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Abstract. This paper analyzes stochastic networks consisting of a set of finite capacity sites where different classes of individuals move according to some routing policy. The associated (non-reversible) Markov jump processes are analyzed under a *thermodynamic limit* regime, i.e. when the networks have some symmetry properties and when the number of nodes goes to infinity. A *metastability property* is proved: under some conditions on the parameters, it is shown that, in the limit, several equilibrium points coexist for the empirical distribution. The key ingredient of the proof of this property is a dimension reduction achieved by the introduction of two energy functions and a convenient mapping of their local minima and saddle points. Cases with a unique equilibrium point are also presented.

Contents

1. Introduction

This paper studies the asymptotic behavior of a class of stochastic networks. A general description of the basic mechanisms of these systems is given below in terms of a finite particle system or in terms of a queueing network.

These networks are analyzed under a thermodynamic scaling, i.e. when the number of nodes goes to infinity. It is shown that the process of the empirical distribution of the system converges to some dynamical system $(y(t))$ in the set of probability distributions on some finite set $\mathcal{X}$,

$$\frac{d}{dt}y(t) = V\big(y(t)\big), \qquad t \geq 0, \tag{1}$$

where the vector field $(V(y), y \in \mathcal{P}(\mathcal{X}))$ driving the dynamical system $(y(t))$ exhibits a quadratic dependence on the variable $y = (y_n, n \in \mathcal{X})$. The main results of this paper concern the analysis of the equilibrium points of this limiting process, that

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is the set of solutions $y \in \mathcal{P}(\mathcal{X})$ of the equation $V(y) = 0$. More specifically, the following problems are investigated:

a) Analysis of the asymptotic behavior of $(y(t))$;
b) Number of zeroes of V;
c) Classification of equilibrium points for the limiting dynamics.

As it will be seen, an equilibrium point is not always unique. When there are more than two classes of particles/requests, it is shown that there may be at least *three zeroes* for V, two of them being "stable", i.e. at the bottom of two valleys of an energy landscape associated with this dynamics. The other zero being a *saddle point* of this landscape between the two valleys. Qualitatively, this can be described as a bistability property of the network. It can be (roughly) described as follows: the state of the network lives for a very long time around one of the stable points and, due to some rare event, it then reaches, via a saddle point, the region of another stable point and so on.

Metastability and Local Dynamics in Stochastic Networks. Metastability results are quite rare in queueing networks. Gibbens *et al.* [11] has shown, via an approximated model, that such an interesting phenomenon may occur in a loss network with a rerouting policy. Marbukh [17] analyzes, also through some approximation, the bistability properties of similar loss networks. The dynamics of the networks of Gibbens *et al.* [11], Marbukh [17] and of this paper are local in the following sense: The interaction between two nodes only depends on the states of these two nodes and not on the state of the whole network.

A key feature of metastability property in this context is the subtle interplay between the local description of the dynamics and its impact on the macroscopic state of the network. In statistical physics, the transition from one stable point to another one through the saddle point is, in general, difficult to describe rigorously. This is related to the determination of the geometry of the *critical droplet*, i.e. the exit path from a stable point. See den Hollander [8], Olivieri and Vares [18] and the references therein for a general presentation of these questions. See also Bovier [2, 3] for a potential theoretical approach in the case of reversible Markov processes and Catoni and Cerf [5] for a study of the saddle points of perturbed Markov chains. For the more classical setting of global dynamics, a large deviation approach is developed in Freidlin and Wentzell [10] to describe the transitions between two stable points.

Description of the stochastic networks. As for some classical processes, like the zero range process, see Liggett [16], one can give two alternative presentations for these networks.

A particle system. Such a network can be thought of as a set of sites where different types $k = 1, \ldots, K$ of particles coexist. At a given site, external type k particles with mass $A_k \in \mathbb{R}_+$ arrive at rate λ_k. A k particle stays an exponential time with parameter γ_k at a site and then moves randomly to another site. A type k particle leaves the network at rate μ_k. Mass constraint: The total mass of particles at a given site must be less than C, so that a particle arriving at a site is accepted only if this constraint is satisfied, otherwise the particle is rejected from the network.

A queueing network. It can be described as a set of identical finite capacity nodes where customers move from one node to another node, being accepted if there is enough room and, otherwise, being rejected. If he is not rejected during his travel through the network, the customer leaves the network after his total service time. Different classes of customers access the network: Classes differ by their arrival rates, total service times, residence times at the nodes and also by the capacities they require at the nodes they visit. For example, a "light" customer will require one unit of capacity while a "heavy" customer may ask for a significant proportion of the total capacity of the node. External class k customers arrive at rate λ_k at any node. During their total service time, which ends at rate μ_k, class k customers undergo transfers from a node to another one occurring at rate γ_k . A class k customer occupies $A_k \in \mathbb{N}$ units of capacity at any visited node, and any class k customer arriving at a node where this amount of capacity is not available is rejected.

These stochastic networks have been introduced in Antunes *et al.* [1] to represent the time evolution of a wireless network. In [1] it is proved that, for the heavy traffic scaling, there is a unique equilibrium point. Contrary to the model considered here, the capacity requirement of a customer in [1] does not depend on his class. On the other hand, the networks analyzed here have a symmetrical structure: all the nodes have the same capacity and the routing is uniform among all the other nodes. In Antunes *et al.* [1], the routing mechanisms are quite general.

Assuming Poisson arrivals and exponential distributions for the various service times and residence times, the time evolution of such a network is described by a Markov jump process with values in some finite (but large) state space. It turns out that, contrary to loss networks, this Markov process is in general *not reversible* and, furthermore, that it does not have a stationary distribution with a product form.

Outline of the Paper. Under the thermodynamic limit scaling, it is shown that the empirical distribution of the state of the network converges, through some mean field convergence, to the continuous dynamical system $(y(t))$ defined by Equation (1) in the space $\mathcal{P}(\mathcal{X})$ of probability distributions on the finite subset $\mathcal{X}$ of $\mathbb{N}^K$ defined by

$$\mathcal{X} = \{n = (n_k) \in \mathbb{N}^K : A_1 n_1 + \cdots + A_K n_K \le C\}.$$

The stability of the possible equilibrium points of this dynamical system is the main point of the paper. The complexity of the state space and the quadratic dependence on y of the vector field $V(y)$ make this problem quite difficult to tackle. The equilibrium points can be identified as elements of a family of probability distributions ν_ρ on $\mathcal{X}$ indexed by $\rho = (\rho_k) \in \mathbb{R}_+^K$,

$$\nu_\rho(n) = \frac{1}{Z(\rho)} \prod_{k=1}^{K} \frac{\rho_k^{n_k}}{n_k!}, \quad n \in \mathcal{X},$$

where $Z(\rho)$ is the partition function and ρ satisfies some fixed point equation. Unfortunately, $(y(t))$ does not reduce to a dynamical system on the sub-manifold $\{\nu_\rho : \rho \in \mathbb{R}_+^K\}$ of $\mathcal{P}(\mathcal{X})$. In other words $(y(t))$ cannot be written as $(\nu_{\rho(t)})$ for some suitable dynamical system $(\rho(t))$ on $\mathbb{R}_+^K$. The equation for the equilibrium points of $(y(t))$ is nevertheless translated into a fixed point equation with respect to $\rho \in \mathbb{R}_+^K$. Whereas this considerably reduces the dimension of the state space, the problem of uniqueness of a solution to fixed point equations is still not straightforward (the existence follows from Brouwer's Theorem). Moreover, if $\bar{\rho}$ is such a solution, the problem of the stability of the equilibrium point $\nu_{\bar\rho}$ for $(y(t))$ still has to be analyzed. A similar situation occurs in Gibbens *et al.* [11] where the equilibrium points are also indexed by the solutions $\rho \in \mathbb{R}_+$ of some fixed point equations and the bistability properties of the system are analyzed through numerical estimates. Here, a detailed study of the stability properties of the equilibrium points is achieved.

In Section 3, convergence results for the empirical distribution process are proved and the equations for the equilibrium points of the asymptotic dynamical system $(y(t))$ are obtained. The correspondence between these equilibrium points and some of the probability distributions ν_ρ on $\mathcal{X}$, $\rho \in \mathbb{R}_+^K$ is also established in this section.

Section 4 is devoted to the networks with a unique equilibrium point. It is shown that this is the case when all classes of requests require the same room at each node. Notice that the arrival rates and the distributions of service times and residence times may be different. The second example is provided by networks with large capacities. A limiting regime of the fixed point equations with respect to ρ is analyzed: The common capacity C of the nodes goes to infinity and the arrival rates are proportional to C. In this context Theorem 2 shows that there is essentially one unique solution: If $\bar\rho_C$ is a solution for capacity C then, in the limit, $\bar\rho_C \sim \eta C$, where η is some vector with an explicit representation in terms of the parameters of the network. It is worth noting that, although the fixed point equations are deterministic equations, the analysis uses probabilistic arguments to get the convergence of these solutions.

Sections 5 and 6 contain the key ingredients of the main results of the paper. In Section 5, an energy function g on $\mathcal{P}(\mathcal{X})$ is introduced for the dynamical system $(y(t))$, so that the equilibrium points are identified with the zeroes of the constrained gradient ∇g of g on $\mathcal{P}(\mathcal{X})$. In Section 6 an energy function ϕ on $\mathbb{R}_+^K$ is introduced and analyzed. It is shown that the zeroes of $\nabla\phi$ in $\mathbb{R}_+^K$ are in a one to one correspondence with the zeroes of the constrained gradient ∇g on $\mathcal{P}(\mathcal{X})$. Moreover, this is *the crucial property*, Theorem 3 proves that the stability properties of these fixed points are also in correspondence. Roughly speaking, the dimension reduction from the space $\mathcal{P}(\mathcal{X})$ of probability distributions on $\mathcal{X}$ to $\mathbb{R}_+^K$ is not achieved through some associated dynamical system on $\mathbb{R}_+^K$ but via an energy function.

