

Manuscript submitted for publication in :

**ESAIM: Control, Optimisation and Calculus of Variations**

URL: <http://www.edpsciences.org/cocv/>

**FLATNESS AND MONGE PARAMETERIZATION  
OF TWO-INPUT SYSTEMS,  
CONTROL-AFFINE WITH 4 STATES  
OR GENERAL WITH 3 STATES**

DAVID AVANESSOFF AND JEAN-BAPTISTE POMET

*INRIA, B.P. 93, 06902 Sophia Antipolis cedex, France*

David.Avanessoff@sophia.inria.fr, Jean-Baptiste.Pomet@sophia.inria.fr

**ABSTRACT.** This paper studies Monge parameterization, or differential flatness, of control-affine systems with four states and two controls. Some of them are known to be flat, and this implies admitting a Monge parameterization. Focusing on systems outside this class, we describe the only possible structure of such a parameterization for these systems, and give a lower bound on the order of this parameterization, if it exists. This lower-bound is good enough to recover the known results about “ $(x, u)$ -flatness” of these systems, with much more elementary techniques.

**RÉSUMÉ.** On s'intéresse aux paramétrisations de Monge, ou à la platitude, des systèmes affines à quatre états et deux entrées. Il est connu que les systèmes appartenant à une certaine classe parmi ceux de ces dimensions sont plats, ce qui entraîne qu'ils admettent une paramétrisation de Monge; on sait aussi que seuls les systèmes de cette classe sont “ $(x, u)$ -plats”. Pour les systèmes n'appartenant pas à cette classe, on montre que toute paramétrisation est d'un type particulier, et on donne une borne inférieure sur l'ordre de cette paramétrisation, suffisante pour retrouver, de manière beaucoup plus élémentaire, le fait, connu, que les systèmes n'appartenant pas à cette classe ne sont pas “ $(x, u)$ -plats”.

## 1. INTRODUCTION

In control theory, after a line of research on exact linearization by dynamic state feedback [13, 5, 6], the concept of differential flatness was introduced in 1992 in [7] (see also [8, 9]). Flatness is equivalent to exact linearization by dynamic state feedback of a special type, called “endogenous” [7], but, as pointed out in that reference, it has its own interest, maybe more important than linearity. An interpretation and framework for that notion is also proposed in [1, 17, 22]; see [15] for a recent review.

The *Monge problem* (see the the survey article [23], published in 1932, that mentions the prominent contributions [12] and [4], and others) is the one of finding explicit formulas giving the “general solution” of an under-determined system of ODEs as functions of some arbitrary functions of time and a certain number of their time-derivatives (in fact [23] allows to change the independent variable, but

---

*Date:* April 21, 2005.

*1991 Mathematics Subject Classification.* 93B18, 93B29, 34C20.

*Key words and phrases.* Dynamic feedback linearization, Flat control systems, Monge problem, Monge equations.

we keep it to be time). Let us call such formulas a *Monge parameterization*, its *order* being the number of time-derivatives.

The authors of [7] already made the link with the above mentioned work on under-determined systems of ODEs dating back from the beginning of 20<sup>th</sup> century; for instance, they used [12, 4] to obtain, in [20, 16] some results on flatness or linearizability of control systems.

Let us precise the relation between flatness and Monge parameterizability : flatness is existence of some functions —we call this collection of functions a *flat output*— of the state, the controls and a certain number  $j$  of time-derivatives of the control, that “invert” the formulas of a Monge parameterization, i.e. a solution  $t \mapsto (x(t), u(t))$  of the control system corresponds to only one choice of the arbitrary functions of time appearing in the parameterization, given by these functions. Let us call  $j$  the *order* of the flat output.

