

***An efficient discretisation scheme for one  
dimensional SDEs with a diffusion coefficient  
function of the form  $|x|^\alpha$ ,  $\alpha \in [1/2, 1)$***

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## An efficient discretisation scheme for one dimensional SDEs with a diffusion coefficient function of the form

$$|x|^\alpha, \alpha \in [1/2, 1)$$

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**Abstract:** In this paper, we study the rate of convergence of a symmetrized version of the classical Euler scheme, applied to the discretisation of the solution of a stochastic differential equation with a diffusion coefficient function of the form  $|x|^\alpha$ ,  $\alpha \in [1/2, 1)$ . For smooth test functions, we show that the weak error is of order one as for the classical Euler scheme. More over, the symmetrized version is very easy to simulate on a computer.

**Key-words:** Euler scheme, weak error, CIR process, Hull-White process, SABR processes.

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# Un schéma de discrétisation d'ordre un pour les EDS unidimensionnelles ayant un coefficient de diffusion de la forme $|x|^\alpha$ , $\alpha \in [1/2, 1[$

**Résumé :** Nous étudions la vitesse de convergence d'une version symétrisée du schéma d'Euler, appliquée à la discrétisation en temps de la solution d'une équation différentielle stochastique dont le coefficient de diffusion n'est pas Lipschitzien mais de la forme  $|x|^\alpha$  avec  $\alpha \in [1/2, 1)$ . Nous montrons que l'erreur faible, pour des fonctions tests régulières, reste d'ordre un comme pour le schéma d'Euler classique. De plus, comme le schéma d'Euler classique, la version symétrisée reste très simple à simuler sur un ordinateur.

**Mots-clés :** Schéma d'Euler, erreur faible, processus de CIR, processus d'Hull-White, processus SABR.

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## 1 Introduction

We consider  $(X_t)_{t \geq 0}$ , the  $\mathbb{R}$ -valued process solution to the following one-dimensional Itô stochastic differential equation

$$X_t = x_0 + \int_0^t b(X_s) ds + \sigma \int_0^t |X_s|^\alpha dW_s, \quad (1.1)$$

where  $x_0$  and  $\sigma$  are given constants,  $\sigma > 0$  and  $(W_t)_{t \geq 0}$  is a one-dimensional Brownian motion defined on a given probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . We denote by  $(\mathcal{F}_t)_{t \geq 0}$  the Brownian filtration. To ensure the existence of such process, we state the following hypotheses:

**(H0)**  $\alpha \in [1/2, 1)$ .

**(H1)** *The drift function  $b$  satisfies*

$$b(0) > 0,$$

$$\text{and } |b(x) - b(y)| \leq K|x - y|, \quad x \in \mathbb{R}, \quad \forall y \in \mathbb{R},$$

*where  $K$  is a positive constant.*

Under hypotheses (H0) and (H1), strong existence and uniqueness holds for equation (1.1) (see e.g. the result from Yamada & Watanabe cited in [9] p.291 ).

Moreover, when  $x_0 \geq 0$  and  $b(0) > 0$ , the process  $(X_t)_{t \geq 0}$  is valued in  $[0, +\infty)$ . Indeed, as  $b(x) \geq b(0) - Kx \geq -Kx$ , comparison Theorems for solutions of one-dimensional stochastic differential equations give that  $X_t \geq Z_t$ , for all  $t \geq 0$ , where

$$Z_t = -K \int_0^t Z_s ds + \sigma \int_0^t |Z_s|^\alpha dW_t.$$

$Z_t \equiv 0$  stands for the unique strong solution of the above equation (see e.g. [9]).  $(X_t)_{t \geq 0}$  is then the unique strong solution to

$$X_t = x_0 + \int_0^t b(X_s) ds + \sigma \int_0^t X_s^\alpha dW_s. \quad (1.2)$$

For the study of simulation scheme of models like (1.1), one can find motivations and applications in Financial mathematics. Indeed, in [4], Cox, Ingersoll and Ross proposed to model the dynamics of the short term interest rate as the solution of (1.1) with  $\alpha = 1/2$  and  $b(x) = a - bx$ . In [8], Hull and White proposed the following mean-reverting diffusion process

$$dr_t = (a(t) - b(t)r_t)dt + \sigma(t)r_t^\alpha dW_t$$

with  $0 \leq \alpha \leq 1$ , to model the short term rate. More recently, (see [7]) the  $\alpha\beta\gamma$ -model or *SABR*-model have been proposed as stochastic volatility model in which the asset price and the volatility are correlated:

$$dX_t = \alpha_t X_t^\beta dW_t^1$$

$$d\alpha_t = \nu \alpha_t dB_t$$

where  $B_t = \gamma W_t^1 + \sqrt{(1 - \gamma^2)} W_t^2$  and  $(W^1, W^2)$  is a standard two dimensional Brownian motion.

When the drift  $b(x)$  and the diffusion coefficient  $\sigma(x)$  are smooth functions, at least with bounded first order derivatives, one can apply the Euler scheme to the process solution of

$$X_t = x_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s$$

and convergence rate results are known, depending on the regularity on  $b(x)$  and  $\sigma(X)$  and/or on the regularity of the test function  $f$ , when one consider the weak error of the scheme (see e.g. [12] and the references therein).

The classical Euler scheme could not be applied in our situation since it does not conserve the sign of the process. Similar situations occur when one consider discretisation scheme of a reflected stochastic differential equation. To maintain the approximation process in a given domain, some strategies are now well-known. One consists in take the projection on the domain of the value obtained by the Euler scheme at each time step (see e.g. [3]). A more efficient strategy consists in symmetrized the value obtained by the Euler scheme with respect to the boundary of the domain (see e.g. [2]).

Here, our preoccupation is quite similar. We want to maintain the positive value of the approximation. Moreover we have to deal with a just locally Lipschitz diffusion coefficient.

In section 2, we construct a time discretisation scheme for  $(X_t, t \in [0, T])$ , based on the symmetrized Euler scheme and which can be simulated easily.

We prove a theoretical rate of convergence of order one for the weak approximation error. Convergence results when  $\alpha = 1/2$  and  $1/2 < \alpha < 1$  are given in the next section in Theorems 2.3 and 2.5 respectively. The sections 3 and 4 are devoted to the proofs in this two respective situations  $\alpha = 1/2$  and  $1/2 < \alpha < 1$ . We denote  $(\bar{X}_t, t \in [0, T])$  the discrete time process. To study the weak error  $\mathbb{E}f(X_T) - \mathbb{E}f(\bar{X}_T)$ , we will use the Feynman–Kac representation  $\mathbb{E}f(X_{T-t}^x) = u(t, x)$  where  $u(t, x)$  is the solution of the associated Kolmogorov Equation. One of the main difficulties of the proofs consist in obtain sufficiently strong upper-bound on the spatial derivatives of  $u$  up to the order four. To our knowledge, for this kind of Cauchy problems there is no generic results. As we cannot hope to obtain uniform upper-bound in  $x$ , we will also need to study the behavior of the approximation process at the origin.

The case  $\alpha = 1/2$  is a little easier: this can be seen intuitively on the associated infinitesimal generator which has unbounded but smooth coefficients. In the case  $1/2 < \alpha < 1$ , the derivatives of the diffusion coefficient of the associated infinitesimal generator are degenerated functions at point 0. This is why in this last case, we enforced the hypotheses in order to obtain stronger controls on the behavior of the approximation process at the origin.

## 2 A modified Euler scheme solving (1.2)

For a fixed  $x_0 > 0$ , let  $(X_t)_{t \geq 0}$  given by (1.1) or (1.2). For a fixed time  $T > 0$ , we define a discretisation scheme  $(\bar{X}_{t_k}, k = 0, \dots, N)$  by

$$\begin{cases} \bar{X}_0 = x_0 > 0, \\ \bar{X}_{t_{k+1}} = \left| \bar{X}_{t_k} + b(\bar{X}_{t_k})\Delta t + \sigma \bar{X}_{t_k}^\alpha (W_{t_{k+1}} - W_{t_k}) \right|, \quad k = 0, \dots, N-1, \end{cases} \quad (2.1)$$

where  $N$  denotes the number of discretisation times  $t_k = k\Delta t$  and  $\Delta t > 0$  is a constant time step such that  $N\Delta t = T$ .

In the sequel we will use the time continuous version  $(\bar{X}_t)_{0 \leq t \leq T}$  of the discrete time process, which consists in freezing the coefficients on each interval  $[t_k, t_{k+1})$ :

$$\bar{X}_t = \left| \bar{X}_{\eta(t)} + (t - \eta(t))b(\bar{X}_{\eta(t)}) + \sigma \bar{X}_{\eta(t)}^\alpha (W_t - W_{\eta(t)}) \right|, \quad (2.2)$$

where  $\eta(s) = \sup_{k \in \{1, \dots, N\}} \{t_k; t_k \leq s\}$ . The process  $(\bar{X}_t)_{0 \leq t \leq T}$  is valued in  $[0, +\infty)$ . By induction on each subinterval  $[t_k, t_{k+1})$ , for  $k = 0$  to  $N - 1$ , using the Tanaka's formula, we can easily show that  $(\bar{X}_t)$  is a continuous semi martingale with a continuous local time  $(L_t^0(\bar{X}))$  at point 0. Indeed, for any  $t \in [0, T]$ , if we set

$$\bar{Z}_t = \bar{X}_{\eta(t)} + b(\bar{X}_{\eta(t)})(t - \eta(t)) + \sigma \bar{X}_{\eta(t)}^\alpha (W_t - W_{\eta(t)}), \quad (2.3)$$

then,  $\bar{X}_t = |\bar{Z}_t|$  and

$$\bar{X}_t = x_0 + \int_0^t \text{sgn}(\bar{Z}_s) b(\bar{X}_{\eta(s)}) ds + \sigma \int_0^t \text{sgn}(\bar{Z}_s) \bar{X}_{\eta(s)}^\alpha dW_s + L_t^0(\bar{X}), \quad (2.4)$$

where  $\text{sgn}(x) := 1 - 2\mathbf{1}_{(x \leq 0)}$ .

The following Lemma ensure the existence of positives moments of  $(X_t)_{0 \leq t \leq T}$  starting at  $x_0$  at time 0 and of the associated discrete time process  $(\bar{X}_t)_{0 \leq t \leq T}$ :

**Lemma 2.1** *Assume (H0) and (H1). For any  $x_0 \geq 0$ , for any  $p \geq 1$ , there exists a positive constant  $C(p)$ , depending on  $p$ , but also on the parameters  $b(0)$ ,  $K$ ,  $\sigma$ ,  $\alpha$  and  $T$ , such that*

$$\mathbb{E} \left( \sup_{t \in [0, T]} X_t^{2p} \right) + \mathbb{E} \left( \sup_{t \in [0, T]} \bar{X}_t^{2p} \right) \leq C(p)(1 + x_0^{2p}). \quad (2.5)$$

**Proof.** We prove (2.5) for  $(\bar{X}_t)_{0 \leq t \leq T}$  only, the case of process  $(X_t)_{0 \leq t \leq T}$  could be deduced by similar arguments. By the Itô's formula, noting that for any  $t \in [0, T]$   $\int_0^t \bar{X}_s^{2p-1} dL_s^0(\bar{X}) = 0$ , we have

$$\begin{aligned} \bar{X}_t^{2p} &= x_0^{2p} + 2p \int_0^t \bar{X}_s^{2p-1} \text{sgn}(\bar{Z}_s) b(\bar{X}_{\eta(s)}) ds + 2p\sigma \int_0^t \bar{X}_s^{2p-1} \text{sgn}(\bar{Z}_s) \bar{X}_{\eta(s)}^\alpha dW_s \\ &\quad + \sigma^2 p(2p-1) \int_0^t \bar{X}_s^{2p-2} \bar{X}_{\eta(s)}^{2\alpha} ds. \end{aligned} \quad (2.6)$$

To prove (2.5), we start to show that

$$\sup_{t \in [0, T]} \mathbb{E} \left( \bar{X}_t^{2p} \right) \leq C(p)(1 + x_0^{2p}) \quad (2.7)$$

(2.5) follows from (2.7), (2.6) and the Burkholder-Davis-Gundy Inequality. Let  $\tau_n$  the stopping time defined by  $\tau_n = \inf\{0 < s < T; \bar{X}_s \leq n\}$ , with the convention  $\inf\{\emptyset\} = 0$ . Then,

$$\mathbb{E} \bar{X}_{t \wedge \tau_n}^{2p} \leq x_0^{2p} + 2p \int_0^{t \wedge \tau_n} \mathbb{E} \left( \bar{X}_s^{2p-1} b(\bar{X}_{\eta(s)}) \right) ds + \sigma^2 p(2p-1) \int_0^{t \wedge \tau_n} \mathbb{E} \left( \bar{X}_s^{2p-2} \bar{X}_{\eta(s)}^{2\alpha} \right) ds.$$



Using (H0), (H1) and the Young Inequality, we get

$$\begin{aligned} \mathbb{E} \bar{X}_{t \wedge \tau_n}^{2p} &\leq x_0^{2p} + Tb(0)^{2p} + (2p - 1 + \sigma^2 p(2p - 1)) \int_0^{t \wedge \tau_n} \mathbb{E} \left( \bar{X}_s^{2p} \right) ds \\ &\quad + 2pK \int_0^{t \wedge \tau_n} \mathbb{E} \left( \bar{X}_s^{2p-1} \bar{X}_{\eta(s)} \right) ds + \sigma^2 p(2p - 1) \int_0^{t \wedge \tau_n} \mathbb{E} \left( \bar{X}_s^{2p-2} \bar{X}_{\eta(s)}^{2\alpha} \right) ds. \end{aligned}$$

Replacing  $\bar{X}_s$  by its expression (2.2) in the integrals above, we easily obtain, using another time (H1) and the Young Inequality, that for any  $t \in [0, T]$ ,

$$\begin{aligned} \mathbb{E} \bar{X}_{\eta(t) \wedge \tau_n}^{2p} &\leq x_0^{2p} + C(p) \left( 1 + \int_0^{\eta(t) \wedge \tau_n} \mathbb{E} \left( \bar{X}_{\eta(s)}^{2p} \right) ds \right) \\ &\leq x_0^{2p} + C(p) \left( 1 + \int_0^{\eta(t)} \mathbb{E} \left( \bar{X}_{\eta(s) \wedge \tau_n}^{2p} \right) ds \right), \end{aligned}$$

where  $C(p) > 0$  depends on  $p$ ,  $b(0)$ ,  $K$ ,  $\sigma$ ,  $\alpha$  and  $T$ . A discrete version of the Gronwall Lemma allows us to conclude that

$$\sup_{k=0, \dots, N} \mathbb{E} \left( \bar{X}_{t_k \wedge \tau_n}^{2p} \right) \leq C(p)(1 + x_0^{2p}),$$

for another constant  $C(p)$ , which does not depend on  $n$ . Taking the limit  $n \rightarrow +\infty$ , we get  $\sup_{k=0, \dots, N} \mathbb{E}(\bar{X}_{t_k}^{2p}) \leq C(p)(1 + x_0^{2p})$ , from which we can easily deduce (2.7) using (2.2). ■

In addition of hypotheses (H0) and (H1), we will analyze the convergence rate of (2.1) under the following hypothesis:

**(H2)**  $b$  is a  $C^4$  function, with bounded derivatives up to the order 4.

## 2.1 Main results

### 2.1.1 Convergence rate when $\alpha = 1/2$

In that case  $(X_t)_{0 \leq t \leq T}$  satisfies the equation

$$X_t = x_0 + \int_0^t b(X_s) ds + \sigma \int_0^t \sqrt{X_s} dW_s, \quad 0 \leq t \leq T. \quad (2.8)$$

When  $b(x)$  is of the form  $a - bx$ , with  $a > 0$ ,  $(X_t)$  is the classical CIR process used in Financial mathematics to model the short interest rate. When  $b(x) = a > 0$ ,  $(X_t)$  is the square of a Bessel process.

Here, we restrict yourself to the case

(H3)  $b(0) \geq 2\sigma^2$ .

**Remark 2.2** When  $x_0 > 0$ , one can show using the Feller's test (see e.g. [9]) that, if  $b(0) \geq \sigma^2/2$ , then  $\mathbb{P}(\tau_0 = \infty) = 1$  where  $\tau_0 = \inf\{t \geq 0; X_t = 0\}$ .

When  $b(0) \geq \sigma^2$ , we are able to control the inverse moments and exponential inverse moment of the CIR-like process  $(X_t)$  (see Lemma 3.1).

Hypothesis (H3) is a stronger condition under which we can prove that the derivative (in the sense of the quadratic mean) of  $X_t^x$  with respect to  $x$  is well defined (see Proposition B.1 in Appendix B).

**Theorem 2.3** Let  $f$  be a  $\mathbb{R}$ -valued  $C^4$  bounded function, with bounded spatial derivatives up to the order 4. Let  $\alpha = \frac{1}{2}$  and  $x_0 > 0$ . Assume (H1), (H2) and (H3). Choose  $\Delta t$  sufficiently small in (2.1), that is  $\Delta t \leq 1/(2K) \wedge x_0$ . Then there exists a positive constant  $C$  depending on  $f$ ,  $b$ ,  $T$  and  $x_0$  such that

$$|\mathbb{E}f(X_T) - \mathbb{E}f(\bar{X}_T)| \leq C \left( \Delta t + \left( \frac{\Delta t}{x_0} \right)^{\frac{b(0)}{\sigma^2}} \right).$$

### 2.1.2 Convergence rate when $1/2 < \alpha < 1$ .

In that case  $(X_t)_{0 \leq t \leq T}$  satisfies

$$X_t = x_0 + \int_0^t b(X_s) ds + \sigma \int_0^t X_s^\alpha dW_s, \quad 0 \leq t \leq T. \quad (2.9)$$

We restrict yourself to the case

(H4)  $x_0 > \frac{b(0)}{2} \Delta t$ .

**Remark 2.4** When  $x_0 > 0$ , Feller's test on process  $(X_t)$  show that it enough to suppose  $b(0) > 0$ , as in (H1), to ensure that  $\mathbb{P}(\tau_0 = \infty) = 1$ , for  $\tau_0 = \inf\{t \geq 0; X_t = 0\}$ .

**Theorem 2.5** Let  $f$  be a  $\mathbb{R}$ -valued bounded  $C^4$  function, with bounded spatial derivatives up to the order 4. Assume (H1), (H2) and (H4). Let  $\frac{1}{2} < \alpha < 1$ .

Consider  $(X_t)_{0 \leq t \leq T}$ , solution of (2.9) and  $(\bar{X}_t)_{0 \leq t \leq T}$ , the corresponding symmetrized Euler scheme given by (2.1). Choose  $\Delta t$  sufficiently small in (2.1), that is  $\Delta t \leq 1/(2K)$ . Then there exists a positive constant  $C$  depending on  $f$ ,  $\alpha$ ,  $\sigma$ ,  $b$ ,  $T$  and  $x_0$  such that

$$|\mathbb{E}f(X_T) - \mathbb{E}f(\bar{X}_T)| \leq \left( 1 + \frac{1}{x_0^{q(\alpha)}} \right) \Delta t,$$

where  $q(\alpha)$  is a positive constant depending only on  $\alpha$ .

### 3 The case of processes with $\alpha = 1/2$ .

#### 3.1 Preliminary results

In this section  $(X_t)$  denotes the solution of (2.8) and  $(X_t^x)$  the solution of (2.8) with the deterministic initial condition  $x$  at time 0, that is the unique strong solution to

$$X_t^x = x + \int_0^t b(X_s^x) ds + \sigma \int_0^t \sqrt{X_s^x} dW_s. \quad (3.1)$$

##### 3.1.1 On the exact solution

We have the following

**Lemma 3.1** *Assume (H1) and (H3). We set  $\nu = \frac{2b(0)}{\sigma^2} - 1$ , so that by (H3),  $\nu \geq 1$ . For any  $p$  such that  $1 < p < \nu$ , for any  $t \in [0, T]$  and any  $x > 0$ ,*

$$\begin{aligned} \mathbb{E}(X_t^x)^{-1} &\leq C(T)x^{-1} \quad \text{and} \quad \mathbb{E}(X_t^x)^{-p} \leq C(T)t^{-p}. \\ \text{Moreover for all } \mu &\leq \frac{\nu^2 \sigma^2}{8}, \quad \mathbb{E} \exp \left( \mu \int_0^t (X_s^x)^{-1} ds \right) \leq C(T) \left( 1 + x^{-\nu/2} \right), \end{aligned} \quad (3.2)$$

where the positive constant  $C(T)$  is a nondecreasing function of  $T$  and does not depend on  $x$ .

