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On the Constructions of the Skew Brownian Motion*

Antoine Lejay[†]

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[†]Projet OMEGA (INRIA/IECN)
IECN, Campus scientifique
BP 239,
54506 Vandœuvre-lès-Nancy CEDEX, France
E-mail: Antoine.Lejay@iecn.u-nancy.fr

Abstract

This article summarizes the various ways one may use to construct the Skew Brownian motion, and shows their connections. Recent applications of this process in modelling and numerical simulation motivates this survey. This article ends with a brief account of related results, extensions and applications of the Skew Brownian motion.

Keywords

skew Brownian motion, PDE with singular drift, PDE with transmission condition, SDE with local time, excursions of Brownian motion, scale function and speed measure, mathematical modelling, Monte Carlo methods

AMS Classification: 60J60 (60H10, 60J55)

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1 Introduction

The Skew Brownian motion appeared in the '70 in [IM74, Wal78] as a natural generalization of the Brownian motion: it is a process that behaves like a Brownian motion except that the sign of each excursion is chosen using an independent Bernoulli random variable of parameter p . As a sequel, if X is a Skew Brownian motion of parameter p (that is, p is the probability that the excursion is positive), then $\mathbb{P}[X_t > 0] = p$ for any $t > 0$, while outside 0, X_t behaves like a Brownian motion. As shown in [HS81], this process is a semi-martingale which is a strong solution to some Stochastic Differential Equation (SDE) with local time

$$X_t = x + B_t + (2p - 1)L_t^0(X),$$

where $L_t^0(X)$ is the local time at 0 of X .

In the same time, N. Portenko constructed in [Por79a, Por79b] a process whose infinitesimal generator has a singular drift concentrated on some hypersurface. This is a way to model some permeable barrier. Indeed, he considers diffusion whose infinitesimal generator is of type

$$L = \sum_{i,j=1}^d \frac{1}{2} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i \frac{\partial}{\partial x_i} + \sum_{i=1}^d q \delta_S N_i(x) \frac{\partial}{\partial x_i}$$

where $N = (N_1, \dots, N_d)$ is the vector conormal to a smooth hyper-surface S , a , b and q are continuous and $|q(x)| < 1$ for all $x \in S$. It is shown that the solution to $\frac{\partial u}{\partial t} = Lu$ is also solution to the PDE

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = Lu(t, x) \text{ on } \mathbb{R}^d \setminus S, \\ \frac{1 + q(x)}{2} N_+(x) \cdot \nabla u(t, x) = \frac{1 - q(x)}{2} N_-(x) \cdot \nabla u(t, x) \\ u(t, \cdot) \text{ is continuous on } S, \end{cases}$$

where N_+ and N_- are the inner and outer conormal to S . The condition on the left and right flux on the surface S is called a *transmission condition* or a *flux condition*.

In dimension one with $a = 1$ and $b = 0$, this process is the Skew Brownian motion. Thus, the infinitesimal generator of this diffusion process has a rather natural interpretation, since it corresponds to half the Laplace operator plus a ‘‘generalized drift’’ given by a Dirac mass at 0 with a coefficient that corresponds to its skewness. Besides, the effect of the drift is translated into the PDE as a transmission condition. It was also noted in [Wei84] that

SDEs with local time may be used to model diffusion processes in a media with permeable barrier.

Since, a few works have used the Skew Brownian motion as a tool for solving applied problems: in astrophysics [Zha00], in ecology [CC99], in homogenization [Wei84, Lan95], in geophysics [Lej03, Lej04] and more recently in finance [DGS05, DdSG04, DdSGS05],...

A recent series of works [Lej03, Mar04, LM06, Éto05] shows that the properties of the Skew Brownian motion may be used in a systematic way to provides different schemes to simulate diffusion processes generated by

$$L = \frac{\rho}{2} \frac{d}{dx} \left(a \frac{d}{dx} \right) + b \frac{d}{dx}, \quad (1)$$

with a and ρ have discontinuous points of first kind. All these Monte Carlo methods rely on a probabilistic interpretation of the transmission conditions at the points where the coefficients are discontinuous and appropriate changes of scales.

Indeed, the diffusion process X generated by L is solution to an SDE with local time of type

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt + \int_{\mathbb{R}} \nu(dx) dL_t^x(X), \quad (2)$$

where ν is a finite, signed measure that has a mass at the points where a or ρ are discontinuous.

SDEs of type (2) have been studied first in [LG82, LG85, ES85] (see also [EW99, BC05] for some extensions), where existence and uniqueness of a strong solution is proved under some rather general conditions (σ shall be uniformly elliptic, bounded and of finite variation, b shall be bounded, and ν shall be of finite mass with $|\nu(\{x\})| < 1$ for any point $x \in \mathbb{R}$).

This type of SDE offers a generalization of the usual SDEs, and allows to deal for example with Dirichlet processes [EW99, BC02] were studied with hypotheses on the coefficients that are as weak as possible: [BC03, FRW03, BC05], ... In addition, it is worth noting that the articles [LG85, ES85] also give an account on the “critical cases” between strong and weak solutions of SDEs, in the sense they deal with the weakest possible conditions on the coefficients to ensure the existence of a strong solution (see also [Bar88] for an example of application to the theory of SDEs).

Another natural extension of the Skew Brownian motion is the *Walsh Brownian motion*. It consists on a diffusion process that moves on rays emanating from a single point and was introduced by J. Walsh in [Wal78]. This diffusion process can be used locally as a description of a *spider martingale*

[Yor97, Sect. 17, p. 103], and is a special case of a diffusion on a graph, since we recover a transmission condition at each vertex. Diffusions on graphs are particularly important in the study of dynamical Hamiltonian systems as shown in the pioneering work of M. Freidlin and M. Weber [FW98], but could be useful for modelling many physical or biological diffusion phenomena. It is then possible to use Monte Carlo methods to solve that type of problems.

One has also to note that Walsh Brownian motion was useful to provide a counter-example to a natural question on Brownian filtration, as B. Tsirelson shown it in [Tsi97].

The goal of this article is then to summarize the different ways to construct a Skew Brownian motion (using PDEs, Dirichlet forms, approximations by diffusions with smooth coefficients and by random walks, scale functions and speed measures, SDE, excursions theory...) and their relationships. The last section presents quickly some extensions of the Skew Brownian motion and their applications to various fields.

2 Differential operators with generalized coefficients

Let δ denotes the Dirac function at 0. In a first time, we are interested in solving the parabolic PDE for some $q \in [-1, 1]$,

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} = \frac{1}{2}\Delta u(t, x) + q\delta\nabla u(t, x), \\ u(0, x) = \varphi(x), \end{cases} \quad (3)$$

which is equivalent to construct the semi-group associated to the operator with a singular first-order differential term

$$L = \frac{1}{2}\Delta + q\delta\nabla. \quad (4)$$

In [Por79a, Por79b], N. Portenko constructed the semi-group generated by L and showed it is a Feller semi-group. In fact, N. Portenko works in \mathbb{R}^m instead to \mathbb{R} , where δ is a Dirac mass of a surface S smooth enough. Here, we restrict ourself to the more simple form of L given by (4).

2.1 Perturbation of the heat kernel

One wants to construct the semi-group $(Q_t)_{t>0}$, if it exists, generated by L . Let us denote by $(P_t)_{t>0}$ the semi-group generated by $\frac{1}{2}\Delta$ over \mathbb{R} . It is well

known that P_t has a density $p(t, x, y)$ given by the heat (or Gaussian) kernel: for any continuous, bounded function f ,

$$P_t\varphi(x) = \int_{\mathbb{R}} p(t, x, y)\varphi(y) dy \text{ with } p(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{|x-y|^2}{2t}\right).$$

It is natural to construct Q_t as a perturbation of P_t , that is

$$Q_t\varphi(x) = P_t\varphi(x) + q \int_0^t p(t-\tau, x, 0)\nabla Q_\tau\varphi(0) d\tau. \quad (5)$$

The problem here is to give a meaning to $\nabla Q_t\varphi(0)$, since we will see that $\nabla Q_t\varphi$ is discontinuous at 0. However, let us remark that for any continuous function f and any $\tau > 0$, $x \mapsto p(\tau, x, 0)f(0)$ is of class $\mathcal{C}^1(\mathbb{R}; \mathbb{R})$ and

$$\frac{\partial}{\partial x} p(\tau, x, 0)f(0)|_{x=0} = 0. \quad (6)$$

So, if we inject in the right-hand side of (5) the value of $Q_t\varphi(x) \stackrel{\text{def.}}{=} u(t, x)$ given by (5), one gets with (6) that

$$Q_t\varphi(x) = u(t, x) = P_t\varphi(x) + q \int_0^t p(t-\tau, x, 0)\nabla P_\tau\varphi(0) d\tau. \quad (7)$$

We use (7) as the definition of Q_t .

2.2 Single layer potential

Another idea to construct the semi-group is to consider that the PDE (3) corresponds to the potential generated by a charge whose value is

$$V(t, \varphi) = q \frac{\nabla u(t, 0+) + \nabla u(t, 0-)}{2}$$

at the point 0. From standard results (See [Fol95, Fri83] for example), one knows that the solution $Q_t\varphi(x) \stackrel{\text{def.}}{=} u(t, x)$ of such a problem is given on \mathbb{R}_+^* and \mathbb{R}_-^* by

$$Q_t\varphi(x) = P_t\varphi(x) + \int_0^t p(t-\tau, x, 0)V(\tau, \varphi) d\tau.$$

Yet $V(\tau, \varphi)$ also involves the value of u . Indeed, a computation similar to the one done previously leads also to (7), and thus

$$V(t, \varphi) = \nabla P_t\varphi(0). \quad (8)$$

The advantage of this formulation is that one also knows from standard results on potential theory (see [Fri83] for example) that

$$\nabla Q_t \varphi(0\pm) = \int_{\mathbb{R}} \nabla p(t, 0, y) \varphi(y) dy \mp qV(\tau, \varphi)$$

which means that with (8),

$$\nabla Q_t \varphi(0+) = (1 - q)\nabla P_t \varphi(0) \text{ and } \nabla Q_t \varphi(0-) = (1 + q)\nabla P_t \varphi(0). \quad (9)$$

Then it is immediate that $(t, x) \mapsto Q_t \varphi(x)$ is of class $\mathcal{C}^{1,2}$ on $\mathbb{R}_+^* \times \mathbb{R}^*$, is continuous on $\mathbb{R}^* \times \mathbb{R}$, but $x \mapsto \nabla Q_t \varphi(x)$ is discontinuous at $x = 0$. From (9), one gets

$$\alpha \nabla Q_t \varphi(0+) = (1 - \alpha) \nabla Q_t \varphi(0-) \quad (10)$$

with

$$\alpha = \frac{1 + q}{2}. \quad (11)$$

Hence, one can give a proper meaning to (3), since $(t, x) \mapsto Q_t \varphi(x)$ is a continuous solution to

$$\begin{cases} (t, x) \mapsto u(t, x) \in \mathcal{C}^{1,2}(\mathbb{R}_+^* \times \mathbb{R}^*; \mathbb{R}) \cap \mathcal{C}(\mathbb{R}_+^* \times \mathbb{R}; \mathbb{R}), \\ \frac{\partial u(t, x)}{\partial t} = \frac{1}{2} \Delta u(t, x) \text{ for any } (t, x) \in \mathbb{R}_+^* \times \mathbb{R}^*, \\ \alpha \nabla u(t, 0+) = (1 - \alpha) \nabla u(t, 0-), \\ u(t, 0+) = u(t, 0-), \\ u(0, x) = \varphi(x). \end{cases} \quad (12)$$

Proposition 1. *When φ is continuous and bounded, and $\alpha \in [0, 1]$, there exists a unique solution to (12) for which the maximum principle holds.*

Proof. The existence and uniqueness of the solution of (12) may be proved the equivalence between this problem and the same PDE written in a variational form: see Section 3.1.

But we could also prove this result without using the notion of weak solutions. The existence of u is proved, since we have constructed such a solution with the help of the semi-group $(Q_t)_{t>0}$.

If $\alpha = 0$ or $\alpha = 1$, then (12) is equivalent in solving two heat equations, one with a Neumann boundary condition on $\mathbb{R}_+^* \times \{0\}$, and the other with a lateral Dirichlet boundary on $\mathbb{R}_+^* \times \{0\}$ that follows from the continuity of u . Hence, the maximum principle holds.