Section 7 analyzes the minima of the function ϕ. It is shown that, for some set of parameters, a bistability phenomenon may occur. An example is provided with three equilibrium points: two local minima and one saddle point.

2. The Stochastic Model

The queueing network terminology is used in the sequel. The network has N nodes and there are K classes of customers circulating through the nodes according to the following rules: For $1 \le k \le K$,

— *Arrivals.* External class k customers arrive at each node according to a Poisson process with parameter λ_k.

— *Total service time.* A class k customer who is never rejected during his travel through the network spends an exponentially distributed time with rate μ_k in the network (call duration in the context of a cellular network). The case $\mu_k = 0$ is not excluded and corresponds to the case of customers who leave the network only when they arrive at some saturated node.

— *Residence time.* During his stay in the network, each customer visits a sequence of nodes. The residence time of a customer of class k at a node is exponentially distributed with parameter γ_k. It can nevertheless be short cut due to the end of his total service time with rate μ_k.

— *Routing.* When a class k customer leaves a node to another one, the next visited node is chosen uniformly among the other nodes.

— *Capacity requirements.* A class k customer requires A_k units of capacity at each node along his route. If this amount of capacity is not available then he is rejected.

All random variables used for arrivals, residence times and service times are assumed to be independent.

Notations. For $t \geq 0$, $i \in \{1, \ldots, N\}$ and $1 \leq k \leq K$, let $X_{i,k}^N(t)$ denote the number of customers of class k present at node i at time t. The state of node i at time t is given by the vector $X_i^N(t) = (X_{i,1}^N(t), \ldots, X_{i,K}^N(t))$ and the capacity constraints imply that $X_i^N(t) \in \mathcal{X}$ with

$$\mathcal{X} = \{n = (n_k) \in \mathbb{N}^K : A_1 n_1 + \cdots + A_K n_K \leq C\}$$

One thus gets a Markov process $X^N(t) = (X_i^N(t), 1 \leq i \leq N)$ with values in $\mathcal{X}^N$ describing the total network at time t. Since the number N of nodes of the network (and therefore the number of coordinates of the vector $X^N(t)$) will go to infinity, it is more convenient to look at the *empirical distribution* associated to this Markov process:

$$\Gamma_N(t) = \frac{1}{N}\left(\delta_{X_1^N(t)} + \delta_{X_2^N(t)} + \cdots + \delta_{X_N^N(t)}\right)$$

where δ_n is the Dirac distribution at $n \in \mathcal{X}$. The process $(\Gamma_N(t))$ has values in the set $\mathcal{P}(\mathcal{X})$ of probability distributions on $\mathcal{X}$, defined by

$$\mathcal{P}(\mathcal{X}) = \left\{(y_n, n \in \mathcal{X}) \in \mathbb{R}_+^{\mathcal{X}} : \sum_{n \in \mathcal{X}} y_n = 1\right\}.$$

It is not difficult to check that, due to the symmetrical structure of the network, it has the Markov property. For $n = (n_k) \in \mathcal{X}$, one denotes by

$$Y_n^N(t) = \Gamma_N(t)(\{n\}) = \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{X_i^N(t)=n\}},$$

the variable $Y_n^N(t)$ is simply the proportion of nodes in state $n = (n_1, \ldots, n_K) \in \mathcal{X}$ at time t or, equivalently, the proportion of nodes having n_k customers of class k at time t for all $1 \leq k \leq K$. The two processes $t \to (\Gamma_N(t))$ and $t \to (Y_n^N(t), n \in \mathcal{X})$ are clearly equivalent.

The Q-matrix $(\Omega_N(y,z), y, z \in \mathcal{P}(\mathcal{X}))$ of $(Y^N(t))$ is given by

$$\begin{aligned}
&\Omega_N(y, y + \frac{1}{N}(e_{n+f_k} - e_n)) = \lambda_k y_n N \mathbb{1}_{\{n+f_k \in \mathcal{X}\}}\\
&\Omega_N(y, y + \frac{1}{N}(e_{n-f_k} - e_n)) = \mu_k n_k y_n N\\
&\Omega_N\left(y, y + \frac{1}{N}\left(e_{n-f_k} - e_n + (e_{m+f_k} - e_m)\mathbb{1}_{\{m+f_k \in \mathcal{X}\}}\right)\right)\\
&\qquad\qquad\qquad\qquad = \frac{\gamma_k N}{N-1} n_k y_n (N y_m - \mathbb{1}_{\{m=n\}}),
\end{aligned}$$

for $n \in \mathcal{X}$, and $k \in \{1, \ldots, K\}$, where e_n is the nth unit vector of $\mathbb{R}^{\mathcal{X}}$ and, f_k denotes the kth unit vector of $\mathbb{R}^K$.

3. The Asymptotic Dynamical System

Two nodes i, $j \in \{1, \ldots, N\}$ of the network interact through the exchange of customers at rate of the order of $1/N$. Due to the symmetrical structure of the network, a stronger statement holds: the impact on i of all nodes different from i appears only through some averaged quantity. For $1 \le k \le K$, the input rate of class k customers at node i from the other nodes is

$$\frac{1}{N} \sum_{1 \le j \le N,\, j \ne i} \gamma_k X_{j,k}^N(t).$$

If this quantity is close to $\gamma_k \mathbb{E}(X_{1,k}^N(t))$, a *mean field property* is said to hold. Note that, if the network starts from some symmetrical initial state, the random variables $X_{j,k}^N(t)$, $j = 1, \ldots, N$, have the same distribution.

Theorem 1. *If $Y^N(0)$ converges weakly to $z \in \mathcal{P}(\mathcal{X})$ as N tends to infinity, then $(Y^N(t))$ converges in the Skorohod topology to the solution $(y(t))$ of the ordinary differential equation*

$$y'(t) = V(y(t)), \tag{2}$$

where $(y(t))$ is the solution starting from $y(0) = z$ and, for $y \in \mathcal{P}(\mathcal{X})$, the vector field $V(y) = (V_n(y), n \in \mathcal{X})$ on $\mathcal{P}(\mathcal{X})$ is defined by

$$\begin{aligned} V_n(y) = &\sum_{k=1}^K (\lambda_k + \gamma_k \langle \mathbb{I}_k, y\rangle) \left(y_{n-f_k} \mathbb{1}_{\{n_k \ge 1\}} - y_n \mathbb{1}_{\{n+f_k \in \mathcal{X}\}} \right) \\ &+ \sum_{k=1}^K (\gamma_k + \mu_k) \left((n_k + 1) y_{n+f_k} \mathbb{1}_{\{n+f_k \in \mathcal{X}\}} - n_k y_n \right) \end{aligned} \tag{3}$$

with $\langle \mathbb{I}_k, y\rangle = \sum_{m \in \mathcal{X}} m_k y_m$ and f_k is the kth unit vector of $\mathbb{R}^K$.

By convergence in the Skorohod topology, one means the convergence in distribution for Skorohod topology on the space of trajectories.

Note that Equation (3) gives the derivative $dy_n(t)/dt = V_n(y(t))$ of $y_n(t)$ as increasing proportionally to the difference $y_{n-f_k} - y_n$ by some factor $\lambda_k + \gamma_k \langle \mathbb{I}_k, y\rangle$ which measures the speed at which nodes in state $n - f_k$ turn to state n (due to an arrival of some type k customer). In this factor, $\gamma_k \langle \mathbb{I}_k, y\rangle$ is added to the external arrival rate λ_k of class k customers at any node, and hence appears as the internal arrival rate of class k customers at any node. This feature characterizes the mean field property. Indeed, $\langle \mathbb{I}_k, y\rangle$ is the mean number of class k customers per node when the empirical distribution of the N nodes is y; so that $\gamma_k \langle \mathbb{I}_k, y\rangle$ is the mean emission rate per node of class k customers to the rest of the network.

Proof. The martingale characterization of the Markov jump process $(Y_n^N(t))$, see Rogers and Williams [19], shows that

$$M_n^N(t) = Y_n^N(t) - Y_n^N(0) - \int_0^t \sum_{w \in \mathcal{P}(\mathcal{X}) \setminus \{Y_n^N(s)\}} \Omega_N\left(Y_n^N(s), w\right) \left(w - Y_n^N(s)\right) ds$$

is a martingale with respect to the natural filtration associated to the Poisson processes involved in the arrival processes, service times and residence times. By using the explicit expression of the Q-matrix Ω_N, trite (and careful) calculations finally show that the following relation holds

$$
\begin{aligned}
(4)\quad & Y_n^N(t) = Y_n^N(0) + M_n^N(t)/N \\
&+ \int_0^t \sum_{k=1}^K \left(\lambda_k + \frac{\gamma_k N}{N-1}\sum_{m\in\mathcal{X}} m_k Y_m^N(s)\right)\left(Y_{n-f_k}^N(s)\mathbb{1}_{\{n_k\geq 1\}} - Y_n^N(s)\mathbb{1}_{\{n+f_k\in\mathcal{X}\}}\right)\,ds \\
&\qquad + \int_0^t \sum_{k=1}^K (\gamma_k+\mu_k)\left((n_k+1)Y_{n+f_k}^N(s)\mathbb{1}_{\{n+f_k\in\mathcal{X}\}} - n_k Y_n^N(s)\right)\,ds \\
&\qquad + \int_0^t \sum_{k=1}^K \frac{\gamma_k}{N-1}\left(n_k Y_n^N(s) - (n_k-1)Y_{n-f_k}^N(s)\mathbb{1}_{\{n_k\geq 1\}}\right)\,ds.
\end{aligned}
$$