**Proof.** As  $b(x) \geq b(0) - Kx$ , comparison Theorem gives that, a.s,  $X_t^x \geq Y_t^x$ , for all  $t \geq t_0$ , where  $(Y_t^x, t \leq T)$  is a CIR process solving

$$Y_t^x = x + \int_0^t (b(0) - KY_s^x) ds + \sigma \int_0^t \sqrt{Y_s^x} dW_s. \quad (3.3)$$

In particular,  $\mathbb{E} \exp \left( \mu \int_0^t (X_s^x)^{-1} ds \right) \leq \mathbb{E} \exp \left( \mu \int_0^t (Y_s^x)^{-1} ds \right)$ . As  $b(0) \geq \sigma^2$  by (H3), one can derive (3.2) from the Lemma A.2. Similarly, for upper bounds on inverse moments of  $X_t^x$ , we apply the Lemma A.1.  $\blacksquare$

##### 3.1.2 On the associated Kolmogorov PDE

**Proposition 3.2** *Let  $f$  be a  $\mathbb{R}$ -valued  $C^4$  bounded function, with bounded spatial derivatives up to the order 4. We consider the  $\mathbb{R}$ -valued function  $u$ , defined on  $[0, T] \times [0, +\infty)$  by  $u(t, x) = \mathbb{E}f(X_{T-t}^x)$ . Let  $\alpha = 1/2$ . Then under (H1), (H2) and (H3),  $u$  is in  $C^{1,4}([0, T] \times [0, +\infty))$ . That is,  $u$  has derivatives up to order 4 in the space variable and a derivative in*

the time variable and there exists a positive constant  $C$  depending on  $f$ ,  $b$  and  $T$  such that, for all  $x \in [0, +\infty)$ ,

$$\sup_{t \in [0, T]} \left| \frac{\partial u}{\partial t}(t, x) \right| \leq C(1 + x) \quad (3.4)$$

and

$$\|u\|_{L^\infty([0, T] \times [0, +\infty))} + \sum_{k=1}^4 \left\| \frac{\partial^k u}{\partial x^k} \right\|_{L^\infty([0, T] \times [0, +\infty))} \leq C. \quad (3.5)$$

Moreover  $u$  satisfies

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) + b(x) \frac{\partial u}{\partial x}(t, x) + \frac{\sigma^2}{2} x \frac{\partial^2 u}{\partial x^2}(t, x) = 0, & (t, x) \in [0, T] \times [0, +\infty), \\ u(T, x) = f(x), & x \in [0, +\infty). \end{cases} \quad (3.6)$$

Before to prove the above Proposition, let us introduce some notations. For any  $\lambda \geq 0$ , and any  $x > 0$ , we denote by  $(X_t^x(\lambda))_{t_0 \leq t \leq T}$ , the  $\mathbb{R}^+$ -valued process which solves

$$X_t^x(\lambda) = x + \lambda \sigma^2 t + \int_0^t b(X_s^x(\lambda)) ds + \sigma \int_0^t \sqrt{X_s^x(\lambda)} dW_s. \quad (3.7)$$

Equation (3.7) has a non-exploding unique strong solution. Moreover, for all  $t \geq 0$ ,  $X_t^x(\lambda) \geq X_t^x$ . The coefficients are locally Lipschitz on  $(0, +\infty)$ , with locally Lipschitz first order derivatives. Then  $X_t^x(\lambda)$  is continuously differentiable (see Theorem V.39 in [11]), and if we denote  $J_t^x(\lambda) = \frac{dX_t^x(\lambda)}{dx}(\lambda)$ , the process  $(J_t^x(\lambda))_{0 \leq t \leq T}$  satisfies the linear equation

$$J_t^x(\lambda) = 1 + \int_0^t J_s^x(\lambda) b'(X_s^x(\lambda)) ds + \int_0^t J_s^x(\lambda) \frac{\sigma dW_s}{2\sqrt{X_s^x(\lambda)}}.$$

By Lemma 3.1, the process  $(\int_0^t \frac{dW_s}{\sqrt{X_s^x(\lambda)}})_{t \geq 0}$  is a locally square integrable martingale. Then, (see Theorem V.51 in [11]),  $J_t^x(\lambda)$  is given by

$$J_t^x(\lambda) = \exp \left( \int_0^t b'(X_s^x(\lambda)) ds + \frac{\sigma}{2} \int_0^t \frac{dW_s}{\sqrt{X_s^x(\lambda)}} - \frac{\sigma^2}{8} \int_0^t \frac{ds}{X_s^x(\lambda)} \right). \quad (3.8)$$

**Lemma 3.3** Assume (H3). The process  $(M_t^x(\lambda))_{t \in [0, T]}$  defined by

$$M_t^x(\lambda) = \exp \left( \frac{\sigma}{2} \int_0^t \frac{dW_s}{\sqrt{X_s^x(\lambda)}} - \frac{\sigma^2}{8} \int_0^t \frac{ds}{X_s^x(\lambda)} \right)$$

is a martingale. Moreover,  $\sup_{t \in [0, T]} \mathbb{E}(J_t^x(\lambda)) \leq \exp(CT)$ .

**Proof.** By Lemma 3.1,  $(M_t^x(\lambda))_{t \in [0, T]}$  satisfies the Novikov criteria. Under (H2),  $b'$  is a bounded function and

$$\mathbb{E}[J_t^x(\lambda)] = \mathbb{E} \left[ \exp \left( \int_0^t b'(X_s^x(\lambda)) ds \right) M_t^x(\lambda) \right] \leq \exp(\|b'\|_\infty T).$$

■

Let  $(\mathcal{Z}_t^{(\lambda, \lambda + \frac{1}{2})})_{t \in [0, T]}$  defined by

$$\mathcal{Z}_t^{(\lambda, \lambda + \frac{1}{2})} = \exp \left( -\frac{\sigma}{2} \int_0^t \frac{1}{\sqrt{X_s^x(\lambda)}} \left( \frac{dX_s^x(\lambda)}{\sigma \sqrt{X_s^x(\lambda)}} - \frac{(b(X_s^x(\lambda)) + (\lambda + \frac{1}{2})\sigma^2)}{\sigma \sqrt{X_s^x(\lambda)}} ds \right) - \frac{\sigma^2}{8} \int_0^t \frac{ds}{X_s^x(\lambda)} \right).$$

By the Girsanov Theorem, under the probability  $\mathbb{Q}^{\lambda + \frac{1}{2}}$  such that  $\frac{d\mathbb{Q}^{\lambda + \frac{1}{2}}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \frac{1}{\mathcal{Z}_t^{(\lambda, \lambda + \frac{1}{2})}}$ ,  $B_t^{\lambda + \frac{1}{2}} = \int_0^t \frac{dX_s^x(\lambda)}{\sigma \sqrt{X_s^x(\lambda)}} - \frac{(b(X_s^x(\lambda)) + (\lambda + \frac{1}{2})\sigma^2)}{\sigma \sqrt{X_s^x(\lambda)}} ds$  is a Brownian motion on  $(\Omega, \mathcal{F}_T, \mathbb{Q}^{\lambda + \frac{1}{2}})$ . Indeed, we have that

$$X_t^x(\lambda) = x + (\lambda + \frac{1}{2})\sigma^2 t + \int_0^t b(X_s^x(\lambda)) ds + \sigma \int_0^t \sqrt{X_s^x(\lambda)} dB_s^{\lambda + \frac{1}{2}}.$$

Hence,  $\mathcal{L}^{\mathbb{Q}^{\lambda + \frac{1}{2}}}(X^x(\lambda)) = \mathcal{L}^{\mathbb{P}}(X^x(\lambda + \frac{1}{2}))$  and from Lemma 3.3,  $\mathcal{Z}_t^{(\lambda, \lambda + \frac{1}{2})} = \exp(-\frac{\sigma}{2} \int_0^t \frac{dB_s^{\lambda + \frac{1}{2}}}{\sqrt{X_s^x(\lambda)}} - \frac{\sigma^2}{8} \int_0^t \frac{ds}{X_s^x(\lambda)})$  is a  $\mathbb{Q}^{\lambda + \frac{1}{2}}$ -martingale.

**Proof of Proposition 3.2.** First, we note that  $|u(t, x)|$  is bounded as  $|f(x)|$ , by hypothesis. Let us show that  $u$  is in  $C^{1,4}([0, T] \times [0, +\infty))$ .  $f$  being in  $C^4(\mathbb{R})$ , by the Itô formula,

$$\begin{aligned} u(t, x) = \mathbb{E}f(X_{T-t}^x) &= f(x) + \int_0^{T-t} \mathbb{E}(b(X_s^x)f'(X_s^x)) ds + \frac{\sigma^2}{2} \int_0^{T-t} \mathbb{E}(X_s^x f''(X_s^x)) ds \\ &\quad + \sigma \mathbb{E} \left( \int_0^{T-t} \sqrt{X_s^x} f'(X_s^x) dW_s \right). \end{aligned}$$

$f'$  being bounded and  $(X_t^x)$  having moments of any order, the above stochastic integral is a martingale. Its expectation is nil and we obtain that

$$\frac{\partial u}{\partial t}(t, x) = -\mathbb{E} \left( b(X_{T-t}^x) f'(X_{T-t}^x) + \frac{\sigma^2}{2} X_{T-t}^x f''(X_{T-t}^x) \right).$$

Thus,  $\frac{\partial u}{\partial t}$  is a continuous function on  $[0, T] \times [0, +\infty)$  and (3.4) follows by Lemma 2.1.

In Proposition B.1 of the Appendix B, we prove that  $u(t, x) = \mathbb{E}f(X_{T-t}^x)$  is differentiable and if we denote  $J_t^x(0) = \frac{dX_t^x}{dx}$  then

$$\frac{\partial u}{\partial x}(t, x) = \mathbb{E} \left( f'(X_{T-t}^x(0)) J_{T-t}^x(0) \right).$$

Hence, using Lemma 3.3, we have

$$\left| \frac{\partial u}{\partial x}(t, x) \right| \leq \|f'\|_\infty \mathbb{E} \left( J_{T-t}^x(0) \right) \leq \|f'\|_\infty \exp(CT). \quad (3.9)$$

We introduce the probability  $\mathbb{Q}^{\frac{1}{2}}$  such that  $\frac{d\mathbb{Q}^{\frac{1}{2}}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \frac{1}{\mathcal{Z}_t^{(0, \frac{1}{2})}}$ . Denoting by  $\mathbb{E}^{\frac{1}{2}}$  the expectation under the probability  $\mathbb{Q}^{\frac{1}{2}}$ , we have

$$\frac{\partial u}{\partial x}(t, x) = \mathbb{E}^{\frac{1}{2}} \left( f'(X_{T-t}^x(0)) \mathcal{Z}_{T-t}^{(0, \frac{1}{2})} J_{T-t}^x(0) \right)$$

By the Girsanov Theorem,  $W_t = B_t^{\frac{1}{2}} + \int_0^t \frac{\sigma}{2\sqrt{X_s^x(0)}} ds$  and

$$J_t^x(0) = \exp \left( \int_0^t b'(X_s^x(0)) ds + \frac{\sigma}{2} \int_0^t \frac{dB_s^{\frac{1}{2}}}{\sqrt{X_s^x(0)}} + \frac{\sigma^2}{8} \int_0^t \frac{ds}{X_s^x(0)} \right).$$

We can deduce from the definition of  $\mathcal{Z}_t^{(0, \frac{1}{2})}$  that  $\mathcal{Z}_{T-t}^{(0, \frac{1}{2})} J_{T-t}^x(0)$  is reduce to  $\exp \left( \int_0^{T-t} b'(X_s^x(0)) ds \right)$  and hence,

$$\frac{\partial u}{\partial x}(t, x) = \mathbb{E}^{\frac{1}{2}} \left[ f'(X_{T-t}^x(0)) \exp \left( \int_0^{T-t} b'(X_s^x(0)) ds \right) \right].$$

As  $\mathcal{L}^{\mathbb{Q}^{\frac{1}{2}}}(X^x(0).) = \mathcal{L}^{\mathbb{P}}(X^x(\frac{1}{2}).)$ , we finally obtain the following expression for  $\frac{\partial u}{\partial x}(t, x)$ :

$$\frac{\partial u}{\partial x}(t, x) = \mathbb{E} \left[ f'(X_{T-t}^x(\frac{1}{2})) \exp \left( \int_0^{T-t} b'(X_s^x(\frac{1}{2})) ds \right) \right], \quad (3.10)$$

from which we deduce that  $\frac{\partial u}{\partial x}(t, x)$  is continuously differentiable (see Proposition B.1) and

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(t, x) = & \mathbb{E} \left[ f''(X_{T-t}^x(\frac{1}{2})) J_{T-t}^x(\frac{1}{2}) \exp \left( \int_0^{T-t} b'(X_s^x(\frac{1}{2})) ds \right) \right] \\ & + \mathbb{E} \left[ f'(X_{T-t}^x(\frac{1}{2})) \exp \left( \int_0^{T-t} b'(X_s^x(\frac{1}{2})) ds \right) \left( \int_0^{T-t} b''(X_s^x(\frac{1}{2})) J_s^x(\frac{1}{2}) ds \right) \right]. \end{aligned} \quad (3.11)$$

It follows that

$$\left| \frac{\partial^2 u}{\partial x^2}(t, x) \right| \leq (\|f''\|_\infty + \|f'\|_\infty \|b''\|_\infty T) \exp(2TK) = C \quad (3.12)$$

and  $u \in C^{1,2}([0, T] \times [0, +\infty))$ . We want to compute the third derivative. Before to do so, we transform the expression (3.11) in order to avoid the appearance of the derivative of  $J_t^x(\frac{1}{2})$  that we do not control. Thank to the Markov property and the time homogeneity of the process  $X_t^x(\frac{1}{2})$ , for any  $s \in [0, T - t]$ ,

$$\begin{aligned} & \mathbb{E} \left[ f'(X_{T-t}^x(\tfrac{1}{2})) \exp \left( \int_s^{T-t} b'(X_u^x(\tfrac{1}{2})) du \right) \middle/ \mathcal{F}_s \right] \\ &= \mathbb{E} \left[ f'(X_{T-t-s}^y(\tfrac{1}{2})) \exp \left( \int_0^{T-t-s} b'(X_u^y(\tfrac{1}{2})) du \right) \right] \Big|_{y=X_s^x(\frac{1}{2})}. \end{aligned}$$

Using (3.10), we get

$$\mathbb{E} \left[ f'(X_{T-t}^x(\tfrac{1}{2})) \exp \left( \int_s^{T-t} b'(X_u^x(\tfrac{1}{2})) du \right) \middle/ \mathcal{F}_s \right] = \frac{\partial u}{\partial x}(t + s, X_s^x(\tfrac{1}{2})).$$

We introduce this last equality in the second term of the right-hand side of (3.11):

$$\begin{aligned} & \mathbb{E} \left[ f'(X_{T-t}^x(\tfrac{1}{2})) \exp \left( \int_0^{T-t} b'(X_u^x(\tfrac{1}{2})) du \right) \left( \int_0^{T-t} b''(X_s^x(\tfrac{1}{2})) J_s^x(\tfrac{1}{2}) ds \right) \right] \\ &= \mathbb{E} \left[ \int_0^{T-t} \mathbb{E} \left( f'(X_{T-t}^x(\tfrac{1}{2})) \exp \left( \int_s^{T-t} b'(X_u^x(\tfrac{1}{2})) du \right) \middle/ \mathcal{F}_s \right) \right. \\ & \quad \left. \exp \left( \int_0^s b'(X_u^x(\tfrac{1}{2})) du \right) b''(X_s^x(\tfrac{1}{2})) J_s^x(\tfrac{1}{2}) ds \right] \\ &= \mathbb{E} \left[ \int_0^{T-t} \frac{\partial u}{\partial x}(t + s, X_s^x(\tfrac{1}{2})) \exp \left( \int_0^s b'(X_u^x(\tfrac{1}{2})) du \right) b''(X_s^x(\tfrac{1}{2})) J_s^x(\tfrac{1}{2}) ds \right] \\ &= \int_0^{T-t} \mathbb{E} \left[ \frac{\partial u}{\partial x}(t + s, X_s^x(\tfrac{1}{2})) \exp \left( \int_0^s b'(X_u^x(\tfrac{1}{2})) du \right) b''(X_s^x(\tfrac{1}{2})) J_s^x(\tfrac{1}{2}) \right] ds. \end{aligned}$$

Coming back to (3.11), this leads to the following expression for  $\frac{\partial^2 u}{\partial x^2}(t, x)$ :

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(t, x) &= \mathbb{E} \left[ f''(X_{T-t}^x(\tfrac{1}{2})) J_{T-t}^x(\tfrac{1}{2}) \exp \left( \int_0^{T-t} b'(X_s^x(\tfrac{1}{2})) ds \right) \right] \\ & \quad + \int_0^{T-t} \mathbb{E} \left[ \frac{\partial u}{\partial x}(t + s, X_s^x(\tfrac{1}{2})) \exp \left( \int_0^s b'(X_u^x(\tfrac{1}{2})) du \right) b''(X_s^x(\tfrac{1}{2})) J_s^x(\tfrac{1}{2}) \right] ds. \end{aligned}$$

We introduce the probability  $\mathbb{Q}^1$  such that  $\frac{d\mathbb{Q}^1}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \frac{1}{Z_t^{(\frac{1}{2},1)}}$ . Then,

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(t, x) = & \mathbb{E}^1 \left[ Z_{T-t}^{(\frac{1}{2},1)} f''(X_{T-t}^x(\frac{1}{2})) J_{T-t}^x(\frac{1}{2}) \exp \left( \int_0^{T-t} b'(X_s^x(\frac{1}{2})) ds \right) \right] \\ & + \int_0^{T-t} \mathbb{E}^1 \left[ Z_s^{(\frac{1}{2},1)} \frac{\partial u}{\partial x}(t+s, X_s^x(\frac{1}{2})) \exp \left( \int_0^s b'(X_u^x(\frac{1}{2})) du \right) b''(X_s^x(\frac{1}{2})) J_s^x(\frac{1}{2}) \right] ds. \end{aligned}$$

By the Girsanov Theorem, for all  $\theta \in [0, T]$ ,  $Z_\theta^{(\frac{1}{2},1)} J_\theta^x(\frac{1}{2}) = \exp \left( \int_0^\theta b'(X_u^x(\frac{1}{2})) du \right)$  and

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(t, x) = & \mathbb{E}^1 \left[ f''(X_{T-t}^x(\frac{1}{2})) \exp \left( 2 \int_0^{T-t} b'(X_s^x(\frac{1}{2})) ds \right) \right] \\ & + \int_0^{T-t} \mathbb{E}^1 \left[ \frac{\partial u}{\partial x}(t+s, X_s^x(\frac{1}{2})) \exp \left( 2 \int_0^s b'(X_u^x(\frac{1}{2})) du \right) b''(X_s^x(\frac{1}{2})) \right] ds. \end{aligned}$$

As  $\mathcal{L}^{\mathbb{Q}^1}(X^x(\frac{1}{2})) = \mathcal{L}^{\mathbb{P}}(X^x(1))$ , we finally obtain the following expression for  $\frac{\partial^2 u}{\partial x^2}(t, x)$ :

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(t, x) = & \mathbb{E} \left[ f''(X_{T-t}^x(1)) \exp \left( 2 \int_0^{T-t} b'(X_s^x(1)) ds \right) \right] \\ & + \int_0^{T-t} \mathbb{E} \left[ \frac{\partial u}{\partial x}(t+s, X_s^x(1)) \exp \left( 2 \int_0^s b'(X_u^x(1)) du \right) b''(X_s^x(1)) \right] ds. \end{aligned} \quad (3.13)$$

As  $J_s^x(1) = \frac{dX_s^x(1)}{dx}$  exists and is given by (3.8),  $\frac{\partial^2 u}{\partial x^2}(t, x)$  is continuously differentiable (see again Proposition B.1) and

$$\begin{aligned} \frac{\partial^3 u}{\partial x^3}(t, x) = & \mathbb{E} \left\{ \exp \left( 2 \int_0^{T-t} b'(X_s^x(1)) ds \right) \right. \\ & \left[ f^{(3)}(X_{T-t}^x(1)) J_{T-t}^x(1) + 2f''(X_{T-t}^x(1)) \int_0^{T-t} b''(X_s^x(q)) J_s^x(1) ds \right] \Big\} \\ & + \int_0^{T-t} \mathbb{E} \left\{ \exp \left( 2 \int_0^s b'(X_u^x(1)) du \right) \right. \\ & \left[ J_s^x(1) \left( \frac{\partial^2 u}{\partial x^2}(t+s, X_s^x(1)) b''(X_s^x(1)) + \frac{\partial u}{\partial x}(t+s, X_s^x(1)) b^{(3)}(X_s^x(1)) \right) \right. \\ & \left. \left. + 2 \frac{\partial u}{\partial x}(t+s, X_s^x(1)) b''(X_s^x(1)) \int_0^s b''(X_u^x(1)) J_u^x(1) du \right] \right\} ds. \end{aligned} \quad (3.14)$$

By Lemma 3.3, the derivatives of  $f$  and  $b$  being bounded up to the order 3, we get immediately that  $\frac{\partial^3 u}{\partial x^3}(t, x) \leq C$ .



The computation of the fourth derivative follows some similar arguments, We detail it at the end of the Appendix B.