Characterizing differential flatness, or dynamic state feedback linearizability is still an open problem [10], apart from the case of single-input systems [5, 4]. The main difficulty is that the order of a parameterization, or the order a flat output, is not known beforehand. Of course, if one can construct a parameterization, or a flat output, it must have some definite finite order, but if one can prove that a system admits no parameterization of some order, then it might admit parameterization of some higher order, and we do not know a way to bound a priori the possible order of a parameterization for a given system. In the present paper, we consider systems of the smallest dimensions for which the answer is not know; we do not really overcome the above mentioned “main difficulty”, in the sense that we only say that our class of systems does not admit a parameterization of order less than some numbers, but the description of the parameterization that we give, and the resulting system of PDEs is valid at any order.

Consider a general control-affine system in  $\mathbb{R}^4$  with two controls, where  $\xi \in \mathbb{R}^4$  is the state,  $\tilde{w}_1$  and  $\tilde{w}_2$  are the two scalar controls and  $X_0$ ,  $X_1$  and  $X_2$  are three smooth vector fields :

$$\dot{\xi} = X_0(\xi) + \tilde{w}_1 X_1(\xi) + \tilde{w}_2 X_2(\xi) .$$

In [18], one can find a necessary and sufficient condition on these vector fields for this system to admit a flat output depending on the state and control only ( $j = 0$  according to the above notations). Systems who do *not* satisfy this conditions may or may not admit flat outputs depending also on some time-derivatives of the control ( $j > 0$ ). Our ultimate goal was to prove (it is our conjecture) that these systems do not admit a Monge parameterization, of any order. We do not go that far here, but we establish that, if they admit a Monge parameterization, it must assume a special form, and it must be of order larger than some number.

In fact, instead of the above control system, we study a reduced equation (3); let us explain very briefly why it represents, modulo a possibly dynamic feedback transformation, all the relevant cases. Systems for which the iterated Lie brackets of  $X_1$  and  $X_2$  do not have maximum rank can be treated in a rather simple manner [18, first cases of Theorem 3.1]; if on the contrary iterated Lie brackets do have maximum rank, it is well known (Engel normal form for distributions of rank 2 in  $\mathbb{R}^4$ , see [3]) that, after a nonsingular feedback ( $\tilde{w}_i = \beta^{i,0}(\xi) + \beta^{i,1}(\xi)w_1 + \beta^{i,2}(\xi)w_2$ ,  $i = 1, 2$ , with  $\beta^{1,1}\beta^{2,2} - \beta^{1,2}\beta^{2,1} \neq 0$ ), there are coordinates such that the system reads

$$\dot{\xi}_1 = w_1, \quad \dot{\xi}_2 = \gamma(\xi_1, \xi_2, \xi_3, \xi_4) + \xi_3 w_1, \quad \dot{\xi}_3 = \delta(\xi_1, \xi_2, \xi_3, \xi_4) + \xi_4 w_1, \quad \dot{\xi}_4 = w_2 \quad (1)$$

with some smooth functions  $\gamma$  and  $\delta$ . One can eliminate  $w_1$  and  $w_2$  and, renaming  $\xi_1, \xi_2, \xi_3, \xi_4$  as  $x, y, z, w$ , obtain the two following relations between these four

functions of time :

$$\begin{cases} \dot{y} = \gamma(x, y, z, w) + z\dot{x} \\ \dot{z} = \delta(x, y, z, w) + w\dot{x} \end{cases} \quad (2)$$

(this can also be seen as a control system with state  $(x, y, z)$  and controls  $w$  and  $\dot{x}$ ). It turns out (see [18] or Example 2.4 below) that this system is always parameterizable, and even flat, if  $\gamma$  does not depend on  $w$ . If, on the contrary,  $\gamma$  does depend on its last argument, one can, around a point where the partial derivative is nonzero, invert  $\gamma$  with respect to  $w$ , *i.e.* transform the first equation into  $w = g(x, y, z, \dot{y} - z\dot{x})$  for some function  $g$ , and obtain, substituting into the last equation, a single differential relation between  $x, y, z$  written as (3) in next section.