From there, with a similar method as in Darling and Norris [6], it is not difficult to prove that if $Y^N(0)$ converges to z then

— the sequence $(Y^N(t))$ of process is tight for the Skorohod topology;
— any limit $(y(t))$ of $(Y^N(t))$ is continuous and satisfies the following deterministic differential equation, $y(0) = z$ and

$$
\begin{aligned}
y_n'(t) = \sum_{k=1}^K \left[\lambda_k + \gamma_k \sum_{m\in\mathcal{X}} m_k y_m(t)\right]\left[y_{n-f_k}(t)\mathbb{1}_{\{n_k\geq 1\}} - y_n(t)\mathbb{1}_{\{n+f_k\in\mathcal{X}\}}\right] \\
+ \sum_{k=1}^K (\gamma_k+\mu_k)\left[(n_k+1)y_{n+f_k}(t)\mathbb{1}_{\{n+f_k\in\mathcal{X}\}} - n_k y_n(t)\right].
\end{aligned}
$$

This is exactly Equation (3). The uniqueness of the solution of this differential equation implies that such a limiting point $(y(t))$ is necessarily unique and therefore that $(Y^N(t))$ converges in distribution to $(y(t))$. The proposition is proved. □

The equilibrium points of the dynamical system defined by Equation (3) are the probability distributions $y \in \mathcal{P}(\mathcal{X})$ on $\mathcal{X}$ such that $V(y)$ is zero. This condition can be written as follows: For $n \in \mathcal{X}$,

$$
\begin{aligned}
(5)\quad & \left(\sum_{k=1}^K \left[(\lambda_k + \gamma_k\langle \mathbb{I}_k, y\rangle)\,\mathbb{1}_{\{n+f_k\in\mathcal{X}\}} + (\gamma_k+\mu_k)n_k\right]\right) y_n \\
&= \sum_{k=1}^K (\lambda_k + \gamma_k\langle \mathbb{I}_k, y\rangle)\, y_{n-f_k}\mathbb{1}_{\{n_k\geq 1\}} + (\gamma_k+\mu_k)(n_k+1)y_{n+f_k}\mathbb{1}_{\{n+f_k\in\mathcal{X}\}}.
\end{aligned}
$$

These equations are equivalent to local balance equations for the numbers of customers of a classical $M/M/C/C$ queue with K classes of customers such that, for $1 \leq k \leq K$, class k customers

— arrive at rate $\lambda_k + \gamma_k\langle \mathbb{I}_k, y\rangle$;
— are served at rate $\gamma_k + \mu_k$;
— require capacity A_k.

Consequently, (y_n) is the invariant distribution of this queue. It is well known, see Kelly [13] for example, that necessarily

$$y_n = \nu_\rho(n) \stackrel{\text{def.}}{=} \frac{1}{Z(\rho)} \prod_{k=1}^{K} \frac{\rho_k^{n_k}}{n_k!}, \qquad n \in \mathcal{X}, \tag{6}$$

where, for $1 \le k \le K$, ρ_k is the ratio of the kth arrival and service rates,

$$\rho_k = \frac{\lambda_k + \gamma_k \langle \mathbb{I}_k, \nu_\rho \rangle}{\gamma_k + \mu_k}, \tag{7}$$

where $\langle \mathbb{I}_k, \nu_\rho \rangle$ is the average value of the kth component under the probability distribution ν_ρ on $\mathcal{X}$ and $Z(\rho)$ is the normalization constant, in other words the partition function,

$$Z(\rho) = \sum_{n \in \mathcal{X}} \prod_{k=1}^{K} \frac{\rho_k^{n_k}}{n_k!}.$$

The equilibrium points of the dynamical system $(y(t))$ are indexed by $\mathbb{R}_+^K$ whose dimension is much smaller than $\mathcal{P}(\mathcal{X})$ thereby suggesting a possible simpler description of the asymptotic behavior of the network. Despite this quite appealing perspective, it turns out that such a dimension reduction cannot be achieved directly since the subset $\{\nu_\rho,\ \rho{\in}\mathbb{R}_+^K\}$ of $\mathcal{P}(\mathcal{X})$ is not left invariant by the dynamical system $(y(t))$.

Denote by $B_k(\rho)$ the blocking probability of a class k customer in an $M/M/C/C$ queue at equilibrium with K classes and loads $\rho_1, \ldots, \rho_K$, i.e.

$$B_k(\rho) = \frac{1}{Z(\rho)} \sum_{n: n+f_k \notin \mathcal{X}} \prod_{h=1}^{K} \frac{\rho_h^{n_h}}{n_h!},$$

it is easily checked that $\langle \mathbb{I}_k, \nu_\rho \rangle = \rho_k(1 - B_k(\rho))$, so Equation (7) becomes then

$$\mu_k \rho_k = \lambda_k - \gamma_k \rho_k B_k(\rho).$$

The following proposition summarizes these results.

Proposition 1. *The equilibrium points of the dynamical system $(y(t))$ defined by Equation (2) are exactly the probability distributions ν_ρ on $\mathcal{X}$,*

$$\nu_\rho(n) = \prod_{\ell=1}^{K} \frac{\rho_\ell^{n_\ell}}{n_\ell!} \Bigg/ \sum_{m \in \mathcal{X}} \prod_{\ell=1}^{K} \frac{\rho_\ell^{m_\ell}}{m_\ell!}, \quad n = (n_\ell) \in \mathcal{X}, \tag{8}$$

where $\rho = (\rho_k, 1 \le k \le K)$ is a vector of $\mathbb{R}_+^K$ satisfying the system of equations

$$\lambda_k = \rho_k \left(\mu_k + \gamma_k \sum_{n: n+f_k \notin \mathcal{X}} \prod_{\ell=1}^{K} \frac{\rho_\ell^{n_\ell}}{n_\ell!} \Bigg/ \sum_{n \in \mathcal{X}} \prod_{\ell=1}^{K} \frac{\rho_\ell^{n_\ell}}{n_\ell!} \right), \quad 1 \le k \le K. \tag{9}$$

There always exists at least one equilibrium point.

Proof. Only the existence result has to be proved. According to Equations (6) and (7), y is an equilibrium point if and only if it is a fixed point of the function

$$\begin{aligned} \mathcal{P}(\mathcal{X}) &\longrightarrow \mathcal{P}(\mathcal{X}) \\ y &\longrightarrow \nu_{\rho(y)}, \end{aligned}$$

with $\rho(y) = (\rho_k(y))$ and, for $1 \le k \le K$, $\rho_k(y) = (\lambda_k + \gamma_k\langle \mathbb{I}_k, y\rangle)/(\mu_k + \gamma_k)$. This functional being continuous on the convex compact set $\mathcal{P}(\mathcal{X})$, it necessarily has a fixed point by Brouwer's Theorem. □

Notation. In the following, one will denote

$$\frac{\rho^n}{n!} = \prod_{k=1}^{K} \frac{\rho_k^{n_k}}{n_k!},$$

for $n = (n_k) \in \mathcal{X}$ and $\rho = (\rho_k) \in \mathbb{R}_+^K$. The system of equations (9) can then be rewritten as,

$$\lambda_k = \rho_k \left(\mu_k + \gamma_k \frac{\sum_{n:n+f_k \notin \mathcal{X}} \rho^n/n!}{\sum_{n\in\mathcal{X}} \rho^n/n!} \right), \quad 1 \le k \le K.$$

4. Uniqueness Results

In this section, several situations in which the asymptotic dynamical system (2) has a unique equilibrium point, i.e. when the fixed point Equations (9) have a unique solution, are presented.

4.1. **Networks with Constant Requirements.** It is assumed that all classes of customers require the same capacity, i.e. that $A_k = A$ for $k = 1, \dots, K$. By replacing C by $\lfloor C/A \rfloor$, it can be assumed that $A = 1$. In this case, if $|n|$ denotes the sum of the coordinates of $n \in \mathcal{X}$,

$$\begin{aligned} B_k(\rho) &= \sum_{n:n+f_k\notin\mathcal{X}} \frac{\rho^n}{n!} \Big/ \sum_{n\in\mathcal{X}} \frac{\rho^n}{n!} = \sum_{|n|=C} \frac{\rho^n}{n!} \Big/ \sum_{n\in\mathcal{X}} \frac{\rho^n}{n!} \\ &= \frac{1}{C!}\left(\sum_{k=1}^{K} \rho_k\right)^C \Big/ \sum_{\ell=0}^{C} \frac{1}{\ell\,!}\left(\sum_{k=1}^{K}\rho_k\right)^\ell \stackrel{\text{def.}}{=} B_1\left(\sum_k \rho_k\right), \end{aligned}$$

$B_1(\theta)$ can be represented as the stationary blocking probability of an $M/M/C/C$ queue with one class of customers and arrival rate θ and service rate 1.

In this case, fixed point Equations (9) are

$$\rho_k = \lambda_k / (\mu_k + \gamma_k B_1(S)), \qquad k = 1, \dots, K, \tag{10}$$

with $S{=}\rho_1 + \cdots + \rho_K$. By summing up these equations, one gets that S is the solution of the equation

$$S = \sum_{k=1}^{K} \frac{\lambda_k}{\mu_k + \gamma_k B_1(S)}.$$

It is easily checked that $S \to B_1(S)$ is non-decreasing and therefore that the above equation has a unique solution. The uniqueness of the vector (ρ_k) follows from Equations (10). The following proposition has been proved.

Proposition 2. *The asymptotic dynamical system $(y(t))$ has a unique equilibrium point when capacity requirements are equal.*

In particular, when there is only one class of customers, there is a unique solution to Equations (9).