To end the proof, it remains to show that  $u(t, x)$  solve the Cauchy problem (3.6). This can be done by using the same arguments than in the proof of Theorem 6.1 in [5]. ■

### 3.1.3 On the approximation process

For  $\alpha = 1/2$ , recall that the discrete time process  $(\bar{X})$  associated to  $(X)$  is defined by

$$\begin{cases} \bar{X}_0 = x_0, \\ \bar{X}_{t_{k+1}} = \left| \bar{X}_{t_k} + b(\bar{X}_{t_k})\Delta t + \sigma\sqrt{\bar{X}_{t_k}}(W_{t_{k+1}} - W_{t_k}) \right|, \quad k = 0, \dots, N-1, \end{cases} \quad (3.15)$$

where  $N$  is such that  $N\Delta t = T$  and  $t_k = k\Delta t$ . The time continuous version  $(\bar{X}_t)_{0 \leq t \leq T}$  is defined on each interval  $[t_k, t_{k+1})$  by

$$\bar{X}_t = \left| \bar{X}_{\eta(t)} + (t - \eta(t))b(\bar{X}_{\eta(t)}) + \sigma\sqrt{\bar{X}_{\eta(t)}}(W_t - W_{\eta(t)}) \right|, \quad (3.16)$$

or equivalently by,

$$\bar{X}_t = x_0 + \int_0^t \text{sgn}(\bar{Z}_s)b(\bar{X}_{\eta(s)})ds + \sigma \int_0^t \text{sgn}(\bar{Z}_s)\sqrt{\bar{X}_{\eta(s)}}dW_s + L_t^0(\bar{X}), \quad (3.17)$$

where for any  $t \in [0, T]$ , we set

$$\bar{Z}_t = \bar{X}_{\eta(t)} + (t - \eta(t))b(\bar{X}_{\eta(t)}) + \sigma\sqrt{\bar{X}_{\eta(t)}}(W_t - W_{\eta(t)}). \quad (3.18)$$

In this section, we are interested in the behavior of the processes  $(\bar{X})$  and  $(\bar{Z})$  visiting the point 0. The main result is the following Proposition, the proof of which is postponed:

**Proposition 3.4** *Let  $\alpha = \frac{1}{2}$ . Assume (H1). For  $\Delta t$  sufficiently small ( $\Delta t \leq 1/(2K)$ ), there exists a constant  $C > 0$ , depending on  $b(0)$ ,  $K$ ,  $\sigma$ ,  $x_0$  and  $T$  but not in  $\Delta t$ , such that*

$$\mathbb{E} \left( L_t^0(\bar{X}) - L_{\eta(t)}^0(\bar{X}) \middle| \mathcal{F}_{\eta(t)} \right) \leq C\Delta t \exp \left( -\frac{\bar{X}_{\eta(t)}}{16\sigma^2\Delta t} \right)$$

and

$$\mathbb{E} L_T^0(\bar{X}) \leq C \left( \frac{\Delta t}{x_0} \right)^{\frac{b(0)}{\sigma^2}}. \quad (3.19)$$

The above upper bounds on the local time  $(L^0(\bar{X}))$  are based on the following technical Lemmas:

**Lemma 3.5** *Let  $(\bar{X})$  defined by (3.15). Assume (H1). Assume also that  $\Delta t$  is sufficiently small ( $\Delta t \leq 1/(2K) \wedge x_0$ ). Then for any  $\gamma \geq 1$ , there exists a positive constant  $C$ , depending on all the parameters  $b(0)$ ,  $K$ ,  $\sigma$ ,  $x_0$ ,  $T$  and also on  $\gamma$ , such that*

$$\sup_{k=0,\dots,N} \mathbb{E} \exp \left( -\frac{\bar{X}_{t_k}}{\gamma \sigma^2 \Delta t} \right) \leq C \left( \frac{\Delta t}{x_0} \right)^{\frac{2b(0)}{\sigma^2} (1 - \frac{1}{2\gamma})}.$$

**Lemma 3.6** *Assume (H1). For  $\Delta t$  sufficiently small ( $\Delta t \leq 1/(2K)$ ), for any  $t \in [0, T]$ ,*

$$\mathbb{P}(\bar{Z}_t \leq 0 / \bar{X}_{\eta(t)}) \leq \frac{1}{2} \exp \left( -\frac{\bar{X}_{\eta(t)}}{2h(\Delta t) \sigma^2 \Delta t} \right),$$

where  $h(\Delta t) := (1 - K\Delta t)^{-2}$  and  $K$  is the Lipschitz constant of  $b$ .

As  $2h(\Delta t) > 1$  when  $\Delta t \leq 1/(2K)$ , the combination of Lemmas 3.5 and 3.6 leads to

$$\mathbb{P}(\bar{Z}_t \leq 0) \leq C \left( \frac{\Delta t}{x_0} \right)^{\frac{b(0)}{\sigma^2}}. \quad (3.20)$$

We give successively the proofs of Lemmas 3.5, 3.6 and Proposition 3.4.

**Proof of Lemma 3.5.** First let us show that there exists a positive sequence  $(\mu_j)_{0 \leq j \leq N}$  such that, for any  $k \in \{1, \dots, N\}$ ,

$$\mathbb{E} \exp \left( -\frac{\bar{X}_{t_k}}{\gamma \sigma^2 \Delta t} \right) \leq \exp \left( -b(0) \sum_{j=0}^{k-1} \mu_j \Delta t \right) \exp(-x_0 \mu_k).$$

We set  $\mu_0 = \frac{1}{\gamma \sigma^2 \Delta t}$ . By (3.15), as  $\exp(-\alpha|x|) \leq \exp(-\alpha x)$ , for any  $\alpha > 0$ , we have that

$$\begin{aligned} \mathbb{E} \exp(-\mu_0 \bar{X}_{t_k}) &\leq \mathbb{E} \exp \left( -\mu_0 \left( \bar{X}_{t_{k-1}} + b(\bar{X}_{t_{k-1}}) \Delta t + \sigma \sqrt{\bar{X}_{t_{k-1}}} \Delta W_{t_k} \right) \right) \\ &\leq \mathbb{E} \exp \left( -\mu_0 \left( \bar{X}_{t_{k-1}} + (b(0) - K \bar{X}_{t_{k-1}}) \Delta t + \sigma \sqrt{\bar{X}_{t_{k-1}}} \Delta W_{t_k} \right) \right), \end{aligned}$$

where we used that  $-b(x) \leq -b(0) + Kx$ , for all  $x \in \mathbb{R}$ .  $\Delta W_{t_k}$  and  $\bar{X}_{t_{k-1}}$  being independent,

$$\mathbb{E} \exp \left( -\mu_0 \sigma \sqrt{\bar{X}_{t_{k-1}}} \Delta W_{t_k} \right) = \mathbb{E} \exp \left( \frac{\sigma^2}{2} \mu_0^2 \Delta t \bar{X}_{t_{k-1}} \right). \text{ Thus}$$

$$\begin{aligned} \mathbb{E} \exp(-\mu_0 \bar{X}_{t_k}) &\leq \exp(-\mu_0 b(0) \Delta t) \mathbb{E} \exp \left( -\mu_0 \bar{X}_{t_{k-1}} \left( 1 - K \Delta t - \frac{\sigma^2}{2} \mu_0 \Delta t \right) \right) \\ &= \exp(-\mu_0 b(0) \Delta t) \mathbb{E} \exp(-\mu_1 \bar{X}_{t_{k-1}}) \end{aligned}$$

where we set  $\mu_1 = \mu_0 \left(1 - K\Delta t \frac{\sigma^2}{2} \mu_0 \Delta t\right)$ . Consider now the sequence  $(\mu_j)_{0 \leq j \leq k}$  defined by

$$\begin{cases} \mu_0 = \frac{1}{\gamma \sigma^2 \Delta t}, \\ \mu_j = \mu_{j-1} \left(1 - K\Delta t - \frac{\sigma^2}{2} \Delta t \mu_{j-1}\right), j = 1, \dots, N. \end{cases} \quad (3.21)$$

For any  $j \in \{0, \dots, k-1\}$ , by the same computation we have

$$\mathbb{E} \exp(-\mu_j \bar{X}_{t_{k-j}}) \leq \exp(-b(0)\mu_j \Delta t) \mathbb{E} \exp(-\mu_{j+1} \bar{X}_{t_{k-j-1}})$$

and it follows by induction that

$$\mathbb{E} \exp(-\mu_0 \bar{X}_{t_k}) \leq \exp\left(-b(0) \sum_{j=0}^{k-1} \mu_j \Delta t\right) \exp(-x_0 \mu_k). \quad (3.22)$$

Now, we study the sequence  $(\mu_j)_{0 \leq j \leq N}$ . First, an easy computation shows that if  $\gamma \geq 1$  and  $\Delta t \leq \frac{1}{2K}$ , then  $(\mu_j)$  is positive and decreasing sequence, that is  $0 \leq \mu_N \leq \mu_{N-1} \leq \dots \leq \mu_0$ . For any  $\alpha > 0$ , we consider the nondecreasing function  $f_\alpha(x) := \frac{x}{1+\alpha x}$ ,  $x \in \mathbb{R}$  and we note that  $(f_\alpha \circ f_\beta)(x) = f_{\alpha+\beta}(x)$ . Then, for any  $j \geq 1$ , as the sequence  $(\mu_j)$  is decreasing,

$$\mu_j \leq f_{\frac{\sigma^2}{2} \Delta t}(\mu_{j-1}) \leq f_{\frac{\sigma^2}{2} \Delta t}(f_{\frac{\sigma^2}{2} \Delta t}(\mu_{j-2})) \leq \dots \leq f_{\frac{\sigma^2}{2} j \Delta t}(\mu_0). \quad (3.23)$$

The next step consists in get a lower-bound of the  $\mu_j$ . By induction, we prove that for any  $j \geq 1$

$$\mu_j \geq \mu_1 \left( \frac{1}{1 + \frac{\sigma^2}{2} \Delta t (j-1) \mu_0} \right) - K \left( \frac{\mu_0 \Delta t (j-1)}{1 + \frac{\sigma^2}{2} \Delta t (j-1) \mu_0} \right). \quad (3.24)$$

(3.24) is clearly true for  $j = 1$ . Suppose (3.24) holds for  $j$ . By (3.21) and (3.23)

$$\begin{aligned} \mu_{j+1} &= \mu_j \left(1 - \frac{\sigma^2}{2} \Delta t \mu_j\right) - K \Delta t \mu_j \geq \mu_j \left(1 - \frac{\sigma^2}{2} \Delta t f_{\frac{\sigma^2}{2} j \Delta t}(\mu_0)\right) - K \Delta t f_{\frac{\sigma^2}{2} j \Delta t}(\mu_0) \\ &\geq \mu_j \left( \frac{1 + \frac{\sigma^2}{2} \Delta t (j-1) \mu_0}{1 + \frac{\sigma^2}{2} \Delta t j \mu_0} \right) - K \left( \frac{\Delta t \mu_0}{1 + \frac{\sigma^2}{2} \Delta t j \mu_0} \right) \end{aligned}$$

and we conclude by using (3.24) for  $\mu_j$ . Then, by definition of  $\mu_1$  and  $\mu_0$ , for any  $j \geq 0$

$$\begin{aligned} \mu_j &\geq \mu_0 \left( \frac{1 - \frac{\sigma^2}{2} \Delta t \mu_0}{1 + \frac{\sigma^2}{2} \Delta t (j-1) \mu_0} \right) - K \left( \frac{\mu_0 \Delta t j}{1 + \frac{\sigma^2}{2} \Delta t (j-1) \mu_0} \right) \\ &\geq \frac{2\gamma - 1}{\Delta t \gamma \sigma^2 (2\gamma - 1 + j)} - \frac{2K}{\sigma^2} \end{aligned}$$

and hence,

$$\begin{aligned} \sum_{j=0}^{k-1} \Delta t \mu_j &\geq \frac{1}{\gamma \sigma^2} \sum_{j=0}^{k-1} \frac{2\gamma - 1}{2\gamma - 1 + j} - t_k \frac{2K}{\sigma^2} \geq \frac{1}{\gamma \sigma^2} \int_0^k \frac{2\gamma - 1}{2\gamma - 1 + u} du - t_k \frac{2K}{\sigma^2} \\ &\geq \frac{2\gamma - 1}{\gamma \sigma^2} \ln \left( \frac{2\gamma - 1 + k}{2\gamma - 1} \right) - \frac{2TK}{\sigma^2}. \end{aligned}$$

Coming back to (3.22), we obtain

$$\begin{aligned} \mathbb{E} \exp(-\mu_0 \bar{X}_{t_k}) &\leq \exp \left( b(0) \frac{2TK}{\sigma^2} \right) \exp \left( x_0 \frac{2K}{\sigma^2} \right) \\ &\quad \times \exp \left( -b(0) \frac{2\gamma - 1}{\gamma \sigma^2} \ln \left( \frac{2\gamma - 1 + k}{2\gamma - 1} \right) \right) \exp \left( -\frac{x_0}{\Delta t \gamma \sigma^2} \frac{2\gamma - 1}{2\gamma - 1 + k} \right) \\ &\leq \exp \left( b(0) \frac{2TK}{\sigma^2} \right) \exp \left( x_0 \frac{2K}{\sigma^2} \right) \\ &\quad \times \left( \frac{2\gamma - 1}{2\gamma - 1 + k} \right)^{\frac{2b(0)}{\sigma^2} (1 - \frac{1}{2\gamma})} \exp \left( -\frac{x_0}{\Delta t \gamma \sigma^2} \frac{2\gamma - 1}{2\gamma - 1 + k} \right). \end{aligned}$$

To conclude, we use that for all  $\alpha > 0$ ,  $x^\alpha \exp(-x) \leq \alpha^\alpha \exp(-\alpha)$ . It comes that

$$\begin{aligned} \mathbb{E} \exp \left( -\frac{\bar{X}_{t_k}}{\gamma \sigma^2 \Delta t} \right) &\leq \exp \left( b(0) \frac{2TK}{\sigma^2} \right) \exp \left( x_0 \frac{2K}{\sigma^2} \right) \\ &\quad \times \left( 2b(0) \frac{\Delta t \gamma}{x_0} \left( 1 - \frac{1}{2\gamma} \right) \right)^{\frac{2b(0)}{\sigma^2} (1 - \frac{1}{2\gamma})} \exp \left( -\frac{2b(0)}{\sigma^2} \left( 1 - \frac{1}{2\gamma} \right) \right) \\ &\leq C(\gamma, x_0, b, T, \sigma) \left( \frac{\Delta t}{x_0} \right)^{\frac{2b(0)}{\sigma^2} (1 - \frac{1}{2\gamma})} \end{aligned}$$

where we set

$$C(\gamma, x_0, b, T, \sigma) = (b(0)(2\gamma - 1))^{\frac{2b(0)}{\sigma^2} (1 - \frac{1}{2\gamma})} \exp \left( \frac{2}{\sigma^2} \left( b(0)(TK - 1 + \frac{1}{2\gamma}) + x_0 K \right) \right).$$

■

**Proof of Lemma 3.6.** For any  $x \geq 0$ , by the Hypothesis (H1),  $b(x) \geq b(0) - Kx$ . Thus, by (3.18),

$$\begin{aligned} \mathbb{P}(\bar{Z}_t \leq 0) &\leq \mathbb{P} \left( \bar{X}_{\eta(t)} + (b(0) - K\bar{X}_{\eta(t)})(t - \eta(t)) + \sigma \sqrt{\bar{X}_{\eta(t)}} (W_t - W_{\eta(t)}) \leq 0 \right) \\ &= \mathbb{P} \left( W_t - W_{\eta(t)} \leq \frac{-\bar{X}_{\eta(t)}(1 - K(t - \eta(t))) - b(0)(t - \eta(t))}{\sigma \sqrt{\bar{X}_{\eta(t)}}}, \bar{X}_{\eta(t)} > 0 \right). \end{aligned}$$

Using the Gaussian inequality  $\mathbb{P}(G \leq \beta) \leq 1/2 \exp(-\beta^2/2)$  when  $G$  is standard normal distributed and  $\beta > 0$ , we get

$$\mathbb{P}(\bar{Z}_t \leq 0) \leq \frac{1}{2} \mathbb{E} \left[ \exp \left( - \frac{(\bar{X}_{\eta(t)}(1 - K(t - \eta(t))) + b(0)(t - \eta(t)))^2}{2\sigma^2(t - \eta(t))\bar{X}_{\eta(t)}} \right) \mathbb{1}_{\{\bar{X}_{\eta(t)} > 0\}} \right]$$

from which we finally obtain that  $\mathbb{P}(\bar{Z}_t \leq 0) \leq \frac{1}{2} \mathbb{E} \left[ \exp \left( - \frac{\bar{X}_{\eta(t)}}{2(1 - K\Delta t)^{-2}\sigma^2\Delta t} \right) \right]$ .  $\blacksquare$

**Proof of Proposition 3.4.** For any  $t \in [t_k, t_{k+1})$ ,  $0 \leq k \leq N$ , for any bounded Borel-measurable function  $\phi$ , by the occupation time formula,  $\mathbb{P}$  a.s

$$\begin{aligned} \int_{\mathbb{R}} \phi(z) (L_t^z(\bar{X}) - L_{t_k}^z(\bar{X})) dz &= \int_{\mathbb{R}} \phi(z) (L_t^z(\bar{Z}) - L_{t_k}^z(\bar{Z})) dz = \int_{t_k}^t \phi(\bar{Z}_s) d\langle \bar{Z}, \bar{Z} \rangle_s \\ &= \sigma^2 \int_{t_k}^t \phi(\bar{Z}_s) \bar{X}_{t_k} ds \end{aligned}$$

and hence, for any  $x > 0$ ,

$$\begin{aligned} &\int_{\mathbb{R}} \phi(z) \mathbb{E} (L_t^z(\bar{X}) - L_{t_k}^z(\bar{X}) / \{\bar{X}_{t_k} = x\}) dz \\ &= \sigma^2 \int_{t_k}^t \left( \int_{\mathbb{R}} \phi(x + b(x)(s - t_k) + \sigma y \sqrt{x}) \frac{x}{\sqrt{2\pi(s - t_k)}} \exp \left( - \frac{y^2}{2(s - t_k)} \right) dy \right) ds \\ &= \sigma \int_{\mathbb{R}} \phi(z) \int_{t_k}^t \frac{\sqrt{x}}{\sqrt{2\pi(s - t_k)}} \exp \left( - \frac{(z - x - b(x)(s - t_k))^2}{2\sigma^2 x(s - t_k)} \right) ds dz. \end{aligned}$$

Then, for any  $z \in \mathbb{R}$ ,

$$\mathbb{E} (L_t^z(\bar{X}) - L_{t_k}^z(\bar{X}) / \{\bar{X}_{t_k} = x\}) = \sigma \int_{t_k}^t \frac{\sqrt{x}}{\sqrt{2\pi(s - t_k)}} \exp \left( - \frac{(z - x - b(x)(s - t_k))^2}{2\sigma^2 x(s - t_k)} \right) ds.$$

In particular for  $z = 0$  and  $t = t_{k+1}$ ,

$$\mathbb{E} (L_{t_{k+1}}^0(\bar{X}) - L_{t_k}^0(\bar{X}) / \{\bar{X}_{t_k} = x\}) = \sigma \int_0^{\Delta t} \frac{\sqrt{x}}{\sqrt{2\pi s}} \exp \left( - \frac{(x + b(x)s)^2}{2\sigma^2 x s} \right) ds.$$

From (H1),  $b(x) \geq -Kx$ , with  $K \geq 0$ . Then,

$$\mathbb{E} (L_t^0(\bar{X}) - L_{t_k}^0(\bar{X}) / \mathcal{F}_{t_k}) \leq \sigma \int_0^{\Delta t} \frac{\sqrt{\bar{X}_{t_k}}}{\sqrt{2\pi s}} \exp \left( - \frac{\bar{X}_{t_k}(1 - Ks)^2}{2\sigma^2 s} \right) ds.$$

For  $\Delta t$  sufficiently small,  $1 - Ks \geq 1 - K\Delta t \geq 1/2$  and

$$\mathbb{E} \left( L_{t_{k+1}}^0(\bar{X}) - L_{t_k}^0(\bar{X}) / \mathcal{F}_{t_k} \right) \leq \sigma \int_0^{\Delta t} \frac{\sqrt{\bar{X}_{t_k}}}{\sqrt{2\pi s}} \exp \left( -\frac{\bar{X}_{t_k}}{8\sigma^2 \Delta t} \right) ds.$$

Now we use the inequality  $y \exp(-\frac{y^2}{2}) \leq 1$  for all  $y \geq 0$ , to get

$$\begin{aligned} \mathbb{E} \left( L_{t_{k+1}}^0(\bar{X}) - L_{t_k}^0(\bar{X}) \right) &\leq \sigma^2 \int_0^{\Delta t} \frac{2\sqrt{\Delta t}}{\sqrt{\pi s}} \mathbb{E} \left[ \exp \left( -\frac{\bar{X}_{t_k}}{16\sigma^2 \Delta t} \right) \right] ds \\ &\leq \frac{4\sigma^2 \Delta t}{\sqrt{\pi}} \sup_{k=0, \dots, N} \mathbb{E} \exp \left( -\frac{\bar{X}_{t_k}}{16\sigma^2 \Delta t} \right). \end{aligned}$$

We sum over  $k$  and apply the Lemma 3.5 to end the proof. ■

### 3.2 Proof of Theorem 2.3

We are now in position to prove the Theorem 2.3. To study the weak error  $\mathbb{E}f(X_T) - \mathbb{E}f(\bar{X}_T)$ , we follow the now classical technique due to Talay and Tubaro [13]: the main idea consists in using the Feynman–Kac representation of the solution of the Cauchy problem (3.6) studied in the Proposition 3.2: for all  $(t, x) \in [0, T] \times (0, +\infty)$ ,  $\mathbb{E}f(X_{T-t}^x) = u(t, x)$ . Thus the weak error becomes

$$\mathbb{E}f(X_T) - \mathbb{E}f(\bar{X}_T) = \mathbb{E} \left( u(0, x_0) - u(T, \bar{X}_T) \right).$$

Using equation (3.17) satisfied by  $(\bar{X})$ , we apply the Itô's formula a first time to obtain that

$$\begin{aligned} &\mathbb{E} \left[ u(0, x_0) - u(T, \bar{X}_T) \right] \\ &= - \int_0^T \mathbb{E} \left[ \frac{\partial u}{\partial t}(s, \bar{X}_s) + \text{sgn}(\bar{Z}_s) b(\bar{X}_{\eta(s)}) \frac{\partial u}{\partial x}(s, \bar{X}_s) + \frac{\sigma^2}{2} \bar{X}_{\eta(s)} \frac{\partial^2 u}{\partial x^2}(s, \bar{X}_s) \right] ds \\ &\quad - \mathbb{E} \int_0^T \text{sgn}(\bar{Z}_s) \sqrt{\bar{X}_{\eta(s)}} \frac{\partial u}{\partial x}(s, \bar{X}_s) dW_s - \mathbb{E} \int_0^T \frac{\partial u}{\partial x}(s, \bar{X}_s) dL^0(\bar{X})_s. \end{aligned}$$

By Proposition 3.2,  $\frac{\partial u}{\partial x}$  is a bounded function and by Lemma 2.1,  $\mathbb{E} \sup_{t \in [0, T]} \bar{X}_t$  is also bounded.