Note that (3) also represents the general (non-affine) systems in  $\mathbb{R}^3$  with two controls that satisfy the necessary condition given in [20, 21], *i.e.* they are “ruled”; we do not develop this here, see [2] or a future publication.

The paper technically focuses on Monge parameterizations of (3) when  $g$  and  $h$  are such that system (1) does not satisfy the above mentioned necessary and sufficient condition. The main results are to show that

- if they do admit a parameterization, it must be of a special form, and a system of partial differential relations must have some solutions,
- this is impossible at orders smaller than some precise numbers.

Our results, besides being more general, are strong enough to recover these of [18], thanks to the relation (see section 7) between the orders of a flat output for system (1) and the corresponding parameterization for (3). The techniques used in the present paper, derived from the original proof of non-parameterizability of some special systems in [12], are much simpler and elementary than these of [18]; hence recovering the results from [18] in this way does have some interest. See section 2 for a more precise setting, and the organization of the paper.

This paper owes a lot not only to the original article [12], but also to the careful re-reading of that article by *P. Rouchon* in [19] (see also in [20]).

## 2. PROBLEM STATEMENT

**2.1. The systems under consideration.** This paper studies the solutions  $t \mapsto (x(t), y(t), z(t))$  of the scalar differential equation

$$\dot{z} = h(x, y, z, \lambda) + g(x, y, z, \lambda)\dot{x} \quad \text{with} \quad \lambda = \dot{y} - z\dot{x} \quad (3)$$

where  $g$  and  $h$  are two real analytic functions  $\Omega \rightarrow \mathbb{R}$ ,  $\Omega$  being an open connected subset of  $\mathbb{R}^4$ . We assume that  $g$  does depend on its fourth argument; more precisely, associating to  $g$  a map  $G : \Omega \rightarrow \mathbb{R}^4$  defined by

$$G(x, y, z, \lambda) = (x, y, z, g(x, y, z, \lambda)),$$

and denoting by  $g_4$  the partial derivative of  $g$  with respect to its fourth argument,

$$g_4 \text{ does not vanish on } \Omega \quad \text{and} \quad G \text{ defines a diffeomorphism } \Omega \rightarrow G(\Omega). \quad (4)$$

We denote by  $\widehat{\Omega}$  the open connected subset of  $\mathbb{R}^5$  defined from  $\Omega$  by :

$$(x, y, z, \dot{x}, \dot{y}) \in \widehat{\Omega} \Leftrightarrow (x, y, z, \dot{y} - z\dot{x}) \in \Omega. \quad (5)$$

From  $g$  and  $h$  one may define  $\gamma$  and  $\delta$ , two real analytic functions  $G(\Omega) \rightarrow \mathbb{R}$ , such that  $G^{-1}(x, y, z, w) = (x, y, z, \gamma(x, y, z, w))$  and  $\delta = h \circ G^{-1}$ , *i.e.*

$$w = g(x, y, z, \lambda) \Leftrightarrow \lambda = \gamma(x, y, z, w), \quad (6)$$

$$h(x, y, z, \lambda) = \delta(x, y, z, g(x, y, z, \lambda)), \quad \text{i.e.} \quad \delta(x, y, z, w) = h(x, y, z, \gamma(x, y, z, w)) \quad (7)$$

Then, one may associate to (3) the control-affine system (1) in  $\mathbb{R}^4$  with two controls, that can also be written as (2); our interest however focuses on system (3) defined by  $g$  and  $h$  as above. Let us set some conventions :

**The functions  $\gamma$  and  $\delta$ :** when using the notations  $\gamma$  and  $\delta$ , *it is not assumed that they are related to  $g$  and  $h$*  by (6) and (7), unless this is explicitly stated.