4.2. **A Limiting Regime of Fixed Point Equations.** Here, the fixed point equations (9) are analyzed under a heavy traffic regime, i.e. when the capacity C goes to infinity and the arrival rates are proportional to C, of the order of $\lambda_k C$ for the kth class. It will be shown that, in this case, there is a unique equilibrium point. Let $(\mathcal{N}_k,\, 1\leq k\leq K)$ be a sequence of K independent Poisson processes with intensity 1. As usual $\mathcal{N}_k(A)$ will denote the number of points of the kth process in the subset A of $\mathbb{R}_+$. For $C>0$, denote by $\rho_C=(\rho_k(C),\, 1\leq k\leq K)$, a solution of the fixed point equations

$$\lambda_k C=\rho_k(C)\left(\mu_k+\gamma_k\sum_{n:n+f_k\notin\mathcal{X}}\frac{\rho_C^n}{n!}\Bigg/\sum_{n\in\mathcal{X}}\frac{\rho_C^n}{n!}\right),\quad 1\leq k\leq K,$$

this can be rewritten as

$$\lambda_k=\frac{\rho_k(C)}{C}\left(\mu_k+\gamma_k-\gamma_k\frac{\mathbb{P}\left(C-\sum_{i=1}^K A_i\mathcal{N}_i([0,\rho_i(C)])\geq A_k\right)}{\mathbb{P}\left(C-\sum_{i=1}^K A_i\mathcal{N}_i([0,\rho_i(C)])\geq 0\right)}\right). \tag{11}$$

This problem is related to the limit of the loss probabilities investigated and solved by Kelly [14] in a general setting in terms of an optimization problem. The proposition below gives an alternative probabilistic proof of this result in the case of a single server queue. It uses a change of probability similar to the one used in the proof of Bahadur-Rao's Theorem. See Dembo and Zeitouni [7].

Proposition 3 (Heavy Traffic)**.** *If $(\rho_k(C), 1\leq k\leq K)\in\mathbb{R}_+^K$ is such that*

$$\lim_{C\to+\infty}\rho_k(C)/C=\overline{\rho}_k,\qquad 1\leq k\leq K,$$

with $(\overline{\rho}_k)\in\mathbb{R}_+^K$ and

$$\overline{\rho}_1A_1+\overline{\rho}_2A_2+\cdots+\overline{\rho}_KA_K\geq 1, \tag{12}$$

then, for $a\in\mathbb{N}$,

$$\lim_{C\to+\infty}\frac{\mathbb{P}\left(C-\sum_{k=1}^K A_k\mathcal{N}_k([0,\rho_k(C)])\geq a\right)}{\mathbb{P}\left(C-\sum_{k=1}^K A_k\mathcal{N}_k([0,\rho_k(C)])\geq 0\right)}=e^{-\omega a}$$

where ω is the unique non-negative solution of the equation

$$\overline{\rho}_1A_1e^{-\omega A_1}+\overline{\rho}_2A_2e^{-\omega A_2}+\cdots+\overline{\rho}_KA_Ke^{-\omega A_K}=1. \tag{13}$$

Proof. It is first assumed that, for $1\leq k\leq K$, $\rho_k(C)=\overline{\rho}_kC$. Let $(Z(t))$ the process with independent increments defined by

$$Z(t)=t-\sum_1^K A_k\mathcal{N}_k([0,t\overline{\rho}_k]),\quad t\geq 0.$$

For $\theta\geq 0$, its exponential moment function is given by

$$\Phi(\theta,t)=\mathbb{E}\left(\exp\left[\theta Z(t)\right]\right)=\exp\left[t\left(\theta-\sum_1^K\overline{\rho}_k\left(1-e^{-\theta A_k}\right)\right)\right].$$

Since

$$\frac{\partial\Phi}{\partial\theta}(\theta,1)=\left(1-\sum_1^K\overline{\rho}_kA_ke^{-\theta A_k}\right)\exp\left[\theta-\sum_1^K\overline{\rho}_k\left(1-e^{-\theta A_k}\right)\right],$$

Condition (12) implies that there exists a unique $\omega > 0$ such that $\dfrac{\partial \Phi}{\partial \theta}(\omega, 1) = 0$ or, equivalently, $\overline{\rho}_1 A_1 e^{-\omega A_1} + \overline{\rho}_2 A_2 e^{-\omega A_2} + \cdots + \overline{\rho}_K A_K e^{-\omega A_K} = 1$.

A change of probability measure. For $t \geq 0$, the σ-algebra generated by the random variables $\mathcal{N}_k([0, s\overline{\rho}_k])$, $1 \leq k \leq K$ and $s \leq t$ is denoted by $\mathcal{F}_t$. By using the fact that (M_t)= $(\exp\left[\omega Z(t)\right]/\Phi(\omega, t))$ is a non-negative martingale whose expected value is 1, there exists a unique probability distribution $\widetilde{\mathbb{P}}$ such that $d\widetilde{\mathbb{P}} = M_t \, d\mathbb{P}$ on the σ-algebra $\mathcal{F}_t$. See Rogers and Williams [20] for example. It is easily checked that, under the probability $\widetilde{\mathbb{P}}$, the process $(Z(t))$ has the same distribution as

$$\left(t - \sum_{k=1}^{K} A_k \mathcal{N}_k([0, t\overline{\rho}_k e^{-\omega A_k}])\right),$$

in particular it has independent increments and, due to Equation (13), $\widetilde{\mathbb{E}}(Z(t)) = 0$. The centered renormalized sum is defined by

$$\widehat{S}_C = \frac{1}{\sigma\sqrt{C}} \left(C - \sum_{i=1}^{K} A_i \mathcal{N}_i([0, C\overline{\rho}_i])\right),$$

with σ^2 is the variance of $1 - A_1\mathcal{N}_1([0, \overline{\rho}_1]) - \cdots - A_1\mathcal{N}_K([0, \overline{\rho}_K])$ under $\widetilde{\mathbb{P}}$. For a fixed $a \in \mathbb{N}$, let

$$\begin{aligned}
\Delta_C(a) &\stackrel{\text{def.}}{=} \frac{1}{\Phi(\omega, C)} \mathbb{P}\left(C - \sum_{i=1}^{K} A_i \mathcal{N}_i([0, C\overline{\rho}_i]) \geq a\right) = \widetilde{\mathbb{E}}\left(e^{-\omega\sigma\sqrt{C}\hat{S}_C} \mathbb{1}_{\{\sigma\sqrt{C}\hat{S}_C \geq a\}}\right) \\
&= \widetilde{\mathbb{E}}\left(\int_{\sigma\sqrt{C}\hat{S}_C}^{+\infty} \omega e^{-\omega u} \, du \, \mathbb{1}_{\{\sigma\sqrt{C}\hat{S}_C \geq a\}}\right) \\
&= \int_a^{+\infty} \widetilde{\mathbb{P}}\left(\frac{u}{\sigma\sqrt{C}} \geq \hat{S}_C \geq \frac{a}{\sigma\sqrt{C}}\right) \omega e^{-\omega u} \, du.
\end{aligned}$$

From the expansions related to the central limit theorem, see Gnedenko and Kolmogorov [12] Theorem 1, page 213 (see also Feller [9] page 540), one gets

$$\begin{aligned}
\widetilde{\mathbb{P}}\left(\hat{S}_C \leq x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-v^2/2} \, dv \\
+ \frac{1}{\sigma\sqrt{2\pi C}} e^{-x^2/2} \left(Q(x) + S\left(x\sigma\sqrt{C}\right)\right) + o\left(\frac{1}{\sqrt{C}}\right),
\end{aligned}$$

uniformly in $x \in \mathbb{R}$, where $Q(x) = \alpha(1 - x^2)$ for some constant $\alpha > 0$ and S is the periodic function, $S(x) = \lceil x \rceil - x + 1/2$. Therefore,

$$\Delta_C(a) = \Delta_C^1(a) + \Delta_C^2(a) + \Delta_C^3(a) + o\left(1/\sqrt{C}\right),$$

with

$$\begin{aligned}
\Delta_C^1(a) &= \frac{1}{\sqrt{2\pi}} \int_a^{+\infty} \left(\int_{a/\left(\sigma\sqrt{C}\right)}^{u/\left(\sigma\sqrt{C}\right)} e^{-v^2/2} \, dv\right) \omega e^{-\omega u} \, du \\
&= \frac{1}{\sigma\sqrt{2\pi C}} \int_a^{+\infty} \int_a^{u} e^{-v^2/\left(2\sigma^2 C\right)} \, dv \, \omega e^{-\omega u} \, du = \frac{1}{\omega\sigma\sqrt{2\pi C}} e^{-\omega a} + o\left(\frac{1}{\sqrt{C}}\right),
\end{aligned}$$

and

$$\Delta_C^2(a) = \frac{1}{\sigma\sqrt{2\pi C}} \int_a^{+\infty} \left[Q\left(\frac{u}{\sigma\sqrt{C}}\right) e^{-u^2/(2\sigma^2 C)} - Q\left(\frac{a}{\sigma\sqrt{C}}\right) e^{-a^2/(2\sigma^2 C)} \right] \omega e^{-\omega u}\, du = o\left(\frac{1}{\sqrt{C}}\right).$$

Finally,

$$\begin{aligned}
\Delta_C^3(a) &= \frac{1}{\sigma\sqrt{2\pi C}} \int_a^{+\infty} \left(S\left(u\right) e^{-u^2/(2\sigma^2 C)} - S\left(a\right) e^{-a^2/(2\sigma^2 C)} \right) \omega e^{-\omega u}\, du \\
&= \frac{1}{\sigma\sqrt{2\pi C}} \int_a^{+\infty} \left(S\left(u\right) - S\left(a\right)\right) \omega e^{-\omega u}\, du + o\left(\frac{1}{\sqrt{C}}\right) \\
&= \frac{e^{-\omega a}}{\sigma\sqrt{2\pi C}} \int_0^{+\infty} \left(S\left(u\right) - S\left(0\right)\right) \omega e^{-\omega u}\, du + o\left(\frac{1}{\sqrt{C}}\right),
\end{aligned}$$

by periodicity of S. These estimations give the relation

$$\lim_{C\to+\infty} \frac{\Delta_C(a)}{\Delta_C(0)} = e^{-\omega a}.$$

which is precisely the proposition in this case.