Hence, the stochastic integral  $(\int_0^t \text{sgn}(\bar{Z}_s) \sqrt{\bar{X}_{\eta(s)}} \frac{\partial u}{\partial x}(s, \bar{X}_s) dW_s)$  is a martingale. Moreover, in the first integral of the term above, noting that  $\text{sgn}(x) = 1 - 2\mathbb{1}_{(x \leq 0)}$ , we use that  $u$  solves

the Cauchy problem (3.6) to get

$$\begin{aligned} & \mathbb{E} [u(0, x_0) - u(T, \bar{X}_T)] \\ &= - \int_0^T \mathbb{E} \left[ \left( b(\bar{X}_{\eta(s)}) - b(\bar{X}_s) \right) \frac{\partial u}{\partial x}(s, \bar{X}_s) + \frac{\sigma^2}{2} (\bar{X}_{\eta(s)} - \bar{X}_s) \frac{\partial^2 u}{\partial x^2}(s, \bar{X}_s) \right] ds \\ & \quad - \mathbb{E} \int_0^T \frac{\partial u}{\partial x}(s, \bar{X}_s) dL^0(\bar{X})_s + \int_0^T 2\mathbb{E} \left( \mathbb{1}_{\{\bar{Z}_s \leq 0\}} b(\bar{X}_{\eta(s)}) \frac{\partial u}{\partial x}(s, \bar{X}_s) \right) ds. \end{aligned}$$

From Proposition 3.4,

$$\left| \mathbb{E} \int_0^T \frac{\partial u}{\partial x}(s, \bar{X}_s) dL^0(\bar{X})_s \right| \leq \left\| \frac{\partial u}{\partial x} \right\|_\infty \mathbb{E} (L_T^0(\bar{X})) \leq C \left( \frac{\Delta t}{x_0} \right)^{\frac{b(0)}{\sigma^2}}.$$

On the other hand, by Lemma 3.6 for any  $s \in [0, T]$ ,

$$\mathbb{E} \left( \mathbb{1}_{\{\bar{Z}_s \leq 0\}} b(\bar{X}_{\eta(s)}) \frac{\partial u}{\partial x}(s, \bar{X}_s) \right) \leq \frac{1}{2} \left\| \frac{\partial u}{\partial x} \right\|_{L^\infty} \mathbb{E} \left[ \left( b(0) + K \bar{X}_{\eta(s)} \right) \exp \left( -\frac{\bar{X}_{\eta(s)}}{2h(\Delta t)\sigma^2\Delta t} \right) \right]$$

so that by Lemma 3.5

$$\left| \int_0^T 2\mathbb{E} \left( \mathbb{1}_{\{\bar{Z}_s \leq 0\}} b(\bar{X}_{\eta(s)}) \frac{\partial u}{\partial x}(s, \bar{X}_s) \right) ds \right| \leq C(1 + |x_0|) \left( \frac{\Delta t}{x_0} \right)^{\frac{b(0)}{\sigma^2}}.$$

The next step consists in applying the Itô's formula a second time:

$$\begin{aligned} & \mathbb{E} \left[ \left( b(\bar{X}_s) - b(\bar{X}_{\eta(s)}) \right) \frac{\partial u}{\partial x}(s, \bar{X}_s) \right] \\ &= \mathbb{E} \int_{\eta(s)}^s \text{sgn}(\bar{Z}_\theta) b(\bar{X}_{\eta(s)}) \left[ \left( b(\bar{X}_\theta) - b(\bar{X}_{\eta(s)}) \right) \frac{\partial^2 u}{\partial x^2}(s, \bar{X}_\theta) + b'(\bar{X}_\theta) \frac{\partial u}{\partial x}(s, \bar{X}_\theta) \right] d\theta \\ & \quad + \mathbb{E} \int_{\eta(s)}^s \frac{\sigma^2}{2} \bar{X}_{\eta(s)} \left[ \left( b(\bar{X}_\theta) - b(\bar{X}_{\eta(s)}) \right) \frac{\partial^3 u}{\partial x^3}(s, \bar{X}_\theta) + 2b'(\bar{X}_\theta) \frac{\partial^2 u}{\partial x^2}(s, \bar{X}_\theta) + b''(\bar{X}_\theta) \frac{\partial u}{\partial x}(s, \bar{X}_\theta) \right] d\theta \\ & \quad + \mathbb{E} \int_{\eta(s)}^s \left[ \left( b(\bar{X}_\theta) - b(\bar{X}_{\eta(s)}) \right) \frac{\partial^2 u}{\partial x^2}(s, \bar{X}_\theta) + b'(\bar{X}_\theta) \frac{\partial u}{\partial x}(s, \bar{X}_\theta) \right] dL_\theta^0(\bar{X}) \end{aligned}$$

Then,

$$\begin{aligned} & \left| \mathbb{E} \left[ \left( b(\bar{X}_s) - b(\bar{X}_{\eta(s)}) \right) \frac{\partial u}{\partial x}(s, \bar{X}_s) \right] \right| \\ & \leq \Delta t C \sup_{0 \leq \theta \leq T} \mathbb{E} |\bar{X}_\theta|^2 + C \mathbb{E} \left[ (1 + |\bar{X}_{\eta(s)}|) \mathbb{E} (L_s - L_{\eta(s)} / \mathcal{F}_{\eta(s)}) \right] \end{aligned}$$

and we conclude by Lemma 2.1 and Proposition 3.4 that

$$\left| \mathbb{E} \left[ \left( b(\bar{X}_s) - b(\bar{X}_{\eta(s)}) \right) \frac{\partial u}{\partial x}(s, \bar{X}_s) \right] \right| \leq C \left( \Delta t + \left( \frac{\Delta t}{x_0} \right)^{\frac{b(0)}{\sigma^2}} \right).$$

With similar arguments, we can also show that

$$\left| \mathbb{E} \left[ \left( \bar{X}_s - \bar{X}_{\eta(s)} \right) \frac{\partial^2 u}{\partial x^2}(s, \bar{X}_s) \right] \right| \leq C \left( \Delta t + \left( \frac{\Delta t}{x_0} \right)^{\frac{b(0)}{\sigma^2}} \right)$$

which ends the proof.

## 4 The case of processes with $1/2 < \alpha < 1$ .

### 4.1 Preliminary results

In this section  $(X_t)$  denotes the solution of (2.9) and  $(X_t^x)$  the solution of (2.9), with the deterministic initial condition  $x$  at time 0, that is the unique strong solution to

$$X_t^x = x + \int_0^t b(X_s^x) ds + \sigma \int_0^t (X_s^x)^\alpha dW_s. \quad (4.1)$$

#### 4.1.1 On the exact solution

We have the following

**Lemma 4.1** *Assume (H1). Let  $x > 0$ . For any  $1/2 < \alpha < 1$ , for any  $p > 0$ , there exists a positive constant  $C(\alpha, b(0), \sigma, p)$ , depending on the parameters of the model (4.1) and on  $p$  such that*

$$\sup_{t \in [0, T]} \mathbb{E} \left( \frac{1}{(X_t^x)^p} \right) \leq \left( \frac{1}{x^p} + C(\alpha, b(0), \sigma, p) \right) \exp(pKT),$$

where  $K > 0$  is the Lipschitz constant of  $b(x)$  in (H1). The constant  $C(\alpha, b(0), \sigma, p)$  is given by

$$C(\alpha, b(0), \sigma, p) = p(2\alpha - 1) \frac{\sigma^2}{2} \left[ (p + 2(1 - \alpha)) \frac{\sigma^2}{2b(0)} \right]^{\frac{p+2(1-\alpha)}{2\alpha-1}}. \quad (4.2)$$

**Proof.** Let  $\tau_n$  be the stopping time defined by  $\tau_n = \inf\{0 < s \leq T; X_s^x \geq 1/n\}$ . By the



Itô's formula,

$$\begin{aligned} \mathbb{E} \left( \frac{1}{(X_{t \wedge \tau_n}^x)^p} \right) &= \frac{1}{x^p} - p \mathbb{E} \left[ \int_0^{t \wedge \tau_n} \frac{b(X_s^x)}{(X_s^x)^{p+1}} ds \right] + p(p+1) \frac{\sigma^2}{2} \mathbb{E} \left[ \int_0^{t \wedge \tau_n} \frac{1}{(X_s^x)^{p+2(1-\alpha)}} ds \right] \\ &\leq \frac{1}{x^p} + pK \int_0^{t \wedge \tau_n} \mathbb{E} \left( \frac{1}{(X_{s \wedge \tau_n}^x)^p} \right) ds \\ &\quad + \mathbb{E} \left[ \int_0^{t \wedge \tau_n} \left( p(p+1) \frac{\sigma^2}{2} \frac{1}{(X_s^x)^{p+2(1-\alpha)}} - p \frac{b(0)}{(X_s^x)^{p+1}} \right) ds \right]. \end{aligned}$$

It is possible to found a positive constant  $C$  such that for any  $x > 0$ ,

$$\left( p(p+1) \frac{\sigma^2}{2} \frac{1}{x^{p+2(1-\alpha)}} - p \frac{b(0)}{x^{p+1}} \right) \leq C.$$

An easy computation shows that  $C(\alpha, b(0), \sigma, p)$  defined in (4.2) is the smallest constant  $C$  satisfying the upper-bound above. Hence,

$$\mathbb{E} \left( \frac{1}{(X_{t \wedge \tau_n}^x)^p} \right) \leq \frac{1}{x^p} + TC(\alpha, b(0), \sigma, p) + pK \int_0^t \sup_{\theta \in [0, s]} \mathbb{E} \left( \frac{1}{(X_{\theta \wedge \tau_n}^x)^p} \right) ds$$

and by the Gronwall Lemma

$$\sup_{t \in [0, T]} \mathbb{E} \left( \frac{1}{(X_{t \wedge \tau_n}^x)^p} \right) \leq \left( \frac{1}{x^p} + TC(\alpha, b(0), \sigma, p) \right) \exp(pKT).$$

We end the proof, by taking the limit  $n \rightarrow +\infty$ . ■

**Lemma 4.2** Assume (H1).  $K$  denotes the Lipschitz constant of the drift function  $b(\cdot)$ .

i) For any  $a \geq 0$ , for all  $t \geq 0$  a.s.  $(X_t^x)^{2(1-\alpha)} \geq \mathfrak{r}_t(a)$  where  $(\mathfrak{r}_t(a))_{0 \leq t \leq T}$  is the solution of the CIR Equation:

$$\mathfrak{r}_t(a) = x^{2(1-\alpha)} + \int_0^t (a - \lambda(a)\mathfrak{r}_s(a)) ds + 2\sigma(1-\alpha) \int_0^t \sqrt{\mathfrak{r}_s(a)} dW_s \quad (4.3)$$

with

$$\lambda(a) = 2(1-\alpha)K + \left( \frac{(2\alpha-1)^{2\alpha-1} (a + \sigma^2(1-\alpha)(2\alpha-1))}{b(0)^{2(1-\alpha)}} \right)^{\frac{1}{2\alpha-1}}. \quad (4.4)$$

ii) For all  $\mu \geq 0$ , there exists a constant  $C(T, \mu) > 0$  with a non decreasing dependency on  $T$  and  $\mu$ , depending also on  $K, b(0), \sigma, \alpha$  and  $x$  such that

$$\mathbb{E} \exp \left( \mu \int_0^T \frac{ds}{(X_s^x)^{2(1-\alpha)}} \right) \leq C(T, \mu). \quad (4.5)$$

iii) The process  $(M_t)_{0 \leq t \leq T}$  defined by

$$M_t = \exp \left( \alpha \sigma \int_0^t \frac{dW_s}{X_s^{1-\alpha}} - \alpha^2 \frac{\sigma^2}{2} \int_0^t \frac{ds}{X_s^{2(1-\alpha)}} \right) \quad (4.6)$$

is a martingale. Moreover for all  $p \geq 1$ , there exists a positive constant  $C(T, p)$  depending also on  $b(0)$ ,  $\sigma$  and  $\alpha$  such that

$$\mathbb{E} \left( \sup_{t \in [0, T]} M_t^p \right) \leq C(T, p) \left( 1 + \frac{1}{x^{\alpha p}} \right). \quad (4.7)$$

**Proof.** Let  $Z_t = X_t^{2(1-\alpha)}$ . By the Itô formula,

$$Z_t = x^{2(1-\alpha)} + \int_0^t \beta(Z_s) ds + 2(1-\alpha)\sigma \int_0^t \sqrt{Z_s} dW_s,$$

with the drift coefficient  $\beta(x)$  defined for all  $x > 0$  by

$$\beta(x) = 2(1-\alpha) \frac{b(x^{\frac{1}{2(1-\alpha)}})}{x^{\frac{2\alpha-1}{2(1-\alpha)}}} - \sigma^2(1-\alpha)(2\alpha-1).$$

From (H1),  $b(x) \geq b(0) - Kx$  and for all  $x > 0$ ,  $\beta(x) \geq \bar{\beta}(x)$ , where we set

$$\bar{\beta}(x) = 2(1-\alpha) \frac{b(0)}{x^{\frac{2\alpha-1}{2(1-\alpha)}}} - 2(1-\alpha)Kx - \sigma^2(1-\alpha)(2\alpha-1).$$

For all  $a \geq 0$  and  $\lambda(a)$  given by (4.4), we consider  $f(x) = \bar{\beta}(x) - a + \lambda(a)x$ . An easy computation shows that  $f(x)$  has one minimum at the point  $x^* = \left( \frac{b(0)(2\alpha-1)}{\lambda(a)-2(1-\alpha)K} \right)^{2(1-\alpha)}$ . Moreover,

$$f(x^*) = \frac{b(0)^{2(1-\alpha)}}{(2\alpha-1)^{2\alpha-1}} (\lambda(a) - 2(1-\alpha)K)^{2\alpha-1} - (a + \sigma^2(1-\alpha)(2\alpha-1))$$

and when  $\lambda(a)$  is given by (4.4),  $f(x^*) = 0$ . We conclude that  $\beta(x) \geq a - \lambda(a)x$  and *i*) holds by the comparison Theorem for solutions of one-dimensional SDE. As a consequence,

$$\mathbb{E} \exp \left( \mu \int_0^T \frac{ds}{X_s^{2(1-\alpha)}} \right) \leq \mathbb{E} \left[ \exp \left( \mu \int_0^T \frac{ds}{\mathfrak{r}_s(a)} \right) \right].$$

In order to apply Lemma A.2, we must choose the constant  $a$  such that  $a \geq 4(1-\alpha)^2\sigma^2$  and  $\mu \leq \frac{\nu^2(a)(1-\alpha)^2 4\sigma^2}{8}$ , for  $\nu(a)$  as in Lemma A.2. An easy computation shows that  $a = 4(1-\alpha)^2\sigma^2 \vee \left( 2(1-\alpha)^2\sigma^2 + (1-\alpha)\sigma 2\sqrt{2\mu} \right)$  is convenient. *ii*) follows by applying Lemma

A.2 to the process  $(\mathbf{r}_t(a))_{0 \leq t \leq T}$ . Now from *ii*), the Novikov Criteria applied to  $M_t$  is clearly satisfied and  $(M_t)_{t \geq 0}$  is a martingale. Moreover, by the integration by part formula.

$$M_t = \left(\frac{X_t}{x}\right)^\alpha \exp\left(\int_0^t \left(-\alpha \frac{b(X_s)}{X_s} + \alpha(1-\alpha) \frac{\sigma^2}{2} \frac{1}{X_s^{2(1-\alpha)}}\right) ds\right).$$

By (H1), for all  $x \geq 0$ ,  $-b(x) \leq -b(0) + Kx$  and

$$M_t \leq \left(\frac{X_t}{x}\right)^\alpha \exp(KT) \exp\left(\int_0^t \left(-\alpha \frac{b(0)}{X_s} + \alpha(1-\alpha) \frac{\sigma^2}{2} \frac{1}{X_s^{2(1-\alpha)}}\right) ds\right). \quad (4.8)$$

To end the proof, we note that it is possible to found a positive constant  $\lambda(\alpha, \sigma, b(0))$  such that  $-\frac{b(0)\alpha}{x} + \frac{\sigma^2\alpha(1-\alpha)}{2} \frac{1}{x^{2(1-\alpha)}} \leq \lambda(\alpha, \sigma, b(0))$ . An easy compilation (similar to the one used to prove *i*)), shows that

$$\lambda(\alpha, \sigma, b(0)) = \frac{\alpha}{2}(2\alpha - 1) \left[ \frac{(1-\alpha)^{3-2\alpha}\sigma^2}{b(0)^{2(1-\alpha)}} \right]^{\frac{1}{2\alpha-1}}.$$

is convenient. Thus,  $M_t \leq \left(\frac{X_t}{x}\right)^\alpha \exp((K + \lambda(\alpha, \sigma, b(0)))T)$ . We take the expectation and we conclude by the Cauchy-Schwarz Inequality and the Lemma 2.1.  $\blacksquare$

#### 4.1.2 On the associated Kolmogorov PDE

**Proposition 4.3** *Let  $f$  be a  $\mathbb{R}$ -valued  $C^4$  bounded function, with bounded spatial derivatives up to the order 4. We consider the  $\mathbb{R}$ -valued function  $u$ , defined on  $[0, T] \times [0, +\infty)$  by  $u(t, x) := \mathbb{E}f(X_{T-t}^x)$ . Let  $1/2 < \alpha \leq 1$ . Then under (H2),  $u$  is in  $C^{1,4}([0, T] \times [0, +\infty))$ . That is,  $u$  has derivatives up to order 4 in the space variable and a derivative in the time variable and there exists a positive constant  $C$  depending on  $f$ ,  $b$  and on  $T$  such that*

$$\sup_{t \in [0, T]} \left| \frac{\partial u}{\partial t}(t, x) \right| \leq C(1 + x^{2\alpha})$$

and

$$\|u\|_{L^\infty([0, T] \times [0, +\infty))} + \left\| \frac{\partial u}{\partial x} \right\|_{L^\infty([0, T] \times [0, +\infty))} \leq C.$$

Then  $u$  satisfies the equation

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) + b(x) \frac{\partial u}{\partial x}(t, x) + \frac{\sigma^2}{2} x^{2\alpha} \frac{\partial^2 u}{\partial x^2}(t, x) = 0, & (t, x) \in [0, T] \times (0, +\infty), \\ u(T, x) = f(x), & x \in [0, +\infty). \end{cases} \quad (4.9)$$

Moreover, for  $k \in \{2, 3, 4\}$ ,

$$\left| \frac{\partial^k u(t, x)}{\partial x^k} \right| \leq C(T, k) \left( 1 + \frac{1}{x^{q(\alpha)}} \right), \quad (4.10)$$

where the constant  $q(\alpha) > 0$  depends only on  $\alpha$ , the positive constants  $C_1(T)$  and  $C_2(T, k)$  are nondecreasing in  $T$ .

**Proof.** In this proof many arguments are similar to those used in Proposition 3.2. We restrict our attention on the main difficulty which consists on the upper-bound of the spatial derivatives up to the order 4, of  $u(t, x)$ . Equation (4.1) has locally Lipschitz coefficients on  $(0, +\infty)$ , with locally Lipschitz first order derivatives. Then  $X_t^x$  is continuously differentiable and if we denote  $J_t^x = \frac{dX_t^x}{dx}$ , the process  $(J_t^x)_{0 \leq t \leq T}$  satisfies the linear equation

$$J_t^x = 1 + \int_0^t J_s^x b'(X_s^x) ds + \int_0^t \alpha \sigma J_s^x \frac{dW_s}{(X_s^x)^{1-\alpha}}. \quad (4.11)$$

By Lemma 4.1, the process  $(\int_0^t \frac{dW_s}{(X_s^x)^{1-\alpha}})_{t \geq 0}$  is a locally square integrable martingale. Then, (see Theorem V.51 in [11])  $J_t^x$  is given by

$$\begin{aligned} J_t^x &= \exp \left( \int_0^t b'(X_s^x) ds + \alpha \sigma \int_0^t \frac{dW_s}{(X_s^x)^{1-\alpha}} - \frac{\sigma^2 \alpha^2}{2} \int_0^t \frac{ds}{(X_s^x)^{2(1-\alpha)}} \right) \\ &= \exp \left( \int_0^t b'(X_s) ds \right) M_t, \end{aligned} \quad (4.12)$$

where  $M_t$  is the martingale defined by (4.6).  $b'$  being bounded, we get  $\mathbb{E}[J_t^x] \leq \exp(KT)$  and by (4.7), for all  $p \geq 1$

$$\mathbb{E} \left( \sup_{t \in [0, T]} (J_t^x)^p \right) \leq C(T) \left( 1 + \frac{1}{x^{\alpha p}} \right). \quad (4.13)$$

We prove in Appendix C that the first derivative of  $u$  is given by

$$\frac{\partial u}{\partial x}(t, x) = \mathbb{E} [f'(X_{T-t}^x) J_{T-t}^x].$$

Then,

$$\left| \frac{\partial u}{\partial x}(t, x) \right| = |\mathbb{E} [f'(X_{T-t}^x) J_{T-t}^x]| \leq \|f'\|_\infty \exp(KT).$$

Now we show (4.10) for the second derivative. By the integration by part formula,

$$\frac{\partial u}{\partial x}(t, x) = \frac{1}{x^\alpha} \mathbb{E} \left[ f'(X_{T-t}^x) (X_{T-t}^x)^\alpha \exp \left( \int_0^{T-t} \left( b'(X_s^x) - \frac{\alpha b(X_s^x)}{X_s^x} + \frac{\sigma^2 \alpha (1-\alpha)}{2(X_s^x)^{2(1-\alpha)}} \right) ds \right) \right].$$

Thus, (the permutation between  $\frac{\partial}{\partial x}$  and  $\mathbb{E}$  is discussed in Proposition C.1)

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2}(t, x) &= -\frac{\alpha}{x} \frac{\partial u}{\partial x}(t, x) + \mathbb{E} [f''(X_{T-t}^x)(J_{T-t}^x)^2] + \alpha \mathbb{E} \left[ \frac{1}{X_{T-t}^x} f'(X_{T-t}^x)(J_{T-t}^x)^2 \right] \\ &\quad + \mathbb{E} \left[ f'(X_{T-t}^x) J_{T-t}^x \int_0^{T-t} \left( b''(X_s^x) - \frac{\alpha b'(X_s^x)}{X_s^x} + \frac{\alpha b(X_s^x)}{(X_s^x)^2} + \frac{\sigma^2 \alpha (1-\alpha)^2}{(X_s^x)^{3-2\alpha}} \right) J_s^x ds \right] \end{aligned}$$

and

$$\begin{aligned} &\mathbb{E} \left[ f'(X_{T-t}^x) J_{T-t}^x \int_0^{T-t} \left( b''(X_s^x) - \frac{\alpha b'(X_s^x)}{X_s^x} + \frac{\alpha b(X_s^x)}{(X_s^x)^2} + \frac{\sigma^2 \alpha (1-\alpha)^2}{(X_s^x)^{3-2\alpha}} \right) J_s^x ds \right] \\ &\leq \|f'\|_\infty \mathbb{E} \left[ \sup_{t \in [0, T]} (J_t^x)^2 \int_0^{T-t} \left| b''(X_s^x) - \frac{\alpha b'(X_s^x)}{X_s^x} + \frac{\alpha b(X_s^x)}{(X_s^x)^2} + \frac{\sigma^2 \alpha (1-\alpha)^2}{(X_s^x)^{3-2\alpha}} \right| ds \right] \\ &\leq C(T) \left( 1 + \frac{1}{x^{2(1+\alpha)}} \right), \end{aligned}$$

the last upper-bounded is obtained using the Cauchy-Schwarz Inequality with Lemma 4.1 and estimate (4.13). Using similar arguments, it comes that

$$\left| \frac{\partial^2 u}{\partial x^2}(t, x) \right| \leq C(T) \left( 1 + \frac{1}{x^{2+2\alpha}} \right).$$

We postpone the upper-bound of the high order derivatives in Appendix C. ■

#### 4.1.3 On the approximation process

For  $1/2 < \alpha < 1$ , recall that the discrete time process  $(\bar{X})$  associated to  $(X)$  is defined by

$$\begin{cases} \bar{X}_0 = x_0, \\ \bar{X}_{t_{k+1}} = \left| \bar{X}_{t_k} + b(\bar{X}_{t_k})\Delta t + \sigma \bar{X}_{t_k}^\alpha (W_{t_{k+1}} - W_{t_k}) \right|, \quad k = 0, \dots, N-1, \end{cases} \quad (4.14)$$

where  $N$  is such that  $N\Delta t = T$  and  $t_k = k\Delta t$ . The time continuous version  $(\bar{X}_t)_{0 \leq t \leq T}$  is defined on each interval  $[t_k, t_{k+1})$  by

$$\bar{X}_t = \left| \bar{X}_{\eta(t)} + (t - \eta(t))b(\bar{X}_{\eta(t)}) + \sigma \bar{X}_{\eta(t)}^\alpha (W_t - W_{\eta(t)}) \right|, \quad (4.15)$$

or equivalently by

$$\bar{X}_t = x_0 + \int_0^t \text{sgn}(\bar{Z}_s) b(\bar{X}_{\eta(s)}) ds + \sigma \int_0^t \text{sgn}(\bar{Z}_s) \bar{X}_{\eta(s)}^\alpha dW_s + L_t^0(\bar{X}), \quad (4.16)$$

where for any  $t \in [0, T]$ , we set

$$\bar{Z}_t = \bar{X}_{\eta(t)} + (t - \eta(t))b(\bar{X}_{\eta(t)}) + \sigma \bar{X}_{\eta(t)}^\alpha (W_t - W_{\eta(t)}). \quad (4.17)$$

Note that for all  $t \in [0, T]$ ,  $\bar{X}_t = \lfloor \bar{Z}_t \rfloor$ . We work under the hypothesis (H4):  $x_0 > \frac{b(0)}{2}\Delta t$ .