**Notations for the derivatives:** We denote partial derivatives by subscript indexes. For functions of many variables, like  $\varphi(u, \dots, u^{(k)}, v, \dots, v^{(\ell)})$  in (10), we use the name of the variable as a subscript :  $\varphi_{v^{(\ell)}}$  means  $\partial\varphi/\partial v^{(\ell)}$ ;  $p_{xu^{(k-1)}}$  means  $\partial^2 p/\partial x \partial u^{(k-1)}$  in (16-b). Since the arguments of  $g$ ,  $h$ ,  $\gamma$ ,  $\delta$  and a few other functions will sometimes be intricate functions of other variables, we use numeric subscripts for their partial derivatives :  $h_2$  stands for  $\partial h/\partial y$ , or  $g_{4,4,4}$  for  $\partial^3 g/\partial \lambda^3$ . To avoid confusions, we will not use numeric subscripts for other purposes than partial derivatives, except the subscript 0, as in  $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0)$  for a reference point.

For derivatives with respect to time, we use the standard with the dot and  $^{(j)}$  for the  $j^{\text{th}}$  time-derivatives; sometimes  $d/dt$ .

The following elementary lemma —we do write it for the argument is used repeatedly throughout the paper— states that no differential equation independent from (3) can be satisfied identically by *all* solutions of (3).

**Lemma 2.1.** *For  $M \in \mathbb{N}$ , let  $W$  be an open subset of  $\mathbb{R}^{3+2M}$  and  $R : W \rightarrow \mathbb{R}$  be a smooth function. If any solution  $(x(\cdot), y(\cdot), z(\cdot))$  of system (3), defined on some time-interval  $I = ]-\varepsilon, \varepsilon[$  and such that  $(z(t), x(t), \dots, x^{(M)}(t), y(t), \dots, y^{(M)}(t)) \in W$  for all  $t \in I$  satisfies*

$$R(z(t), y(t), \dots, y^{(M)}(t), x(t), \dots, x^{(M)}(t)) = 0,$$

for all  $t \in I$ , then  $R$  is identically zero.

*Proof.* For any  $\mathcal{X} \in W$  there is a germ of solution of (3) such that  $(z(0), x(0), \dots, x^{(M)}(0), y(0), \dots, y^{(M)}(0)) = \mathcal{X}$ . Indeed, take e.g. for  $x(\cdot)$  and  $y(\cdot)$  the polynomials in  $t$  of degree  $M$  that have these derivatives at time zero; Cauchy-Lipschitz theorem then yields a (unique)  $z(\cdot)$  solution of (3) with the prescribed  $z(0)$ .  $\square$

**2.2. The notion of parameterization.** In order to give somehow rigorous definitions without taking care of time-intervals of definition of the solutions, we consider germs of solutions at time 0, instead of solutions themselves. For  $O$  an open subset of  $\mathbb{R}^n$ , the notation  $\mathcal{C}_0^\infty(\mathbb{R}, O)$  stands for the set of germs at  $t = 0$  of smooth functions of one variable with values in  $O$ , see e.g. [11].

Let  $k, \ell, L$  be some non negative integers,  $U$  an open subset of  $\mathbb{R}^{k+\ell+2}$  and  $V$  an open subset of  $\mathbb{R}^{2L+3}$ . We denote by  $\mathcal{U} \subset \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^2)$  (resp.  $\mathcal{V} \subset \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^3)$ ) the set of germs of smooth functions  $t \mapsto (u(t), v(t))$  (resp.  $t \mapsto (x(t), y(t), z(t))$ ) such that their jets at  $t = 0$  to the order precised below are in  $U$  (resp. in  $V$ ) :

$$\mathcal{U} = \{(u, v) \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^2) | (u(0), \dot{u}(0), \dots, u^{(k)}(0), v(0), \dots, v^{(\ell)}(0)) \in U\}, \quad (8)$$

$$\mathcal{V} = \{(x, y, z) \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^3) | (x(0), y(0), z(0), \dot{x}(0), \dot{y}(0), \dots, x^{(L)}(0), y^{(L)}(0)) \in V\}. \quad (9)$$

These are open sets for the Whitney  $\mathcal{C}^\infty$  topology [11, p. 42].

