Lines tangent to four triangles in three-dimensional space

Hervé Brönnimann, Olivier Devillers, Sylvain Lazard, Frank Sottile

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Abstract

We investigate the lines tangent to four triangles in $\mathbb{R}^3$. By a construction, there can be as many as 62 tangents. We show that there are at most 162 connected components of tangents, and at most 156 if the triangles are disjoint. In addition, if the triangles are in (algebraic) general position, then the number of tangents is finite and it is always even.

1 Introduction

Motivated by visibility problems, we investigate lines tangent to four triangles in $\mathbb{R}^3$. In computer graphics and robotics, scenes are often represented as unions of not necessarily disjoint polygonal or polyhedral objects. The objects that can be seen in a particular direction from a moving viewpoint may change when the line of sight becomes tangent to one or more objects in the scene. Since this line of sight is tangent to a subset of the edges of the polygons and polyhedra representing the scene, we are also led to questions about lines tangent to segments and to polygons. Four polygons will typically have finitely many common tangents, while five or more will have none and three or fewer will have either none or infinitely many.

This paper follows a series of papers by the authors and their collaborators investigating such questions. The paper [4] investigated the lines of sight tangent to four convex polyhedra in a scene of k convex but not necessarily disjoint polyhedral objects, and proved that there could be up to but no more than $\Theta(n^2k^2)$ connected components of such lines. The same bound for the considerably easier case of disjoint convex polyhedra in algebraic general position was proved earlier [3, 10]. The paper [6] offers a

*A preliminary version appeared in Proc. 16th Canad. Conf. Comput. Geom. (CCCG), Montreal, pp. 184–187, 2004.

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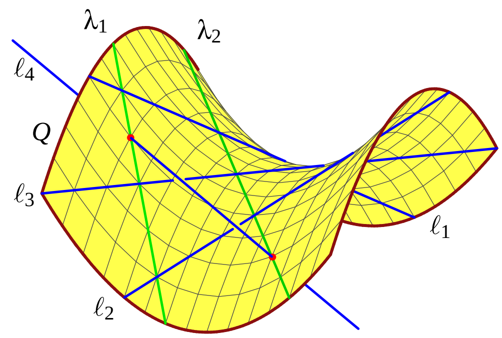


Figure 1: The lines ℓ_1, ℓ_2 and ℓ_3 span a hyperbolic paraboloid Q which meets line ℓ_4 in two points. The two lines λ_1 and λ_2 are the transversals to the lines ℓ_1 ℓ_2, ℓ_3, and ℓ_4.

detailed study of transversals to n line segments in $\mathbb{R}^3$ and proved that although there are at most two such transversals for four segments in (algebraic) general position, there are at most n such connected components of transversals in any case. Dealing with curved objects in $\mathbb{R}^3$, the paper [2] studies the tangent lines to four arbitrary spheres and [8] shows that there is a linear expected number of maximal non-occluded line segments tangent to four among n uniformly distributed unit balls.

Halperin and Sharir [12], and Pellegrini [13], proved that, in a polyhedral terrain, the set of free lines with n edges has near-cubic complexity. De Berg, Everett and Guibas [7] showed a $\Omega(n^3)$ lower bound on the complexity of the set of free lines (and thus free segments) among n disjoint homothetic convex polyhedra. Recently, Agarwal et al. [1] proved that the set of free lines among n unit balls has complexity $O(n^{3+\varepsilon})$. For related books and surveys, see [9, 11, 14, 15].

In this paper, we consider the case of four triangles in $\mathbb{R}^3$, and establish lower and upper bounds on the number of tangent lines.