In the sequel, we will use the following notation:

$$\mathcal{O}_{exp}(\Delta t) = C(T) \exp\left(\frac{C}{\Delta t^{\alpha - \frac{1}{2}}}\right),$$

where the positive constants  $C$  and  $C(T)$  are independent of  $\Delta t$  but can depend on  $\alpha$ ,  $\sigma$  and  $b(0)$ .  $C(T)$  is nondecreasing in  $T$ . The quantity  $\mathcal{O}_{exp}(\Delta t)$  decreases exponentially fast with  $\Delta t$ .

In this section, we are interested in the behavior of the processes  $(\bar{X})$  and  $(\bar{Z})$  near 0. We introduce the stopping time  $\tau$  defined by

$$\tau = \inf \left\{ s \geq 0; \bar{X}_s < \frac{b(0)}{2}\Delta t \right\}. \quad (4.18)$$

At the contrary of the case  $\alpha = 1/2$ , under (H4), we are able to control some probabilities like  $\mathbb{P}(\tau \leq T)$ .

**Proposition 4.4** *Assume (H1), (H2) and (H4).*

$$\mathbb{P}(\tau \leq T) \leq \mathcal{O}_{exp}(\Delta t). \quad (4.19)$$

**Proof.** The first step of the proof consists in obtaining the following estimate:

$$\forall k \in \{1, \dots, N\}, \mathbb{P}\left(\bar{X}_{t_k} \leq \frac{b(0)}{\sqrt{2}}\Delta t\right) \leq \mathcal{O}_{exp}(\Delta t). \quad (4.20)$$

Indeed,

$$\begin{aligned} \mathbb{P}\left(\bar{X}_{t_k} \leq \frac{b(0)}{\sqrt{2}}\Delta t\right) &\leq \mathbb{P}\left(\bar{X}_{t_{k-1}} + b(\bar{X}_{t_{k-1}})\Delta t + \sigma \bar{X}_{t_{k-1}}^\alpha (W_{t_k} - W_{t_{k-1}}) \leq \frac{b(0)}{\sqrt{2}}\Delta t\right) \\ &\leq \mathbb{P}\left(W_{t_k} - W_{t_{k-1}} \leq \frac{-\bar{X}_{t_{k-1}}(1 - K\Delta t) - b(0)(1 - \frac{1}{\sqrt{2}})\Delta t}{\sigma \bar{X}_{t_{k-1}}^\alpha}; \bar{X}_{t_{k-1}} > 0\right) \end{aligned}$$

where we have used  $b(x) \geq b(0) - Kx$  for  $x \geq 0$ . We integrate with respect to the law of  $(W_{t_k} - W_{t_{k-1}})$  and use that  $\mathbb{P}(G \leq \beta) \leq 1/2 \exp(-\beta^2/2)$  when  $G$  is a Normal random

variable and  $\beta \leq 0$ . When  $\Delta t$  is sufficiently small, say  $1 - K\Delta t \geq 1/2$ , we get

$$\begin{aligned} \mathbb{P}\left(\bar{X}_{t_k} \leq \frac{b(0)}{\sqrt{2}}\Delta t\right) &\leq \mathbb{E} \exp\left(-\frac{\left(\bar{X}_{t_{k-1}}(1 - K\Delta t) + b(0)(1 - \frac{1}{\sqrt{2}})\Delta t\right)^2}{2\sigma^2 \bar{X}_{t_{k-1}}^{2\alpha} \Delta t}\right) \\ &\leq \mathbb{E} \left[ \exp\left(-\frac{\bar{X}_{t_{k-1}}^{2(1-\alpha)}}{8\sigma^2 \Delta t}\right) \exp\left(-\frac{b(0)(1 - \frac{1}{\sqrt{2}})}{2\sigma^2 \bar{X}_{t_{k-1}}^{2\alpha-1}}\right) \right]. \end{aligned}$$

Separating the events  $\{\bar{X}_{t_{k-1}} \geq \sqrt{\Delta t}\}$  and  $\{\bar{X}_{t_{k-1}} < \sqrt{\Delta t}\}$  in the expectation above, we obtain

$$\mathbb{P}\left(\bar{X}_{t_k} \leq \frac{b(0)}{\sqrt{2}}\Delta t\right) \leq \exp\left(-\frac{1}{8\sigma^2 \Delta t^\alpha}\right) + \exp\left(-\frac{b(0)(1 - \frac{1}{\sqrt{2}})}{2\sigma^2 (\Delta t)^{\alpha-\frac{1}{2}}}\right) \leq \mathcal{O}_{exp}(\Delta t).$$

Now we prove (4.19).

$$\begin{aligned} \mathbb{P}(\tau \leq T) &= \mathbb{P}\left(\exists k \in \{0, 1, \dots, N-1\} \text{ s.t. } \bar{X}_{t_k} > \frac{b(0)}{2}\Delta t \text{ and } \inf_{t_k < s \leq t_{k+1}} \bar{Z}_s \leq \frac{b(0)}{2}\Delta t\right) \\ &\leq \sum_{k=0}^{N-1} \mathbb{P}\left(\inf_{t_k < s \leq t_{k+1}} \bar{Z}_s \leq \frac{b(0)}{2}\Delta t; \bar{X}_{t_k} > \frac{b(0)}{2}\Delta t\right) \end{aligned} \quad (4.21)$$

For each  $k \in \{0, 1, \dots, N-1\}$ ,

$$\begin{aligned} &\mathbb{P}\left(\inf_{t_k < s \leq t_{k+1}} \bar{Z}_s \leq \frac{b(0)}{2}\Delta t; \bar{X}_{t_k} > \frac{b(0)}{2}\Delta t\right) \\ &= \mathbb{P}\left(\inf_{t_k < s \leq t_{k+1}} \bar{Z}_s \leq \frac{b(0)}{2}\Delta t; \bar{X}_{t_k} > \frac{b(0)}{2}\Delta t; \bar{X}_{t_k} \leq \frac{b(0)}{\sqrt{2}}\Delta t\right) \\ &\quad + \mathbb{P}\left(\inf_{t_k < s \leq t_{k+1}} \bar{Z}_s \leq \frac{b(0)}{2}\Delta t; \bar{X}_{t_k} > \frac{b(0)}{\sqrt{2}}\Delta t\right). \end{aligned}$$

By using (4.20) and  $b(x) \leq b(0) - Kx$

$$\begin{aligned} &\mathbb{P}\left(\inf_{t_k < s \leq t_{k+1}} \bar{Z}_s \leq \frac{b(0)}{2}\Delta t; \bar{X}_{t_k} > \frac{b(0)}{2}\Delta t\right) \\ &\leq \mathbb{P}\left(\bar{X}_{t_k} \leq \frac{b(0)}{\sqrt{2}}\Delta t\right) + \mathbb{P}\left(\inf_{t_k < s \leq t_{k+1}} \bar{Z}_s \leq \frac{b(0)}{2}\Delta t; \bar{X}_{t_k} > \frac{b(0)}{\sqrt{2}}\Delta t\right) \\ &\leq \mathcal{O}_{exp}(\Delta t) + \mathbb{E}\left\{\mathbb{P}\left(\inf_{0 < s \leq \Delta t} \frac{x^{1-\alpha}}{\sigma} + \frac{b(0) - Kx}{\sigma x^\alpha} s + W_s \leq \frac{b(0)\Delta t}{2\sigma x^\alpha}\right) \Big|_{x=\bar{X}_{t_k}} \mathbb{1}_{\left(\bar{X}_{t_k} > \frac{b(0)}{\sqrt{2}}\Delta t\right)}\right\}. \end{aligned}$$

The proof is ended if we show that

$$\mathbb{P}\left(\inf_{0 < s \leq \Delta t} \frac{x^{1-\alpha}}{\sigma} + \frac{b(x)}{\sigma x^\alpha} s + W_s \leq \frac{b(0)\Delta t}{2\sigma x^\alpha}\right) \leq \mathcal{O}_{exp}(\Delta t), \text{ for } x \geq \frac{b(0)}{\sqrt{2}}\Delta t.$$

We use the following formula (see [1]): if  $(W_t^\mu)_{0 \leq t \leq T}$  denotes a driven Brownian motion starting at  $x$ , with drift  $\mu$ , then for all  $y \leq x$

$$\mathbb{P} \left( \inf_{0 < s < t} W_s^\mu \leq y \right) = \frac{1}{2} \operatorname{erfc} \left( \frac{x-y}{\sqrt{2t}} + \frac{\mu\sqrt{t}}{\sqrt{2}} \right) + \frac{1}{2} \exp(\mu(y-x)) \operatorname{erfc} \left( \frac{x-y}{\sqrt{2t}} - \frac{\mu\sqrt{t}}{\sqrt{2}} \right),$$

where  $\operatorname{erfc}(z) = \frac{\sqrt{2}}{\pi} \int_{\sqrt{2}z}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy$  for all  $z \in \mathbb{R}$ . Note that, if  $x > \frac{b(0)}{\sqrt{2}} \Delta t$ , then  $\frac{x^{1-\alpha}}{\sigma} \geq \frac{b(0)\Delta t}{2\sigma x^\alpha}$ . Thus,

$$\begin{aligned} & \mathbb{P} \left( \inf_{0 < s \leq \Delta t} \frac{x^{1-\alpha}}{\sigma} + \frac{(b(0) - Kx)}{\sigma x^\alpha} s + W_s \leq \frac{b(0)\Delta t}{2\sigma x^\alpha} \right) \\ &= \frac{1}{2} \operatorname{erfc} \left( \frac{x - \frac{b(0)\Delta t}{2}}{\sigma x^\alpha \sqrt{2\Delta t}} + \frac{(b(0) - Kx) \sqrt{\Delta t}}{\sigma x^\alpha \sqrt{2}} \right) \\ &+ \frac{1}{2} \exp \left( -\frac{2(b(0) - Kx)}{\sigma^2 x^{2\alpha}} \left( x - \frac{b(0)\Delta t}{2} \right) \right) \operatorname{erfc} \left( \frac{x - \frac{b(0)\Delta t}{2}}{\sigma x^\alpha \sqrt{2\Delta t}} - \frac{(b(0) - Kx) \sqrt{\Delta t}}{\sigma x^\alpha \sqrt{2}} \right) \\ &:= A(x) + B(x). \end{aligned}$$

For any  $z \geq 0$ ,  $\operatorname{erfc}(z) \leq \exp(-z^2)$ . If  $\Delta t$  is sufficiently small, say  $\Delta t \leq \frac{1}{2K}$ , we have

$$A(x) = \frac{1}{2} \operatorname{erfc} \left( \frac{x(1 - K\Delta t) + \frac{b(0)}{2} \Delta t}{\sigma x^\alpha \sqrt{2\Delta t}} \right) \leq \exp \left( -\frac{\left( x(1 - K\Delta t) + \frac{b(0)}{2} \Delta t \right)^2}{2\sigma^2 x^{2\alpha} \Delta t} \right) \leq \exp \left( -\frac{x^{2(1-\alpha)}}{8\sigma^2 \Delta t} \right)$$

and then  $A(x) \leq \exp \left( -\frac{2^\alpha b(0)^{2(1-\alpha)}}{16\sigma^2 \Delta t^{2\alpha-1}} \right)$  for  $x \geq \frac{b(0)}{\sqrt{2}} \Delta t$ .

Now consider  $B(x)$ .

$$B(x) = \frac{1}{2} \exp \left( -\frac{2(b(0) - Kx)}{\sigma^2 x^{2\alpha}} \left( x - \frac{b(0)\Delta t}{2} \right) \right) \operatorname{erfc} \left( \frac{x(1 + K\Delta t) - \frac{3}{2}b(0)\Delta t}{\sigma x^\alpha \sqrt{2\Delta t}} \right).$$

If  $x \geq \frac{\frac{3}{2}b(0)\Delta t}{(1+K\Delta t)}$ , then, as for  $A(x)$  we have

$$\begin{aligned} B(x) &\leq \exp \left( -\frac{2(b(0) - Kx)}{\sigma^2 x^{2\alpha}} \left( x - \frac{b(0)\Delta t}{2} \right) \right) \exp \left( -\frac{\left( \left( x - \frac{b(0)\Delta t}{2} \right) - (b(0) - Kx)\Delta t \right)^2}{\sigma^2 x^{2\alpha} 2\Delta t} \right) \\ &= \exp \left( -\frac{\left( x(1 - K\Delta t) + \frac{b(0)\Delta t}{2} \right)^2}{\sigma^2 x^{2\alpha} 2\Delta t} \right) \leq \exp \left( -\frac{x^{2(1-\alpha)}}{8\sigma^2 \Delta t} \right) \end{aligned}$$



and  $B(x) \leq \exp\left(-\frac{2^\alpha b(0)^{2(1-\alpha)}}{16\sigma^2 \Delta t^{2\alpha-1}}\right)$ , for  $x \geq \frac{\frac{3}{2}b(0)\Delta t}{(1+K\Delta t)}$ .

If  $\frac{b(0)}{\sqrt{2}}\Delta t \leq x < \frac{\frac{3}{2}b(0)\Delta t}{(1+K\Delta t)}$ , then  $\frac{2(b(0)-Kx)}{\sigma^2 x^{2\alpha}} \left(x - \frac{b(0)\Delta t}{2}\right) \geq \frac{b(0)^2 \Delta t (\frac{1}{\sqrt{2}} - \frac{1}{2})}{\sigma^2 x^{2\alpha}}$  and

$$B(x) \leq \exp\left(-\frac{2(b(0)-Kx)}{\sigma^2 x^{2\alpha}} \left(x - \frac{b(0)\Delta t}{2}\right)\right) \leq \exp\left(-\frac{b(0)^2 \Delta t (\frac{1}{\sqrt{2}} - \frac{1}{2})}{\sigma^2 x^{2\alpha}}\right).$$

For  $x \geq \frac{b(0)}{\sqrt{2}}\Delta t$ , we get  $B(x) \leq \exp\left(-\frac{2^{2\alpha} b(0)^{2(1-\alpha)} (\frac{1}{\sqrt{2}} - \frac{1}{2})(1+K\Delta t)^{2\alpha}}{3^{2\alpha} \sigma^2 (\Delta t)^{2\alpha-1}}\right) \leq \mathcal{O}_{exp}(\Delta t)$ .  $\blacksquare$

**Proposition 4.5** Assume (H1), (H2) and (H4). For all  $p \geq 0$ , there exists a positive constant  $C$  depending on  $b(0)$ ,  $\sigma$ ,  $\alpha$ ,  $T$  and  $p$  but not on  $\Delta t$ , such that

$$\forall t \in [0, T], \quad \mathbb{E}\left(\frac{1}{\bar{Z}_{t \wedge \tau}^p}\right) \leq C \left(1 + \frac{1}{x_0^p}\right). \quad (4.22)$$

**Proof.** The first step of the proof consists in obtain the following estimate:

$$\forall t \in [0, T], \quad \mathbb{P}\left(\bar{Z}_t \leq \frac{\bar{X}_{\eta(t)}}{2}\right) \leq \mathcal{O}_{exp}(\Delta t). \quad (4.23)$$

Indeed, as in the proof of Proposition 4.4 we have

$$\mathbb{P}\left(\bar{Z}_t \leq \frac{\bar{X}_{\eta(t)}}{2}\right) \leq \mathbb{E} \exp\left(-\frac{(\bar{X}_{\eta(t)}(1-2K(t-\eta(t))) + 2b(0)(t-\eta(t)))^2}{8\sigma^2(t-\eta(t))\bar{X}_{\eta(t)}^{2\alpha}}\right).$$

Using  $(a+b)^2 \geq a^2 + 2ab$ , with  $a = \bar{X}_{\eta(t)}(1-2K(t-\eta(t)))$  and  $b = 2b(0)(t-\eta(t))$ ,

$$\mathbb{P}\left(\bar{Z}_t \leq \frac{\bar{X}_{\eta(t)}}{2}\right) \leq \mathbb{E}\left(\exp\left(-\frac{\bar{X}_{\eta(t)}^{2(1-\alpha)}(1-2K\Delta t)^2}{8\sigma^2\Delta t}\right) \exp\left(-\frac{b(0)(1-K\Delta t)}{2\sigma^2\bar{X}_{\eta(t)}^{2\alpha-1}}\right)\right).$$

If  $\Delta t$  is sufficiently small, say  $(1-2K\Delta t)^2 \geq \frac{1}{4}$ ,

$$\mathbb{P}\left(\bar{Z}_t \leq \frac{\bar{X}_{\eta(t)}}{2}\right) \leq \mathbb{E}\left(\exp\left(-\frac{\bar{X}_{\eta(t)}^{2(1-\alpha)}}{32\sigma^2\Delta t}\right) \exp\left(-\frac{b(0)}{4\sigma^2\bar{X}_{\eta(t)}^{2\alpha-1}}\right)\right).$$

Separating the events  $\{\bar{X}_{\eta(t)} \geq \sqrt{\Delta t}\}$  and  $\{\bar{X}_{\eta(t)} < \sqrt{\Delta t}\}$  in the expectation above, we obtain

$$\mathbb{P}\left(\bar{Z}_t \leq \frac{\bar{X}_{\eta(t)}}{2}\right) \leq \exp\left(-\frac{1}{32\sigma^2(\Delta t)^\alpha}\right) + \exp\left(-\frac{b(0)}{4\sigma^2(\Delta t)^{\alpha-\frac{1}{2}}}\right) \leq \mathcal{O}_{exp}(\Delta t).$$

Now we prove 4.22. Note that  $\bar{Z}_{t \wedge \tau} = \bar{X}_{t \wedge \tau}$ . By the Itô formula

$$\frac{1}{\bar{Z}_{t \wedge \tau}^p} = \frac{1}{x_0^p} - p \int_0^{t \wedge \tau} \frac{b(\bar{X}_{\eta(s)})}{\bar{Z}_s^{p+1}} ds - p\sigma \int_0^{t \wedge \tau} \frac{\bar{X}_{\eta(s)}^\alpha}{\bar{Z}_s^{p+1}} dW_s + p(p+1) \frac{\sigma^2}{2} \int_0^{t \wedge \tau} \frac{\bar{X}_{\eta(s)}^{2\alpha}}{\bar{Z}_s^{p+2}} ds.$$

Taking the expectation and using again  $b(x) \geq b(0) - Kx$ ,

$$\mathbb{E} \left( \frac{1}{\bar{Z}_{t \wedge \tau}^p} \right) \leq \frac{1}{x_0^p} - p \mathbb{E} \int_0^{t \wedge \tau} \frac{b(0)}{\bar{Z}_s^{p+1}} ds + pK \mathbb{E} \int_0^{t \wedge \tau} \frac{\bar{X}_{\eta(s)}}{\bar{Z}_s^{p+1}} ds + p(p+1) \frac{\sigma^2}{2} \mathbb{E} \int_0^{t \wedge \tau} \frac{\bar{X}_{\eta(s)}^{2\alpha}}{\bar{Z}_s^{p+2}} ds$$

By definition of  $\tau$ ,

$$\begin{aligned} \mathbb{E} \int_0^{t \wedge \tau} \frac{\bar{X}_{\eta(s)}}{\bar{Z}_s^{p+1}} ds &= \mathbb{E} \int_0^{t \wedge \tau} \mathbb{1}_{(\bar{Z}_{s \wedge \tau} \leq \frac{\bar{X}_{\eta(s \wedge \tau)}}{2})} \frac{\bar{X}_{\eta(s \wedge \tau)}}{\bar{Z}_{s \wedge \tau}^{p+1}} ds + \mathbb{E} \int_0^{t \wedge \tau} \mathbb{1}_{(\bar{Z}_{s \wedge \tau} > \frac{\bar{X}_{\eta(s \wedge \tau)}}{2})} \frac{\bar{X}_{\eta(s \wedge \tau)}}{\bar{Z}_{s \wedge \tau}^{p+1}} ds \\ &\leq \left( \frac{2}{b(0)\Delta t} \right)^{p+1} T \sup_{t \in [0, T]} \left[ \mathbb{P} \left( \bar{Z}_t \leq \frac{\bar{X}_{\eta(t)}}{2} \right) \right]^{1/2} \sup_{t \in [0, T]} \left[ \mathbb{E} \left( \bar{X}_{\eta(t)}^2 \right) \right]^{1/2} + 2 \mathbb{E} \int_0^t \frac{ds}{\bar{Z}_{s \wedge \tau}^p}. \end{aligned}$$

By Lemma 2.1 and by the upper-bound (4.23), we have  $\mathbb{E} \int_0^{t \wedge \tau} \frac{\bar{X}_{\eta(s)}}{\bar{Z}_s^{p+1}} ds \leq C + 2 \mathbb{E} \int_0^t \frac{ds}{\bar{Z}_{s \wedge \tau}^p}$ .