The general case with $\rho_k(C)$ instead of $C\overline{\rho}$ is treated similarly by using the generalization, see Feller [9] page 546, of the expansions of the central limit theorem to the case of independent but non-identically distributed random variables. □

The main result of this section can now be stated. Basically, it states that, under a heavy traffic limit, the fixed point equations (9) have a unique solution when the capacity gets large.

Theorem 2. *If $\mu_k > 0$ for all $1 \leq k \leq K$ and if for any $C > 0$ the vector $(\rho_k(C, \lambda C))$ is* any *solution of Equation (9) then, for $1 \leq k \leq K$,*

$$\lim_{C\to+\infty} \frac{\rho_k(C,\lambda C)}{C} = \frac{\lambda_k}{\mu_k + \gamma_k - \gamma_k e^{-\omega A_k}},$$

where $\omega \geq 0$ is defined as

$$\omega = \inf\left\{ x \geq 0 : \sum_{k=1}^{K} \frac{\lambda_k A_k e^{-x A_k}}{\mu_k + \gamma_k - \gamma_k e^{-x A_k}} \leq 1 \right\}.$$

Proof. For $1 \leq k \leq K$ the function $C \to \overline{\rho}_k(C) = \rho_k(C, \lambda C)/C$ is bounded by λ_k/μ_k. By taking a subsequence, it can be assumed that $\overline{\rho}_k(C)$ converges to some finite $\overline{\rho}_k$ as C goes to infinity. Under the condition

$$A_1 \frac{\lambda_1}{\mu_1} + A_2 \frac{\lambda_2}{\mu_2} + \cdots + A_K \frac{\lambda_K}{\mu_K} \geq 1,$$

then necessarily $\overline{\rho}_1 A_1 + \overline{\rho}_2 A_2 + \cdots + \overline{\rho}_K A_K \geq 1$, otherwise one would have, via the law of large numbers for Poisson processes, for $a \geq 0$,

$$\lim_{C\to+\infty} \mathbb{P}\left(C - \sum_{i=1}^{K} A_i \mathcal{N}_i([0, C\overline{\rho}_i(C)]) \geq a \right) = 1, \tag{14}$$

and Equation (11) would then give the relation $\overline{\rho}_k = \lambda_k/\mu_k$ for $1 \leq k \leq K$, so that

$$\overline{\rho}_1 A_1 + \overline{\rho}_2 A_2 + \cdots + \overline{\rho}_K A_K \geq 1,$$

contradiction. From Proposition 3 and Equation (11), one gets that

$$\lambda_k = \overline{\rho}_k \left(\mu_k + \gamma_k - \gamma_k e^{-\omega A_k}\right), \quad 1 \leq k \leq K,$$

were ω is the solution of Equation (13) associated to $(\overline{\rho}_k)$. Equation (13) can then be rewritten as

$$\sum_1^K \frac{\lambda_k A_k e^{-\omega A_k}}{\mu_k + \gamma_k - \gamma_k e^{-\omega A_k}} = 1.$$

The statement of the theorem is proved in this case.

Now, if it is assumed that $A_1\lambda_1/\mu_1 + A_2\lambda_2/\mu_2 + \cdots + A_K\lambda_K/\mu_K < 1$ then, since $\overline{\rho}_k \leq \lambda_k/\mu_k$ for all k, Relation (14) holds and Equation (11) finally gives that $\overline{\rho}_k = \lambda_k/\mu_k$, $1 \leq k \leq K$. The theorem is proved. $\square$

5. An Energy Function on $\mathcal{P}(\mathcal{X})$

In this section, a Lyapunov function is introduced. As it will be seen, it plays a key role in the analysis of the fixed points of the asymptotic dynamical system. Define the function g on the set $\mathcal{P}(\mathcal{X})$ of probability distributions on $\mathcal{X}$,

$$g(y) = \sum_{n\in\mathcal{X}} y_n \log(n! y_n) - \sum_{k=1}^K \int_0^{\langle \mathbb{I}_k, y\rangle} \log \frac{\lambda_k + \gamma_k x}{\mu_k + \gamma_k}\, dx, \quad y \in \mathcal{P}(\mathcal{X}). \tag{15}$$

Recall that, for $y \in \mathcal{P}(\mathcal{X})$, $\langle \mathbb{I}_k, y\rangle = \sum_{m\in\mathcal{X}} m_k y_m$.

Proposition 4. *The function g is* a Lyapunov function *for the asymptotic dynamical system $(y(t))$, that is,*

$$\langle V(y), \nabla g(y)\rangle = \sum_{n\in\mathcal{X}} V_n(y) \frac{\partial g}{\partial y_n}(y) \leq 0, \qquad \forall y \in \mathring{\mathcal{P}}(\mathcal{X}),$$

and, for $y \in \mathring{\mathcal{P}}(\mathcal{X})$, the following assertions are equivalent:

(a) $\langle V(y), \nabla g(y)\rangle = 0$;

(b) *The linear functional $h \to \langle \nabla g(y), h\rangle$ is constant on $\mathcal{P}(\mathcal{X})$;*

(c) *y is an equilibrium point of $(y(t))$, i.e. $V(y) = 0$.*

Proof. The vector field $V(y) = (V_n(y))$ can be written as

$$\begin{aligned} V_n(y) = \sum_{k=1}^K \Big[&\Big(\lambda_k + \gamma_k \langle \mathbb{I}_k, y\rangle\Big) y_{n-f_k} \mathbb{1}_{\{n_k\geq 1\}} + (\mu_k+\gamma_k)(n_k+1) y_{n+f_k} \mathbb{1}_{\{n+f_k\in\mathcal{X}\}} \\ &- \Big((\lambda_k + \gamma_k\langle \mathbb{I}_k, y\rangle)\, \mathbb{1}_{\{n+f_k\in\mathcal{X}\}} + (\mu_k+\gamma_k) n_k\Big)\, y_n \Big] \\ = \sum_{k=1}^K &\left(F^k_{n+f_k}(y) - F^k_n(y)\right) \end{aligned}$$

where, for $n \in \mathcal{X}$, $F^k_n(y) = (\mu_k + \gamma_k) n_k y_n - \big(\lambda_k + \gamma_k\langle \mathbb{I}_k, y\rangle\big) y_{n-f_k} \mathbb{1}_{\{n_k\geq 1\}}$ and $F^k_n(y) = 0$ when $n \notin \mathcal{X}$; note that $F^k_n = 0$ whenever $n_k = 0$.

For $y \in \overset{\circ}{\mathcal{P}}(\mathcal{X})$

$$\begin{aligned}\langle V(y), \nabla g(y)\rangle &= \sum_{n\in\mathcal{X}}\sum_{k=1}^{K} \frac{\partial g}{\partial y_n}(y)\left(F^k_{n+f_k}(y) - F^k_n(y)\right)\\ &= \sum_{k=1}^{K}\sum_{n\in\mathcal{X}} F^k_n(y)\left(\frac{\partial g}{\partial y_{n-f_k}}(y) - \frac{\partial g}{\partial y_n}(y)\right).\end{aligned}$$

Since, for $n \in \mathcal{X}$,

$$\frac{\partial g}{\partial y_n}(y) = 1 + \log(n! y_n) - \sum_{k=1}^{K} n_k \log\frac{\lambda_k + \gamma_k\langle \mathbb{I}_k, y\rangle}{\mu_k + \gamma_k},$$

one finally gets that the relation

$$\langle V(y), \nabla g(y)\rangle = \sum_{k=1}^{K}\sum_{n\in\mathcal{X}} F^k_n(y)\log\frac{\Big(\lambda_k + \gamma_k\langle \mathbb{I}_k, y\rangle\Big) y_{n-f_k}}{(\mu_k+\gamma_k) n_k y_n} \tag{16}$$

holds. The quantity $\langle V(y), \nabla g(y)\rangle$ is thus clearly non-positive. On the other hand, for k and n such that $n_k \geq 1$,

$$\frac{\partial g}{\partial y_n}(y) - \frac{\partial g}{\partial y_{n-f_k}}(y) = \log\left(\frac{\Big(\mu_k+\gamma_k\Big) n_k y_n}{\Big(\lambda_k + \gamma_k\langle \mathbb{I}_k, y\rangle\Big) y_{n-f_k}}\right),$$

hence $\langle V(y), \nabla g(y)\rangle$ is zero if and only if the coordinates of $\nabla g(y)$ are equal and this is equivalent to the system of equations

$$\Big(\lambda_k + \gamma_k\langle \mathbb{I}_k, y\rangle\Big) y_{n-f_k} = (\mu_k+\gamma_k) n_k y_n$$

for all k and n such that $n_k \geq 1$, so that y is an equilibrium point of the asymptotic dynamical system. The equivalence a), b) and c) is proved. □