**Definition 2.2** (Monge parameterization). Let  $k, \ell, L$  be non negative integers,  $L > 0$ ,  $k \leq \ell$ , and  $\mathcal{X} = (x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dots, x_0^{(L)}, y_0^{(L)})$  be in  $\widehat{\Omega} \times \mathbb{R}^{2L-2}$  ( $\widehat{\Omega}$  is defined in (5)). A *parameterization of order  $(k, \ell)$  at  $\mathcal{X}$*  for system (3) is defined by

- a neighborhood  $V$  of  $\mathcal{X}$  in  $\widehat{\Omega} \times \mathbb{R}^{2L-2}$ ,
- an open subset  $U \subset \mathbb{R}^{k+\ell+2}$  and
- three real analytic functions  $U \rightarrow \mathbb{R}$ , denoted  $\varphi, \psi, \chi$ ,

such that, with  $\mathcal{U}$  and  $\mathcal{V}$  defined from  $U$  and  $V$  according to (8)-(9), and  $\Gamma : \mathcal{U} \rightarrow \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^3)$  the map that assigns to  $(u, v) \in \mathcal{U}$  the germ  $\Gamma(u, v)$  at  $t = 0$  of

$$t \mapsto \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \varphi(u(t), \dot{u}(t), \dots, u^{(k)}(t), v(t), \dot{v}(t), \dots, v^{(\ell)}(t)) \\ \psi(u(t), \dot{u}(t), \dots, u^{(k)}(t), v(t), \dot{v}(t), \dots, v^{(\ell)}(t)) \\ \chi(u(t), \dot{u}(t), \dots, u^{(k)}(t), v(t), \dot{v}(t), \dots, v^{(\ell)}(t)) \end{pmatrix}, \quad (10)$$

the following three properties hold :

- (1) for all  $(u, v)$  belonging to  $\mathcal{U}$ ,  $\Gamma(u, v)$  is a solution of system (3),
- (2) the map  $\Gamma$  is open and  $\Gamma(\mathcal{U}) \supset \mathcal{V}$ ,
- (3) the two maps  $U \rightarrow \mathbb{R}^3$  defined by the triples  $(\varphi_{u^{(k)}}, \psi_{u^{(k)}}, \chi_{u^{(k)}})$  and  $(\varphi_{v^{(\ell)}}, \psi_{v^{(\ell)}}, \chi_{v^{(\ell)}})$  are identically zero on no open subset of  $U$ .

**Remark 2.3** (On ordering the pairs  $(k, \ell)$ ). Since  $u$  and  $v$  play a symmetric role, they can always be exchanged, and there is no lack of generality in assuming  $k \leq \ell$ . This convention is useful only when giving bounds on  $(k, \ell)$ . For instance,  $k \geq 2$  means that both integers are no smaller than 2.

**Example 2.4.** Suppose that the function  $\gamma$  in (2) depends on  $x, y, z$  only (this is treated in [18, case 6 in Theorem 3.1]). For such systems, eliminating  $w$  does not lead to (3), but to the simpler relation  $\dot{y} - z\dot{x} = \gamma(x, y, z)$ . One can easily adapt the above definition replacing (3) by this relation. This system  $\dot{y} - z\dot{x} = \gamma(x, y, z)$  admits a parameterization of order (1,1) at any  $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0)$  such that  $\dot{x}_0 + \gamma_3(x_0, y_0, z_0) \neq 0$ .

*Proof.* In a neighborhood of such a point,, the map  $(x, \dot{x}, y, z) \mapsto (x, \dot{x}, y, \gamma(x, y, z) + z\dot{x})$  is a local diffeomorphism, whose inverse can be written as  $(x, \dot{x}, y, \dot{y}) \mapsto (x, \dot{x}, y, \chi(x, \dot{x}, y, \dot{y}))$ , thus defining a map  $\chi$ . Then  $x = u, y = v, z = \chi(u, \dot{u}, v, \dot{v})$  defines a parameterization of order (1,1) in a neighborhood of these points.

