\Omega \setminus S(t)} |\nabla \varphi|^2 dx = \frac{1}{2} \rho_F |\dot{h}|^2 \int_{\mathcal{A}_\sigma} |\nabla \phi|^2 d\mathbf{w}.$$

Let us define  $\mathcal{E}(\sigma) = \int_{\mathcal{A}_\sigma} |\nabla \phi|^2(u_1, u_2) d\mathbf{w}$ . Since  $|\nabla \phi(u_1, u_2)|^2 = |\mathbf{F}'(\mathbf{w})|^2$ , we have in cylindrical coordinates:

$$\mathcal{E}(\sigma) = 2\pi \int_\sigma^1 \left[ \frac{1}{2\pi} \int_0^{2\pi} |\mathbf{F}'|^2(\rho \cos(\theta), \rho \sin(\theta)) d\theta \right] \rho d\rho.$$

In order to apply the Parseval Theorem, we need to prove that for all  $\rho \in [\sigma, 1]$  the function  $\theta \mapsto \mathbf{F}'(\rho \cos(\theta), \rho \sin(\theta))$  is in  $L^2(0, 2\pi)$ . We have the following estimates:

$$\begin{aligned} |\mathbf{F}'(\rho \cos(\theta), \rho \sin(\theta))| &\leq \frac{a}{\sigma} \sum_{n \geq 1} \frac{n\sigma^{2n}}{1 - \sigma^{2n}} (1 + \sigma^{-n}) \\ &\leq \frac{a}{\sigma(1 - \sigma)} \sum_{n \geq 1} n\sigma^n = \frac{a}{(1 - \sigma)^3}. \end{aligned}$$

So we can apply the Parseval equality to  $|\mathbf{F}'|$  and we obtain:

$$\mathcal{E}(\sigma) = 2\pi \int_\sigma^1 \rho d\rho \sum_{n \geq 1} \left[ \frac{an\sigma^{2n}}{1 - \sigma^{2n}} \right]^2 [\rho^{2n-2} + \rho^{-2n-2}] \quad (4.10a)$$

$$= \pi a^2 \sum_{n \geq 1} n\sigma^{2n} \frac{1 + \sigma^{2n}}{1 - \sigma^{2n}}. \quad (4.10b)$$

The well known identities  $x/(1-x) = \sum_{n \geq 1} x^n$  and  $x/(1-x)^2 = \sum_{n \geq 1} nx^n$ , for all  $|x| < 1$  together with (4.10b), allow us to turn the expression of  $\mathcal{E}$  into:

$$\mathcal{E}(\sigma) = \pi a^2 \sum_{n \geq 1} n(1 + \sigma^{2n}) \sum_{k \geq 1} \sigma^{2nk}.$$



We can invert the order of both sums because all the terms are positive.

$$\begin{aligned}\mathcal{E}(\sigma) &= \pi a^2 \sum_{k \geq 1} \sum_{n \geq 1} n \sigma^{2nk} + n \sigma^{2n(k+1)} \\ &= \pi a^2 \sum_{k \geq 1} \frac{\sigma^{2k}}{(1 - \sigma^{2k})^2} + \frac{\sigma^{2(k+1)}}{(1 - \sigma^{2(k+1)})^2}.\end{aligned}$$

The similarity of both terms in the above series allows us to get the final expression for  $\mathcal{E}$ :

$$\mathcal{E}(\sigma) = \frac{\pi}{2} \left[ \sum_{k \geq 1} \sigma^{2k-2} \left[ \frac{1 - \sigma^2}{1 - \sigma^{2k}} \right]^2 \right] - \frac{\pi}{4}.$$

We show that the coefficients in this series are uniformly bounded by the terms of a convergent series. By the way we prove the continuity of the energy up to  $\sigma = 1$  (i.e.  $h = r$ ). We have the following inequality:

$$\frac{1 - \sigma^k}{1 - \sigma} = \sum_{p=0}^{k-1} \sigma^p \geq k \sigma^{k-1} \quad \text{for all } 0 < \sigma < 1,$$

and then

$$\sigma^{2k-2} \left[ \frac{1 - \sigma^2}{1 - \sigma^{2k}} \right]^2 = \sigma^{2k-2} \left[ \frac{1 + \sigma}{1 + \sigma^k} \frac{1 - \sigma}{1 - \sigma^k} \right]^2 \leq \frac{4}{k^2} \quad \text{for all } 0 < \sigma < 1.$$

Finally the terms of  $\mathcal{E}$  are uniformly bounded by a convergent series for all  $0 < \sigma < 1$ . Then  $\mathcal{E}$  is a continuous function of  $\sigma$  (and therefore of  $h$ ). We can pass to the limit in the series as  $\sigma$  goes to 1 and as  $\sigma$  goes to 0 and we get the values for the energy given by (4.9). ■