Convergence of the Stationary Distribution. If F is some real valued function on $\mathbb{R}^{\mathcal{X}}$ and $y \in \mathcal{P}(\mathcal{X})$, the functional operator associated to the Q-matrix Ω_N is given by

$$\begin{aligned}\Omega_N(F)(y) &= \sum_{z\in\mathcal{P}(\mathcal{X})\setminus\{y\}} \Omega_N(y,z)(F(z) - F(y))\\ &= \sum_{n\in\mathcal{X}}\left[\sum_{k=1}^{K}\lambda_k y_n N \mathbb{1}_{\{n+f_k\in\mathcal{X}\}}\left(F\left(y + \frac{1}{N}(e_{n+f_k} - e_n)\right) - F(y)\right)\right.\\ &\quad + \sum_{k=1}^{K}\mu_k n_k y_n N\left(F\left(y + \frac{1}{N}(e_{n-f_k} - e_n)\right) - F(y)\right)\\ &\quad + \sum_{\substack{1\leq k\leq K\\ m\in\mathcal{X}}} \frac{\gamma_k N}{N-1} n_k y_n\left(N y_m - \mathbb{1}_{\{m=n\}}\right)\\ &\qquad\left.\times\left(F\left(y + \frac{e_{n-f_k} - e_n}{N} + \frac{e_{m+f_k} - e_m}{N}\mathbb{1}_{\{m+f_k\in\mathcal{X}\}}\right) - F(y)\right)\right].\end{aligned}$$

If it is assumed that the function F is of class C^2 on $\mathbb{R}^K$, then it is easy to check that the sequence $(\Omega_N(F)(y))$ converges to the following expression

$$\sum_{n\in\mathcal{X}}\left[\sum_{k=1}^{K}\lambda_k y_n \mathbb{1}_{\{n+f_k\in\mathcal{X}\}}\left\langle \nabla F(y), e_{n+f_k}-e_n\right\rangle + \sum_{k=1}^{K}\mu_k n_k y_n \left\langle \nabla F(y), e_{n-f_k}-e_n\right\rangle\right.$$

$$\left. + \sum_{1\le k\le K, m\in\mathcal{X}} \gamma_k n_k y_n y_m \left(\left\langle \nabla F(y), e_{n-f_k}-e_n\right\rangle + \left\langle \nabla F(y), e_{m+f_k}-e_m\right\rangle \mathbb{1}_{\{m+f_k\in\mathcal{X}\}}\right)\right]$$

which is defined as $\Omega_\infty(F)(y)$. Moreover, this convergence is *uniform with respect to $y\in\mathcal{P}(\mathcal{X})$* by using Taylor's Formula at the second order. By Theorem 1, one necessarily has

$$\Omega_\infty(F)(y) = \langle \nabla F(y), V(y)\rangle, \quad y\in\mathcal{P}(\mathcal{X}).$$

Note that this identity can also be checked directly with the above equation.

Proposition 5. *If π_N denotes the invariant probability distribution of $(Y^N(t))$ on $\mathcal{P}(\mathcal{X})$, then any limiting point of (π_N) is a probability distribution carried by the equilibrium points of the asymptotic dynamical system $(y(t))$ of Theorem 1.*

In particular, if $(y(t))$ has a unique equilibrium point y_∞, then the sequence of invariant distributions (π_N) converges to the Dirac mass at y_∞.

Proof. The set $\mathcal{P}(\mathcal{X})$ being compact, the sequence of distributions (π_N) is relatively compact. Let $\widetilde{\pi}$ be the limit of some subsequence (π_{N_p}). If F is a function of class C^2 on $\mathbb{R}^{\mathcal{X}}$, then for $p\ge 0$,

$$\int_{\mathcal{P}(\mathcal{X})}\Omega_{N_p}(F)(y)\,\pi_{N_p}(dy) = 0.$$

The uniform convergence of $\Omega_{N_p}(F)$ to $\Omega_\infty(F)$ implies that

$$0 = \int_{\mathcal{P}(\mathcal{X})}\Omega_\infty(F)(y)\,\widetilde{\pi}(dy) = \int_{\mathcal{P}(\mathcal{X})}\langle \nabla F(y), V(y)\rangle\,\widetilde{\pi}(dy),$$

so that $\widetilde{\pi}$ is an invariant distribution of the (deterministic) Markov process associated to the infinitesimal generator Ω_∞.

For $t\ge 0$, denote (temporarily) by $(y(x,t))$ the dynamical system starting from $y(0)=x\in\mathcal{P}(\mathcal{X})$. Assume that there exist $x\in\mathcal{P}(\mathcal{X})$ and $s>0$ such that $y(x,s)\in\partial\mathcal{P}(\mathcal{X})$, i.e. there exists $n\in\mathcal{X}$ such that $y_n(x,s)=0$. Since $(y_n(x,t))$ is non-negative and since the function $t\to y(x,t)$ is of class C^1, it implies that $V_n(y(x,s)) = \dot{y}_n(x,s) = 0$. This last relation, Relation (3) defining the vector field $(V_n(y))$ and the equation $y_n(x,s)=0$ give that $y_{n\pm f_k}(x,s)=0$ for any k such that $n\pm f_k\in\mathcal{X}$ and consequently, by repeating the argument, all the coordinates of $y(x,s)$ are null. Contradiction since $y(x,s)$ is a probability distribution on $\mathcal{X}$. Hence, the boundary $\partial\mathcal{P}(\mathcal{X})$ of $\mathcal{P}(\mathcal{X})$ cannot be reached in positive time by $(y(x,t))$. This entails, in particular, that $\partial\mathcal{P}(\mathcal{X})$ is negligible for any invariant distribution of $(y(x,t))$.

For $x\in\mathcal{P}(\mathcal{X})$ and $0<s'<s$, since the function $g(y(x,\cdot))$ is of class C^1 on $[s',s]$ and its derivative is $\langle\nabla(g)(y(x,\cdot)), V(y(x,\cdot))\rangle$, one has

$$g(y(x,s)) - g(y(x,s')) = \int_{s'}^{s}\Omega_\infty(g)(y(x,u))\,du, \tag{17}$$

By integrating with respect to $\widetilde{\pi}$ this relation, the invariance of $\widetilde{\pi}$ for the process $(y(x,t))$ and Fubini's Theorem show that

$$\int_{\mathcal{P}(\mathcal{X})} g(y(x,s))\,\widetilde{\pi}(dx) - \int_{\mathcal{P}(\mathcal{X})} g(y(x,s'))\,\widetilde{\pi}(dx) = 0$$

$$= \int_{\mathcal{P}(\mathcal{X})} \int_{s'}^{s} \Omega_\infty(g)(y(x,u))\,du\,\widetilde{\pi}(dx) = (s-s') \int_{\mathcal{P}(\mathcal{X})} \Omega_\infty(g)(x)\,\widetilde{\pi}(dx).$$

The integrand having a constant sign by Proposition 4, one deduces that $\widetilde{\pi}$-almost surely, $\Omega_\infty(g)(x) = \langle \nabla g(x), V(x)\rangle = 0$. The probability $\widetilde{\pi}$ is thus carried by the equilibrium points of the dynamical system. The proposition is proved. □

Asymptotic Independence. In the case where $(y(t))$ has a unique equilibrium point y_∞, by using the convergence of the invariant distributions π_N to the Dirac distribution δ_{y_∞} and the fact that the coordinates of $(X_i^N(t))$ are exchangeable, it is easy (and quite classical) to show that for any subset I of coordinates, the random variables $(X_i^N(t), i \in I)$ at equilibrium are asymptotically independent with y_∞ as a common limiting distribution. To summarize, the uniqueness of an equilibrium point implies that, asymptotically, the invariant distribution of the Markov process $(X_i^N(t))$ has a product form.

6. A Dimension Reduction on $\mathbb{R}^K$

In this section, a function ϕ on $\mathbb{R}_+^K$ is introduced such that $\rho \in \mathbb{R}_+^K$ is a zero of $\nabla\phi$ if and only if the corresponding probability distribution ν_ρ is an equilibrium point of $(y(t))$. Furthermore, it is shown that ρ is a local minimum of ϕ if and only if ν_ρ is a local minimum of g on the set of probability distributions on $\mathcal{X}$. In the next section, the function ϕ will be used to exhibit the metastable behavior of the network in some cases.

For $\rho = (\rho_k) \in \mathbb{R}_+^K$, define

$$\phi(\rho) = -\log Z(\rho) + \sum_{k=1}^{K} (\beta_k \rho_k - \alpha_k \log(\rho_k)) \tag{18}$$

with $\alpha_k = \lambda_k/\gamma_k$, $\beta_k = (\gamma_k + \mu_k)/\gamma_k$, and Z is the partition function

$$Z(\rho) = \sum_{n\in\mathcal{X}} \frac{\rho^n}{n!}.$$

Proposition 6. *The probability distribution ν_ρ on $\mathcal{X}$ is an equilibrium point of the asymptotic dynamical system $(y(t))$ if and only if $\nabla\phi(\rho) = 0$.*

Proof. Remark that, for $1 \le k \le K$,

$$\frac{\partial Z}{\partial \rho_k}(\rho) = \sum_{n: n+f_k \in \mathcal{X}} \frac{\rho^n}{n!},$$

so that

$$\frac{\partial \phi}{\partial \rho_k}(\rho) = \frac{\mu_k}{\gamma_k} - \frac{\lambda_k}{\rho_k \gamma_k} + \sum_{n: n+f_k \notin \mathcal{X}} \frac{\rho^n}{n!} \Bigg/ \sum_{n\in\mathcal{X}} \frac{\rho^n}{n!}\,,$$

hence this quantity is 0 if and only if the fixed point equation (9) holds. The proposition is proved. □

Local minima of ϕ and g. Proposition 1 has shown that an equilibrium point is necessarily a probability vector ν_ρ on $\mathcal{X}$ for some $\rho \in \mathbb{R}_+^K$. It has been shown that the function g defined in Section 5 decreases along any trajectory of the dynamical system $(y(t))$ by Proposition 4 so that if it starts in the neighborhood of a local minimum of g, ultimately it reaches this point. At the normal scale, i.e. for a finite network, it implies that, with an appropriate initial state, the state of the network $(X^N(t))$ will live for some (likely long) time in a subset of the states corresponding, up to a scaling, to this local minimum. For this reason, it is important to be able to distinguish stable from unstable equilibrium points of $(y(t))$. Due to the quite complicated expression defining g, it is not clear how the stability properties of the equilibrium points can be established directly with g. The function ϕ plays a key role in this respect, it reduces the complexity of the classification of the equilibrium points according to their stability properties.