We deal now with the case $\alpha \in (0, 1)$.

The uniqueness follows from the maximum principle. Assume that $\varphi \leq 0$ and φ has a compact support. If the maximum principle holds, that is

$u(t, x) \leq 0$ for all $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}$, then one deduces that $\varphi = 0$ implies that $u = 0$. Indeed, if u is a solution of (12), then u is solution to $\frac{\partial u}{\partial t}(t, x) = \frac{1}{2}\Delta u(t, x)$ both on $\mathbb{R}_+^* \times \mathbb{R}_+$ and $\mathbb{R}_+^* \times \mathbb{R}_-$, with the boundary condition φ on \mathbb{R}_+ and \mathbb{R}_- , and the lateral boundary condition $u(t, 0)$ on $\mathbb{R}_+^* \times \{0\}$. The solution u for each of these equations satisfies $u(t, x) \rightarrow 0$ when $|x| \rightarrow +\infty$ or $t \rightarrow +\infty$. Thus, if $u > 0$ inside $\mathbb{R}_+^* \times \mathbb{R}$, the maximum of u is at a point (t_0^\pm, x_0^\pm) of $\mathbb{R}_+^* \times \mathbb{R}_\pm$. If $x_0^\pm \neq 0$, then it follows from Theorem 2 in [Fri83, Sect. 1, p. 38] that $u(t_0^\pm, 0) = u(t_0, x_0^\pm)$. Let t_0 be the time such that $u(t_0, 0)$ is maximum. As $\alpha \nabla u(t_0, 0+) = (1 - \alpha) \nabla u(t_0, 0-)$, then if $\nabla u(t_0, 0+) \neq 0$, $u(t_0, x) > u(t_0, 0)$ for some $x \neq 0$ in a neighborhood of 0, as $\nabla u(t_0, 0+)$ and $\nabla u(t_0, 0-)$ have the same sign. But unless that u is constant (which implies that $\varphi = 0$, since $u(t, x) \rightarrow 0$ when (t, x) goes to infinity), Theorem 14 in [Fri83, Sect. 1, p. 49] asserts that $\nabla u(t_0, 0\pm) \neq 0$. Hence, u has no maximum in $\mathbb{R}_+^* \times \mathbb{R}$ and thus $u \leq 0$.

If $\varphi \geq 0$, then $-\varphi \leq 0$ and then $u \geq 0$, since $-u$ is solution to (12) with $-\varphi$ as an initial condition. Similarly, if u solves (12). To conclude, let us note that if $\varphi(x) = C$ for all $x \in \mathbb{R}$, then $u(t, x) = C$. Besides, from (7), if φ is continuous, $|\varphi|$ is bounded and $\varphi(x) = 0$ on $(-R, R)$, then $\sup_{(t,x) \in \mathbb{R}_+^* \times \mathbb{R}} |Q_t \varphi(x)|$ converges to 0 as $R \rightarrow \infty$. Hence, this proves that if $\varphi \leq C$ (resp. $\varphi \geq C$), then the solution $Q_t \varphi(x)$ of (12) satisfies $Q_t \varphi(x) \leq C$ (resp. $Q_t \varphi(x) \geq C$) for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$. \square

The next proposition follows from the previous facts.

Proposition 2 ([Por79a]). *If $|q| \leq 1$ (or equivalently $\alpha \in [0, 1]$), then $(Q_t)_{t>0}$ is a Feller semi-group, where Q_t is defined by (7) for any function φ which is continuous and bounded on \mathbb{R} .*

2.3 Construction of a Skew Brownian motion

The consequences of the construction given by N. Portenko are summarized in the following theorem.

Theorem 1. *If $|q| \leq 1$, then $(Q_t)_{t>0}$ is the infinitesimal generator of a strong Markov process $(X_t, \mathcal{F}_t, \mathbb{P}_x; t \geq 0, x \in \mathbb{R})$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which we call a Skew Brownian motion of parameter α (abbreviated by SBM(α)) with $\alpha = (1 + q)/2$. This process is continuous and conservative. Besides, there exists a $(\mathcal{F}_t, \mathbb{P})$ -Brownian motion B and a continuous additive functional η such that*

$$X_t = x + B_t + \eta_t, \quad \forall t \geq 0, \mathbb{P}_x\text{-a.s.} \quad (13)$$

The additive functional η is of finite variation, and its variation increases only when the process is at 0: $\int_0^t |d\eta_s| = \int_0^t \mathbf{1}_{\{X_s=0\}} |d\eta_s|$ \mathbb{P}_x -a.s..

Of course, it remains to prove that the infinitesimal generator of X is L given by (4). It is clear from (7) that Q_t has a density $q(t, x, y)$ given by

$$q(t, x, y) = p(t, x, y) + \frac{1}{\pi} \int_0^t \frac{y}{\sqrt{\tau} \sqrt{t-\tau}} \exp\left(-\frac{x^2}{t-\tau} - \frac{y^2}{\tau}\right) d\tau. \quad (14)$$

In fact, a more convenient expression of $q(t, x, y)$ will be given later. For that, it could be shown that for any function ψ continuous and bounded,

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}} \psi(x) \left(\frac{1}{t} \int_{\mathbb{R}} q(t, x, y)(y-x) dy \right) dx = q\psi(0), \quad (15)$$

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}} \psi(x) \left(\frac{1}{t} \int_{\mathbb{R}} q(t, x, y)(y-x)^2 dy \right) dx = \int_{\mathbb{R}} \psi(x) dx. \quad (16)$$

The equality (16) identifies the diffusion coefficient, which is equal to 1 here, while (15) allows us to identify the drift term of X with $q\delta_0$. This is a generalization of the way of characterizing the drift term and the diffusion coefficient of a diffusion process, as presented for example in the book [Por90].

2.4 An explicit construction of the semi-group

We have construct a semi-group $(Q_t)_{t>0}$ such that $Q_t\varphi(x)$ is a solution to the PDE (12). It is possible to look for a solution to (12) using an explicit computation.

Let φ be a continuous, bounded function. Let us remark that

$$\begin{aligned} \nabla P_t \varphi(0) &= \int_{\mathbb{R}} \frac{\sqrt{2}y}{\sqrt{\pi t^{3/2}}} \exp\left(-\frac{y^2}{2t}\right) \varphi(y) dy \\ &= \int_0^{+\infty} \frac{\sqrt{2}y}{\sqrt{\pi t^{3/2}}} \exp\left(-\frac{y^2}{2t}\right) (\varphi(y) - \varphi(-y)) dy \\ &= \nabla P_t \widehat{\varphi}(0) \end{aligned}$$

where $\widehat{\varphi}(x) = \mathbf{1}_{\{x \geq 0\}}(\varphi(x) - \varphi(-x))$. Let us also remark that $P_t \widehat{\varphi}(x) \xrightarrow[t \rightarrow 0]{} 0$ when $x \leq 0$. We set $\psi(x) = \widehat{\varphi}(-|x|)$. Thus, $P_t \psi(x)$ is solution to the heat equation on $\mathbb{R}^* \times \mathbb{R}^*$ with the initial condition equal to 0. Moreover, $\nabla P_t \psi(0+) = -\nabla P_t \varphi(0)$ and $\nabla P_t \psi(0-) = \nabla P_t \varphi(0)$. Thus, we are looking for a solution $T_t \varphi(x)$ written under the form

$$T_t \varphi(x) = P_t \varphi(x) + \lambda P_t \psi(x).$$

where λ is chosen in order that $\alpha \nabla T_t \varphi(0+) = (1 - \alpha) \nabla T_t \varphi(0-)$. Hence, is is always true if

$$\lambda = 2\alpha - 1 = q.$$

This result is a direct consequence of Proposition 1.

Proposition 3. *The semi-groups $(T_t)_{t>0}$ and $(Q_t)_{t>0}$ are equal.*

Hence, one gets the following expression for Q_t , which is more tractable than (14):

$$Q_t\varphi(x) = \int_{\mathbb{R}} p(t, x, y)\varphi(y) dy + q \int_0^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(y+|x|)^2}{2t}\right) (\varphi(y) - \varphi(-y)) dy. \quad (17)$$

3 Using Dirichlet forms

The idea is to get rid of the singular first-order term by transforming L into a symmetric operator in a proper Hilbert space.

3.1 Weak solutions of PDE

Here, we assume that $\alpha = (q+1)/2 \notin \{0, 1\}$. The problem with the parabolic PDE (11) is to deal with the transmission condition at 0 given by

$$\alpha \nabla u(t, 0+) = (1 - \alpha) \nabla u(t, 0-). \quad (18)$$

So, an alternative way to consider (11) is to look for a weak solution of the following PDE

$$\begin{cases} u(t, x) \in \mathcal{C}(0, T; L^2(\mathbb{R})) \cap L^2(0, T; H^1(\mathbb{R})) \\ \frac{\partial u(t, x)}{\partial t} = Au(t, x), \\ u(0, x) = \varphi(x) \in L^2(\mathbb{R}), \end{cases} \quad (19)$$

where A is the divergence form operator

$$A = \frac{1}{2a(x)} \nabla(a(x) \nabla \cdot) \text{ with } a(x) = \begin{cases} \alpha & \text{if } x \geq 0, \\ 1 - \alpha & \text{otherwise.} \end{cases} \quad (20)$$

By a weak solution to (19), we mean a function u such that for all $\psi \in \mathcal{C}^\infty([0, T]; \mathbb{R})$ with $\psi(T) = 0$, then, integrating formally (19) with respect to $\psi(t, x)\rho(x) dx$ with $\rho(x) = 1/2a(x)$ and using integrations by parts,

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \frac{\partial \psi(t, x)}{\partial t} u(t, x) \rho(x) dx dt \\ & + \int_0^T \int_{\mathbb{R}} a(x) \frac{\partial u(t, x)}{\partial x} \frac{\partial \psi(t, x)}{\partial x} dx dt = - \int_{\mathbb{R}} \varphi(x) \psi(0, x) \rho(x) dx. \end{aligned}$$

As u is smooth on the domain where the coefficient a is smooth, it is easily deduced that u is smooth on $(0, T] \times \mathbb{R}_+^*$ and $(0, T] \times \mathbb{R}_-^*$. Let us choose $\psi(t, x) = \psi_1(x)\psi_2(t)$ for two functions ψ_1 and ψ_2 smooth enough and such that $\psi_2(T) = 0$. Using an integration by parts on \mathbb{R}_+^* and \mathbb{R}_-^* and the freedom of choice of ψ_1 and ψ_2 , we are led to (18). Moreover, one knows that $u(t, x)$ is continuous on $(0, T] \times \mathbb{R}$. Thus, the weak solution of (19) is also a solution of (12) and the converse is also true. The article [LRU66] and [LSU68, § III.13, p. 224] contains an account about the properties of the solution of the transmission problem.

The domain $\text{Dom}(A)$ of A is the set of functions f of $H^1(\mathbb{R})$ such that Af belongs to $L^2(\mathbb{R})$. With this domain, A is a self-adjoint (hence closed) operator with respect to the scalar product $\langle f, g \rangle_{L^2(\mathbb{R}; a)} = \int_{\mathbb{R}} f(x)g(x)a(x) dx$ of $L^2(\mathbb{R}; a)$. It is also well known that a divergence-form operator as A is the infinitesimal generator of a Feller semi-group $(\widehat{P}_t)_{t>0}$ with a density transition function $(t, x, y) \mapsto \Gamma(t, x, y)$ with respect to $a(x) dx$, that is $\widehat{P}_t f(x) = \int_{\mathbb{R}} \Gamma(t, x, y)f(y)a(y) dy$ for any continuous and bounded function f on \mathbb{R} . Of course, $u(t, x) = \widehat{P}_t \varphi(x)$, and from the self-adjointness of A , $\Gamma(t, x, y) = \Gamma(t, y, x)$ for all $(t, x, y) \in \mathbb{R}_+^* \times \mathbb{R}^2$.

The Proposition 4 is a direct consequence of Proposition 1.

Proposition 4. *The semi-groups $(\widehat{P}_t)_{t>0}$ and $(Q_t)_{t>0}$ are equal.*

The next proposition follow from general results about divergence form operators: See [Str88, Lej00] for example.