**Remark 2.5.** The integer  $L$  characterizes the number of derivatives needed to describe the open set in which the parameterization is valid. This is illustrated in the Example 2.4, where  $L$  must be taken at least equal to 1. Obviously, a parameterization of order  $(k, \ell)$  at  $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dots, x_0^{(L)}, y_0^{(L)})$  is also, for  $K > L$  and any value of  $(x_0^{(L+1)}, y_0^{(L+1)}, \dots, x_0^{(K)}, y_0^{(K)})$ , a parameterization of the same order at  $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dots, x_0^{(K)}, y_0^{(K)})$ .

The above definition is local around some jet of solutions of (3). In general, the idea of a global parameterization, meaning that  $\Gamma$  would be defined globally, is not realistic; it is not realistic either to require that there exists a parameterization around all jets (this would be “everywhere local” rather than “global”) : the systems in example 2.4 admit a local parameterization around “almost every” jets, meaning jets outside the zeroes of a real analytic function (namely jets such that  $\dot{x} + \gamma_3(x, y, z) \neq 0$ ). We shall not define more precisely the notion of “almost everywhere local” parameterizability, but rather the following (sloppier) one.

**Definition 2.6.** We say that system (3) admits a parameterization of order  $(k, \ell)$  somewhere in  $\Omega$  if there exists an integer  $L$ , and at least one jet  $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dots, x_0^{(L)}, y_0^{(L)}) \in \widehat{\Omega} \times \mathbb{R}^{2L-2}$  with a parameterization of order  $(k, \ell)$  at this jet in the sense of Definition 2.2

In a colloquial way this is a “somewhere local” property. Using real analyticity, “somewhere local” should imply “almost everywhere local”, but we do not investigate this.

**2.3. The functions  $S, T$  and  $J$ .** Given  $g, h$ , let us define three functions  $S, T$  and  $J$ , to be used to discriminate different cases. They were already more or less

present in [18]. The most compact way is as follows : let  $\omega$ ,  $\omega^1$  and  $\eta$  be the following differential forms in the variables  $x, y, z, \lambda$  :

$$\begin{aligned}\omega^1 &= dy - z dx \\ \omega &= -2g_4^2 dx + (g_{4,4}h_4 - g_4 h_{4,4})\omega^1 - g_{4,4}(dz - g dx) , \\ \eta &= dz - g dx - h_4 \omega^1 .\end{aligned}\tag{11}$$

From (4),  $\omega \wedge \omega^1 \wedge \eta = 2g_4^2 dx \wedge dy \wedge dz \neq 0$ . Decompose  $d\omega \wedge \omega$  on the basis  $\omega, \omega^1, \eta, d\lambda$ , thus defining the functions  $S, T$  and  $J$  :

$$d\omega \wedge \omega = - \left( \frac{S}{2g_4} d\lambda \wedge \eta + \frac{T}{2} d\lambda \wedge \omega^1 + J \omega^1 \wedge \eta \right) \wedge \omega .\tag{12}$$

The expression of  $J$  is complicated, but  $S$  and  $T$  are the following (note that  $S/(g_4)^2$  is the Schwartzian derivative of  $g$  with respect to its fourth argument) :

$$S = 2g_4 g_{4,4,4,4} - 3g_{4,4}^2 , \quad T = 2g_4 h_{4,4,4,4} - 3g_{4,4} h_{4,4} .\tag{13}$$

**2.4. Contributions of the paper.** Our (unachieved) goal is to give necessary and sufficient conditions on  $g$  and  $h$  for system (3) to admit a parameterization of some order.

If  $S = T = J = 0$ , system (3) admits a parameterization of order (1,2), at all points except some singularities. This is Theorem 4.2, adapted from [18].