Let $y \in \mathring{\mathcal{P}}(\mathcal{X})$, Taylor's formula for g gives the relation, for $y' \in \mathcal{P}(\mathcal{X})$,

$$g(y') = g(y) + \langle \nabla g(y), y' - y \rangle + {}^t(y' - y)\, \mathcal{H}_g^y\, (y' - y) + o(\|y' - y\|^2)$$

where $\mathcal{H}_g^y$ is the Hessian matrix of g,

$$\mathcal{H}_g^y = \left(\frac{\partial^2 g}{\partial y_m \partial y_n}(y), m, n \in \mathcal{X} \right),$$

and ${}^t z$ is the transpose of vector z.
Propositions 1, 4 and 6 give the equivalence between

— $y \in \mathcal{P}(\mathcal{X})$ is an equilibrium point;
— $y = \nu_\rho$ with $\nabla\phi(\rho) = 0$.
— $y \in \mathring{\mathcal{P}}(\mathcal{X})$ and $\langle \nabla g(y), y' - y \rangle = 0$, $\forall y' \in \mathcal{P}(\mathcal{X})$;

hence the relation

$$g(y') = g(\nu_\rho) + {}^t(y' - \nu_\rho)\, \mathcal{H}_g^{\nu_\rho}\, (y' - \nu_\rho) + o(\|y' - \nu_\rho\|^2).$$

holds. It is assumed throughout this section that the Hessian matrix has non-zero eigenvalues at ν_ρ such that $\nabla\phi(\rho) = 0$. Consequently, for ρ such that $\nabla\phi(\rho) = 0$, the probability vector ν_ρ is a local minimum of g, i.e. a stable equilibrium point of $(y(t))$ if and only if the quadratic form associated to $\mathcal{H}_g^{\nu_\rho}$ satisfies the following property

$$ {}^t h\, \mathcal{H}_g^{\nu_\rho}\, h \geq 0 \text{ for all } h = (h_n) \text{ with } \sum_{n \in \mathcal{X}} h_n = 0. \tag{19}$$

It will be shown in the following theorem that Relation (19) is equivalent to the fact that the Hessian of ϕ at ρ is a positive quadratic form, thereby establishing the dimension reduction for the problem of classification.

Theorem 3 (Correspondence between the extrema of g and ϕ)**.**

(1) *A vector $\rho \in \mathbb{R}_+^K$ is a local minimum of the function ϕ if and only if ν_ρ is a local minimum of the Lyapunov function g.*
(2) *If ρ is a saddle point for ϕ, then ν_ρ is a saddle point for g.*

Proof. According to the above remarks, one has to study, on one hand, the sign of the quadratic form $h \to {}^t h\, \mathcal{H}_g^y\, h$ associated to g at $y = \nu_\rho$, $\rho \in \mathbb{R}_+^K$, on the vector space of elements $h = (h_n) \in \mathbb{R}^{\mathcal{X}}$ such that the sum of the coordinates of h is 0; And on the other hand, the sign of the quadratic form ϕ at ρ.

The Hessian of g and its quadratic form. It is easily checked that

$$\frac{\partial^2 g}{\partial y_n \partial y_m}(y) = \frac{1}{y_n}\mathbb{1}_{\{n=m\}} - \sum_{k=1}^{K} n_k m_k \frac{\gamma_k}{\lambda_k + \gamma_k \langle \mathbb{I}_k, y\rangle}.$$

The quadratic form can be expressed as

$${}^t h\,\mathcal{H}_g^y\,h = \sum_{n\in\mathcal{X}} \frac{h_n^2}{y_n} - \sum_{k=1}^{K} \frac{\gamma_k}{\lambda_k + \gamma_k \langle \mathbb{I}_k, y\rangle} \left(\sum_{n\in\mathcal{X}} n_k h_n\right)^2.$$

The change of variable $(h_n) \to (h_n/\sqrt{y_n})$ shows that if

$$H = \left\{ h = (h_n) \in \mathbb{R}^{\mathcal{X}} : \sum_{n\in\mathcal{X}} \sqrt{y_n} h_n = 0 \right\},$$

then it is enough to study the sign of the quadratic form G_y on H given by

$$\begin{aligned} G_y(h) &= \sum_{n\in\mathcal{X}} h_n^2 - \sum_{k=1}^{K} \frac{\gamma_k}{\lambda_k + \gamma_k \langle \mathbb{I}_k, y\rangle} \left(\sum_{n\in\mathcal{X}} n_k \sqrt{y_n} h_n\right)^2 \\ &= \langle h, h\rangle - \sum_{k=1}^{K} \langle v_k^y, h\rangle^2, \end{aligned}$$

where, for $1 \le k \le K$, $v_k^y \in R_+^{\mathcal{X}}$ is defined as

$$v_k^y = \frac{\sqrt{\gamma_k}}{\sqrt{\lambda_k + \gamma_k \langle \mathbb{I}_k, y\rangle}} \left(n_k \sqrt{y_n}, n \in \mathcal{X}\right).$$

Set

$$w_k^y \stackrel{\text{def.}}{=} \frac{\sqrt{\gamma_k}}{\sqrt{\lambda_k + \gamma_k \langle \mathbb{I}_k, y\rangle}} \left(\sqrt{y_n}(n_k - \langle \mathbb{I}_k, y\rangle), n \in \mathcal{X}\right),$$

then it is easy to check that w_k^y is the orthogonal projection of v_k^y in the vector space H, therefore

$$G_y(h) = \langle h, h\rangle - \sum_{k=1}^{K} \langle w_k^y, h\rangle^2.$$

If W is the sub-vector space of H generated by the vectors w_k^y, $1 \le k \le K$ and P_W [resp. $P_{W^\perp}$] is the orthogonal projection on W [resp. on the orthogonal of W], then

$$\begin{aligned} G_y(h) &= \langle P_{W^\perp}(h), P_{W^\perp}(h)\rangle + \langle P_W(h), P_W(h)\rangle - \sum_{k=1}^{K} \langle w_k^y, P_W(h)\rangle^2 \\ &= \langle P_{W^\perp}(h), P_{W^\perp}(h)\rangle + G_y(P_W(h)). \end{aligned} \tag{20}$$

To determine the sign G_y on H, it is thus enough to have the sign of $G_y(h)$ for $h \in W$. An element $h \in W$ can be written as $h = a_1 w_1^y + \cdots + a_K w_K^y$ with $(a_k) \in \mathbb{R}^K$,

$$G_y(h) = \sum_{1\le i,j\le K} a_i a_j \left(\langle w_i^y, w_j^y\rangle - \sum_{k=1}^{K} \langle w_k^y, w_i^y\rangle \langle w_k^y, w_j^y\rangle \right)$$

hence, if $\mathcal{W}^y$ is the $K\times K$ matrix defined by $\mathcal{W}^y=(\langle w_k^y,w_l^y\rangle\,,1\le k,\,l\le K)$,

$$G_y(h)={}^t a\,\mathcal{W}^y(I-\mathcal{W}^y)\,a,\ \text{ for } h=\sum_{k=1}^K a_k w_k^y. \tag{21}$$

The eigenvalues of the matrix $\mathcal{W}^{\nu_\rho}$ being all real (it is symmetrical) and non-negative since its associated quadratic form is non-negative, therefore G_y is positive on W if and only if all the eigenvalues of $\mathcal{W}^y$ are in the interval $(0,1)$.

The Hessian of ϕ and its quadratic form. For $\rho\in\mathbb{R}_+^K$,

$$\frac{\partial^2\phi}{\partial\rho_k\partial\rho_l}(\rho)=-\frac{\partial^2\log Z}{\partial\rho_k\partial\rho_l}(\rho)+\frac{\alpha_k}{\rho_k^2}\mathbb{1}_{\{k=l\}}$$

with $(\alpha_k)=(\lambda_k/\gamma_k)$, for $1\le k,l\le K$. The derivatives of $\log Z$ have the following properties,

$$\rho_k\frac{\partial\log Z}{\partial\rho_k}(\rho)=\frac{1}{Z(\rho)}\sum_{n\in\mathcal{X}}n_k\frac{\rho^n}{n!}=\langle\mathbb{I}_k,\nu_\rho\rangle, \tag{22}$$

and

$$\begin{aligned}-\rho_k\rho_l\frac{\partial^2\log Z}{\partial\rho_k\partial\rho_l}(\rho)&=-\frac{1}{Z(\rho)}\sum_{n\in\mathcal{X}}n_kn_l\frac{\rho^n}{n!}\\&+\left(\frac{1}{Z(\rho)}\sum_{n\in\mathcal{X}}n_k\frac{\rho^n}{n!}\right)\left(\frac{1}{Z(\rho)}\sum_{n\in\mathcal{X}}n_l\frac{\rho^n}{n!}\right)+\mathbb{1}_{\{k=l\}}\frac{1}{Z(\rho)}\sum_{n\in\mathcal{X}}n_k\frac{\rho^n}{n!},\end{aligned}$$

hence

$$-\rho_k\rho_l\frac{\partial^2\log Z}{\partial\rho_k\partial\rho_l}(\rho)=\langle\mathbb{I}_k,\nu_\rho\rangle\langle\mathbb{I}_l,\nu_\rho\rangle-\langle\mathbb{I}_k\mathbb{I}_l,\nu_\rho\rangle+\mathbb{1}_{\{k=l\}}\langle\mathbb{I}_k,\nu_\rho\rangle.$$