Proposition 5. (i) *As $(\widehat{P})_{t>0}$ is a Feller semi-group, it is the generator of a strong Markov stochastic process which is continuous and conservative (which is then a SBM(α) thanks to Proposition 4).*

(ii) *There exists a constant $\delta \in (0, 1)$ depending only on α such that the function $(t, x, y) \mapsto \Gamma(t, x, y)$ is $\delta/2$ -Hölder continuous with respect to t and δ -Hölder continuous with respect to (x, y) .*

(iii) *There exists some constants $C_1, C_2, C_3, C_4 > 0$ depending only on α such that for all $(t, x, y) \in \mathbb{R}_+^* \times \mathbb{R}^2$,*

$$C_1 g(tC_2, x, y) \leq \Gamma(t, x, y) \leq C_3 g(tC_4, x, y), \quad (21)$$

where $g(t, x, y) = \frac{1}{\sqrt{2\pi t}} \exp(-(x - y)^2/2t)$ is the heat kernel.

The inequality (21) is called the *Aronson inequality*. It allows to deduce some general results about the tightness of a family of processes generated by divergence form operators, or a bound on the probability to leave a sphere centered on x prior some time T (see for example [Str88, Roz96, Lej00] for applications).

Of course, the transmission condition (18) implies that

$$\alpha \nabla_x \Gamma(t, 0-, y) = (1 - \alpha) \nabla_x \Gamma(t, 0+, y), \quad \forall (t, y) \in \mathbb{R}_+ \times \mathbb{R}.$$

A note on the adjoint of A with respect to $L^2(\mathbb{R})$. When one uses the scalar product $\langle f, g \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x)g(x) dx$, it is easily seen that the adjoint A^* of A is given by $A^* = \frac{1}{a}A(\frac{1}{a}\cdot)$. This implies that $\text{Dom}(A^*) = \{ f \in L^2(\mathbb{R}) \mid f/a \in \text{Dom}(A) \}$. Thus, any function f in the domain of A^* shall satisfies the condition

$$\alpha f(0-) = (1 - \alpha)f(0+) \text{ and } \nabla f \text{ is continuous at } 0.$$

This also explains why the density $p(t, x, y)$ of the SBM(α) with respect to the Lebesgue measure is discontinuous as a function of y , in account of the Forward Kolmogorov equation. Of course, this can also be deduce from Proposition 5(ii).

3.2 Dirichlet forms

Let $(\mathcal{E}, \text{Dom}(\mathcal{E}))$ be the quadratic form

$$\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}} a(x) \nabla f(x) \nabla g(x) dx$$

for any $(f, g) \in \text{Dom}(\mathcal{E}) = H^1(\mathbb{R})$. Let $\langle \cdot, \cdot \rangle_{L^2(a, \mathbb{R})}$ be the scalar product $\langle f, g \rangle_{L^2(a, \mathbb{R})} = \int_{\mathbb{R}} f(x)g(x)a(x) dx$. Then $(\mathcal{E}, \text{Dom}(\mathcal{E}))$ is the bilinear form associated to $(A, \text{Dom}(A))$ by

$$\langle Af, g \rangle_{L^2(a, \mathbb{R})} = \mathcal{E}(f, g), \text{ for all } (f, g) \in \text{Dom}(A) \times H^1(\mathbb{R}).$$

It is indeed easily checked that $(\mathcal{E}, \text{Dom}(\mathcal{E}))$ is a *Dirichlet form* [MR91, FOT94], that is a symmetric bilinear form that is closed and contractive.

We can now give a third construction (however with somewhat complicated tools that we use here in a simple case) of a SBM(α).

Proposition 6. *The Dirichlet form $(\mathcal{E}, \text{Dom}(\mathcal{E}))$ generates a continuous, Hunt (hence strong Markov) process, which is a SBM(α).*

Proof. This follows from standard results in the theory of Dirichlet forms since $(\mathcal{E}, \text{Dom}(\mathcal{E}))$ is a regular, local Dirichlet form. Indeed, the infinitesimal generator of the process it generates is $(A, \text{Dom}(A))$, hence it is a SBM(α). \square

Remark 1. In general, the process we construct using the previously invoked results on Dirichlet form is only defined only for *quasi-every* starting point but not for any starting point. Yet in dimension one and under our assumptions on \mathcal{E} , any set of zero capacity is empty.

3.3 The Itô-Fukushima decomposition

The theory of Dirichlet form gives us another way to prove Theorem 1. We denote by $(X, \mathcal{F}_t, \mathbb{P}_x; t \geq 0, x \in \mathbb{R})$ the SBM(α) and by $(\theta_t)_{t \geq 0}$ its shift operator.

A *continuous additive function* (CAF) is a continuous process Y such that $Y_{t+s}(\omega) = Y_t(\theta_s \omega) + Y_s(\omega)$. A typical CAF is given by $\int_0^t f(X_s) ds$ for some bounded, measurable function f .

A CAF Y is characterized by a Radon measure ν , called its *Revuz measure* by the following way: If f is solution to

$$\langle f, \varphi \rangle_{L^2(a, \mathbb{R})} + \mathcal{E}(f, \varphi) = \int_{\mathbb{R}} \varphi(x) d\nu(x), \quad \forall \varphi \in \mathcal{C}_c^1(\mathbb{R}, \mathbb{R}),$$

then $\mathbb{E}_x[\int_0^{+\infty} e^{-t} dY_t]$ is a continuous¹ version of f .

Let G_α be the resolvent of $(\mathcal{E}, \text{Dom}(\mathcal{E}))$, that is the operator giving the unique solution $u \in H^1(\mathbb{R})$ to

$$\alpha \langle u, \varphi \rangle_{L^2(a, \mathbb{R})} + \mathcal{E}(u, \varphi) = \langle f, \varphi \rangle_{L^2(a, \mathbb{R})}, \quad \forall \varphi \in H^1(\mathbb{R}).$$

Using the resolvent of the Dirichlet form, we see immediately that if $\nu(dx) = h(x) dx$, then the corresponding CAF is $\int_0^t h(X_s)/a(X_s) ds$.

We say that a CAF Y is of *zero quadratic variation* if

$$e(Y) \stackrel{\text{def.}}{=} \sup_{t \rightarrow 0} \frac{1}{2t} \int_{\mathbb{R}} a(x) \mathbb{E}_x[(Y_t - Y_0)^2] dx = 0.$$

Of course, a CAF is said to be *locally of zero quadratic variation* if there exists a sequence $\{\tau^n\}_{n \in \mathbb{N}}$ of stopping times (with respect to the minimal filtration generated by X satisfying the usual hypothesis) converging almost surely to $+\infty$ and such that $N_{\cdot \wedge \tau^n}$ is of zero quadratic variation.

Theorem 2 (Itô-Fukushima decomposition). *Let f be a function locally in $H^1(\mathbb{R})$. Then there exists a local, square-integrable martingale M^f with $\langle M^f \rangle_t = \int_0^t |\nabla f(X_s)|^2 ds$, and a CAF N^f locally of zero quadratic variation such that*

$$f(X_t) = f(X_0) + M_t^f + N_t^f.$$

Proof. This theorem follows from the results in [FOT94]. □

¹In the general case, it is only a quasi-continuous version, since a CAF may be only defined for any starting point except for those in a set of zero capacity. However, as noted in Remark 1, in dimension one and under our assumptions on \mathcal{E} , there are no such sets.

Corollary 1. *If f belongs locally to $H^2(\mathbb{R})$ (we choose a version of f such that f' is continuous), then*

$$N_t^f = \int_0^t f''(X_s) ds + \beta_f \ell_t \text{ with } \beta_f = \frac{\alpha f'(0+) - (1-\alpha)f'(0-)}{2},$$

where $(\ell_t)_{t \geq 0}$ is the CAF associated to the Revuz measure δ_0 .

If $f(x) = x$, then $N_t^f = \frac{(2\alpha-1)}{2} \ell_t$.

Remark 2. Of course, up to some multiplicative constant, the CAF ℓ is the local time in 0 of the process X (see Proposition VI.45.10 in [RW00]).

Proof. Using a localization argument, we may assume that f belongs to $H^2(\mathbb{R})$, and we choose a version of f such that f' is continuous. By an integration by part, for any smooth function φ with compact support,

$$\begin{aligned} \int_{\mathbb{R}} a(x) f'(x) \varphi'(x) dx &= - \int_0^{+\infty} \alpha f''(x) \varphi(x) dx - \int_{-\infty}^0 (1-\alpha) f''(x) \varphi(x) dx \\ &\quad - \alpha f'(0+) \varphi(0+) + (1-\alpha) f'(0-) \varphi(0-). \end{aligned}$$

Hence,

$$\langle f, \varphi \rangle_{L^2(a, \mathbb{R})} + \mathcal{E}(f, \varphi) = \int_{\mathbb{R}} \varphi(x) \nu(dx)$$

with $\nu(dx) = a(x)(f(x) - f''(x)/2) dx - \beta_f \delta_0$. It follows that $f = G_1 \nu$ and that

$$N_t^f = \int_0^t f(X_s) ds - Y_t,$$

where Y is a CAF associated to ν . But the CAF associated to ν is $Y_t = \int_0^t (f(X_s) - f''(X_s)/2) ds - \beta_f \ell_t$, which yields the result. \square

3.4 Approximations of the coefficients

We give now two results about the approximation of the coefficients.

Let (ρ, a, b) be measurable functions from \mathbb{R} to \mathbb{R} . We assume that there exist some constants λ and Λ for which

$$\lambda \leq a(x) \leq \Lambda, \quad \lambda \leq \rho(x) \leq \Lambda \text{ and } |b(x)| \leq \Lambda. \quad (22)$$

Let $(L, \text{Dom}(L))$ be the infinitesimal generator

$$L = \frac{1}{2\rho} \frac{d}{dx} \left(a \frac{d}{dx} \right) + b \frac{d}{dx}, \quad \text{Dom}(L) = \left\{ f \in H_1(\mathbb{R}) \mid Lf \in L^2(\mathbb{R}) \right\}.$$

Under (22), $(L, \text{Dom}(L))$ is the infinitesimal generator of a continuous, conservative strong Markov process (See [Str88, Lej00], ... for example).

Proposition 7. *Let $(\rho^n, a^n, b^n)_{n \in \mathbb{N}}$ be a family of measurable coefficients such that (22) holds uniformly and*

$$\frac{1}{a^n} \xrightarrow[n \rightarrow \infty]{L^q(\mathbb{R})} \frac{1}{a(x)}, \quad \frac{1}{\rho^n} \xrightarrow[n \rightarrow \infty]{L^q(\mathbb{R})} \frac{1}{\rho(x)} \quad \text{and} \quad \frac{b^n}{\rho^n a^n} \xrightarrow[n \rightarrow \infty]{L^q(\mathbb{R})} \frac{b}{\rho a} \quad (23)$$

for some $1 \leq q \leq \infty$. Then the stochastic process X^n generated by the operator $\frac{1}{2\rho^n} \frac{d}{dx} \left(a^n \frac{d}{dx} \right) + b^n \frac{d}{dx}$ converges in distribution to the process X generated by L .

Proof. Under (23), the convergence of the resolvent and the semi-group of L^n to that of L follows from results in [ZKOTN79, ZKO81] for example. The hypothesis (22) implies that one may compare the density transition functions of the processes X^n and X uniformly with respect to the Gaussian kernel, and that this functions are locally Hölder continuous in space and time. Thus, this is sufficient to deduce the weak convergence of X^n to X : see [Roz96, Lej00] for example. \square

Corollary 2. *Let a^n and b^n such that $(\rho^n, a^n, b^n)_{n \in \mathbb{N}}$ satisfies (22) and*

$$(\rho^n(x), a^n(x), b^n(x)) \xrightarrow[n \rightarrow \infty]{} (\rho(x), a(x), b(x)) \quad \text{almost everywhere,}$$

then the conclusion of Proposition 7 is true.

In order to approximate the SBM by some classical SDEs, it is then possible to use Proposition 7 or Corollary 2 with the operator A defined in (20). We will see in the next section an application of this result to identify to different constructions of the SBM. We give now another application, in which the SBM is constructed as a renormalized limit of some SDE with a drift. The following result has first appeared in [Ros75] (See also [LG85]).