If  $(S, T, J) \neq (0, 0, 0)$ , we conjecture that system (3) admits no parameterization of any order. We are not able to prove this, but (recall that  $k \leq \ell$ , see Remark 2.3) :

- we prove (Theorem 5.4) that system (3) admits no parameterization of order  $(k, \ell)$  with  $k \leq 2$  or  $k = \ell = 3$ ,
- we prove (Theorem 5.1) that a parameterization of order  $(k, \ell)$  is always related to a solution of a system of PDEs  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ ,
- since a solution of this system of PDEs is also sufficient to construct a parameterization (Theorem 3.7), the conjecture can be entirely re-formulated in terms of this (very compact) system of partial differential relations.
- *Comparison with the results in [18] on  $(x, u)$ -flatness.* To the best of our knowledge, the most general result available up to date on the class of systems studied here is in [18] : the control system (1) associated to  $g$  and  $h$  is “ $(x, u)$ -dynamic linearizable” if and only if  $S = T = J = 0$ . See section 7 for the relation between dynamic linearization or flatness and Monge parameterizations. It turns out (Proposition 7.4) that “ $(x, u)$ -dynamic linearizability” in the sense of [18] implies existence of a parameterization of order  $(k, \ell)$  with  $k$  and  $\ell$  no larger than 3; hence Theorem 5.4, recalled above, allows us to recover all the results from that paper.

The paper is organized as follows. Section 3 introduces this partial differential system  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$  and proves that a “regular solution” of this system induces a parameterization of order  $(k, \ell)$ . Section 4 is devoted to some special constructions for the case where  $S = T = 0$ . The main results are stated in Section 5; necessary conditions are stated locally in a more detailed way, and proved, in Section 6. Section 3.3 is devoted to proving that the partial differential system  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$  cannot have regular solutions unless  $k \geq 3$  and  $\ell \geq 4$ . Finally, Section 7 says a word of how our results translates to flatness instead of parameterization.

### 3. A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

**3.1. The equation  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ , regular solutions.** For  $k$  and  $\ell$  some positive integers, we define a partial differential system in  $k + \ell + 1$  independent variables

and one dependent variable, *i.e.* the unknown is one function of  $k + \ell + 1$  variables. The dependent variable is denoted by  $p$  and the independent variables by  $u, \dot{u}, \dots, u^{(k-1)}, x, v, \dot{v}, \dots, v^{(\ell-1)}$ . Note that, although the name of the variables may suggest some “time-derivatives”, time is not a variable here and there is no relation between the independent variables  $v$  and  $\dot{v}$  for instance.

Denote by  $F$  the following differential operator of order 1 (vector field) in  $\mathbb{R}^{k+\ell+1}$ , with coordinates the  $k + \ell + 1$  independent variables mentioned above :

$$F = \sum_{i=0}^{k-2} u^{(i+1)} \frac{\partial}{\partial u^{(i)}} + \sum_{i=0}^{\ell-2} v^{(i+1)} \frac{\partial}{\partial v^{(i)}} \quad (14)$$

where the first sum is zero if  $k \leq 1$  and the second one is zero if  $\ell \leq 1$ .

Let  $\tilde{\Omega}$  be an open connected subset of  $\mathbb{R}^4$  and  $\gamma, \delta$  two real analytic functions  $\tilde{\Omega} \rightarrow \mathbb{R}$  such that

$$\gamma_4 \neq 0 \quad (15)$$

(partial derivative of  $\gamma$  with respect to its 4<sup>th</sup> argument, see end of section 2.1) at every point in  $\tilde{\Omega}$ . Consider the system of two partial differential equations and three inequations :

$$(\mathcal{E}_{k,\ell}^{\gamma,\delta}) \begin{cases} p_{u^{(k-1)}} (Fp_x - \delta(x, p, p_x, p_{xx})) - p_{xu^{(k-1)}} (Fp - \gamma(x, p, p_x, p_{xx})) = 0, & (a) \\ p_{u^{(k-1)}} p_{xv^{(\ell-1)}} - p_{xu^{(k-1)}} p_{v^{(\ell-1)}} = 0, & (b) \\ p_{u^{(k-1)}} \neq 0, & (c) \\ p_{v^{(\ell-1)}} \neq 0, & (d) \\ \gamma_1 + \gamma_2 p_x + \gamma_3 p_{xx} + \gamma_4 p_{xxx} - \delta \neq 0. & (e) \end{cases} \quad (16)$$