To conclude this section, we show that the disk can touch the half plane boundary with non-zero velocity. We denote merely  $\mathcal{E}(h)$  the function  $\mathcal{E}(1/(h + \sqrt{h^2 - 1}))$  and we compute the Lagrangian  $\mathcal{L}$  of the system and write the Euler-Lagrange principle (1.10). That leads to the following equations:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} m \dot{h}^2 + \frac{\rho_F}{2} |\dot{h}|^2 \mathcal{E}(h), \\ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{h}} - \frac{\partial \mathcal{L}}{\partial h} &= \ddot{h} (m + \rho_F \mathcal{E}(h)) + \frac{\rho}{2} \dot{h} \frac{\partial \mathcal{E}}{\partial h}(h) = 0,\end{aligned}$$

and hence

$$\dot{h} = \dot{h}_0 \sqrt{\frac{m + \rho_F \mathcal{E}(h_0)}{m + \rho_F \mathcal{E}(h)}}, \quad (4.12)$$

where  $h_0$  and  $\dot{h}_0$  are the initial position and velocity of the disk. The ODE (4.12) ensures that for all  $t > 0$ ,  $\dot{h}$  keeps the same signum as  $\dot{h}_0$ . In particular, for  $\dot{h}_0 < 0$  and for any  $h_0 > 1$ , we have  $1 \leq h \leq h_0$ . By continuity of  $\mathcal{E}(h)$  at  $h = 1$  (what corresponds to  $\sigma = 1$ ), there exists  $h_0 > 1$  such that, for all  $h \in ]h_0, 1]$ ,  $\mathcal{E}(h) < 3\mathcal{E}(1)/2$ . We deduce that, for any  $\dot{h}_0 < 0$ :

$$\frac{\dot{h}}{\dot{h}_0} \geq \sqrt{\frac{m}{m + 3\rho_F \mathcal{E}(1)/2}}, \quad \forall h \in ]h_0, 1],$$

what proves that the disk collides the boundary in finite time with non-zero velocity.

## 5 Velocity damping at the collision point

We have seen that our model of fluid-solid interaction, dealing with an ideal fluid flow, allows the rigid body to collide the boundary of the fluid domain. Nevertheless, the fluid has a damping effect on the velocity of the solid. Considering again the configuration of the preceding section, we can rewrite (4.12):

$$\frac{\dot{h}}{\dot{h}_0} = \sqrt{\frac{\pi + \rho_F/\rho_S \mathcal{E}(h_0)}{\pi + \rho_F/\rho_S \mathcal{E}(h)}}. \quad (5.1)$$

We have drawn in the figure 5 the disk velocity damping percentage  $100(1 - \dot{h}/\dot{h}_0)$  with respect to  $h$  for some values of  $\rho_F/\rho_S$ . Note that the d'Alembert's paradox, holding for a disk in the whole space  $\mathbb{R}^2$ , does not apply any more in our case. The resulting hydrodynamic force on the disk is non-null and the disk slows down when approaching the fluid boundary.