A *triangle* in $\mathbb{R}^3$ is the convex hull of three distinct (and non-collinear) points in $\mathbb{R}^3$. A line is *tangent* to a triangle if it meets an edge of the triangle. Note that a line tangent to each of four triangles forming a scene corresponds to an unoccluded line of sight in that scene. If there are $k > 4$ triangles, then the bound $\Theta(k^4)$ of [4] stands (as the total number of edges is $n = 3k$ and one of the lower bound examples is made of triangles). We thus investigate the case of four triangles. Let $n(t_1,t_2,t_3,t_4)$ be number of lines tangent to four triangles t_1, t_2, t_3, and t_4 in $\mathbb{R}^3$. This number may be infinite if the lines supporting the edges of the different triangles are not in general position.

Our first step is to consider the algebraic relaxation of this geometric problem in which we replace each edge of a triangle by the line in $\mathbb{CP}^3$ supporting it, and then ask for the set of lines in $\mathbb{CP}^3$ which meet one supporting line from each triangle. Since there are $3^4 = 81$ such quadruples of supporting lines, this is the disjunction of 81 instances of the classical problem of transversals to four given lines in $\mathbb{CP}^3$. As there are two such transversals to four given lines in general position, we expect that this algebraic relaxation has 162 solutions. We say that four triangles t_1,t_2,t_3,t_4 are in (algebraic) *general position* if each of the 81 quadruples of supporting lines have two transversals in $\mathbb{CP}^3$ and all 162 transversals are distinct. Let $\mathcal{T}$ be the configuration space of all quadruples of triangles in $\mathbb{R}^3$ and $T \subset \mathcal{T}$ consist of those quadruples which are in general position. Thus if $(t_1,t_2,t_3,t_4) \in T$, the number $n(t_1,t_2,t_3,t_4)$ is finite and is at most 162.

Our primary interest is the number

$$N \;:=\; \max\{n(t_1,t_2,t_3,t_4) \mid (t_1,t_2,t_3,t_4) \in T\}\,.$$

Our results about this number N are two-fold. First, we show that $N \geqslant 62$.

Theorem 1. *There are four disjoint triangles in T with* 62 *common tangent lines.*

The idea is to perturb a configuration of four lines in $\mathbb{R}^3$ with two real transversals, such as in Figure 1. The triangles in our construction are very 'thin'—the smallest angle among them measures about 10^{-11} degrees. We ran a computer search for 'fatter' triangles having many common tangents, checking the number of tangents to 5 million different quadruples of triangles. It appears that random quadruples of realistic triangles often have a fair number of common tangents. Several had as many as 40 common tangents, and quadruples that admit common tangents have 16 tangents or more with probability at least 15%. This is discussed in Section 5.

We can improve the upper bound on N when the triangles are disjoint.

Theorem 2. *Four triangles in T admit at most* 162 *distinct common tangent lines. This number is at most* 156 *if the triangles are disjoint.*

We believe, however, that the upper bounds we give here are far from optimal. When the four triangles are not in general position, the number of tangent lines can be infinite. In this case, we may group these tangents by connected components: two line tangents are in the same component if one may move continuously between the two lines while staying tangent to the four triangles. Each quadruple of edges may induce up to four components of tangent lines [6], giving a trivial upper bound of 324. This may be improved.

Theorem 3. *Four triangles have at most* 162 *connected components of common tangents. If the triangles are disjoint, then this number is at most* 156*.*

We have one more result which we do not prove in this paper, but is proved in the preprint [5] and is relevant to mention here.

Theorem 4. *If $(t_1,t_2,t_3,t_4) \in T$, then $n(t_1,t_2,t_3,t_4)$ is even.*

This result may not seem surprising as complex roots come in conjugate pairs. However, this usual argument does not apply because we seek tangents to triangles and not transversals to lines. Frequently, only one of two real transversals to a quadruple of supporting lines is tangent to the triangles. The main new idea behind Theorem 4 is that such tangent lines essentially come in pairs.