Similarly,

$$\begin{aligned} \mathbb{E} \int_0^{t \wedge \tau} \frac{\bar{X}_{\eta(s)}^{2\alpha}}{\bar{Z}_s^{p+2}} ds &= \mathbb{E} \int_0^{t \wedge \tau} \mathbb{1}_{(\bar{Z}_{s \wedge \tau} \leq \frac{\bar{X}_{\eta(s \wedge \tau)}}{2})} \frac{\bar{X}_{\eta(s \wedge \tau)}^{2\alpha}}{\bar{Z}_{s \wedge \tau}^{p+2}} ds + \mathbb{E} \int_0^{t \wedge \tau} \mathbb{1}_{(\bar{Z}_{s \wedge \tau} > \frac{\bar{X}_{\eta(s \wedge \tau)}}{2})} \frac{\bar{X}_{\eta(s \wedge \tau)}^{2\alpha}}{\bar{Z}_{s \wedge \tau}^{p+2}} ds \\ &\leq \mathbb{E} \int_0^{t \wedge \tau} \mathbb{1}_{(\bar{Z}_{s \wedge \tau} \leq \frac{\bar{X}_{\eta(s \wedge \tau)}}{2})} \frac{\bar{X}_{\eta(s \wedge \tau)}^{2\alpha}}{\bar{Z}_{s \wedge \tau}^{p+2}} ds + 2^{2\alpha-1} \mathbb{E} \int_0^{t \wedge \tau} \frac{ds}{\bar{Z}_{s \wedge \tau}^{p+2(1-\alpha)}}. \end{aligned}$$

Using again the Lemma 2.1 and the upper-bound (4.23),

$$\begin{aligned} \mathbb{E} \int_0^{t \wedge \tau} \mathbb{1}_{(\bar{Z}_{s \wedge \tau} \leq \frac{\bar{X}_{\eta(s \wedge \tau)}}{2})} \frac{\bar{X}_{\eta(s \wedge \tau)}^{2\alpha}}{\bar{Z}_{s \wedge \tau}^{p+2}} ds &\leq \left( \frac{2}{b(0)\Delta t} \right)^{p+2} \sup_{t \in [0, T]} \left[ \mathbb{P} \left( \bar{Z}_t \leq \frac{\bar{X}_{\eta(t)}}{2} \right) \right]^{1/2} \sup_{t \in [0, T]} \left[ \mathbb{E} \left( \bar{X}_{\eta(t)}^{4\alpha} \right) \right]^{1/2} \\ &\leq \left( \frac{2}{b(0)\Delta t} \right)^{p+2} \mathcal{O}_{exp}(\Delta t) C(T) \leq C. \end{aligned}$$

Finally

$$\mathbb{E} \left( \frac{1}{\bar{Z}_{t \wedge \tau}^p} \right) \leq \frac{1}{x_0^p} - p \mathbb{E} \int_0^{t \wedge \tau} \frac{b(0)}{\bar{Z}_{s \wedge \tau}^{p+1}} ds + 2^{2\alpha-1} p(p+1) \sigma^2 \int_0^{t \wedge \tau} \frac{ds}{\bar{Z}_{s \wedge \tau}^{p+2(1-\alpha)}} + 2pK \mathbb{E} \int_0^t \frac{ds}{\bar{Z}_{s \wedge \tau}^p} + C.$$

Now one can easily check that there exists a positive constant  $c(b(0), \alpha, p)$ , depending only on  $b(0)$ ,  $\alpha$  and  $p$  such that  $\frac{-b(0)}{z^{p+1}} + \frac{(p+1)2^{2\alpha-1}}{z^{p+2(1-\alpha)}} \leq c(\alpha, p)$  and hence

$$\mathbb{E} \left( \frac{1}{\bar{Z}_{t \wedge \tau}^p} \right) \leq \frac{1}{x_0^p} + 2pK \mathbb{E} \int_0^t \frac{1}{\bar{Z}_{s \wedge \tau}^p} ds + C(b(0), p, \alpha).$$

We end the proof by applying the Gronwall Lemma. ■

## 4.2 Proof of Theorem 2.5

We follow the same technique as in the proof of Theorem 2.3. To study the weak error  $\mathbb{E}f(X_T) - \mathbb{E}f(\bar{X}_T)$ , we use the Feynman–Kac representation of the solution of the Cauchy problem (4.9) studied in the Proposition 4.3: for all  $(t, x) \in [0, T] \times (0, +\infty)$ ,  $\mathbb{E}f(X_{T-t}^x) = u(t, x)$ . Thus, the weak error becomes

$$\mathbb{E}f(X_T) - \mathbb{E}f(\bar{X}_T) = \mathbb{E} \left( u(0, x_0) - u(T, \bar{X}_T) \right).$$

Let  $\tau$  the stopping time defined by (4.18).

$$\mathbb{E} \left( u(0, x_0) - u(T, \bar{X}_T) \right) = \mathbb{E} \left( u(0, x_0) - u(T \wedge \tau, \bar{Z}_{T \wedge \tau}) \right) + \mathbb{E} \left( u(T \wedge \tau, \bar{Z}_{T \wedge \tau}) - u(T, \bar{X}_T) \right).$$

But

$$\begin{aligned} \mathbb{E} \left( u(T \wedge \tau, \bar{Z}_{T \wedge \tau}) - u(T, \bar{X}_T) \right) &= \mathbb{E} \left[ \left( u(T \wedge \tau, \bar{Z}_{T \wedge \tau}) - u(T, \bar{X}_T) \right) \mathbb{1}_{\{\tau \leq T\}} \right] \\ &\leq 2 \|u\| \mathbb{P}(\tau \leq T) \leq \mathcal{O}_{exp}(\Delta t). \end{aligned}$$

Let  $L$  and  $\mathcal{L}_z$  the second order differential operators defined for any  $C^2$  function  $g(x)$  by

$$Lg(x) = b(x) \frac{\partial g}{\partial x}(x) + \frac{\sigma^2}{2} x^{2\alpha} \frac{\partial^2 g}{\partial x^2}(x) \quad \text{and} \quad \mathcal{L}_z g(x) = b(z) \frac{\partial g}{\partial x}(x) + \frac{\sigma^2}{2} z^{2\alpha} \frac{\partial^2 g}{\partial x^2}(x).$$

$\frac{\partial u}{\partial x}(t, x)$  is uniformly bounded and  $\bar{X}_t$  has bounded momentums. By the Itô formula,

$$\mathbb{E} \left[ u(T \wedge \tau, \bar{X}_{T \wedge \tau}) - u(0, \bar{X}_0) \right] = \mathbb{E} \int_0^{T \wedge \tau} \left( \frac{\partial u}{\partial s} + \mathcal{L}_z u \right) \Big|_{z=\bar{X}_{\eta(s)}}(s, \bar{X}_s) ds$$

Recall also that  $\frac{\partial u}{\partial s}(t, x) + Lu(t, x) = 0$  by (4.9). Hence,

$$\mathbb{E} \left[ u(T \wedge \tau, \bar{X}_{T \wedge \tau}) - u(0, \bar{X}_0) \right] = \mathbb{E} \int_0^{T \wedge \tau} (\mathcal{L}_z u - Lu) \Big|_{z=\bar{X}_{\eta(s)}}(s, \bar{X}_s) ds$$

But applying the Itô formula between  $s$  and  $\eta(s)$ ,

$$\begin{aligned} & (\mathcal{L}_z u - Lu) \Big|_{z=\bar{X}_{\eta(s)}} (s, \bar{X}_s) \\ &= \int_{\eta(s)}^s \left( \frac{\partial[\mathcal{L}_z u - Lu]}{\partial \theta} + b(\bar{X}_{\eta(s)}) \frac{\partial[\mathcal{L}_z u - Lu]}{\partial x} + \frac{\sigma^2}{2} \bar{X}_{\eta(s)}^{2\alpha} \frac{\partial^2[\mathcal{L}_z u - Lu]}{\partial x^2} \right) \Big|_{z=\bar{X}_{\eta(s)}} (\theta, \bar{X}_\theta) d\theta \\ &+ \int_{\eta(s)}^s \frac{\partial[\mathcal{L}_z u - Lu]}{\partial x} (\theta, \bar{X}_\theta) \sigma \bar{X}_{\eta(s)}^\alpha dW_\theta \end{aligned}$$

But  $\mathbb{E}[\int_0^{T \wedge \tau} \int_{\eta(s)}^s \frac{\partial[\mathcal{L}_z u - Lu]}{\partial x} (\theta, \bar{X}_\theta) \sigma \bar{X}_{\eta(s)}^\alpha dW_\theta] = 0$  and, as  $u$  satisfies Equation (4.9) we have

$$\frac{\partial[\mathcal{L}_z u - Lu]}{\partial \theta} + b(z) \frac{\partial[\mathcal{L}_z u - Lu]}{\partial x} + \frac{\sigma^2}{2} z^{2\alpha} \frac{\partial^2[\mathcal{L}_z u - Lu]}{\partial x^2} = \mathcal{L}_z^2 u - 2\mathcal{L}_z Lu + L^2 u.$$

Then

$$\mathbb{E}[u(T \wedge \tau, \bar{X}_{T \wedge \tau}) - u(0, \bar{X}_0)] = \int_0^T \int_{\eta(s)}^s \mathbb{E} \left[ \mathbb{1}_{(\theta \leq \tau)} (\mathcal{L}_z^2 u - 2\mathcal{L}_z Lu + L^2 u) \Big|_{z=\bar{X}_{\eta(s)}} (\theta, \bar{X}_\theta) \right] d\theta ds.$$

$(\mathcal{L}_z^2 u - 2\mathcal{L}_z Lu + L^2 u) \Big|_z (\theta, x)$  combine with derivatives of  $u$  up to order four,  $b$  and its derivatives up to order two and power functions like  $z^{4\alpha}$  or  $x^{2\alpha-2}$ . When one value this expression at point  $(z, x) = (\bar{X}_{\eta(s)}, \bar{X}_\theta)$ , with the upper-bounds on the derivatives of  $u$  given in (4.10) and Lemmas 4.5 and 2.1 on the positive and negative momentum of  $\bar{X}$  we get

$$\left| \mathbb{E} \left[ \mathbb{1}_{(\theta \leq \tau)} (\mathcal{L}_z^2 u - 2\mathcal{L}_z Lu + L^2 u) \Big|_{z=\bar{X}_{\eta(s \wedge \tau)}} (\theta \wedge \tau, \bar{X}_{\theta \wedge \tau}) \right] \right| \leq C \left( 1 + \frac{1}{x_0^{q(\alpha)}} \right).$$

which implies the result.

## A Some properties of the Cox-Ingersoll-Ross model

In [4], Cox, Ingersoll and Ross proposed to model the dynamics of the short term interest rate as the solution of the following stochastic differential equation

$$dr_t = (a - br_t) + \sigma \sqrt{r_t} dW_t, \quad (\text{A.1})$$

where  $(W_t)_{0 \leq t \leq T}$  is a one-dimensional Brownian motion on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ ,  $a$  and  $\sigma$  are positives constants and  $b \in \mathbb{R}$ . For any  $t \in [0, T]$ , Let  $\mathcal{F}_t = \sigma(s \leq t, W_s)$ .

We denote by  $(r_t^x)$  the solution of (A.1) starting at  $x$  at time 0:

$$\begin{cases} dr_t^x = (a - br_t^x) dt + \sigma \sqrt{r_t^x} dW_t, \\ r_0^x = x \geq 0. \end{cases}$$

**Lemma A.1** For any  $x > 0$  and any  $p > 0$ ,

$$\mathbb{E} \left[ \frac{1}{(r_t^x)^p} \right] = \frac{1}{\Gamma(p)} \left( \frac{2b}{\sigma^2(1 - e^{-bt})} \right)^p \int_0^1 \theta^{p-1} (1 - \theta)^{\frac{2a}{\sigma^2} - p - 1} \exp \left( -\frac{2bx\theta}{\sigma^2(e^{bt} - 1)} \right) d\theta, \quad (\text{A.2})$$

where  $\Gamma(p) = \int_0^{+\infty} u^{p-1} \exp(-u) du$ ,  $p > 0$ , denotes the Gamma function.

Moreover, if  $a > \sigma^2$ , then, for any  $p$  such that  $1 < p < \frac{2a}{\sigma^2} - 1$ :

$$\mathbb{E} \left[ \frac{1}{(r_t^x)^p} \right] \leq \frac{1}{\Gamma(p)} \left( \frac{2e^{|b|t}}{\sigma^2 t} \right)^p \quad \text{or} \quad \mathbb{E} \left[ \frac{1}{(r_t^x)^p} \right] \leq C(p, T) \frac{1}{x^p} \quad (\text{A.3})$$

where  $C(p, T)$  is a positive constant depending on  $p$  and  $T$ . For  $p = 1$ , we have

$$\mathbb{E} \left[ \frac{1}{r_t^x} \right] \leq \frac{e^{bt}}{x}. \quad (\text{A.4})$$

**Proof.** By definition of the Gamma function, we note that, for all  $x > 0$  and  $p > 0$ ,  $x^{-p} = \Gamma(p)^{-1} \int_0^{+\infty} u^{p-1} \exp(-ux) du$ , so that

$$\mathbb{E} \left[ \frac{1}{(r_t^x)^p} \right] = \frac{1}{\Gamma(p)} \int_0^{+\infty} u^{p-1} \mathbb{E} \exp(-ur_t^x) du.$$

The Laplace transform of  $r_t^x$  is given by

$$\mathbb{E} \exp(-ur_t^x) = \frac{1}{(2uL(t) + 1)^{2a/\sigma^2}} \exp \left( -\frac{uL(t)\zeta(t, x)}{2uL(t) + 1} \right),$$

(see e.g. [10]), where  $L(t) = \frac{\sigma^2}{4b}(1 - \exp(-bt))$  and  $\zeta(t, x) = \frac{4xb}{\sigma^2(\exp(bt) - 1)} = xe^{-bt}/L(t)$ . Hence,

$$\mathbb{E} \left[ \frac{1}{(r_t^x)^p} \right] = \frac{1}{\Gamma(p)} \int_0^{+\infty} \frac{u^{p-1}}{(2uL(t) + 1)^{2a/\sigma^2}} \exp \left( -\frac{uL(t)\zeta(t, x)}{2uL(t) + 1} \right) du$$

We set  $\theta = \frac{uL(t)}{2uL(t) + 1}$ , so that  $u = \frac{\theta}{2L(t)(1-\theta)}$ ,  $2uL(t) + 1 = 1/(\theta - 1)$  and  $du = \frac{d\theta}{2L(1-\theta)^2}$ , it gives

$$\begin{aligned} \mathbb{E} \left[ \frac{1}{(r_t^x)^p} \right] &= \frac{1}{2^p \Gamma(p) L(t)^p} \int_0^1 \theta^{p-1} (1 - \theta)^{\frac{2a}{\sigma^2} - p - 1} \exp \left( -\frac{\zeta(t, x)\theta}{2} \right) d\theta \\ &= \frac{1}{2^p \Gamma(p) L(t)^p} \int_0^1 \theta^{p-1} (1 - \theta)^{\frac{2a}{\sigma^2} - p - 1} \exp \left( -\frac{xe^{-bt}\theta}{2L(t)} \right) d\theta, \end{aligned}$$

from which, we deduce (A.2). Now if  $a > \sigma^2$ , we have that

$$\mathbb{E} \left[ \frac{1}{r_t^x} \right] \leq \frac{1}{2L(t)} \int_0^1 \exp \left( -\frac{xe^{-bt}\theta}{2L(t)} \right) d\theta \leq \frac{e^{bt}}{x}$$

and if  $1 < p < \frac{2a}{\sigma^2} - 1$ ,

$$\mathbb{E} \left[ \frac{1}{(r_t^x)^p} \right] \leq \frac{1}{2^p \Gamma(p) L(t)^p} = \frac{2^p |b|^p}{\sigma^{2p} \Gamma(p) (1 - e^{-|b|t})^p}$$

which gives (A.3), noting that  $(1 - e^{-|b|t}) \geq |b|te^{-|b|t}$ . ■

**Lemma A.2** *If  $a \geq \sigma^2/2$  and  $b \geq 0$ , there exists a constant  $C$  depending on  $a, b, \sigma$  and  $T$ , such that*

$$\sup_{t \in [0, T]} \mathbb{E} \exp \left( \frac{\nu^2 \sigma^2}{8} \int_0^t \frac{ds}{r_s^x} \right) \leq C(T) (1 + x^{-\frac{\nu}{2}}), \quad (\text{A.5})$$

where  $\nu = \frac{2a}{\sigma^2} - 1 \geq 0$  and the positive constant  $C(T)$  is a nondecreasing function of  $T$ .

**Proof.** For any  $t \in [0, T]$ , we set  $H_t = \frac{2}{\sigma} \sqrt{r_t^x}$ , so that  $\mathbb{E} \exp(\frac{\nu^2 \sigma^2}{8} \int_0^t \frac{ds}{r_s^x}) = \mathbb{E} \exp(\frac{\nu^2}{2} \int_0^t \frac{ds}{H_s^2})$ . Then the process  $(H_t)_{t \in [0, T]}$  solves

$$dH_t = \left( \frac{2a}{\sigma^2} - \frac{1}{2} \right) \frac{dt}{H_t} - \frac{b}{2} H_t dt + dW_t, \quad H_0 = \frac{2}{\sigma} \sqrt{x}$$

For any  $t \in [0, T]$ , we set  $B_t = H_t - H_0 - \int_0^t \left( \frac{2a}{\sigma^2} - \frac{1}{2} \right) \frac{ds}{H_s}$ . Let  $(Z_t)_{t \in [0, T]}$  defined by

$$Z_t = \exp \left( - \int_0^t \frac{b}{2} H_s dB_s - \frac{b^2}{8} \int_0^t H_s^2 ds \right).$$

Then, by the Girsanov Theorem, under the probability  $\mathbb{Q}$  such that  $\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \frac{1}{Z_t}$ ,  $(B_t)_{t \in [0, T]}$  is a Brownian motion. Indeed  $(H_t)_{t \in [0, T]}$  solves

$$dH_t = \left( \frac{2a}{\sigma^2} - \frac{1}{2} \right) \frac{dt}{H_t} + dB_t, \quad t \leq T, \quad H_0 = \frac{2}{\sigma} \sqrt{x}.$$

We note that, under  $\mathbb{Q}$ ,  $(H_t)$  is a Bessel process with index  $\nu = \frac{2a}{\sigma^2} - 1$ . Moreover, by the integration by parts formula,  $\int_0^t 2H_s dB_s = H_t^2 - H_0^2 - \frac{4a}{\sigma^2} t$  and

$$Z_t = \exp \left( -\frac{b}{4} H_t^2 - \frac{b^2}{8} \int_0^t H_s^2 ds + \frac{b}{\sigma^2} x + t \frac{ba}{\sigma^2} \right) \leq \exp \left( \frac{b}{\sigma^2} x + T \frac{ba}{\sigma^2} \right).$$