Proposition 8. *Let X be the solution to the SDE*

$$X_t = B_t + \int_0^t b(X_s) ds \quad (24)$$

where B is a Brownian motion and $b \in L^1(\mathbb{R}; \mathbb{R})$. If $X_t^n = n^{-1} X_{tn^2}$, then X^n converges in distribution to a SBM(α) with $\alpha = 1/(1 + e^\kappa)$ and $\kappa = 2 \int_{-\infty}^{+\infty} b(y) dy$.

Proof. In dimension one, the infinitesimal generator $A = \frac{1}{2\rho} \frac{d}{dx} \left(a \frac{d}{dx} \right) + b \frac{d}{dx}$ may be be written

$$A = \frac{e^{-\Phi}}{2\rho} \frac{d}{dx} \left(a e^{\Phi} \frac{d}{dx} \right) \quad \text{with} \quad \Phi(x) = \int_{\zeta}^x \frac{2b(y)}{a(y)\rho(y)} dy$$

for any $\zeta \in \mathbb{R}$. As b belongs to $L^1(\mathbb{R}; \mathbb{R})$, then one can set $\zeta = -\infty$. Assume that $a = \rho = 1$. The process (X, \mathbb{P}_x) generated by A is then the solution to (24). The infinitesimal generator of X^n is $\frac{1}{2}\Delta + nb(\cdot)n\frac{d}{dx}$.

It follows that, if $\Phi^n(x) = 2 \int_{-\infty}^x nb(ny) dy$, then $\Phi^n(x) = 2 \int_{-\infty}^{nx} b(y) dy$ and converges to κ when $x > 0$ and to 0 when $x < 0$. Thus, it is easily checked that X^n converges in distribution to the Brownian motion of parameter $\alpha = 1/(1 + e^\kappa)$. \square

4 Using the scale function and the speed measure

4.1 The scale function and the speed measure

Studying a Markov process $(X, \mathbb{P}_x)_{x \in \mathbb{R}}$ with infinitesimal generator $(L, \text{Dom}(L))$ is much more simpler in dimension one, since its behaviour can generally be described by two unique (up to multiplicative and additive constants) strictly increasing, convex functions S and V . For more results about this description, see [IM74, Bre81, RW00],...

The function S , defined on the state space of the process, is called the *scale function* and satisfies

$$\mathbb{P}_y[\tau_x < \tau_z] = \frac{S(z) - S(y)}{S(x) - S(z)} \text{ for all } x < y < z.$$

The function S is harmonic for L , and the scale function of the Brownian motion is $S(x) = x$. A process for which $S(x) = x$ is said to be *on its natural scale*. Of course, if S^{-1} is the right-continuous inverse of S , $Y = S^{-1}(X)$ is also a Markov process and is on its natural scale.

The function V is a primitive of a measure m on the state space of the process, which is called the *speed measure*. Indeed, this measure characterizes the average exit time from any given interval for the process. If $a < x < b$, then

$$\mathbb{E}_x[\tau_{(a,b)}] = \int_a^b G_{(a,b)}(x, y)m(dy)$$

where $G_{(a,b)}$ is the Green function

$$G_{(a,b)}(x, y) = \begin{cases} \frac{2(S(x)-S(a))(S(b)-S(y))}{S(b)-S(a)} & \text{if } x \leq y, \\ \frac{2(S(y)-S(a))(S(b)-S(x))}{S(b)-S(a)} & \text{if } x > y \end{cases}$$

and $\tau_{(a,b)} = \inf \{ t > 0 \mid X_t \notin (a, b) \}$. Any Markov process on its natural scale may be written as $B_{T(\cdot)}$, where B is a Brownian motion and $T(\cdot)$ is a random

time change computed from B and m (or equivalently V). If m is absolutely continuous with respect to the Lebesgue measure, then the time spent by the process at any point has zero Lebesgue measure.

The boundary conditions are also coded by S and m , but from now, we only deal with processes whose state-space is \mathbb{R} and that does not explode.

The infinitesimal generator $(L, \text{Dom}(L))$ can be described by

$$L = \frac{1}{2} \frac{d}{dV} \frac{d}{dS}, \quad (25)$$

where

$$\frac{df}{d\varphi}(x) = \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon) - f(x)}{\varphi(x + \varepsilon) - \varphi(x)}$$

for $\varphi = S$ ou V . At a point x where φ is differentiable, $\frac{df}{d\varphi}(x) = f'(x)/\varphi'(x)$.

The following theorem provides a way to prove the convergence of stochastic processes by looking at their scale functions and speed measures. It allows us to identify the process constructed by its speed measure and its scale function with a process constructed by Dirichlet forms or PDE's results.

Theorem 3 ([FW94a]). *Let $(X, \mathbb{P}_x; x \in \mathbb{R})$ be a conservative, continuous process defined by the functions (S, V) , and let $(X, \mathbb{P}_x^n; x \in \mathbb{R})_{n \in \mathbb{N}}$ be a family of conservative, continuous processes defined by (S^n, V^n) . If for all $x \in \mathbb{R}$, $S^n(x) \xrightarrow[n \rightarrow \infty]{} S(x)$ and $V^n(x) \xrightarrow[n \rightarrow \infty]{} V(x)$, then \mathbb{P}_x^n converges weakly to \mathbb{P}_x in the space of continuous functions for any starting point x .*

4.2 The Skew Brownian motion

Comparing (25) and (20) suggests to construct the process $(X, \mathbb{P}_x)_{x \in \mathbb{R}}$ with scale function and integrated speed measures

$$S(x) = \begin{cases} \alpha^{-1}x & \text{if } x \geq 0, \\ (1 - \alpha)^{-1}x & \text{if } x < 0, \end{cases} \text{ and } V(x) = \begin{cases} \alpha x & \text{if } x \geq 0, \\ 1 - \alpha x & \text{if } x < 0. \end{cases} \quad (26)$$

Remark 3. If (S, V) are the scale function and the integrated speed measure of a diffusion process, then $(\kappa S + \lambda, \kappa^{-1}V + \lambda)$ for $\kappa > 0$ and $\lambda \in \mathbb{R}$ are also possible scale functions and integrated speed measure.

The question is to know if $(X, \mathbb{P}_x)_{x \in \mathbb{R}}$ is equal to the process previously constructed with Feller semi-groups.

Proposition 9. *The scale function S and the integrated speed measure V of the SBM(α) are given by (26).*

Proof. Let $(a^n, \rho^n, b^n)_{n \in \mathbb{N}}$ be a family of measurable functions satisfying (22). Let us consider

$$A^n = \frac{1}{2\rho^n} \frac{d}{dx} \left(a^n \frac{d}{dx} \right) + b^n \frac{d}{dx}. \quad (27)$$

If a^n , ρ^n and b^n are smooth enough, then it is known that A^n is the infinitesimal generator of a continuous stochastic process $(X, \mathbb{P}_x^n)_{x \in \mathbb{R}}$ that can be described either by the functions

$$S^n(x) = \int_0^x \frac{\exp(-h^n(y))}{a^n(y)} dy \text{ and } V^n(x) = \int_0^x \frac{\exp(-h^n(y))}{\rho^n(y)} dy$$

$$\text{with } h^n(x) = 2 \int_0^x \frac{b^n(y)}{\rho^n(y)a^n(y)} dy,$$

or by its semi-group, which is a Feller semi-group. Hence, if almost everywhere, $a^n(x) \xrightarrow[n \rightarrow \infty]{} a(x)$, $\rho^n(x) \xrightarrow[n \rightarrow \infty]{} 1/a(x)$ and $b^n(x) \xrightarrow[n \rightarrow \infty]{} 0$, then Proposition 7 together with Theorem 3 proves that the processes generated by A via its Feller semi-group and the one by its scale function and speed measure are equal in distribution. \square

5 The SBM as solution of a SDE with local time

Combining the description of the Skew Brownian motion together with the Itô-Tanaka formula allows us to strengthen the decomposition in Theorems 1 and 2 and to express the Skew Brownian motion as the strong solution of some SDE with its local time.

Let B be a standard Brownian motion with its natural filtration (transformed to satisfy the standard hypotheses), and let Y be the process $Y_t = B_{T(t)}$, where $T(t)$ is the random time change defined by

$$t = \int_0^{T(t)} \frac{ds}{\sigma(B_s)^2}$$

and $\sigma(x) = \alpha^{-1}$ if $x \geq 0$ and $\sigma(x) = (1 - \alpha)^{-1}$ if $x < 0$. Thus, the inverse $r(x)$ of $\sigma(x)$ is $r(x) = \alpha$ if $x \geq 0$ and $1 - \alpha$ if $x < 0$. It follows that $X = r(Y)$ is a diffusion process whose scale function and integrated speed measure are given by (S, V) defined in (26).

With the symmetric Itô-Tanaka formula, it follows that

$$X_t = X_0 + B_t + \beta L_t^0(X) \text{ with } \beta = 2\alpha - 1, \quad (28)$$

where $L_t^0(X)$ is the symmetric local time of X at 0, which is defined by

$$L_t^0(X) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{X_s \in [-\varepsilon, \varepsilon]\}} ds.$$

Theorem 4 ([HS81]). *The SDE (28) has a unique strong solution if and only if $\alpha \in [0, 1]$. If the strong solution exists, then it is the Skew Brownian motion of parameter α .*

Remark 4. In [LG85], J.-F. Le Gall extended results on strong existence and uniqueness to SDEs of type $dX_t = \sigma(X_t) dB_t + \int_{\mathbb{R}} \nu(dx) dL_t^x(X)$ under rather general conditions on the coefficient σ and the measure ν . In this article, we reader will also find another way to prove that the diffusion process generated by (27) also converges to the solution of (28).

More recently, R. Bass and Z.-Q. Chen also considered this kind of equation in [BC05] and extended some of the results of [LG85].

About the left, right and symmetric local time. As $t \mapsto L_t^0(X)$ is a continuous additive functional that increases only when t belongs to $\mathcal{Z} = \overline{\{s \geq 0 \mid X_s = 0\}}$, any continuous additive functional $t \mapsto \eta_t$ with the same property is almost surely equal to $C_\eta L^0(X)$, where C_η is a constant that depends only on η (see Proposition VI.45.10 in [RW00]).

Among the continuous additive functionals having this property, two are of importance: the left and the right local time at 0 that are defined respectively by

$$L_t^{0-}(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{X(s) \in [-\varepsilon, 0]\}} ds \quad \text{and} \quad L_t^{0+}(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{\{X(s) \in [0, \varepsilon]\}} ds.$$

By definition,

$$L_t^0(x) = \frac{L_t^{0-}(X) + L_t^{0+}(X)}{2}, \quad (29)$$

and it follows from the Itô-Tanaka formula that

$$L_t^{0+}(X) - L_t^{0-}(X) = 2 \int_0^t \mathbf{1}_{\{X_s=0\}} dX_s. \quad (30)$$

As the quadratic variation of $\int_0^\cdot \mathbf{1}_{\{X_s=0\}} dB_s$ is equal to 0, this term is also equal to 0. But $\int_0^t \mathbf{1}_{\{X_s=0\}} dL_s^0(X) = L_t^0(X)$. It follows that

$$L_t^{0+}(X) = 2\alpha L_t^0(X) \quad \text{and} \quad L_t^{0-}(X) = 2(1 - \alpha)L_t^0(X). \quad (31)$$

For all $t \geq 0$, $x \mapsto L_t^{x+}(X)$ is right-continuous with a left-limit at each point. Moreover, $L_t^{x-}(X) = \lim_{\varepsilon \rightarrow 0, \varepsilon < 0} L_t^{(x-\varepsilon)^+}(X)$. Outside 0, the process X

behaves like the Brownian motion and thus $x \mapsto L_t^{x+}(X)$ is continuous on $\mathbb{R} \setminus \{0\}$. With equalities (31), this proves the following theorem, initially due to J. Walsh.

Theorem 5 ([Wal78]). *Unless $\alpha = 1/2$ (the Brownian case), the map $x \mapsto L_t^{x+}(X)$ is discontinuous at 0 and continuous elsewhere.*

6 The martingale problem

Of course, one could wish to describe the distribution of the SBM(α) with the help of the martingale problem (see [SV79, KS91] among many other books).