The quadratic form associated to ϕ at $\rho\in\mathbb{R}_+^K$ is given by, for $a=(a_k)\in\mathbb{R}^K$,

$$\Phi_\rho(a)=\sum_{1\le k,l\le K}\Big(\langle\mathbb{I}_k,\nu_\rho\rangle\langle\mathbb{I}_l,\nu_\rho\rangle-\langle\mathbb{I}_k\mathbb{I}_l,\nu_\rho\rangle\Big)\frac{a_k}{\rho_k}\frac{a_l}{\rho_l}+\sum_{k=1}^K(\alpha_k+\langle\mathbb{I}_k,\nu_\rho\rangle)\frac{a_k^2}{\rho_k^2}.$$

By using the change of variable (recall that $\alpha_k=\lambda_k/\gamma_k$),

$$a=(a_k)\to\left(\frac{\sqrt{\lambda_k+\gamma_k\langle\mathbb{I}_k,\nu_\rho\rangle}}{\sqrt{\gamma_k}}\frac{a_k}{\rho_k}\right),$$

one gets that the sign of ϕ_ρ has the same range as the sign of Ψ_ρ, where:

$$\begin{aligned}&\Psi_\rho(a)=\langle a,a\rangle\\&+\sum_{1\le k,l\le K}\frac{\sqrt{\gamma_k}}{\sqrt{\lambda_k+\gamma_k\langle\mathbb{I}_k,\nu_\rho\rangle}}\frac{\sqrt{\gamma_l}}{\sqrt{\lambda_l+\gamma_l\langle\mathbb{I}_l,\nu_\rho\rangle}}\Big(\langle\mathbb{I}_k,\nu_\rho\rangle\langle\mathbb{I}_l,\nu_\rho\rangle-\langle\mathbb{I}_k\mathbb{I}_l,\nu_\rho\rangle\Big)a_ka_l\\&\qquad=\langle a,a\rangle-\sum_{1\le k,l\le K}\left\langle w_k^{\nu_\rho},w_l^{\nu_\rho}\right\rangle a_ka_l,\end{aligned}$$

with the above notations. Therefore, the sign of the quadratic form associated to ϕ at ρ has the same values as the sign of $\Psi_\rho(a)$ defined by

$$\Psi_\rho(a)={}^t a\,\left(I-\mathcal{W}^{\nu_\rho}\right)\,a,\qquad a=(a_k)\in\mathbb{R}^K. \tag{23}$$

Equations (21) and (23) show that G_{ν_ρ} is positive on W if and only if Ψ_ρ is positive on $\mathbb{R}_+^K$ which proves Assertion 1 of the theorem. Similarly, if ρ is a saddle point of ϕ, Equation (23) shows that the matrix $\mathcal{W}^{\nu_\rho}$ has eigenvalues in $(0,1)$ and in $(1,+\infty)$, so that G_{ν_ρ} takes positive and negative values on W, and hence on H, ν_ρ is thus a saddle point of g. The theorem is proved. $\square$

7. Metastability Phenomena

This section gives an example where the asymptotic dynamical system has at least three fixed points: Two of them are stable and the other is a saddle point. The corresponding stochastic network therefore exhibits a metastability property. In the limit, it suggests that its state switches from one stable point to the other after a long residence time. The problem of estimating the residence time in the neighborhood of a stable point is not addressed here. According to examples from statistical physics, the expected value of this residence time should be of exponential order with respect to the size N of the network. For reversible Markov processes, Bovier [2, 3, 4] present a potential theoretical approach to get lower and upper bounds for this expected value. These tools do not seem to apply here since the Markov process under study is not reversible.

A Network with Two Classes. A simple setting is considered here: There are two classes of customers, $K=2$, the capacity requirements are $A_1=1$ (small customers) and $A_2=C$ (large ones) so that, at a given node, there may be n class 1 customers, $0\leq n\leq C$, or only one class 2 customer. It is assumed that $\gamma_1=\gamma_2=1$ and $\mu_1=\mu_2=0$ so that a customer leaves the network only when it is rejected at some node.

The two classes cannot coexist at a given node and, moreover, when a node contains class 1 customers, it has to get completely empty before accommodating a class 2 customer. Starting from an initial state where all the nodes contain only class 1 customers, the network will, very likely, evolve on a subspace where the states have few class 2 customers. If λ_2 is sufficiently large, intuitively, class 2 customers may, in the end, occupy a non-negligible proportion of the nodes. Similarly, starting from this state, due to the pressure of class 1 customers, class 2 customers may progressively disappear from the network to go back to the original situation. This phenomenon does not always happen, there may be only one stable region, depending on the values of the parameters.

Proposition 7. *For a network with two classes of customers such that $A_1=1$, $A_2=C$, $\gamma_1=\gamma_2=1$, $\mu_1=\mu_2=0$, for C sufficiently large, there exist λ_1 and $\lambda_2\in\mathbb{R}_+$ such that the corresponding energy function ϕ has at least one saddle point and two local minima.*

From Theorem 3, one deduces that there exists a stochastic network whose asymptotic dynamical system has at least two stable points.

Proof. Fix $\rho\in\mathbb{R}_+^2$ and choose $(\lambda_1,\lambda_2)\in\mathbb{R}_+^2$ so that ρ satisfies Equations (7), i.e.

$$\lambda_k=\rho_k-\langle\mathbb{I}_k,\nu_\rho\rangle=\rho_k\left(1-\frac{\partial\log Z}{\partial\rho_k}(\rho)\right),\ k=1,\,2, \tag{24}$$

by Relation (22), so that ν_ρ is an equilibrium point for the limiting dynamics. It will be assumed for the moment that $C=+\infty$. The corresponding function ϕ is

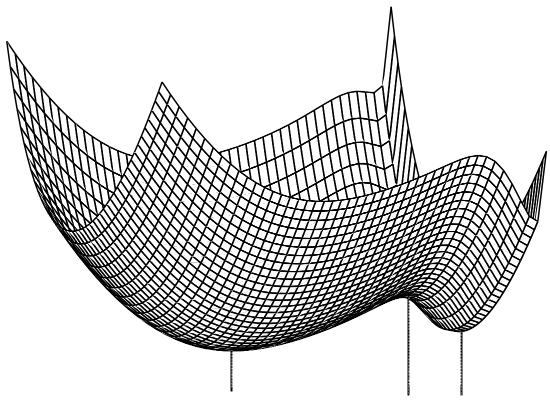

FIGURE 1. Function ϕ with one Saddle Point and Two Stable Points. Two classes with $\lambda_1 = 0.68$, $\lambda_2 = 9.0$, $A_1 = 1$ and $A_2 = C = 20$.

then given by

$$\tilde{\phi}(\rho) = -\log\left(\rho_2 + e^{\rho_1}\right) + \rho_1 + \rho_2 - \lambda_1 \log \rho_1 - \lambda_2 \log \rho_2.$$

Using Equation (24), one gets that

$$\frac{\partial^2 \tilde{\phi}}{\partial \rho_1^2}(\rho) = \frac{\lambda_1}{\rho_1^2} - \frac{\rho_2 e^{\rho_1}}{(\rho_2 + e^{\rho_1})^2} = \frac{\rho_2\left(\rho_2 + (1-\rho_1)e^{\rho_1}\right)}{\rho_1(\rho_2 + e^{\rho_1})^2}$$

and

$$\frac{\partial^2 \tilde{\phi}}{\partial \rho_2^2}(\rho) = \frac{\lambda_2}{\rho_2^2} + \frac{1}{(\rho_2 + e^{\rho_1})^2} > 0.$$

If $\bar{\rho} = (\bar{\rho}_1, \bar{\rho}_1)$ is chosen such that the inequality $\bar{\rho}_2 < (\bar{\rho}_1 - 1)\exp(\bar{\rho}_1)$ holds, then

$$\frac{\partial^2 \tilde{\phi}}{\partial \rho_1^2}(\bar{\rho}) < 0 \text{ and } \frac{\partial^2 \tilde{\phi}}{\partial \rho_2^2}(\bar{\rho}) > 0.$$

The constant C is now assumed to be finite and sufficiently large so that the above inequalities with ϕ in place of $\tilde{\phi}$ are satisfied, $\bar{\rho}$ is a saddle point for ϕ. The function ϕ is given by

$$\phi(\rho) = -\log\left(\rho_2 + \sum_{n=0}^{C} \frac{\rho_1^n}{n!}\right) + \rho_1 + \rho_2 - \lambda_1 \log \rho_1 - \lambda_2 \log \rho_2.$$

The function $\rho_2 \to \phi(\bar{\rho}_1, \rho_2)$ is convex, $\bar{\rho}_2$ is a strict local minimum by construction and therefore a *global* minimum. Similarly, the function $\rho_1 \to \phi(\rho_1, \bar{\rho}_2)$ has a strict local maximum at $\bar{\rho}_1$,

$$\begin{aligned}
&\inf\{\phi(\rho) : \rho = (\rho_1, \bar{\rho}_2), \rho_1 < \bar{\rho}_1\} < \phi(\bar{\rho}),\\
&\inf\{\phi(\rho) : \rho = (\rho_1, \bar{\rho}_2), \bar{\rho}_1 < \rho_1\} < \phi(\bar{\rho}) = \inf\{\phi(\rho) : \rho \in \Delta\},
\end{aligned}$$

with $\Delta = \{(\bar{\rho}_1, \rho_2) : \rho_2 \in \mathbb{R}_+ \setminus \{0\}\}$. Since $\phi((\rho_1, \rho_2))$ converges to $+\infty$ when ρ_1 or ρ_2 converges to 0 or $+\infty$, one concludes that the function ϕ has at least two local finite minima, one on each side of Δ. The proposition is proved. Figure 1 gives an example of such a situation. $\square$

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