In our proof of Theorem 4, we consider four *moving* triangles, and show that common tangents are created and destroyed in pairs, and so the parity of $n(t_1,t_2,t_3,t_4)$ does not change. There are two cases to consider. The first is when two real tangents which

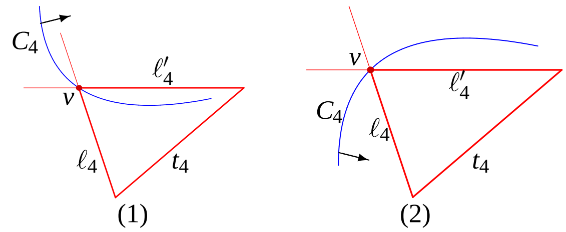


Figure 2: Configuration in plane π_4

are transversal to the same four edges coalesce and become a pair of complex conjugate transversals; this is the 'usual' argument. The second case is when a real transversal to edges e_1, e_2, e_3, and e_4 moves off of e_4 and is thus no longer tangent to the four triangles. In doing so, it must pass through a vertex v of e_4. In this case, there is a real transversal to edges e_1, e_2, e_3, and some *other* edge e'_4 meeting v which simultaneously moves off of e'_4, also passing through the vertex v. Theorem 4 follows as there are triangles in T with no common tangents. We give a complete proof in the preprint version of [5].

Theorems 1, 2, and 3 are proved in Sections 2, 3, and 4, respectively. Section 5 discusses our search for 'fat' triangles with many common tangents.

2 A construction with 62 tangents

Consider the four triangles whose vertices are given in Table 1.

Theorem 1′. *There are exactly* 62 *lines tangent to the four triangles of Table* 1.

This can be verified by a direct computation. Software is provided on this paper's web page[†]. More illuminating perhaps is our construction. The idea is to perturb a configuration of four lines in $\mathbb{R}^3$ with two transversals such as in Figure 1. The

[†] `http://www.math.tamu.edu/~sottile/stories/4triangles/index.html`

t_1	$(-10.5, 1, -10.5)$ $(.5628568345479573470378601, 1, .5628568345479573470378601)$ $(.56285683454726874605620706, .9999999999822994290647247, .56285683454726874605620706)$
t_2	$(-10.5, -1, 10.5)$ $(1.394218989475, -1, -1.394218989475)$ $(1.3942406911811439954597161, -1.0000237884694881275439271, -1.3942406911811439954597161)$
t_3	$(-9.5, -9.5, .25)$ $(.685825, .685825, .25)$ $(.69121730616063647303519136, .69121730616063647303519136, .26069756890079842876805653)$
t_4	$(9.5, 0, 0)$ $(-.511, 0, 0)$ $(-1.0873912730501133759642956, 0, -.51645811088049333541289247)$

Table 1: Four triangles with 62 common tangents

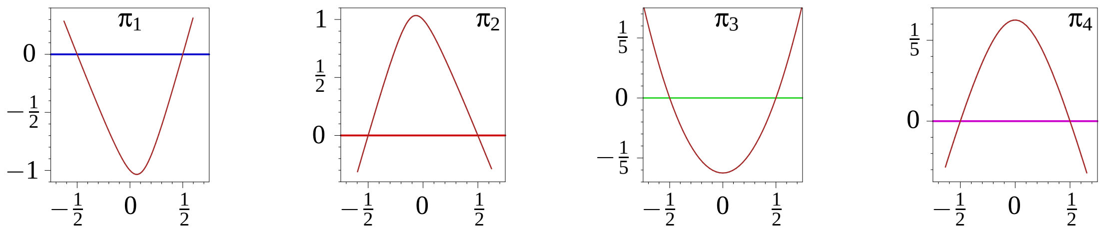


Figure 3: Conics in the planes π_i

resulting triangles of Theorem 1′ are very thin. In degrees, their smallest angles are

$$t_1\!:6.482\times 10^{-12},\quad t_2\!:8.103\times 10^{-5},\quad t_2\!:4.253\times 10^{-2},\quad \text{and}\quad t_4\!:2.793\,.$$

The construction. The lines given parametrically by

$$\ell_1 : (t,1,t)\,,\quad \ell_2 : (t,-1,-t)\,,\quad \ell_3 : (t,t,\tfrac{1}{4})\,,\quad \text{and}\quad \ell_4 : (t,0,0)\,,$$

have two transversals $\lambda_1 : (\frac{1}{2},2t,t)$ and $\lambda_2 : (-\frac{1}{2},2t,-t)$.