Due to the presence of the local time in the SDE describing the SBM(α), this is not practical here. Yet we will be able to define a martingale problem and show that it is well posed.

Let us denote by $D(\alpha)$ the set of continuous, bounded functions f on \mathbb{R} with two bounded derivatives f' and f'' on \mathbb{R}^* such that $f''(0+)$ and $f''(0-)$ exists, and $\alpha f'(0+) = (1 - \alpha)f'(0-)$.

From the symmetric Itô-Tanaka applied to a SBM(α), the diffusion process $(X, \mathcal{F}_t, \mathbb{P}_x; t \geq 0, x \in \mathbb{R})$ which is solution to $dX_t = dB_t + \beta L_t^0(X)$ with $\beta = 2\alpha - 1$ satisfies \mathbb{P}_x -almost surely

$$\forall f \in D(\alpha), f(X_t) = f(x) + \int_0^t f'(X_s) dB_s + \frac{1}{2} \int_0^t f''(X_s) ds$$

for all $x \in \mathbb{R}$. Thus,

$$M_t^f = f(X_t) - f(x) - \frac{1}{2} \int_0^t f''(X_s) ds \tag{32}$$

is a $(\mathcal{F}_t)_{t \geq 0}$ -martingale under \mathbb{P}_x .

We denote by $(X_t)_{t \geq 0}$ the canonical process on $\mathcal{C}(\mathbb{R}_+; \mathbb{R})$. Let us define by $(\mathcal{B}_t)_{t \geq 0}$ the filtration $\mathcal{B}_t = \sigma(X_s; s \leq t)$.

Thus, let us now state the martingale problem.

Definition 1. A probability measure \mathbb{Q} on $(\mathcal{C}(\mathbb{R}_+; \mathbb{R}), \text{Bor}(\mathcal{C}(\mathbb{R}_+; \mathbb{R})))$ for which for all $0 \leq s \leq t$,

$$\mathbb{E}^{\mathbb{Q}} \left[f(X_t) - f(X_s) - \frac{1}{2} \int_s^t f''(X_r) dr \middle| \mathcal{B}_s \right] = 0, \forall f \in D(\alpha),$$

is said to be a *solution of the martingale problem associated to $D(\alpha)$* .

Proposition 10. *The distribution \mathbb{P}_x of the SBM(α) is the unique solution of the martingale problem associated to $D(\alpha)$ satisfying $\mathbb{P}_x[X_0 = x] = 1$ (In other words, the martingale problem is well posed).*

Proof. We have already seen that the SBM(α) is solution to the martingale problem associated to $D(\alpha)$, except for the replacement of the filtration of the Brownian motion by $(\mathcal{B}_t)_{t \geq 0}$. Yet, this last point may be treated as usual (see for example Remark 5.4.16 in [KS91, p. 320]).

Let \mathbb{Q} be a solution of the martingale problem with $\mathbb{Q}[X_0 = x] = 1$. The uniqueness follows also from the standard considerations, but with two supplementary points: one has to prove that X is a \mathbb{Q} -semi-martingale, which is not automatic since any function f with $f(x) = x$ around 0 does not belongs to $D(\alpha)$ unless $\alpha = 1/2$. And then, one has to use an extension of the Itô-Tanaka formula for the process (t, X_t) .

Let us consider $f \in \mathcal{C}_b^2(\mathbb{R}; \mathbb{R})$ with bounded first and second derivative. For a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of positive real numbers increasing to $+\infty$, we define a sequence of functions $(h_n)_{n \in \mathbb{N}}$ by

$$h_n(x) = \begin{cases} \frac{\gamma}{\lambda_n} e^{-\lambda_n x}, & x > 0, \\ \frac{\gamma}{\lambda_n}, & x \leq 0 \end{cases} \quad \text{with } \gamma = f'(0) \frac{2\alpha - 1}{\alpha}.$$

It is clear that $h_n(x)$ converges uniformly to 0, and that $f_n(x) = f(x) + h_n(x)$ belongs to $D(\alpha)$ for each $n \in \mathbb{N}$. Moreover, $|h'_n(x)| \leq \gamma$ for all $x \in \mathbb{R}$. The second derivative h''_n is discontinuous at 0 but converges uniformly to 0 on $(-\infty, 0) \cup [\varepsilon, +\infty)$ for any $\varepsilon > 0$.

Hence, let us set $V_t^n = \frac{1}{2} \int_0^t f''_n(X_s) ds$ and M_t^n the \mathbb{Q} -martingale $M_t^n = f_n(X_t) - f_n(X_0) - V_t^n$. As f_n^2 also belongs to $D(\alpha)$, it is easily proved by comparing $(f_n(X_t) - f_n(X_0))^2$ and $f_n(X_t)^2 - f_n(X_0)^2$ that for all $\varepsilon > 0$,

$$\langle M^n \rangle_t = \int_0^t f'_n(X_s)^2 ds.$$

Hence, it is clear that for any $T > 0$,

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left[\langle M^n \rangle_T + \frac{1}{2} \int_0^T \mathbf{1}_{\mathbb{R} \setminus [0, \varepsilon]}(X_s) |f''_n(X_s)| ds \right] < +\infty. \quad (33)$$

But

$$\begin{aligned} \int_0^T \mathbf{1}_{[0, \varepsilon]}(X_s) |f''_n(X_s)| ds &\leq |M_T^n| + |f_n(X_t)| + |f_n(X_0)| \\ &\quad + \int_0^T \mathbf{1}_{\mathbb{R} \setminus [0, \varepsilon]}(X_s) |f''_n(X_s)| ds. \end{aligned}$$

With the Burkholder-Davis-Gundy inequality and the boundedness of f_n , one can set $\varepsilon = 0$ in (33). According to standard results on good sequence of semi-martingales (see for example the review article [KP96]), this implies that the limit $f(X)$ of $f_n(X)$ is a semi-martingale with respect to its natural filtration under \mathbb{Q} . Hence, $f(X)$ is a semi-martingale under \mathbb{Q} . With a localization argument, one can use $f(x) = x$ to obtain that X is a semi-martingale.

Now, for $f \in D(\alpha)$, let us consider the solution $u(t, x)$ to $\partial_t u(t, x) + \frac{1}{2}u''(t, x) = 0$ on $[0, T) \times \mathbb{R}^*$, $\alpha u'(t, 0+) = (1 - \alpha)u'(t, 0-)$ for $t \in [0, T)$ and $u(T, x) = f(x)$. As $f \in D(\alpha)$, the solution $u(t, x)$ is continuous up to $t = 0$ and is continuous on $\mathbb{R}_+ \times \mathbb{R}$. Besides, it may be extended to a function of class $\mathcal{C}^{1,2}$ on $\mathbb{R}_+ \times \mathbb{R}_+$ and $\mathbb{R}_- \times \mathbb{R}_-$ (see [LRU66]). Using the extension of the Itô-Tanaka formula given in [Pes05], for $t < T$,

$$u(t, X_t) = u(0, X_0) + \int_0^t \frac{\partial u}{\partial t}(r, X_r) dr + \frac{1}{2} \int_0^t u''(r, X_r) dr, \quad \mathbb{Q}\text{-a.s.}$$

This is also true for the SBM(α) under \mathbb{P}_x . Hence with $t \nearrow T$,

$$u(0, x) = \mathbb{E}_x[f(X_T)] = \mathbb{E}^{\mathbb{Q}}[f(X_T)]$$

and this holds for any $f \in D(\alpha)$. Now, if f belongs to $\mathcal{C}_b^2(\mathbb{R}; \mathbb{R})$, the approximations of f in $D(\alpha)$ precedently constructed allow to show that $\mathbb{E}_x[f(X_T)] = \mathbb{E}^{\mathbb{Q}}[f(X_T)]$ and then that \mathbb{Q} and \mathbb{P}_x have the same one-dimensional distributions. The usual arguments adapt very well to prove then that $\mathbb{Q} = \mathbb{P}_x$. \square

7 Decomposition of the excursions' measure

Another construction is the following: Let Y be a Reflected Brownian motion, and let $\mathcal{Z} = \overline{\{s \geq 0 \mid Y_s = 0\}}$ be the closure of its zeros. The Lebesgue measure of \mathcal{Z} is zero, but this set cannot be ordered. However, the set $\mathbb{R}_+ \setminus \mathcal{Z}$ can be decomposed as a countable union $\cup_{n \in \mathbb{N}} J_n$ of intervals J_n . Each interval J_n corresponds to some excursion of Y , that is, if $J_n = (\ell_n, r_n)$,

$$Y_t > 0 \text{ for } t \in (\ell_n, r_n) \text{ and } Y_{\ell_n} = Y_{r_n} = 0.$$

Fix $\alpha \in [0, 1]$. At each J_n , we associate a Bernoulli random variable e_n which is independent from any other random variables (and the Reflected Brownian motion Y) and such that $\mathbb{P}[e_n = 1] = \alpha$ and $\mathbb{P}[e_n = -1] = 1 - \alpha$. Let X be the process given by

$$X_t = e_n Y_t \text{ if } t \in J_n.$$

Theorem 6 ([IM74]). *The process X is a Skew Brownian motion of parameter α .*

Proof. Obviously, the process X behaves like a Brownian motion on any time interval J_n . By Remark 3, its scale function S and its integrated speed measure V are of type

$$S(x) = \begin{cases} \gamma^{-1}x & \text{if } x \geq 0, \\ (1 - \gamma)^{-1}x & \text{if } x \leq 0 \end{cases} \text{ and } V(x) = \begin{cases} \gamma x & \text{if } x \geq 0, \\ (1 - \gamma)x & \text{if } x \leq 0 \end{cases}$$

where the constant γ has to be specified. If τ_1 (resp. τ_{-1}) are the first time the process X reaches 1 (resp. -1),

$$\mathbb{P}_0[\tau_1 < \tau_{-1}] = \frac{S(-1) - S(0)}{S(-1) - S(1)} = \gamma.$$

On the other hand, using the decomposition by excursions and the independence of the e_n 's with respect to Y ,

$$\begin{aligned} \mathbb{P}_0[\tau_1 < \tau_{-1}] &= \sum_{n \in \mathbb{N}} \mathbb{P}_0[\tau_1 \wedge \tau_{-1} \in J_n; X_t \geq 0 \text{ on } J_n] \\ &= \sum_{n \in \mathbb{N}} \mathbb{P}[e_n = 1] \mathbb{P}_0[\tau_1 \wedge \tau_{-1} \in J_n] = \alpha. \end{aligned}$$

Thus, $\gamma = \alpha$, and X is the Skew Brownian motion of parameter α . □

Corollary 3. *For all $t > 0$, $\mathbb{P}_0[X_t \geq 0] = \alpha$.*

Let τ be the right continuous inverse of the local time $L_t^0(X)$ of a continuous diffusion process X . Each jump of τ corresponds to some excursion of this process, that is for each t such that $\tau(t) = \tau(t-)$, there exists an interval $J_n = (\ell_n, r_n)$ such that $\ell_n = \tau(t-)$ and $r_n = \tau(t)$. Let \mathcal{U} be the set of excursions, that is the set of continuous functions f from \mathbb{R}_+ to \mathbb{R} such that $f(0) = 0$, $f(t) = 0$ for all $t \geq \zeta$ for some $\zeta > 0$ and either $f(t) > 0$ for $t \in (0, \zeta)$ or $f(t) < 0$ for $t \in (0, \zeta)$. For any $n \in \mathbb{N}$, $f(t) = X((t - r_n) \wedge (\ell_n - r_n))$ is some element of \mathcal{U} , with the life-time $\zeta = \ell_n - r_n$. Accordingly, for each $t \in \mathcal{J}$ where $\mathcal{J} = \{t \geq 0 \mid \tau(t) \neq \tau(t-)\}$, one may associate a point $f(t)$ in \mathcal{U} , corresponding to some excursion of X . A striking result due to K. Itô is that the process $(t, f(t))_{t \geq 0}$ is an homogeneous Poisson point process with values on $\mathbb{R}_+ \times \mathcal{U}$ with intensity measure $dt \times \widehat{\mathbb{P}}$. The measure $\widehat{\mathbb{P}}$ is a σ -finite measure, but which is infinite. This measure $\widehat{\mathbb{P}}$, called the excursions' measures, fully characterizes the diffusion process.