Now,

$$\begin{aligned} \mathbb{E} \exp \left( \frac{\nu^2}{2} \int_0^t \frac{ds}{H_s^2} \right) &= \mathbb{E}^{\mathbb{Q}} \left[ \exp \left( \frac{\nu^2}{2} \int_0^t \frac{ds}{H_s^2} \right) Z_t \right] \\ &\leq \exp \left( \frac{b}{\sigma^2} x + T \frac{ba}{\sigma^2} \right) \mathbb{E}^{(\nu)}_{\frac{2}{\sigma} \sqrt{x}} \left[ \exp \left( \frac{\nu^2}{2} \int_0^t \frac{ds}{R_s^2} \right) \right] \end{aligned}$$

where  $\mathbb{E}^{\mathbb{Q}}$  denotes the expectation relative to  $\mathbb{Q}$ ,  $\mathbb{E}_{\frac{2}{\sigma}\sqrt{x}}^{(\nu)}$  the expectation relative to  $\mathbb{P}_{\frac{2}{\sigma}\sqrt{x}}^{(\nu)}$ , the law on  $C(\mathbb{R}^+, \mathbb{R}^+)$  of the Bessel process with index  $\nu$ , starting at  $\frac{2}{\sigma}\sqrt{x}$ . The next step consist in use the following relationship, for  $\nu \geq 0$ ,

$$\mathbb{P}_{\frac{2}{\sigma}\sqrt{x}}^{(\nu)} \Big|_{\sigma(R_s, s \leq t)} = \left( \frac{\sigma R_t}{2\sqrt{x}} \right)^\nu \exp \left( -\frac{\nu^2}{2} \int_0^t \frac{ds}{R_s^2} \right) \cdot \mathbb{P}_{\frac{2}{\sigma}\sqrt{x}}^{(0)} \Big|_{\sigma(R_s, s \leq t)},$$

where  $(R_t)_{t \geq 0}$  denotes the canonical process on  $C(\mathbb{R}^+, \mathbb{R}^+)$  (see e.g. Proposition 2.4 in [6]). Then,

$$\mathbb{E} \exp \left( \frac{\nu^2}{2} \int_0^t \frac{ds}{H_s^2} \right) \leq \exp \left( \frac{b}{\sigma^2} x + T \frac{ba}{\sigma^2} \right) \mathbb{E}_{\frac{2}{\sigma}\sqrt{x}}^{(0)} \left[ \left( \frac{\sigma R_t}{2\sqrt{x}} \right)^\nu \right].$$

It remains to compute  $\mathbb{E}_{\frac{2}{\sigma}\sqrt{x}}^{(0)} \left[ \left( \frac{\sigma R_t}{2\sqrt{x}} \right)^\nu \right]$ , where  $(R_t)$  is a Bessel process with index nil. Let  $(W_t^1, W_t^2)_{t \geq 0}$  be a two dimensional Brownian motion. Then

$$\mathbb{E}_{\frac{2}{\sigma}\sqrt{x}}^{(0)} \left[ \left( \frac{\sigma R_t}{2\sqrt{x}} \right)^\nu \right] = \left( \frac{\sigma}{2\sqrt{x}} \right)^\nu \mathbb{E} \left[ \left( (W_t^1)^2 + (W_t^2 + \frac{2\sqrt{x}}{\sigma})^2 \right)^{\frac{\nu}{2}} \right]$$

and an easy computation shows that  $\mathbb{E}_{\frac{2}{\sigma}\sqrt{x}}^{(0)} \left[ \left( \frac{\sigma R_t}{2\sqrt{x}} \right)^\nu \right] \leq C(T) (1 + x^{-\frac{\nu}{2}})$ . ■

## B End of the proof of Proposition 3.2

**Proposition B.1** Assume (H2) and (H3). For any  $\lambda \geq 0$ , we consider the process  $(X_t^x(\lambda))_{0 \leq t \leq T}$ , solution of Equation (3.7) and the process  $(J_t^x(\lambda))_{0 \leq t \leq T}$ , solution of Equation (3.8). Let  $g(x)$ ,  $h(x)$  and  $k(x)$  be some bounded  $C^1$  functions, with bounded first derivative. Let  $u$  be the  $\mathbb{R}$ -valued function defined on  $[0, T] \times \mathbb{R}_+^*$  by

$$u(t, x) = \mathbb{E} \left[ g(X_t^x(\lambda)) \exp \left( \int_0^t k(X_s^x(\lambda)) ds \right) \right] + \int_0^t \mathbb{E} \left[ h(X_s^x(\lambda)) \exp \left( \int_0^s k(X_\theta^x(\lambda)) d\theta \right) \right] ds$$

Then  $u(t, x)$  is of class  $C^1$  with respect to  $x$  and

$$\begin{aligned} \frac{\partial u}{\partial x}(t, x) = & \mathbb{E} \left[ \exp \left( \int_0^t k(X_s^x(\lambda)) ds \right) \left( g'(X_t^x(\lambda)) J_t^x(\lambda) + g(X_t^x(\lambda)) \int_0^t k'(X_s^x(\lambda)) J_s^x(\lambda) ds \right) \right] \\ & + \int_0^t \mathbb{E} \left[ \exp \left( \int_0^s k(X_\theta^x(\lambda)) d\theta \right) \left( h'(X_s^x(\lambda)) J_s^x(\lambda) + h(X_s^x(\lambda)) \int_0^s k'(X_\theta^x(\lambda)) J_\theta^x(\lambda) d\theta \right) \right] ds \end{aligned}$$

**Proof.** To simplify the notation, we consider only the case when  $k(x)$  and  $h(x)$  are nil. Moreover, we omit the dependency in  $\lambda$  and use  $X_t^x$  instead of  $X_t^x(\lambda)$ . When  $b(0) \geq \sigma^2/2$ , the trajectories of the process  $(X_t^x(0))_{0 \leq t \leq T}$  are strictly positive a.s. (c.f. Remark 2.2). For any  $\epsilon > 0$  and  $x > 0$ , we define for all  $t \in [0, T]$  the process

$$Z_t^{x,\epsilon} = \frac{1}{\epsilon}(X_t^{x+\epsilon} - X_t^x)$$

so that

$$Z_t^{x,\epsilon} = 1 + \int_0^t \phi_s^\epsilon Z_s^{x,\epsilon} ds + \int_0^t \psi_s^\epsilon Z_s^{x,\epsilon} dW_s.$$

with

$$\phi_s^\epsilon = \int_0^1 b'(X_s^x + \alpha \epsilon Z_s^{x,\epsilon}) d\alpha \quad \text{and} \quad \psi_s^\epsilon = \int_0^1 \frac{\sigma d\alpha}{2\sqrt{X_s^x + \alpha \epsilon Z_s^{x,\epsilon}}}.$$

By Lemma A.1,  $\int_0^t \psi_s^\epsilon dW_s$  is a martingale and then  $(Z_t^{x,\epsilon})$  is explicitly given by

$$Z_t^{x,\epsilon} = \exp \left( \int_0^t \phi_s^\epsilon ds + \int_0^t \psi_s^\epsilon dW_s - \frac{1}{2} \int_0^t (\psi_s^\epsilon)^2 ds \right). \quad (\text{B.1})$$

For  $p > 1$ , as  $b'$  is a bounded function,

$$\begin{aligned} \mathbb{E}(Z_t^{x,\epsilon})^2 &\leq C(T) \mathbb{E} \left[ \exp \left( 2 \int_0^t \psi_s^\epsilon dW_s - 2p \int_0^t (\psi_s^\epsilon)^2 ds \right) \exp \left( (2p-1) \int_0^t (\psi_s^\epsilon)^2 ds \right) \right] \\ &\leq C(T) \left[ \mathbb{E} \exp \left( 2p \int_0^t \psi_s^\epsilon dW_s - 2p^2 \int_0^t (\psi_s^\epsilon)^2 ds \right) \right]^{1/p} \\ &\quad \times \left[ \mathbb{E} \exp \left( (2p-1) \left(1 - \frac{1}{p}\right) \int_0^t (\psi_s^\epsilon)^2 ds \right) \right]^{\frac{p-1}{p}}. \end{aligned}$$

The last upper-bound is obtained with the Holder Inequality. Choosing  $p = (\frac{2b(0)}{\sigma^2} - 1)/2 > 1$  under H3, we have  $\frac{p^2 \sigma^2}{2} \leq (\frac{2b(0)}{\sigma^2} - 1)^2 \frac{\sigma^2}{8} = \frac{\nu^2 \sigma^2}{8}$  and by Lemma A.2

$$\mathbb{E} \exp \left( 2p^2 \int_0^t (\psi_s^\epsilon)^2 ds \right) \leq \mathbb{E} \exp \left( \frac{p^2 \sigma^2}{2} \int_0^t \frac{ds}{X_s^x} \right) \leq C(T, x).$$

This implies that  $(\exp(2p \int_0^t \psi_s^\epsilon dW_s - 2p^2 \int_0^t (\psi_s^\epsilon)^2 ds), t \in [0, T])$  is a martingale and hence

$$\begin{aligned} \mathbb{E}(Z_t^{x,\epsilon})^2 &\leq C(T) \left[ \mathbb{E} \exp \left( (2p-1) \left(1 - \frac{1}{p}\right) \int_0^t (\psi_s^\epsilon)^2 ds \right) \right]^{\frac{p-1}{p}} \\ &\leq C(T) \left[ \mathbb{E} \exp \left( (2p-1) \left(1 - \frac{1}{p}\right) \frac{\sigma^2}{4} \int_0^t \frac{ds}{X_s^x} \right) \right]^{\frac{p-1}{p}} \leq C(T, x), \end{aligned}$$



where we have used again Lemma A.2 as, for the same chose of  $p$ ,  $(2p-1)(1-\frac{1}{p})\frac{\sigma^2}{4} \leq \frac{\nu^2\sigma^2}{8}$ . Hence, there exists a positive constant  $C$  such that

$$\mathbb{E}(X_t^{x+\epsilon} - X_t^x)^2 \leq C\epsilon^2, \quad \forall t \in [0, T]$$

and  $X_t^{x+\epsilon}$  tends  $X_t^x$  in probability. One consider now the process

$$J_t^x = 1 + \int_0^t b'(X_s^x) J_s^x ds + \sigma \int_0^t \frac{J_s^x}{2\sqrt{X_s^x}} dW_s.$$

The next step consist in prove that  $\mathbb{E}|J_t^x - Z_t^{x,\epsilon}|$  tends to 0 with  $\epsilon$ . We have

$$\begin{aligned} J_t^x - Z_t^{x,\epsilon} &= \int_0^t (b'(X_s^x) J_s^x - \phi_s^\epsilon Z_s^{x,\epsilon}) ds + \int_0^t \left( \frac{\sigma J_s^x}{2\sqrt{X_s^x}} - \psi_s^\epsilon Z_s^{x,\epsilon} \right) dW_s \\ &= \int_0^t J_s^x \left( (b'(X_s^x) - \phi_s^\epsilon) ds + \left( \frac{\sigma}{2\sqrt{X_s^x}} - \psi_s^\epsilon \right) dW_s \right) + \int_0^t (\phi_s^\epsilon ds + \sigma \psi_s^\epsilon dW_s) (J_s^x - Z_s^{x,\epsilon}). \end{aligned}$$

As the solution of the linear SDE above, the process  $(J_t^x - Z_t^{x,\epsilon})_{0 \leq t \leq T}$  admit the following expression

$$J_t^x - Z_t^{x,\epsilon} = Z_t^{x,\epsilon} \int_0^t (Z_s^{x,\epsilon})^{-1} J_s^x \left( (b'(X_s^x) - \phi_s^\epsilon) ds - \psi_s^\epsilon \left( \frac{\sigma}{2\sqrt{X_s^x}} - \psi_s^\epsilon \right) ds + \left( \frac{\sigma}{2\sqrt{X_s^x}} - \psi_s^\epsilon \right) dW_s \right)$$

and hence, by the Cauchy-Schwarz Inequality

$$\begin{aligned} \mathbb{E} |J_t^x - Z_t^{x,\epsilon}|^2 &\leq C \mathbb{E} (Z_t^{x,\epsilon})^2 \\ &\times \int_0^t \mathbb{E} \left( (Z_s^{x,\epsilon})^{-2} (J_s^x)^2 \left[ (b'(X_s^x) - \phi_s^\epsilon)^2 + (\psi_s^\epsilon \left( \frac{\sigma}{2\sqrt{X_s^x}} - \psi_s^\epsilon \right))^2 + \left( \frac{\sigma}{2\sqrt{X_s^x}} - \psi_s^\epsilon \right)^2 \right] \right) ds. \end{aligned}$$

Moreover an easy computation shows that

$$b'(X_t^x) - \phi_t^\epsilon \leq \epsilon Z_s^{x,\epsilon} \|b''\|_\infty \quad \text{and} \quad \frac{\sigma}{2\sqrt{X_s^x}} - \psi_s^\epsilon \leq \epsilon \frac{\sigma Z_s^{x,\epsilon}}{(X_s^x)^{\frac{3}{2}}}$$

and, by the integration by part formula applied to the expression of  $J_t^x$  given in (3.8), we get

$$\begin{aligned} J_t^x &= \sqrt{\frac{X_t^x}{x}} \exp \left( \int_0^t \left( -\frac{b(X_s^x)}{2X_s^x} + \frac{\sigma^2}{8} \frac{1}{X_s^x} \right) ds \right) \\ &\leq \sqrt{\frac{X_t^x}{x}} \exp \left( -\int_0^t (b(0) - \frac{\sigma^2}{4}) \frac{ds}{2X_\theta^x} \right) \exp(KT) \leq C(T) \sqrt{\frac{X_t^x}{x}}, \end{aligned}$$

Then,

$$\mathbb{E} |J_t^x - Z_t^{x,\epsilon}|^2 \leq \epsilon \frac{C(T)}{x} \int_0^t \mathbb{E} \left( |X_s^x| + \frac{|X_s^x|}{|Z_s^{x,\epsilon}|} + \frac{1}{(X_s^x)^2} \right) ds.$$

and  $\mathbb{E}|J_t^x - Z_t^{x,\epsilon}|$  tends to 0 with  $\epsilon$ . Consider now

$$\begin{aligned} & \frac{1}{\epsilon} (g(X_t^{x+\epsilon}) - g(X_t^x)) - g'(X_t^x) J_t^x \\ &= Z_t^{x,\epsilon} \int_0^1 g'(X_t^x + \epsilon \alpha Z_t^{x,\epsilon}) d\alpha - J_t^x g'(X_t^x) \\ &= (Z_t^{x,\epsilon} - J_t^x) \int_0^1 g'(X_t^x + \epsilon \alpha Z_t^{x,\epsilon}) d\alpha + J_t^x \int_0^1 (g'(X_t^x + \epsilon \alpha Z_t^{x,\epsilon}) - g'(X_t^x)) d\alpha \\ &:= A^\epsilon + B^\epsilon. \end{aligned}$$

$\mathbb{E}A^\epsilon \leq \|g'\|_\infty \times \mathbb{E}|J_t^x - Z_t^{x,\epsilon}|$ , which tends to zero with  $\epsilon$ .  $B^\epsilon$  is a uniformly integrable sequence.  $g'$  is a continuous function. By the Lebesgue Theorem, as  $X_t^{x+\epsilon}$  tends  $X_t^x$  in probability,  $B^\epsilon$  tends to 0 with  $\epsilon$ . As a consequence,  $\mathbb{E} \left( \frac{1}{\epsilon} (g(X_t^{x+\epsilon}) - g(X_t^x)) \right)$  tends to  $\mathbb{E}[g'(X_t^x)J_t^x]$  when  $\epsilon$  tends to 0.  $\blacksquare$

**End of the proof of the Proposition 3.2.** To compute the fourth derivative, we decompose (3.14) in four terms:

$$\begin{aligned} \frac{\partial^3 u}{\partial x^3}(t, x) = & \mathbb{E} \left\{ \exp \left( 2 \int_0^{T-t} b'(X_s^x(1)) ds \right) f^{(3)}(X_{T-t}^x(1)) J_{T-t}^x(1) \right\} \\ & + 2 \mathbb{E} \left\{ \exp \left( 2 \int_0^{T-t} b'(X_s^x(1)) ds \right) f''(X_{T-t}^x(1)) \int_0^{T-t} b''(X_s^x(1)) J_s^x(1) ds \right\} \\ & + \int_0^{T-t} \mathbb{E} \left\{ \exp \left( 2 \int_0^s b'(X_u^x(1)) du \right) J_s^x(1) \right. \\ & \quad \left. \left( \frac{\partial^2 u}{\partial x^2}(t+s, X_s^x(1)) b''(X_s^x(1)) + \frac{\partial u}{\partial x}(t+s, X_s^x(1)) b^{(3)}(X_s^x(1)) \right) \right\} ds \\ & + 2 \mathbb{E} \left\{ \int_0^{T-t} \exp \left( 2 \int_0^s b'(X_u^x(1)) du \right) \frac{\partial u}{\partial x}(t+s, X_s^x(1)) \right. \\ & \quad \left. \times b''(X_s^x(1)) \left( \int_0^s b''(X_u^x(1)) J_u^x(1) du \right) ds \right\} := \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned}$$

Using again the Markov property and the time homogeneity property of the process  $X_t^x(1)$ ,

$$\begin{aligned} & \mathbb{E} \left[ \exp \left( 2 \int_s^{T-t} b'(X_\theta^x(1)) d\theta \right) f''(X_{T-t}^x(1)) \middle/ \mathcal{F}_s \right] \\ &= \mathbb{E} \left[ \exp \left( 2 \int_0^{T-t-s} b'(X_\theta^y(1)) d\theta \right) f''(X_{T-t-s}^y(1)) \right] \Big|_{y=X_s} \end{aligned}$$

and, by using (3.13),

$$\begin{aligned} \text{II} = & 2 \int_0^{T-t} \mathbb{E} \left\{ b''(X_s^x(1)) J_s^x(1) \exp \left( 2 \int_0^s b'(X_\theta^x(1)) d\theta \right) \frac{\partial^2 u}{\partial x^2}(t+s, X_s^x(1)) \right\} ds \\ & - 2 \int_0^{T-t} \mathbb{E} \left\{ b''(X_s^x(1)) J_s^x(1) \exp \left( 2 \int_0^s b'(X_\theta^x(1)) d\theta \right) \right. \\ & \quad \left. \left( \int_0^{T-t-s} \mathbb{E} \left[ \frac{\partial u}{\partial x}(t+s+u, X_u^y(1)) \exp \left( 2 \int_0^u b'(X_\theta^y(1)) d\theta \right) b''(X_u^y(1)) \right] \right) \right|_{y=X_s^x(1)} du \right\} ds \end{aligned}$$

Now we note that

$$\begin{aligned} & \int_0^{T-t-s} \mathbb{E} \left[ \frac{\partial u}{\partial x}(t+s+u, X_u^y(1)) \exp \left( 2 \int_0^u b'(X_\theta^y(1)) d\theta \right) b''(X_u^y(1)) \right] \Big|_{y=X_s^x(1)} du \\ &= \int_s^{T-t} \mathbb{E} \left[ \frac{\partial u}{\partial x}(t+u, X_u^x(1)) \exp \left( 2 \int_s^u b'(X_\theta^x(1)) d\theta \right) b''(X_u^x(1)) \middle/ \mathcal{F}_s \right] du \end{aligned}$$

and then

$$\begin{aligned} \text{II} = & 2 \int_0^{T-t} \mathbb{E} \left\{ b''(X_s^x(1)) J_s^x(1) \exp \left( 2 \int_0^s b'(X_\theta^x(1)) d\theta \right) \frac{\partial^2 u}{\partial x^2}(t+s, X_s^x(1)) \right\} ds \\ & - 2 \mathbb{E} \left\{ \int_0^{T-t} b''(X_s^x(1)) J_s^x(1) \left( \int_s^{T-t} \frac{\partial u}{\partial x}(t+u, X_u^x(1)) \exp \left( 2 \int_0^u b'(X_\theta^x(1)) d\theta \right) b''(X_u^x(1)) du \right) ds \right\} \end{aligned}$$

The integration by parts formula gives immediately that

$$\text{IV} = 2 \mathbb{E} \left\{ \int_0^{T-t} b''(X_s^x(1)) J_s^x(1) \left( \int_s^{T-t} \frac{\partial u}{\partial x}(t+u, X_u^x(1)) \exp \left( 2 \int_0^u b'(X_\theta^x(1)) d\theta \right) b''(X_u^x(1)) du \right) ds \right\}.$$

Finally, summing the expression of II with I, III and IV, we get

$$\begin{aligned} \frac{\partial^3 u}{\partial x^3}(t, x) = & \mathbb{E} \left\{ \exp \left( 2 \int_0^{T-t} b'(X_s^x(1)) ds \right) f^{(3)}(X_{T-t}^x(1)) J_{T-t}^x(1) \right\} \\ & + \int_0^{T-t} \mathbb{E} \left\{ \exp \left( 2 \int_0^s b'(X_u^x(1)) du \right) J_s^x(1) \left( 3 \frac{\partial^2 u}{\partial x^2}(t+s, X_s^x(1)) b''(X_s^x(1)) \right. \right. \\ & \quad \left. \left. + \frac{\partial u}{\partial x}(t+s, X_s^x(1)) b^{(3)}(X_s^x(1)) \right) \right\} ds. \end{aligned}$$

We introduce the probability  $\mathbb{Q}^{3/2}$  such that  $\frac{d\mathbb{Q}^{3/2}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \frac{1}{Z_t^{(1, \frac{3}{2})}}$ . Then

$$\begin{aligned} \frac{\partial^3 u}{\partial x^3}(t, x) = & \mathbb{E}^{3/2} \left\{ \exp \left( 2 \int_0^{T-t} b'(X_s^x(1)) ds \right) f^{(3)}(X_{T-t}^x(1)) Z_{T-t}^{(1, \frac{3}{2})} J_{T-t}^x(1) \right\} \\ & + \int_0^{T-t} \mathbb{E}^{3/2} \left\{ \exp \left( 2 \int_0^s b'(X_u^x(1)) du \right) Z_s^{(1, \frac{3}{2})} J_s^x(1) \left( 3 \frac{\partial^2 u}{\partial x^2}(t+s, X_s^x(1)) b''(X_s^x(1)) \right. \right. \\ & \left. \left. + \frac{\partial u}{\partial x}(t+s, X_s^x(1)) b^{(3)}(X_s^x(1)) \right) \right\} ds. \end{aligned}$$

By the Girsanov Theorem,  $Z_t^{(1, \frac{3}{2})} J_t^x(1) = \exp \left( \int_0^t b'(X_u^x(1)) du \right)$  and

$$\begin{aligned} \frac{\partial^3 u}{\partial x^3}(t, x) = & \mathbb{E}^{3/2} \left\{ \exp \left( 3 \int_0^{T-t} b'(X_s^x(1)) ds \right) f^{(3)}(X_{T-t}^x(1)) \right\} \\ & + \int_0^{T-t} \mathbb{E}^{3/2} \left\{ \exp \left( 3 \int_0^s b'(X_u^x(1)) du \right) \left( 3 \frac{\partial^2 u}{\partial x^2}(t+s, X_s^x(1)) b''(X_s^x(1)) \right. \right. \\ & \left. \left. + \frac{\partial u}{\partial x}(t+s, X_s^x(1)) b^{(3)}(X_s^x(1)) \right) \right\} ds \end{aligned}$$