For each $i=1,2,3,4$, let Q_i be the hyperboloid spanned by the lines other than ℓ_i. For example, Q_3 has equation $z=xy$. The intersection of Q_i with a plane containing ℓ_i will be a conic which meets ℓ_i in two points (corresponding to the common transversals λ_1 and λ_2 at $t=\pm\frac{1}{2}$). We choose the plane π_i so that these two points lie in the same connected component of the conic. Here is one possible choice

$$\pi_1 : x=z\,,\quad \pi_2 : x=-z\,,\quad \pi_3 : x=y\,,\quad \text{and}\quad \pi_4 : y=0\,.$$

For each i, let C_i be the conic $\pi_i\cap Q_i$, shown in the plane π_i in Figure 3. Here, the horizontal coordinate is t, the parameter of the line ℓ_i, while the vertical coordinate is $y-1$ for π_1, $y+1$ for π_2, $z-\frac{1}{4}$ for π_3, and z for π_4.

For each $i=1,\ldots,4$, rotate line ℓ_i in plane π_i very slightly about a point that is far from the conic C_i, obtaining a new line k_i in π_i which also meets C_i in two points. Consider now the transversals to $\ell_i\cup k_i$, for $i=1,\ldots,4$. Because k_i is close to ℓ_i and there were two transversals to $\ell_1,\ell_2,\ell_3,\ell_4$, there will be two transversals to each of the 16 quadruples of lines obtained by choosing one of ℓ_i or k_i for $i=1,\ldots,4$. By our choice of the point of rotation, all of these will meet ℓ_i and k_i in one of the two thin wedges they form. In this wedge, form a triangle by adding a third side so that the edges on ℓ_i and k_i contain all the points where the transversals meet the lines. The resulting triangles will then have at least 32 common tangents. We claim that by carefully choosing the third side (and tuning the rotations) we are able to get 30 additional tangents.

To begin, look at Figure 4 which displays the configuration in π_4 given by the four triangles from Table 1. Since the lines ℓ_i and k_i for $i=1,2$ are extremely close, the four conics given by transversals to them and to ℓ_3 cannot be resolved in these pictures. The same is true for the four conics given by k_3, so that each of the apparent two conics are clusters of four nearby conics. The picture on the left is a view of this configuration

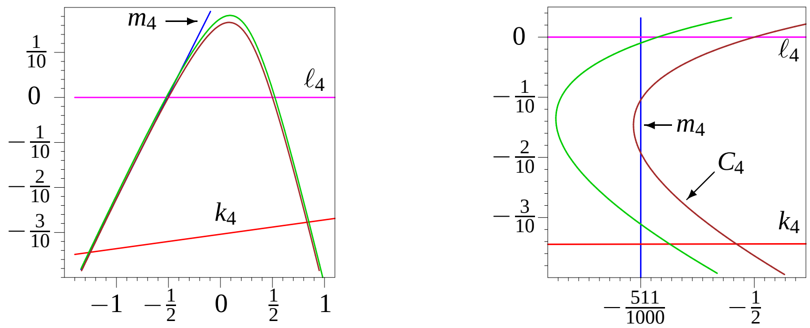


Figure 4: Configuration in plane π_4

in the coordinates for π_4 of Figure 3. It includes a secant line m_4 to the conics. We choose coordinates on the right so that m_4 is vertical, but do not change the coordinates on ℓ_4. The horizontal scale has been accentuated to separate the two clusters of conics. The three lines, ℓ_4, k_4, and m_4 form the triangle t_4. Let its respective edges be e_4, f_4, and g_4. Each edge meets each of the eight conics in two points and these 48 points of intersection give 48 lines tangent to the four triangles.