For a measurable subset Γ of \mathcal{U} such that $\widehat{\mathbb{P}}[\Gamma]$ is finite, $\widehat{\mathbb{P}}[\Gamma]$ denotes the average number of excursions in Γ per unit of local time.

We denote by \mathcal{U}^+ (resp. \mathcal{U}^-) the set of positive (resp. negative) excursions, *i.e.* the excursions f for which $f(t) > 0$ (resp. $f(t) < 0$) on $(0, \zeta)$.

On the excursions theory, see [Blu92, RW00]...

The following proposition is a direct result from the construction of the Skew Brownian motion given by the theory of excursions (see also [Blu92]).

Proposition 11. *For the Skew Brownian motion of parameter α , the measure $\widehat{\mathbb{P}}$ may be written*

$$\widehat{\mathbb{P}} = \alpha \widehat{\mathbb{P}}^+ + (1 - \alpha) \widehat{\mathbb{P}}^-, \quad (34)$$

where $\widehat{\mathbb{P}}^+$ (resp. $\widehat{\mathbb{P}}^-$) is the excursions' measure of the Reflected Brownian motion on \mathbb{R}_+ (resp. on \mathbb{R}_-), that is $(|W_t|)_{t \geq 0}$, (resp. $(-|W_t|)_{t \geq 0}$), where W is a Brownian motion.

Remark 5. This proposition has to be connected with the relations between the left, right and symmetric local times in (31).

Remark 6. For a diffusion Y with a scale function S^Y , The function $1/S^Y(x)$ is also the average number of excursions reaching the point x per unit of local time. Thus, this proposition is also coherent with the fact that if $S^B(x) = x$ is the scale function of the Brownian motion, the average number of excursions of the Skew Brownian motion reaching $x > 0$ (resp. $x < 0$) per unit of local time is equal to α/x (resp. $(1 - \alpha)/x$), and thus the scale function of the Skew Brownian motion is $S(x)$ given by (26).

Remark 7. The article [Lej01] studies the link between the coefficients in a decomposition of type (34) for a general diffusion with coefficients discontinuous at 0, that is related to the derivatives of some Green functions at 0. We recover the decomposition (34) with $L = \frac{1}{2a(x)} \frac{d}{dx} \left(a(x) \frac{d}{dx} \right)$ with $a(x) = \alpha$ if $x \geq 0$ and $a(x) = 1 - \alpha$ if $x < 0$.

Another construction of the density transition function. The reflection principle for the Brownian motion together with the decomposition of excursions allows us to give another construction of density transition function of the Skew Brownian motion, following a result of J. Walsh [Wal78]. Let τ be the first time the Skew Brownian motion hit 0, and B be a Brownian motion. Then

$$\mathbb{P}_x[X_t \in dy] = \mathbb{P}_x[X_t \in dy; \tau \leq t] + \mathbb{P}_x[X_t \in dy; \tau > t].$$

If (x, y) belongs to $\mathbb{R}_+^* \times \mathbb{R}_-$ or to $\mathbb{R}_-^* \times \mathbb{R}_+$, by the continuity of the path, $\mathbb{P}_x[X_t \in dy; \tau > t] = 0$. If (x, y) belongs to $\mathbb{R}_+^* \times \mathbb{R}_+$ or to $\mathbb{R}_-^* \times \mathbb{R}_-$, then

$$\mathbb{P}_x[X_t \in dy; \tau > t] = \mathbb{P}_x[B_t \in dy; \tau > t] = (p(t, x, y) - p(t, x, -y)) dy,$$

where $p(t, x, y)$ is the transition density function of the Brownian motion. On the other hand, by the reflection principle and the construction of the Skew Brownian motion,

$$\mathbb{P}_x[X_t \in dy; \tau \leq t] = \begin{cases} \alpha \mathbb{P}_x[B_t \in -dy] & \text{if } x > 0, y \geq 0, \\ (1 - \alpha) \mathbb{P}_x[B_t \in dy] & \text{if } x \geq 0, y < 0, \\ \alpha \mathbb{P}_x[B_t \in dy] & \text{if } x \leq 0, y \geq 0, \\ (1 - \alpha) \mathbb{P}_x[B_t \in -dy] & \text{if } x < 0, y < 0. \end{cases}$$

Hence, after a short computation, this gives us the expression (17).

8 Approximation by random walks

As one can expect, a Skew Brownian motion may be approximated by the following random walk: Let $(S_k)_{k \geq 0}$ be the random walk with probability transition

$$\begin{cases} \mathbb{P}[S_{k+1} = i + 1 \mid S_k = i] = \mathbb{P}[S_{k+1} = i - 1 \mid S_k = i] = \frac{1}{2} \text{ if } i \neq 0, \\ \mathbb{P}[S_{k+1} = 1 \mid S_k = 0] = \alpha, \\ \mathbb{P}[S_{k-1} = -1 \mid S_k = 0] = 1 - \alpha. \end{cases}$$

The following theorem may be found in [HS81], and then in [LG85] and in [CSY04] in a more general setting. P. Étoré uses in [Éto05] this result to construct a numerical scheme and compute its speed of convergence (see Section 10.8.3).

Theorem 7 ([HS81]). *For all $t \geq 0$, let $X_t^n = \frac{1}{n} S_{\lfloor n^2 t \rfloor}$. Then X^n converges in distribution in the space of continuous functions to the SBM(α) X .*

Proof. We prove only the convergence of the marginals. We remark first that

$$\mathbb{P}[S_k = i] = \begin{cases} \alpha \mathbb{P}[|S_k| = i] & \text{if } i > 0 \\ (1 - \alpha) \mathbb{P}[|S_k| = -i] & \text{if } i < 0. \end{cases}$$

Hence,

$$\mathbb{E}[\exp(i\lambda X_t^n)] = \alpha \psi^n(\lambda) + (1 - \alpha) \psi^n(\lambda) \text{ with } \psi(\lambda) = \mathbb{E}[\exp(i\lambda |X_t^n|)].$$

Using the results of Section 7, one has also that

$$\mathbb{E}[\exp(i\lambda X_t)] = \alpha\psi(\lambda) + (1 - \alpha)\psi(\lambda) \text{ with } \psi(\lambda) = \mathbb{E}[\exp(i\lambda|X_t|)].$$

The results follows from the convergence of the normalized reflected random walk $|X^n|$ to the reflected Brownian motion $|X_t|$ given by the Donsker theorem. \square

Indeed, in [CSY04], it is proved that (X_t^n, Y_t^n) converges in distribution to $(B_t^\alpha, \int_0^t f(B_s^\alpha) dB_s^\alpha)_{t \geq 0}$, where $(X_t^n)_{t \geq 0}$ and $(Y_t^n)_{t \geq 0}$ are the linear interpolations of $X_{k/n}^n = (\sqrt{n})^{-1}S_{k/n}$ and $Y_{k/n}^n = \sum_{i=1}^k f(X_{(i-1)/n}^n)(X_{i/n}^n - X_{(i-1)/n}^n)$, and f is a measurable, bounded function. This allows to prove some results related to the “horizontal-vertical” random walk.

Remark 8. In [HS81], it is also noted that if given $S_i = 0$, the distribution of S_{i+1} has the distribution of an integrable random variable Z with values in \mathbb{Z} , then $n^{-1}S_{[n^2t]}$ converges to a SBM(α) with $\alpha = \mathbb{E}[Z^+]/\mathbb{E}[|Z|]$.

9 A “follow the leader” construction

We present here another construction based on techniques coming from the theory of continuous multi-armed bandits (see [MSV90] for example). Indeed, we present here in a simpler case the proof given in [BBKM00] that deals to a variably skewed Brownian motion (see Section 10.5. Some of the results given in [CSY04] are also related to this construction.

Let us set $\alpha \in (0, 1)$, $\beta = 2\alpha - 1 \in (-1, 1)$ and $\gamma = (1 + \beta)/(1 - \beta) > 0$.

The idea is to construct a process (Z_t^1, Z_t^2, U_t^1) where (Z_t^1, Z_t^2) is a bidimensional process that moves either horizontally or vertically when away from the line $\Upsilon = \{(x, y) \mid y = \gamma x\}$, and U_t^1 is linked to the local time of the SBM(α). The horizontal (resp. vertical) displacements of Z away from $(U_t^1, \gamma U_t^1)$ corresponds to a positive (resp. negative) excursions of the SBM(α) arising when its local times takes the value $L_t = (1 + \gamma)U_t^1$. So, we can read “graphically” the signs and the heights of the excursions on the local time scale: See Figure 1.

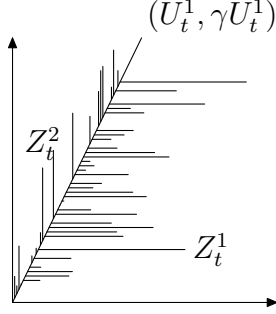


Figure 1: A representation of the excursions of the SBM(α)

The heuristic idea of the construction is the following. We consider two independent Brownian motions B^1 and B^2 whose filtrations are denoted by $(\mathcal{F}_t^1)_{t \geq 0}$ and $(\mathcal{F}_t^2)_{t \geq 0}$, and we construct two time changes $T^1(t)$ and $T^2(t)$. When one of the Brownian motion, say B^1 reaches a local maximum, then we let $T^1(t)$ increasing at speed one until B^1 reaches again its maximum. During this time, the other time change $T^2(t)$ remains constant. By this way, we select excursions of the Brownian motion below a given level, that are equal in distribution to excursions of the Brownian motion below 0. The excursions of Z away from Υ will then be constructed from these excursions of B^1 and B^2 . Of course, the irregularities of the paths of the Brownian motion ensure that either $T^1(t)$ or $T^2(t)$ increases. The skewness is introduced by comparing not B^1 and B^2 , but γB^1 and B^2 .

We now turn to the rigorous construction.

Let us define a subset D of \mathbb{R}_+^2 by

$$D = \left\{ (s_1, s_2) \in \mathbb{R}_+^2 \mid \gamma \sup_{u_1 \leq s_1} B^1(u_1) > \sup_{u_2 > s_2} B^2(u_2) \right\},$$

whose closure \overline{D} has the following properties: (a) $\mathbb{R}_+ \times \{0\} \subset D$; (b) $\{s \in \overline{D} \mid \in\} \mathcal{F}_s$, where $(\mathcal{F}_{(s,t)})_{(s,t) \in \mathbb{R}_+^2}$ is the multi-parameter filtration defined by $\mathcal{F}_{(s,t)} = \mathcal{F}_s^1 \vee \mathcal{F}_t^2$; (c) If $(s_1, s_2) \in D$, then

$$\{(u_1, u_2) \in \mathbb{R}_+^2 \mid u_1 \geq s_1, 0 \leq u_2 \leq s_2\} \subset \overline{D}.$$

It follows from these properties that \overline{D} is described by the part of its boundary that intersects $(\mathbb{R}_+^*)^2$: See Figure 2. In addition, there exists a unique process $T(t) = (T^1(t), T^2(t))$ — called a *strategy* — such that [Wal81] (a) $T(t) = (0, 0)$ and T^1, T^2 are non-decreasing; (b) $T^1(t) + T^2(t) = t$ for $t \geq 0$; (c) for all $s_1, s_2 \geq 0$, the set $\{(s_1, s_2) \mid T^1(t) \leq s_1, T^2(t) \leq s_2\}$ is $\mathcal{F}_{(s_1, s_2)}$ -measurable; (d) the graph of $(T(t))_{t \geq 0}$ parametrizes the boundary of \overline{D} .

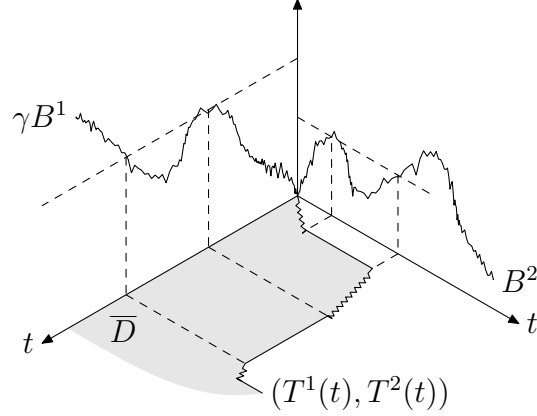


Figure 2: Construction of the time changes

Indeed, $T^1(t)$ increases at rate 1 when $B_{T^2(t)}^2 > \gamma B_{T^1(t)}^1$ and $T^2(t)$ increases at rate 1 when $B_{T^2(t)}^2 < \gamma B_{T^1(t)}^1$. In other words, the strategy “follows the leader” between γB^1 and B^2 .