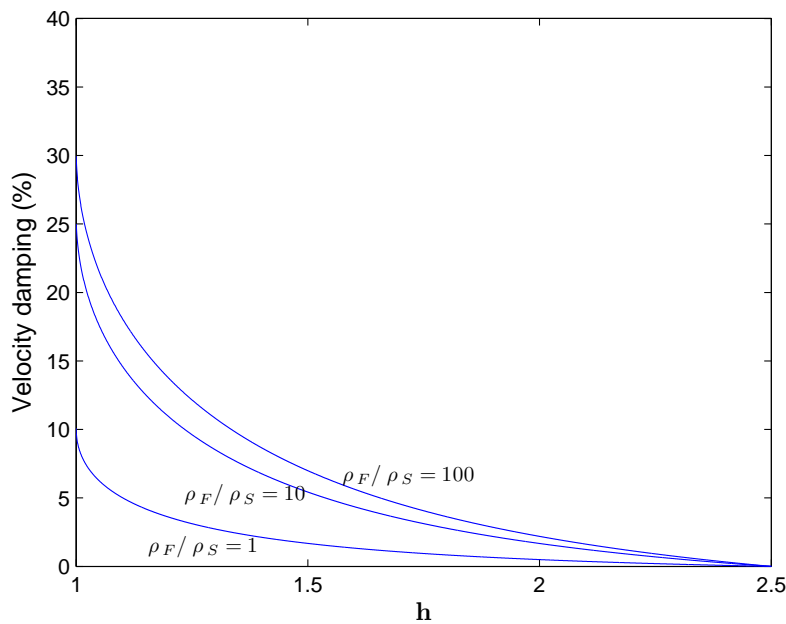


Figure 4: Ball velocity damping for  $h_0 = 2.5$

No surprisingly, the damping effect increases as  $h$  decreases and is maximum at the collision point, that is to say for  $h = 1$ . Since  $\mathcal{E}(1) = \pi^3/12 - \pi/4$  and  $\mathcal{E}(\infty) = \pi/4$ , we can compute that a disk coming from infinity has lost  $100(1 - \sqrt{(\pi + \rho_F/\rho_S \pi/4)/(\pi + \rho_F/\rho_S (\pi^3/12 - \pi/4))})$  percents of its initial velocity at the moment of the impact. This quantity, represented in figure 5, increases with  $\rho_F/\rho_S$  and tends to  $100(1 - \sqrt{1/(\pi^2/3 - 1)}) \sim 34\%$  as  $\rho_F/\rho_S$  goes to infinity.

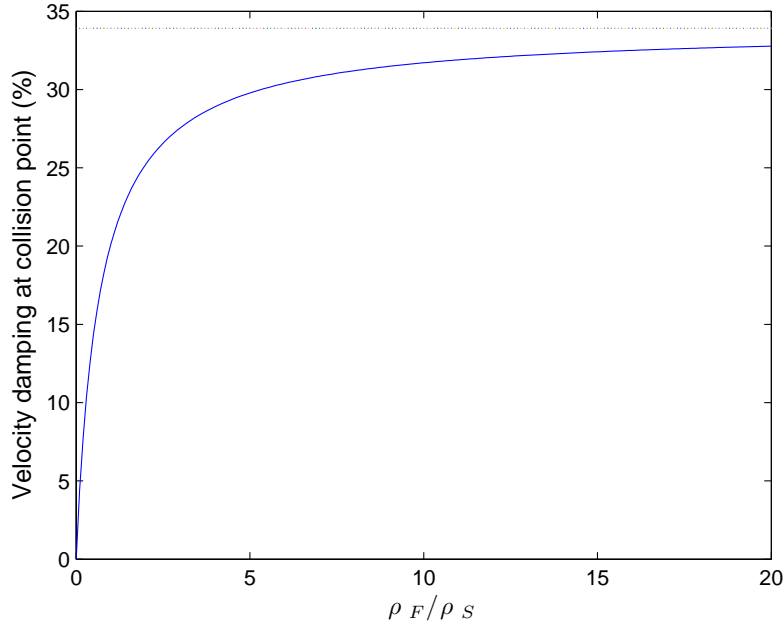


Figure 5: Velocity damping at the collision point with respect to  $\rho_F/\rho_S$

In particular, even with an infinite density fluid, the disk does collide the boundary.

## Acknowledgements

The authors gratefully thank Pr. Marius Tucsnak for his constant help and without who this work would not have been possible.

## References

- [1] V. I. Arnold. *Mathematical methods of classical mechanics*. Springer-Verlag, New York, 1978. Translated from the Russian by K. Vogtmann and A. Weinstein, Graduate Texts in Mathematics, 60.
- [2] H. Brezis. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].
- [3] Earl A. Coddington and Norman Levinson. *Theory of ordinary differential equations*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.
- [4] M.-P. do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.
- [5] V. Girault and P. A. Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms.

- [6] J. Giroire. *Etude de quelques problèmes aux limites extérieurs et résolution par équations intégrales*. PhD thesis, Université Pierre et Marie Curie - Paris 6, September 1987.
- [7] J. Giroire and J.-C. Nédélec. Numerical solution of an exterior Neumann problem using a double layer potential. *Math. Comp.*, 32(144):973–990, 1978.
- [8] H. Goldstein. *Classical Mechanics*. Addison-Wesley Press, Inc., Cambridge, Mass., 1951.
- [9] P. Henrici. *Applied and computational complex analysis. Vol. 1*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. Power series—integration—conformal mapping—location of zeros, Reprint of the 1974 original, A Wiley-Interscience Publication.
- [10] A. Henrot and M. Pierre. *Variation et optimisation de formes : une analyse géométrique*, volume 048 of *Mathématiques et applications*. Springer, Berlin/Heidelberg/New York, 2005.
- [11] H. Lamb. *Hydrodynamics*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, sixth edition, 1993. With a foreword by R. A. Caffisch [Russel E. Caffisch].
- [12] J. H. Ortega, L. Rosier, and T. Takahashi. Classical solutions for the equations modelling the motion of a ball in a bidimensional incompressible perfect fluid. *M2AN Math. Model. Numer. Anal.*, 39(1):79–108, 2005.