This last assertion that the 16 lines transversal to m_4 and to $\ell_i \cup k_i$ for $i = 1, 2, 3$ meet the edges of the triangles t_1, t_2, and t_3 needs justification. Consider for example the transversals to ℓ_1, ℓ_2, and ℓ_3. These form a ruling of the doubly-ruled quadric Q_4 and are parameterized by their point of intersection with ℓ_1. The intersection of Q_4 with π_4 is the conic C_4. Since the intersections of the conic C_4 with the segment g_4 supported on m_4 lie between its intersections with ℓ_4 and k_4, the corresponding transversals to ℓ_1, ℓ_2, ℓ_3, and g_4 meet ℓ_1 between points of ℓ_1 met by common transversals to $\ell_4 \cup k_4$ and ℓ_1, ℓ_2, and ℓ_3. The same argument for the other lines and for all 8 conics justifies the assertion.

Naïvely, we would expect that this same construction (the third side cutting all eight conics in π_i) could work to select each of the remaining sides of the triangles g_3, g_2, and g_1, and that this would give four triangles having $32+16+16+16+16 = 96$ common tangents. Unfortunately this is not the case. In the earlier conference version of this paper [5], we gave a construction that we claimed would yield 88 common tangents. Attempting that construction using Maple revealed a flaw in the argument and the current construction of four triangles with 62 common tangents is the best we can accomplish.

In π_4, the conics come in two clusters, depending upon whether or not they correspond to ℓ_3 or to k_3. In order for the edge g_4 to cut all conics, the angle between ℓ_4 and k_4 has to be large, in fact significantly larger than the angle between ℓ_3 and k_3. Thus in π_3, the conics corresponding to ℓ_4 are quite far from the conics corresponding to k_4, and the side g_3 can only be drawn to cut four of the conics, giving eight additional common tangents. Similarly, g_2 can only cut two conics, and g_1 only one. In this way, we arrive at four triangles having $32+16+8+4+2 = 62$ common tangents, which has been verified by computer.

Figure 5: Stabbing and non-stabbing configurations

3 Upper bound for disjoint triangles in general position

Four triangles in general position have at most 162 common tangents. If the triangles are disjoint, we slightly improve this upper bound to 156. Our method will be to show that not all $81 = 3^4$ quadruples of edges can give rise to a common tangent. Our proof follows that for the upper bound on the number of tangents to four polytopes [3], limiting the number of configurations for disjoint triangles in $\mathbb{R}^3$. We divide the proof into two lemmas, which do not assume general position. The application of the lemmas to the proof of [3], however, requires the general position assumption.

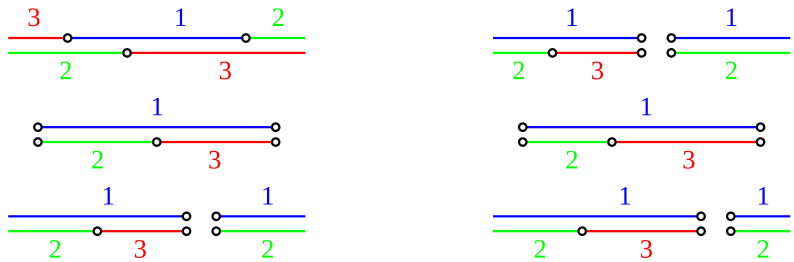
In order for a tangent to meet an edge e, the plane it spans with e must meet one edge from each of the other triangles. A triple of edges, one from each of the other triangles, is *contributing* if there is a plane containing e which meets the three edges. We say that an edge e *stabs* a triangle t if its supporting line meets the interior of t.

Lemma 5. *Let e be an edge of some triangle. If e stabs exactly one of the other triangles, then there are at most* 26 *contributing triples of edges. If e stabs no other triangle, then there are at most* 25 *contributing triples.*

It is not hard to see that if e stabs at least two of the other triangles, then each of the $27 = 3^3$ triples of edges can be contributing.