Let us now consider a maximal time interval (t, t') with $t' > t$ such that T^1 is strictly increasing on it. As $(T^1(t), T^2(t)) \in \overline{D}$ this implies that

$$\gamma \sup_{s_1 \in [0, T^1(t)]} B_{s_1}^1 = \sup_{s_2 \in [0, T^2(t)]} B_{s_2}^2$$

and then B^1 is below $B_{T^1(t)}^1$ on $(T^1(t), T^1(t'))$ with $B_{T^1(t')}^1 = B_{T^1(t)}^1$. Thus, the process $B_{T^1(\cdot)}^1$ performs an excursion below the level $B_{T^1(t)}^1$ on (t, t') . Hence, at this time t , $B_{T^1(t)}^1 = \sup_{s \in [0, T^1(t)]} B_s^1$. Besides, since $T(t)$ belongs to the boundary of \overline{D} , there exists no time $\hat{t} < T^2(t)$ such that $\sup_{s \in [0, \hat{t}]} B_s^2 = B_{T^1(t)}^1$. This means that $B_{T^2(t)}^2 = \sup_{t \in [0, T^2(t)]} B_s^2$. Hence, at time t , $B_{T^1(t)}^1$ has a local maximum and $B_{T^2(t)}^2$ reaches its maximum on $[0, t]$.

We now set for $r \geq 0$, $Z_r^i = B_{T^i(r)}^i$ for $i = 1, 2$ and $U_r^1 = \sup_{s \in [0, r]} Z_s^1$, $U_r^2 = \gamma U_r^1$. From the previous considerations,

$$\gamma U_t^2 = \gamma U_{t'}^1 = \gamma Z_t^1 = \gamma Z_{t'}^1 = Z_t^2 = Z_{t'}^2 = U_t^2 = U_{t'}^2.$$

Thus, on (t, t') , Z^1 performs an excursion below the level U_t^1 and then Z moves horizontally away from Υ , with $Z_t = Z_{t'} = U_t \in \Upsilon$.

A similar study could be done for T^2 , and we see now how to define our candidate to be a SBM:

$$X_t = (\gamma U_t^1 - Z_t^2) - (U_t^1 - Z_t^1) \text{ and } L_t = (1 + \gamma)U_t^1.$$

Proposition 12. *The process X is a SBM(α) and L is its local time.*

Proof. Let $(\mathcal{G}_t)_{t \geq 0}$ be the filtration defined by $\mathcal{G}_t = \mathcal{F}_{T(t)}$. Let us remark first that $B_t = -Z_t^2 + Z_t^1$ has for quadratic variation $\langle B \rangle_t = T^1(t) + T^2(t) = t$ from the very definition of the strategy and is $(\mathcal{G}_t)_{t \geq 0}$ -adapted. By the Paul Lévy's theorem, B is a Brownian motion and then $X_t = B_t + \beta L_t$ with $\beta = 2\alpha - 1$. The process $|X|$ is equal to the distance of Z from Υ and then, by construction,

$$|X_t| = \widehat{B}_t + L_t$$

with $\widehat{B}_t = -B_{T^1(t)}^1 - B_{T^2(t)}^2$. Again with computing the quadratic variation of this later process, \widehat{B} is a $(\mathcal{G}_t)_{t \geq 0}$ -Brownian motion. Besides,

$$-\widehat{B}_t = Z_t^1 + Z_t^2 = \int_0^t \operatorname{sgn}(X_s) dB_s = \int_0^t \operatorname{sgn}(X_s) dX_s$$

where $\operatorname{sgn}(x) = 1$ if $x > 0$, $\operatorname{sgn}(x) = -1$ if $x < 0$ and $\operatorname{sgn}(x) = 0$ if $x = 0$, as $\int_0^t \mathbf{1}_{\{Z_s \notin \Upsilon\}} dL_s = 0$ if $X_t \neq 0$. Thus, with the Itô-Tanaka formula, L_t is the symmetric local time of X . \square

10 Applications and extensions

10.1 General constructions of “Skew” diffusion processes

The general theory of one-dimensional diffusion processes gives all the tools to construct rather easily a “skew” diffusion process. There are basically two ways to do so.

First, consider diffusion processes living on $[0, +\infty)$, or symmetric processes, and the sign of each excursion is changed with independent Bernoulli random variables. A way to do consists in using the description of a strong Markov process in term of a *minimal process* (*i.e.*, a process that is killed when it reaches 0) and an *entrance law* (see [Blu92] for a whole account on this theory). See [VA96] for related results and constructions.

See for example [BPY89a, BPY89b] for the construction of a Skew Bessel process by this way.

Indeed, the symmetry of the process plays an important role in the previous construction, so here is another one. Let X be a process with a scale function S and a speed measure m , x_0 in the state space of X and $\gamma > 0$ let us define a new process X^γ by the scale measure S^γ and speed measure m^γ defined by

$$S^\gamma(x) = \begin{cases} \gamma S(x) & \text{if } x \geq x_0, \\ S(x) & \text{if } x < x_0, \end{cases} \quad \text{and } m^\gamma(dx) = \begin{cases} \gamma^{-1} m(dx) & \text{if } x \geq x_0, \\ m(dx) & \text{if } x < x_0 \end{cases}$$

if S is such that $S(x_0) = 0$. As the behavior of a process around a point x is specified locally by its scale function and speed measure and the process generated by (S, m) is the same as the process generated by $(\lambda S, \lambda^{-1}m)$ for all $\lambda > 0$, the process X^γ behaves like X except at x_0 . The “skewness” of this process may be appreciated by the result stated in Remark 6.

The article [DGS05] shows how to construct an *asymmetric Bessel process* by this method and provides applications to the computation of Asian options with discontinuous volatility.

10.2 Walsh’s Brownian motion

Initially introduced by J. Walsh in [Wal78], the *Walsh’s Brownian motion* is a diffusion process on a set of n rays in \mathbb{R}^2 emanating from 0. To each ray I_i is associated a weight α_i corresponding heuristically to the probability for the process to go in this ray. On each ray, the process behaves like a Brownian motion.

Of course, due to the irregularities of the trajectories of the Brownian motion, this description is a non-sense, but this process may be described by its excursions’ measure

$$\widehat{\mathbb{P}} = \sum_{i=1}^n \alpha_i \widehat{\mathbb{P}}^i, \text{ with } \sum_{i=1}^n \alpha_i = 1 \text{ and } \alpha_i \in (0, 1),$$

where $\widehat{\mathbb{P}}^i$ is the excursion measure of the Reflected Brownian motion on the ray I_i . On this process, see also [BPY89a].

The Skew Brownian motion corresponds to the case $n = 2$. The *spider martingale* generalizes the notion of Walsh’s Brownian motion. This object has given rise to an abundant literature on Brownian filtrations, especially by giving a negative answer to the question “if a Brownian motion is adapted to some filtration, is this filtration generated by a Brownian motion?” (See [Tsi97], [BÉK⁺98], [Yor97, Sect. 17, p. 103] and the cited works within), but this goes far beyond the scope of this article.

10.3 A generalized arcsine law

For the Walsh’s Brownian motion of Section 10.2, let $A^i(t)$ be the occupation time up to time t of the ray I_i , that is $A^i(t) = \int_0^t \mathbf{1}_{\{X(s) \in I_i\}} ds$. As for the Brownian motion, $A^i(t) = tA^i(1)$. The arcsine law for the Brownian motion may be extended to this case.

Theorem 8 ([BPY89b]). *Let T_1, \dots, T_n be independent, stable random variables of index $1/2$. Then*

$$(A_1(1), \dots, A_n(1)) \stackrel{\text{dist.}}{=} \left(\frac{\alpha_i T_i}{\sum_{j=1}^n \alpha_j T_j} \right)_{i=1, \dots, n}.$$

More is proved in [BPY89b], and Skew Bessel processes are also considered.

Assume that $n = 2$ and that T_1, T_2 are two independent copies of a stable random variable with exponent $\gamma \in (0, 1)$ (i.e., $\mathbb{E}[\exp(-\zeta T_i)] = \exp(-\zeta^\gamma)$). Define by $\xi_{\gamma, \alpha}$ the random variables

$$\xi_{\gamma, \alpha} = \frac{\alpha^{1/\gamma} T_1}{\alpha^{1/\gamma} T_1 + (1 - \alpha)^{1/\gamma} T_2}.$$

The distribution function of $\xi_{1/2, \alpha}$ is

$$\mathbb{P}[\xi_{1/2, \alpha} \leq x] = \frac{2}{\pi} \arcsin \sqrt{\frac{x}{x + (\alpha/(1 - \alpha))^2(1 - x)}}.$$

For $\alpha = 1/2$, this is the arcsine distribution.

In Theorem 8, the occupation time of \mathbb{R}_+ or \mathbb{R}_- at time 1 of the Skew Brownian motion is a random variable of type $\xi_{1/2, \alpha}$.

If the Skew Brownian motion is replaced by a Skew Bessel process of dimension $2 - 2\nu$, then the occupation time at time 1 is distributed as one of the $\xi_{\nu, \alpha}$ (See [Wat95, BPY89b]). In [Wat95], it is also proved that for a general one-dimensional diffusion, the possible limit of $A_{\mathbb{R}^+}(t)/t$, where $A_{\mathbb{R}^+}(t)$ is the occupation time of \mathbb{R}^+ , is a random variable of type $\xi_{\nu, \gamma}$.

In the case of a Brownian motion on a graph (see below Section 10.8.2), analytical considerations allows also to compute the Laplace transforms of the occupation time on each of the edges of the graph [Des02].

10.4 The Skew Brownian motion as a flow

In a series of articles [BBKM01, BC01, BK04], K. Burdzy and his co-authors study the Skew Brownian motions as a flow, that is the family $(X^{s,x})_{s \geq 0, x \in \mathbb{R}}$ of stochastic processes defined by

$$X_t^{s,x} = x + B_t - B_s + \beta \ell_t^{s,x}, \quad t \geq s$$

where $\ell_t^{s,x} = L_{t-s}^0(X^{s,x})$ with the same underlying Brownian motion B . Mainly, they have studied coalescence of such solutions, and existence of multiple solutions.

10.5 Generalization of the Skew Brownian motion (I)

A possible generalization of the Skew Brownian motion is to use a variable “skewness coefficients”, which leads to an SDEs with local time of type

$$X_t = x + B_t + \beta(L_t^0(X)),$$

where β is a differentiable function on \mathbb{R}_+ with $\beta'(x) \in (-1, 1)$ for all $x \in \mathbb{R}_+$. This SDE has been studied in [BBKM00].

10.6 Generalization of the Skew Brownian motion (II)

Another natural generalization is the solution of the SDE

$$X_t = x + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds + \int_0^t \alpha(s) dL_s^0(X),$$

whose pathwise uniqueness was proved in [Wei82] under the assumption that σ is uniformly elliptic and bounded, Lipschitz continuous, b is bounded and α is continuous with $|\alpha| \leq 1/2$.

10.7 Generalization of the Skew Brownian motion (III)

The SDE solved by the Skew Brownian motion may be generalized as a SDE of type

$$X_t = x + \int_0^t \sigma(X_s) dB_s + \int_{\mathbb{R}} \nu(dx) L_t^x(X), \quad (35)$$

where ν is a signed measure.

The existence of uniqueness of such SDEs have been studied in [LG82, LG85, ES85]. In particular, it is shown there that there exists a unique strong solution to (35) provided that σ is locally of bounded variation, uniformly elliptic and bounded (that is $0 < \lambda < \sigma(x) < \Lambda, \forall x \in \mathbb{R}$), the total mass of ν is finite and that $|\nu(\{x\})| < 1$ for any point $x \in \mathbb{R}$. A recent article [BC05] of R. Bass and Z.-Q. Chen refines some of the results of J.-F. Le Gall. In particular, it is shown that (35) has no solution if $|\nu(\{x_0\})| > 1$ for some $x_0 \in \mathbb{R}$ (the case $|\nu(\{x_0\})| = 1$ corresponds of course to a reflection).