As  $\mathcal{L}^{\mathbb{Q}^{3/2}}(X^x(1)) = \mathcal{L}^{\mathbb{P}}(X^x(\frac{3}{2}))$ , we finally obtain the following expression for  $\frac{\partial^3 u}{\partial x^3}(t, x)$ :

$$\begin{aligned} \frac{\partial^3 u}{\partial x^3}(t, x) = & \mathbb{E} \left\{ \exp \left( 3 \int_0^{T-t} b'(X_s^x(\frac{3}{2})) ds \right) f^{(3)}(X_{T-t}^x(\frac{3}{2})) \right\} \\ & + \int_0^{T-t} \mathbb{E} \left\{ \exp \left( 3 \int_0^s b'(X_u^x(\frac{3}{2})) du \right) \left( 3 \frac{\partial^2 u}{\partial x^2}(t+s, X_s^x(\frac{3}{2})) b''(X_s^x(\frac{3}{2})) \right. \right. \\ & \left. \left. + \frac{\partial u}{\partial x}(t+s, X_s^x(\frac{3}{2})) b^{(3)}(X_s^x(\frac{3}{2})) \right) \right\} ds. \end{aligned} \quad (\text{B.2})$$

$J_s^x(\frac{3}{2})$  exists and is given by (3.8). By proposition B.1,  $\frac{\partial^3 u}{\partial x^3}(t, x)$  is continuously differentiable and

$$\begin{aligned} \frac{\partial^4 u}{\partial x^4}(t, x) = & \mathbb{E} \left\{ \exp \left( 3 \int_0^{T-t} b'(X_s^x(\frac{3}{2})) ds \right) \right. \\ & \left[ 3f^{(3)}(X_{T-t}^x(\frac{3}{2})) \int_0^{T-t} b''(X_s^x(\frac{3}{2})) J_s^x(\frac{3}{2}) ds + f^{(3)}(X_{T-t}^x(\frac{3}{2})) J_{T-t}^x(\frac{3}{2}) \right] \Big\} \\ & + \int_0^{T-t} \mathbb{E} \left\{ \exp \left( 3 \int_0^s b'(X_u^x(\frac{3}{2})) du \right) \left( 3 \frac{\partial^2 u}{\partial x^2}(t+s, X_s^x(\frac{3}{2})) b''(X_s^x(\frac{3}{2})) \right. \right. \\ & \quad \left. \left. + \frac{\partial u}{\partial x}(t+s, X_s^x(\frac{3}{2})) b^{(3)}(X_s^x(\frac{3}{2})) \right) \int_0^s 3b''(X_u^x(\frac{3}{2})) J_u^x(\frac{3}{2}) du \right\} ds \\ & + \int_0^{T-t} \mathbb{E} \left\{ \exp \left( 3 \int_0^s b'(X_u^x(\frac{3}{2})) du \right) J_s^x(\frac{3}{2}) \left( 3 \frac{\partial^3 u}{\partial x^3}(t+s, X_s^x(\frac{3}{2})) b''(X_s^x(\frac{3}{2})) \right. \right. \\ & \quad \left. \left. + 4 \frac{\partial^2 u}{\partial x^2}(t+s, X_s^x(\frac{3}{2})) b^{(3)}(X_s^x(\frac{3}{2})) \right. \right. \\ & \quad \left. \left. + \frac{\partial u}{\partial x}(t+s, X_s^x(\frac{3}{2})) b^{(4)}(X_s^x(\frac{3}{2})) \right) \right\} ds \end{aligned}$$

from which we can conclude on (3.5). ■

## C End of the proof of Proposition 4.3

**Proposition C.1** *Assume (H2). We consider the process  $(X_t^x)_{0 \leq t \leq T}$ , solution of Equation (4.1) and the process  $(J_t^x)_{0 \leq t \leq T}$ , solution of Equation (4.11). Let  $g(x)$ ,  $h(x)$  and  $k(x)$  be some  $C^1$  functions on  $(0, +\infty)$  such that*

$$\forall x > 0, \quad |g(x)| + |g'(x)| + |h(x)| + |h'(x)| + |k(x)| + |k'(x)| \leq C \left( x^\alpha + \frac{1}{x^2} \right).$$

Let  $u$  be the  $\mathbb{R}$ -valued function defined on  $[0, T] \times \mathbb{R}_+^*$  by

$$u(t, x) = \mathbb{E} \left[ g(X_t^x(\lambda)) \exp \left( \int_0^t k(X_s^x(\lambda)) ds \right) \right] + \int_0^t \mathbb{E} \left[ h(X_s^x(\lambda)) \exp \left( \int_0^s k(X_\theta^x(\lambda)) d\theta \right) \right] ds$$

Then  $u(t, x)$  is of class  $C^1$  with respect to  $x$  and

$$\begin{aligned} \frac{\partial u}{\partial x}(t, x) = & \mathbb{E} \left[ \exp \left( \int_0^t k(X_s^x(\lambda)) ds \right) \left( g'(X_t^x(\lambda)) J_t^x(\lambda) + g(X_t^x(\lambda)) \int_0^t k'(X_s^x(\lambda)) J_s^x(\lambda) ds \right) \right] \\ & + \int_0^t \mathbb{E} \left[ \exp \left( \int_0^s k(X_\theta^x(\lambda)) d\theta \right) \left( h'(X_s^x(\lambda)) J_s^x(\lambda) + h(X_s^x(\lambda)) \int_0^s k'(X_\theta^x(\lambda)) J_\theta^x(\lambda) d\theta \right) \right] ds \end{aligned}$$

**Proof.** The proof is similar to the proof of Proposition B.1. Again, we consider only the case when  $h(x)$  and  $k(x)$  are nil. We consider  $Z_t^{x,\epsilon} = \frac{1}{\epsilon}(X_t^{x+\epsilon} - X_t^x)$ , so that

$$Z_t^{x,\epsilon} = 1 + \int_0^t \phi_s^\epsilon Z_s^{x,\epsilon} ds + \int_0^t \psi_s^\epsilon Z_s^{x,\epsilon} dW_s \quad (\text{C.1})$$

and

$$Z_t^{x,\epsilon} = \exp \left( \int_0^t \phi_s^\epsilon ds + \int_0^t \psi_s^\epsilon dW_s - \frac{1}{2} \int_0^t (\psi_s^\epsilon)^2 ds \right),$$

with, for all  $t \in [0, T]$

$$\phi_t^\epsilon = \int_0^1 b'(X_t^x + \mu(X_s^{x+\epsilon} - X_t^x)) d\mu \quad \text{and} \quad \psi_t^\epsilon = \sigma \alpha \int_0^1 \frac{d\mu}{(X_t^x + \mu(X_s^{x+\epsilon} - X_t^x))^{1-\alpha}}.$$

For any  $C^1$  function  $g(x)$ , we have

$$\begin{aligned} \frac{1}{\epsilon} [\mathbb{E}g(X_t^{x+\epsilon}) - \mathbb{E}g(X_t^x)] - \mathbb{E}[J_t^x g'(X_t^x)] &= \mathbb{E} \left[ Z_t^{x,\epsilon} \int_0^1 g'(X_t^x + \mu \epsilon Z_t^{x,\epsilon}) d\mu \right] - \mathbb{E}[J_t^x g'(X_t^x)] \\ &= \mathbb{E} \left[ (Z_t^{x,\epsilon} - J_t^x) \int_0^1 g'(X_t^x + \mu \epsilon Z_t^{x,\epsilon}) d\mu \right] + \mathbb{E} \left[ J_t^x \int_0^1 (g'(X_t^x + \mu \epsilon Z_t^{x,\epsilon}) - g'(X_t^x)) d\mu \right] = A^\epsilon + B^\epsilon. \end{aligned}$$

We consider first  $B^\epsilon$ . We note that

$$\begin{aligned} \mathbb{E}(Z_t^{x,\epsilon})^2 &\leq \exp(\|b'\|_\infty t) \mathbb{E} \exp \left( 2 \int_0^t \psi_s^\epsilon dW_s - \int_0^t (\psi_s^\epsilon)^2 ds \right) \\ &\leq \exp(\|b'\|_\infty t) \left[ \mathbb{E} \exp \left( 4 \int_0^t \psi_s^\epsilon dW_s - 8 \int_0^t (\psi_s^\epsilon)^2 ds \right) \right]^{1/2} \left[ \mathbb{E} \exp \left( 6 \int_0^t (\psi_s^\epsilon)^2 ds \right) \right]^{1/2}. \end{aligned}$$

By definition of  $\psi_t^\epsilon$  and by applying (4.5)

$$\mathbb{E} \exp \left( 8 \int_0^t (\psi_s^\epsilon)^2 ds \right) \leq \mathbb{E} \exp \left( 8\sigma^2 \alpha^2 \int_0^t \frac{ds}{(X_s^x)^{2(1-\alpha)}} \right) \leq C(T).$$

Hence, the process  $(\exp(4 \int_0^t \psi_s^\epsilon dW_s - 8 \int_0^t (\psi_s^\epsilon)^2 ds), t \in [0, T])$  is a martingale and  $\mathbb{E}(Z_t^{x,\epsilon})^2 \leq C(T)$ . As a consequence,  $X_t^{x+\epsilon}$  tends to  $X_t^x$  in  $L^2(\Omega)$  and  $|B^E\epsilon|$  tends 0 by applying the Lebesgue Theorem. Indeed by Lemmas 2.1 and 4.1 and by (4.13),

$$\mathbb{E} \left[ J_t^x \int_0^1 (g'(X_t^x + \mu \epsilon Z_t^{x,\epsilon}) - g'(X_t^x)) d\mu \right] \leq C [\mathbb{E}(J_t^x)^2]^{\frac{1}{2}} [\mathbb{E}(X_t^x)^{-4} + \mathbb{E}(X_t^{x+\epsilon})^{2\alpha}]^{\frac{1}{2}} < +\infty.$$

Now we consider  $|A^\epsilon| \leq C [\mathbb{E}(X_t^x)^{-4} + \mathbb{E}(X_t^{x+\epsilon})^{2\alpha}]^{\frac{1}{2}} [\mathbb{E}|Z_t^{x,\epsilon} - J_t^x|]^{\frac{1}{2}}$ . From (4.11) and (C.1), we have the following linear SDE for  $Z_t^{x,\epsilon} - J_t^x$

$$\begin{aligned} Z_t^{x,\epsilon} - J_t^x &= \int_0^t (Z_s^{x,\epsilon} - J_s^x) (\phi_s^\epsilon ds + \psi_s^\epsilon dW_s) \\ &\quad + \int_0^t J_s^x \left( (\phi_s^\epsilon - b'(X_s^x)) ds + (\psi_s^\epsilon - \frac{\sigma\alpha}{(X_s^x)^{1-\alpha}}) dW_s \right), \end{aligned}$$

and then

$$Z_t^{x,\epsilon} - J_t^x = Z_t^{x,\epsilon} \int_0^t (Z_s^{x,\epsilon})^{-1} \left[ J_s^x \left( (\phi_s^\epsilon - b'(X_s^x)) ds + (\psi_s^\epsilon - \frac{\sigma\alpha}{(X_s^x)^{1-\alpha}}) dW_s \right) - \psi_s^\epsilon (\psi_s^\epsilon - \frac{\sigma\alpha}{(X_s^x)^{1-\alpha}}) ds \right].$$

By the Cauchy-Schwarz Inequality,

$$\begin{aligned} \mathbb{E}|Z_t^{x,\epsilon} - J_t^x|^2 &\leq C \mathbb{E}(Z_t^{x,\epsilon})^2 \\ &\times \int_0^t \mathbb{E} \left( ((Z_s^{x,\epsilon})^{-1} J_s^x)^2 \left[ \left( \phi_s^\epsilon - b'(X_s^x) + \psi_s^\epsilon (\psi_s^\epsilon - \frac{\sigma\alpha}{(X_s^x)^{1-\alpha}}) \right)^2 + (\psi_s^\epsilon - \frac{\sigma\alpha}{(X_s^x)^{1-\alpha}})^2 \right] \right) ds. \end{aligned}$$

Using the upper-bounds of Lemma 4.2, an easy computation shows that  $\mathbb{E}((Z_s^{x,\epsilon})^{-1} J_s^x)^4 \leq C(T)$ . The convergence of  $A^\epsilon$  follows as in the proof of Proposition B.1.  $\blacksquare$

**End of the proof of the Proposition 4.3.** We have to found an upper-bound for  $\frac{\partial^3 u}{\partial x^3}(t, x)$ . Recall that

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\alpha}{x} \frac{\partial u}{\partial x}(t, x) + \mathbb{E} [f''(X_{T-t}^x)(J_{T-t}^x)^2] + \alpha \mathbb{E} \left[ \frac{1}{X_{T-t}^x} f'(X_{T-t}^x)(J_{T-t}^x)^2 \right] + I(x)$$

where

$$I(x) = \mathbb{E} \left[ f'(X_{T-t}^x) J_{T-t}^x \int_0^{T-t} \left( b''(X_s^x) - \frac{\alpha b'(X_s^x)}{X_s^x} + \frac{\alpha b(X_s^x)}{(X_s^x)^2} + \frac{\sigma^2 \alpha (1-\alpha)^2}{(X_s^x)^{3-2\alpha}} \right) J_s^x ds \right].$$

Thus,

$$\begin{aligned} \frac{\partial^3 u}{\partial x^3}(t, x) &= \frac{\alpha}{x^2} \frac{\partial u}{\partial x}(t, x) - \frac{\alpha}{x} \frac{\partial^2 u}{\partial x^2}(t, x) + \mathbb{E} [f^{(3)}(X_{T-t}^x)(J_{T-t}^x)^3] + 2\mathbb{E} \left[ f''(X_{T-t}^x) J_{T-t}^x \frac{d}{dx} J_{T-t}^x \right] \\ &\quad - \alpha \mathbb{E} \left[ \frac{1}{(X_{T-t}^x)^2} f'(X_{T-t}^x)(J_{T-t}^x)^3 \right] + \alpha \mathbb{E} \left[ \frac{1}{X_{T-t}^x} f''(X_{T-t}^x)(J_{T-t}^x)^4 \right] \\ &\quad + 2\alpha \mathbb{E} \left[ \frac{1}{X_{T-t}^x} f'(X_{T-t}^x) J_{T-t}^x \frac{d}{dx} J_{T-t}^x \right] + \frac{d}{dx} I(x). \end{aligned}$$

Hence,

$$\begin{aligned} \left| \frac{\partial^3 u}{\partial x^3}(t, x) \right| &\leq \frac{\alpha}{x^2} \left| \frac{\partial u}{\partial x}(t, x) \right| + \frac{\alpha}{x} \left| \frac{\partial^2 u}{\partial x^2}(t, x) \right| + \|f^{(3)}\|_\infty \mathbb{E}(J_{T-t}^x)^3 + 2\|f''\|_\infty \mathbb{E} \left[ J_{T-t}^x \left| \frac{d}{dx} J_{T-t}^x \right| \right] \\ &\quad + 2\alpha\|f'\|_\infty \mathbb{E} \left[ \frac{1}{X_{T-t}^x} J_{T-t}^x \left| \frac{d}{dx} J_{T-t}^x \right| \right] + \alpha\|f'\|_\infty \mathbb{E} \left[ \frac{1}{(X_{T-t}^x)^2} (J_{T-t}^x)^3 \right] \\ &\quad + \alpha\|f'\|_\infty \mathbb{E} \left[ \frac{1}{X_{T-t}^x} (J_{T-t}^x)^4 \right] + \left| \frac{d}{dx} I(x) \right|. \end{aligned}$$

But

$$\begin{aligned} \frac{d}{dx} I(x) &= \mathbb{E} \left[ f''(X_{T-t}^x) (J_{T-t}^x)^2 \int_0^{T-t} \left( b''(X_s^x) - \frac{\alpha b'(X_s^x)}{X_s^x} + \sigma^2 \alpha (1-\alpha)^2 \frac{1}{(X_s^x)^{3-2\alpha}} \right) J_s^x ds \right] \\ &\quad + \mathbb{E} \left[ f'(X_{T-t}^x) \left( \frac{d}{dx} J_{T-t}^x \right) \int_0^{T-t} \left( b''(X_s^x) - \frac{\alpha b'(X_s^x)}{X_s^x} + \sigma^2 \alpha (1-\alpha)^2 \frac{1}{(X_s^x)^{3-2\alpha}} \right) J_s^x ds \right] \\ &\quad + \mathbb{E} \left[ f'(X_{T-t}^x) J_{T-t}^x \int_0^{T-t} \left( b^{(3)}(X_s^x) - \frac{\alpha b''(X_s^x)}{X_s^x} + \frac{2\alpha b'(X_s^x)}{(X_s^x)^2} + \frac{2\alpha b(X_s^x)}{(X_s^x)^3} - \frac{\sigma^2 \alpha (1-\alpha)^2 (3-2\alpha)}{(X_s^x)^{2(2-\alpha)}} \right) \right. \\ &\quad \quad \quad \left. \times (J_s^x)^2 ds \right] \\ &\quad + \mathbb{E} \left[ f'(X_{T-t}^x) J_{T-t}^x \int_0^{T-t} \left( b''(X_s^x) - \frac{\alpha b'(X_s^x)}{X_s^x} + \frac{\alpha b(X_s^x)}{(X_s^x)^2} + \frac{\sigma^2 \alpha (1-\alpha)^2}{(X_s^x)^{3-2\alpha}} \right) \frac{d}{dx} J_s^x ds \right] \end{aligned}$$

and

$$\frac{d}{dx} J_{T-t}^x = J_{T-t}^x \left[ -\frac{\alpha}{x} + \int_0^{T-t} \left( b''(X_s^x) - \frac{\alpha b'(X_s^x)}{X_s^x} + \frac{\sigma^2 \alpha (1-\alpha)^2}{(X_s^x)^{3-2\alpha}} \right) J_s^x ds \right]. \quad (\text{C.2})$$

Thus, applying Lemma 4.1 and estimate (4.13), it comes that

$$\left| \frac{\partial^3 u}{\partial x^3}(t, x) \right| \leq C(T) \left( 1 + \frac{1}{x^{6-\alpha}} \right).$$



Now we upper-bound  $\frac{\partial^4 u}{\partial x^4}(t, x)$ :

$$\begin{aligned}
\frac{\partial^4 u}{\partial x^4}(t, x) = & -\frac{\alpha}{x^3} \frac{\partial u}{\partial x}(t, x) + \frac{2\alpha}{x^2} \frac{\partial^2 u}{\partial x^2}(t, x) - \frac{\alpha}{x} \frac{\partial^3 u}{\partial x^3}(t, x) \\
& + \mathbb{E} \left[ f^{(4)}(X_{T-t}^x)(J_{T-t}^x)^4 \right] + 5\mathbb{E} \left[ f^{(3)}(X_{T-t}^x)(J_{T-t}^x)^2 \frac{d}{dx} J_{T-t}^x \right] \\
& + 2\mathbb{E} \left[ f''(X_{T-t}^x) J_{T-t}^x \frac{d^2}{dx^2} J_{T-t}^x \right] + 2\alpha \mathbb{E} \left[ \frac{1}{(X_{T-t}^x)^3} f'(X_{T-t}^x)(J_{T-t}^x)^4 \right] \\
& - \alpha \mathbb{E} \left[ \frac{1}{(X_{T-t}^x)^2} f''(X_{T-t}^x)(J_{T-t}^x)^5 \right] + \alpha \mathbb{E} \left[ \frac{1}{X_{T-t}^x} f^{(3)}(X_{T-t}^x)(J_{T-t}^x)^5 \right] \\
& + 4\alpha \mathbb{E} \left[ \frac{1}{X_{T-t}^x} f''(X_{T-t}^x)(J_{T-t}^x)^3 \frac{d}{dx} J_{T-t}^x \right] - 2\alpha \mathbb{E} \left[ \frac{1}{(X_{T-t}^x)^2} f'(X_{T-t}^x)(J_{T-t}^x)^2 \frac{d}{dx} J_{T-t}^x \right] \\
& + 2\alpha \mathbb{E} \left[ \frac{1}{X_{T-t}^x} f''(X_{T-t}^x)(J_{T-t}^x)^2 \frac{d}{dx} J_{T-t}^x \right] + 2\alpha \mathbb{E} \left[ \frac{1}{X_{T-t}^x} f'(X_{T-t}^x) J_{T-t}^x \frac{d^2}{dx^2} J_{T-t}^x \right] \\
& + 2\mathbb{E} \left[ f''(X_{T-t}^x) \left( \frac{d}{dx} J_{T-t}^x \right)^2 \right] + \frac{d^2}{dx^2} I(x).
\end{aligned}$$

We deduce from (C.2) that

$$\begin{aligned}
\frac{d^2}{dx^2} J_{T-t}^x = & J_{T-t}^x \left( -\frac{\alpha}{x} + \int_0^{T-t} \left( b''(X_s^x) - \frac{\alpha b'(X_s^x)}{X_s^x} + \frac{\sigma^2 \alpha (1-\alpha)^2}{(X_s^x)^{3-2\alpha}} \right) J_s^x ds \right)^2 \\
& + J_{T-t}^x \left[ \frac{\alpha}{x^2} + \int_0^{T-t} \left( b^{(3)}(X_s^x) - \frac{\alpha b''(X_s^x)}{X_s^x} + \frac{\alpha b'(X_s^x)}{(X_s^x)^2} - \frac{\sigma^2 \alpha (1-\alpha)^2 (3-2\alpha)}{(X_s^x)^{2(2-\alpha)}} \right) (J_s^x)^2 ds \right].
\end{aligned}$$

The same arguments as for the derivative of order three allow to conclude that

$$\left| \frac{\partial^4 u}{\partial x^4}(t, x) \right| \leq C(T) \left( 1 + \frac{1}{x^{9-\alpha}} \right).$$

■

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