Proof. Suppose that e is an edge of some triangle. Let $\pi(\alpha)$ be the pencil of planes containing e. (This is parameterized by the angle α.) For each edge f of another triangle t, there is an interval of angles α for which $\pi(\alpha)$ meets f. Figure 5 illustrates the two possible configurations for these intervals, which depend upon whether or not e stabs the triangle t. The intervals are labeled 1, 2, and 3 for the three edges of t. When e stabs t, these intervals cover the entire range of α and the picture is actually wrapped. Call this a *stabbing diagram*. When the supporting line of e does not meet t, these intervals do not cover the entire range of α, and there are two endpoints and one *interior vertex* of the diagram. If the supporting line of e meets an edge of t, then the two endpoints of the non-stabbing diagram wrap around and coincide. Call either of these last two configurations a non-stabbing diagram.

To count contributing triples, we line up (overlay) diagrams from each of the three triangles not containing e and count how many of the 27 triples $\{1,2,3\}^3$, one from each triangle, occur at some value of α. For example, Figure 6 displays a configuration with 26 contributing triples (where e stabs a single triangle) and a configuration with 25 contributing triples (e stabs no other triangles). The configuration on the left is missing the triple $(2,3,3)$, while the configuration on the right is missing the triples $(2,2,3)$ and $(3,3,2)$.



e stabs one triangle   *e* stabs no triangle

Figure 6: Configurations with 26 and 25 contributing triples

These configurations are the best possible. Indeed, begin with two non-stabbing diagrams in which all 9 pairs of edges occur. (If only 8 pairs occurred, there would be at most 24 contributing triples.) The unique way to do this up to relabeling the edges is given by the lower two diagrams in either picture in Figure 6. These two diagrams divide the domain of α into six intervals (the two at the ends are wrapped). The five pairs involving 1 occur in two intervals, but four exceptional pairs $\{(2,2),(2,3),(3,2),(3,3)\}$ occur uniquely in different intervals.

Consider now a third diagram. An exceptional pair extends to three contributing triples only if all three sides in the third diagram meet the interval corresponding to that pair. If the third diagram is stabbing, then one of its three vertices lies in that interval—thus there is at least one triple which does not contribute. If the third diagram is non-stabbing, then either the middle vertex or else both endpoints must lie in that interval—thus there are at least two triples which do not contribute.

Lemma 6. *At most* 78 *quadruples of edges of four disjoint triangles can lead to a common tangent.*

Proof. First consider the maximum number of stabbing edges between two triangles. If the triangles are disjoint, then there are at most three stabbing edges; one triangle could have three edges stabbing the other. Indeed, if at least two supporting lines of a triangle t meet another triangle t' which is disjoint from t, then t lies entirely on one side of the plane supporting t', and thus no supporting lines of t' can meet t. Figure 7(a) shows a configuration in which all three supporting lines of t stab t'.

Consider now the bipartite graph between 12 nodes representing the edges of the four triangles and 4 nodes representing the triangles. This graph has an arc between an edge e and a triangle t if the line supporting e stabs t. (We assume that e is not an edge of t.) We just showed that the edges of one triangle t can have at most three arcs incident on another triangle t', and so this graph has at most 18 edges.