Indeed, as the local time plays a central role in the theory of one-dimensional diffusion processes, SDE with local times are also considered in the study of diffusions with *singular drift*, in order to get some diffusions under weak hypotheses (see for example [BC03, FRW03, BC05], ...), Dirichlet processes [EW99, BC02], ... In addition, they also appear as a tool to study the “critical cases” for results on strong/weak existence and uniqueness of solutions of SDE (see [LG85, ES85] for example).

This kind of process appears naturally when one studies diffusion processes generated by differential operators with discontinuous coefficients: see Section 10.8.

10.7.1 Spatial change of variable

The special case where $\nu = \kappa\delta_0$ and σ is constant on \mathbb{R}_+ and \mathbb{R}_-^* has been studied in [Ouk90]. In this article, the Itô-Tanaka formula applied to functions of type $f(x) = px^+ + qx^-$ plays a key role. Indeed, we obtain a family of diffusion with constant coefficients on \mathbb{R}_+^* and \mathbb{R}_-^* .

Of course, in the same way the Itô formula is a very important tool for SDEs, the Itô-Tanaka formula is essential for studying SDEs with local time. Indeed, the family of SDEs of type (35) remains stable under applications of continuous functions that are piecewise of class $\mathcal{C}^2(\mathbb{R}; \mathbb{R})$ and whose first and second derivative are left-continuous with a right-limit at each point.

In particular, with a good choice of a function f , it is possible to obtain $Y_t = f(X_t)$ either as a solution of some SDE without local time but with discontinuous coefficients, or as a process which may be written locally $dY_t = dB_t + b(Y_t) dt + \kappa dL_t^y(Y)$ around a point y where $\nu(\{y\}) \neq 0$, if $\nu(\{z\}) = 0$ for all z close enough to y .

From a numerical point of view, these two changes of variables are of interest in order to simulate the process X , as shown in Section 10.8.3. In [Bar88], M. Barlow used this kind of transform to construct counter-examples to the uniqueness of strong solutions to some SDEs.

10.7.2 Random time change

Random time change is also a tool to study SDE with local time, as shown first in [ES85]. An application of a random time change where the Skew Brownian motion is used instead of Brownian motion may be found in [Mar04].

10.8 Diffusions with discontinuous coefficients and diffusions on graphs

We present now in this section some results about PDEs with discontinuous coefficients, and diffusion equation on graphs. This gives the general framework to consider both a large subclass of SDEs of type (35) and extensions of the Walsh's Brownian motion presented in Section 10.2.

10.8.1 Infinitesimal generators and associated diffusion processes

Let a and ρ be two measurable functions on \mathbb{R} such that $0 < \lambda \leq a(x) \leq \Lambda$ and $\lambda \leq \rho(x) \leq \Lambda$ for all $x \in \mathbb{R}$. Let also b be a measurable bounded function on \mathbb{R} . We assume that a, b, ρ are of bounded variation on \mathbb{R} and are right-continuous with left-limit at each point.

Let $(L, \text{Dom}(L))$ be the differential operator

$$L = \frac{\rho}{2} \frac{d}{dx} \left(a \frac{d}{dx} \right) + b \frac{d}{dx}, \quad \text{Dom}(L) = \{ f \in \mathcal{C}(\mathbb{R}; \mathbb{R}) \mid Lf \in \mathcal{C}(\mathbb{R}; \mathbb{R}) \}. \quad (36)$$

Using scale functions and speed measures, or analytical results on the fundamental solution of $\frac{\partial}{\partial t} - L$, or Dirichlet forms, it can be proved that $(L, \text{Dom}(L))$ is the infinitesimal generator of a continuous, conservative, strong Markov process $(X_t, t \geq 0; \mathcal{F}_t, t \geq 0; \mathbb{P}_x, x \in \mathbb{R})$. Using the arguments of [LG85] (see also [Lej00, Éto05]), one gets that X is the strong solution to the SDE

$$X_t = x + \int_0^t \sqrt{a(X_s)\rho(X_s)} dB_s + \int_0^t \frac{1}{2} a'(X_s)\rho(X_s) ds + \int_0^t b(X_s) ds + \int_{\mathbb{R}} \nu(dy) L_t^y(X), \quad (37)$$

where $L_t^y(X)$ is the symmetric local time of X at the point y and

$$\nu = \sum_{n \geq 0} \frac{a(x_{n+}) - a(x_{n-})}{a(x_{n+}) - a(x_{n-})} \delta_{x_n} \text{ if } \{x_n\}_{n \in \mathbb{N}} = \{x \in \mathbb{R} \mid a(x) \neq a(x-)\}.$$

This is particular case of (35) encountered in Section 10.7.

The article [GOO87] shows how to compute explicitly the density transition function and the Green function of this process.

10.8.2 Transmission conditions and diffusions on a graph

The solution of the parabolic PDE $\frac{\partial u}{\partial t} = Lu$ with $u(0, x) = f(x)$, where L is given by (36), has to be understood as a *weak solution*, that is as a function $u(t, x) \in \mathcal{C}(0, T; L^2(\mathbb{R})) \cap L^2(0, T; H^1(\mathbb{R}))$ such that for all $\psi \in \mathcal{C}^\infty([0, T]; \mathbb{R})$,

$$\begin{aligned} \int_0^T \int_{\mathbb{R}} \frac{\partial \psi(t, x)}{\partial t} u(t, x) \rho(x)^{-1} dx dt - \int_0^T \int_{\mathbb{R}} b(x) \frac{\partial u(t, x)}{\partial x} \psi(t, x) dx dt \\ + \int_0^T \int_{\mathbb{R}} a(x) \frac{\partial u(t, x)}{\partial x} \frac{\partial \psi(t, x)}{\partial x} dx dt = - \int_{\mathbb{R}} \varphi(x) \psi(0, x) \rho(x)^{-1} dx. \end{aligned}$$

But, as previously, it is also the solution to the transmission problem

$$\begin{cases} \partial_t u(t, x) = Lu(t, x) \text{ on } \mathbb{R}_+^* \times (\mathbb{R} \setminus \{x_n\}_{n \in \mathbb{N}}), \\ u(0, x) = f(x) \text{ on } \mathbb{R}, \\ a(x_i-) \partial_x u(t, x_i-) = a(x_i+) \partial_x u(t, x_i+), \\ u(t, x_i-) = u(t, x_i+) \text{ for } i \in \mathbb{N}, \end{cases} \quad (38)$$

where $u \in \mathcal{C}^{1,2}(\mathbb{R}_+^* \times (\mathbb{R} \setminus \{x_n\}_{n \in \mathbb{N}})) \cap \mathcal{C}(\mathbb{R}_+^* \times \mathbb{R})$.

Remark 9. Note that we can get any arbitrary condition transmission at some point x_i without changing the diffusion coefficient on the intervals (x_{i-1}, x_i) and (x_i, x_{i+1}) . For this, it is sufficient to multiply the coefficients a by a μ_j and ρ by $1/\mu_j$ on (x_j, x_{j+1}) for a well chosen sequence $\{\mu_j\}_{j \in \mathbb{N}}$.

Differential operators on graphs are natural expansions of one-dimensional differential operators with a transmission conditions at some given points. Let G be an oriented graph with a length $\ell_e > 0$ associated to each edge e . Thus each edge e may be seen as a segment of \mathbb{R} and a differential operator $L_e = \frac{1}{2}a_e(x)\Delta + b_e(x)\nabla$ is associated to it. A differential operator $(L, \text{Dom}(L))$ may be constructed from the L_e 's. Yet to be properly defined, one needs to specify the behavior of the functions in the domain $\text{Dom}(L)$ at the vertices. The domain $\text{Dom}(L)$ of L is composed of functions that are continuous on G , of class \mathcal{C}^2 on each edge and such that at any vertex v ,

$$\sum_{e \sim v} \alpha_{v,e} \frac{\partial f}{\partial n_{v,e}}(v) = 0 \text{ with } \alpha_{v,e} \in [0, 1], \sum_{e \sim v} \alpha_{v,e} = 1, \quad (39)$$

where $e \sim v$ means that v is an endpoint of the edge e and $\frac{\partial f}{\partial n_{v,e}}$ is the derivative of f in the direction of the edge e at the vertex v .

The condition (39) prescribes the flux in each direction, and is clearly a transmission condition. The one-dimensional problem (38) can be seen as a diffusion equation on a graph whose vertices are the x_i 's and whose edges are the intervals (x_i, x_{i+1}) .

Indeed this generalization goes farther, since one may derive an Itô formula for this diffusion process with a local time at each vertex [FS00]. Besides, approximations by random walks is considered in [EK01]. The differential operator $(L, \text{Dom}(L))$ is the infinitesimal generator of a stochastic process X , which can be seen as the limit of a diffusion process moving in small tubes around the edges and reflected at the boundary: See [FW93]. Diffusions on graphs are useful to describe the limiting behavior of perturbed Hamiltonian systems (see [FW98] and the subsequent works). Some of the Monte Carlo methods presented below in Section 10.8.3 may be adapted to this case. On some particular graphs, it is also possible to compute explicitly the transition density function of X : see [Oka93] for example.

10.8.3 Numerical simulations

We saw how solutions of SDEs with local times are related to diffusion problems with discontinuous coefficients and/or transmission conditions. It is thus natural to set up Monte Carlo methods to simulate such SDEs or some of its related statistics.

This require to understand the relationship between the “skewness parameter” $(a(x+) - a(x-))/(a(x+) + a(x-))$ and the diffusion coefficients $\sqrt{\rho(x+)a(x+)}$ and $\sqrt{\rho(x-)a(x-)}$ around each point x where a or ρ is discontinuous.

The spatial change of variable introduced in Section 10.7.1 allows us to do so. Different kind of algorithms may then be given: See [LM06] for a variation of the method of random walk on spheres, and [Éto05] for an approximation by random walks.

Another alternative consists in using another spatial change of variable to transform the SDE (37) into a SDE without local times but whose coefficients that are discontinuous. The Euler scheme is known to converge for that kind of SDEs (See [Yan02]). This approach is presented in [Mar04], where the speed of convergence is computed in the case of divergence-form operators.

In [Lej03], we have dealt with an algorithm for simulating a diffusion process on a graph when there is no drift.

In [DdSG04], M. Decamps, A. De Schepper and M. Goovaerts use propagators to compute barrier options in presence of discontinuous coefficients.

10.8.4 Parameter estimation

In [Bar05, Mar04], O. Bardou and M. Martinez have constructed a scheme to estimate both the skewness parameter γ and the position of the point x of the doubly reflected Skew Brownian motion $dX_t = dB_t + \gamma dL_t^x(X) + dL_t^{-1}(X) - dL_t^1(X)$. Their construction relies on the ergodicity of the underlying process, and their connections with PDEs with discontinuous coefficients.

10.9 Domains of applications

Being related to diffusion processes with discontinuous coefficients and to diffusions on graphs, the Skew Brownian motion has potential applications in many fields of physics: in astrophysics [Zha00], in geophysics and study of heterogeneous media [Wei84, Lan95, FW93, Lej03, Lej04], in perturbed Hamiltonian systems [FW94b], in biology (some modelling of nerve impulses involves parabolic PDEs as in Section 10.8.2 [Nic85]), ecology (the dynamic of populations moving between different refuges is studied in [CC99]), in

finance and actuarial sciences [DdSG04, DGS05, DdSGS05], in optimization (see below)...

In all these applications, the properties of the Skew Brownian motion may be used in order to solve the related problems by Monte Carlo methods, or to use its properties to solve such a problem analytically.

Applications to optimal problems

The article [Lee94] looks for a drift $b \geq 0$ with $\int_0^1 b(x) dx < +\infty$ that minimizes $\mathbb{E}_{x_0} \tau$, where $\tau = \inf \{t \geq 0 \mid X_t = 1\}$ and X is the solution to $dX_t = dB_t + b(X_t) dt$ with a reflecting boundary at 0. Indeed, the optimal drift is a measure that can be computed explicitly, and that is of type $b(dx) = \tilde{b}(x)dx + \kappa\delta_{x_0}(dx)$. Hence it involves a process of Skew Brownian motion type.

The constructions in [BBKM00] and in [CSY04] are also related to methods of solving resources' allocations problems (see [MSV90] for example).

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