Let the weight of a triangle be the number of arcs emanating from its edges in this graph. As the graph has at most 18 arcs, at least one triangle has weight less than 5. We argue that there is a triangle of weight at most 3. This is immediate if the graph has 15 or fewer edges. On the other hand, this graph has more structure. If it has 18 edges, then all pairs of triangles are in the configuration of Figure 7(a), and so every

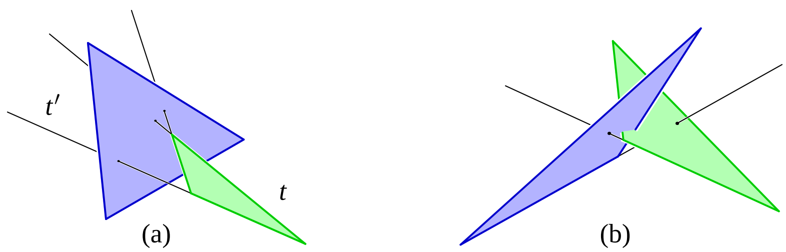


Figure 7: (a) Two disjoint triangles can have at most 3 stabbing lines. (b) Two intersecting triangles may have up to four.

triangle has weight a multiple of 3, which implies that some triangle has weight at most 3. If the graph has 17 edges, then there is exactly one pair of triangles with only two stabbing edges, and so the possible weights less than 5 are 0, 2, and 3. If the graph has 16 edges, then there is one pair with only one edge stabbing, or two pairs with two edges stabbing. There can be at most two triangles of weight 4, and again we conclude that there is a triangle with weight at most 3.

If a triangle has weight at most three, either all three edges stab a unique triangle, or else one edge stabs no triangles and another edge stabs at most one other triangle. We sum the number of contributing triples over the edges of this triangle. By Lemma 5, this sum will be at most 26+26+26=78 if all three edges stab a unique triangle and at most 27+26+25=78 if not. This proves the lemma.

Remark 7. There exist four disjoint triangles whose bipartite graph has exactly 18 edges. Thus the previous argument cannot be improved without additional ideas. It is conceivable that further restrictions to the bipartite graph may exist, leading to a smaller upper bound.

Remark 8. This proof does not enable us to improve the bound when the triangles are not disjoint. Two intersecting triangles can induce up to four arcs (see Figure 7(b)) and thus the total number of arcs is bounded above by 24. The minimal weight of a triangle is then 6, and the edges of such a triangle could all have degree 2, which leads to no restrictions.

4 Upper bounds on the number of components

Let $\mathcal{F}$ and $\mathcal{I}$ be the sets of quadruples of edges, one from each of four triangles, whose supporting lines have finitely and infinitely, respectively, many common transversals. Let $n_{\mathcal{F}}$ and $n_{\mathcal{I}}$ be the sum over all quadruples of edges in $\mathcal{F}$ and $\mathcal{I}$, respectively, of the numbers of connected components of common transversals to each quadruple of edges. Note that the number of quadruples in $\mathcal{F}$ and $\mathcal{I}$ is $|\mathcal{F}|+|\mathcal{I}|=81$.

Consider a connected component c of common transversals to a quadruple of edges $q\in\mathcal{I}$. The arguments of [6] show that c contains a line that meets a vertex of one of the four edges. That line is thus transversal to another quadruple q' of edges. Thus,

Number	0	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40
Frequency	1 515 706	331 443	646 150	403 679	637 202	327 159	358 312	238 913	253 396	114 046	80 199	44 870	27 726	12 426	5 796	2 016	813	111	30	3	4

Table 2: Number of triangles with a given number of tangents, out of 5 000 000 randomly constructed triangles

the connected component c of common transversals to q is connected with a connected component c' of common transversals to q'. If $q' \in \mathcal{F}$ we charge the component $c \cup c'$ to c'. Otherwise q and q' are both in $\mathcal{I}$ and the component $c \cup c'$ is counted twice. The number of connected components of tangents to four triangles is thus at most $n_{\mathcal{F}} + n_{\mathcal{I}}/2$.

Since any four lines admit at most two or infinitely many transversals, $n_{\mathcal{F}} \leq 2|\mathcal{F}|$. Also, any four segments admit at most four connected components of common transversals [6], thus $n_{\mathcal{I}} \leq 4|\mathcal{I}|$. Hence, the number of connected components of tangents to four triangles is at most $2|\mathcal{F}| + 2|\mathcal{I}| = 162$.

This still may overcount the number of connected components of tangents, but further analysis is very delicate. Such complicated arguments are not warranted as we have already obtained the upper bound of 162 common tangents to four triangles in T. As in Section 3, if the triangles are disjoint, then not all quadruples of edges can contribute, which lowers this bound to 156.

5 Random triangles

We proved Theorem 1 by exhibiting four triangles having 62 common tangents. We do not know if that is the best possible. Since the geometric problem of determining the tangents to four triangles is computationally feasible—it is the disjunction of 81 problems with algebraic degree 2 and simple inequalities on the solutions—we investigated it experimentally.

For this, we generated 5 000 000 quadruples of triangles whose vertices were points with integral coordinates chosen uniformly at random from the cube $[-1000, 1000]^3$. For each, we computed the number of tangents. The resulting frequencies are recorded in Table 2. This search consumed over six months of CPU time on 1.2GHz processors at the MSRI and a DEC Alpha machine at the University of Massachusetts in 2004. It is archived on the web page† accompanying this article.

In this search, we found four different quadruples of triangles with 40 common tangents, and none with more. Based on this *uiid* random model, we find that the probabilty that the four triangles have at least one tangent is around 69.7%, and that the expected number of tangents is somewhat around 6.325, with a standard deviation of

†`www.math.tamu.edu/~sottile/stories/4triangles/index.html`

Triangle	Vertices		
t_1	$(-4,-731,-336)$	$(297,-507,978)$	$(824,-62,-359)$
t_2	$(531,-631,-820)$	$(-24,-716,713)$	$(807,377,177)$
t_3	$(586,-205,952)$	$(861,-774,235)$	$(-450,758,161)$
t_4	$(330,-141,-908)$	$(942,-920,651)$	$(-226,489,968)$

Table 3: Four triangles with 40 common tangents

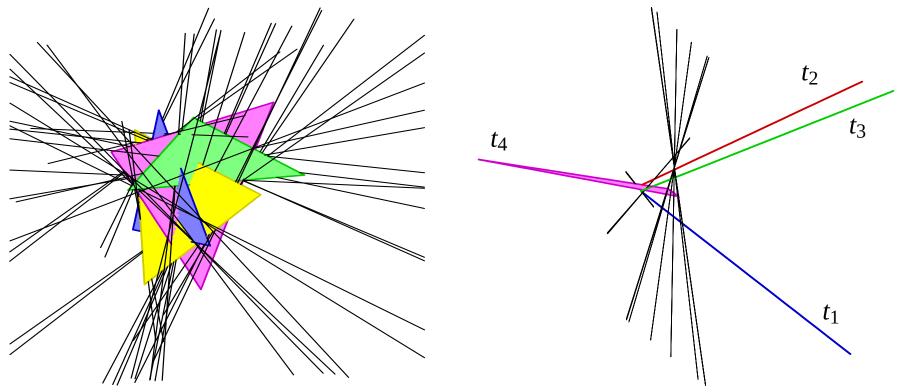


Figure 8: Triangles with many common tangents

about 12.93. The vertices of one are given in Table 3. These triangles are rather 'fat', in that none have very small angles. Contrast that to the triangles of our construction in Section 3. In Figure 8 we compare these two configurations of triangles. On the left is the configuration of triangles from Table 3, together with their 40 common tangents, while on the right is the configuration of triangles having 62 common tangents. The triangles are labeled in the second diagram, as they are hard to distinguish from the lines. As we remarked in Section 3, many of the lines are extremely close and cannot be easily distinguished; that is why one can only count eight lines in this picture.

Acknowledgments

Research on this paper was initiated at the Second McGill-INRIA Workshop on Computational Geometry in Computer Graphics, February 7–14, 2003, co-organized by H. Everett, S. Lazard, and S. Whitesides, and held at the Bellairs Research Institute of McGill University in Holetown, St. James, Barbados, West Indies. We would like to thank the other participants of the workshop for useful discussions.

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