To any  $p$  satisfying these equations and inequations, we may associate two functions  $\sigma$  and  $\tau$ , and a vector field  $E$  defined as follows :

$$\sigma = -\frac{p_{v^{(\ell-1)}}}{p_{u^{(k-1)}}}, \quad \tau = \frac{-Fp + \gamma(x, p, p_x, p_{xx})}{p_{u^{(k-1)}}}, \quad E = \sigma \frac{\partial}{\partial u^{(k-1)}} + \frac{\partial}{\partial v^{(\ell-1)}}. \quad (17)$$

We also introduce the differential operator  $D$  (see Remark 3.2 on the additional variables  $\dot{x}, \dots, x^{(k+\ell-1)}$ ) :

$$D = F + \tau \frac{\partial}{\partial u^{(k-1)}} + \sum_{i=0}^{k+\ell-2} x^{(i+1)} \frac{\partial}{\partial x^{(i)}}. \quad (18)$$

**Definition 3.1** (Regular solutions of  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ ). A *regular* solution of system  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$  is a real analytic function  $p : O \rightarrow \mathbb{R}$ , with  $O$  a connected open subset of  $\mathbb{R}^{k+\ell+1}$ , such that the image of  $O$  by  $(x, p, p_x, p_{xx})$  is contained in  $\tilde{\Omega}$ , (16-a,b) are identically satisfied on  $O$ , the left-hand sides of (16-c,d,e) are not identically zero, and, for at least one integer  $K \in \{1, \dots, k + \ell - 2\}$ ,

$$ED^K p \neq 0 \quad (19)$$

(not identically zero, as a function of  $u, \dots, u^{(k-1)}, x, v, \dots, v^{(\ell-1)}, \dot{x}, \dots, x^{(K)}$  on  $O \times \mathbb{R}^K$ ).

We call it *K-regular* if  $K$  is the smallest such integer, *i.e.* if  $ED^i p = 0$  for all  $i \leq K - 1$ .

**Remark 3.2** (on the additional variables  $\dot{x}, \dots, x^{(k+\ell-1)}$  in  $D$ ). These variables appear in the expression (18). Note that  $D$  is only applied (recursively) to functions of  $u, \dots, u^{(k-1)}, x, v, \dots, v^{(\ell-1)}$  only; hence we view it as a vector field in  $\mathbb{R}^{k+\ell+1}$  with these variables as parameters. In fact,  $D$  is only used in  $ED^i p$ ,  $1 \leq i \leq k + \ell - 1$ . This is a polynomial with respect to the variables  $\dot{x}, \ddot{x}, \dots, x^{(i)}$  with coefficients depending on  $u, \dots, u^{(k-1)}, x, v, \dots, v^{(\ell-1)}$  via the functions  $p, \gamma, \delta$  and their partial derivatives. Hence  $ED^i p = 0$  means that all these coefficients are

zero, *i.e.* it encodes a collection of differential relations on  $p$ , where the spurious variables  $\dot{x}, \ddot{x}, \dots, x^{(i)}$  no longer appear. Likewise,  $ED^i p \neq 0$  means that one of these relations is not satisfied.

**Definition 3.3.** We say that system  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$  admits a regular (resp.  $K$ -regular) solution somewhere in  $\widehat{\Omega}$  if there exists at least an open connected  $O \subset \mathbb{R}^{k+\ell+1}$  and a regular (resp.  $K$ -regular) solution  $p : O \rightarrow \mathbb{R}$ .

**Remark 3.4.** It is easily seen that  $p$  is solution of  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$  if and only if there exists  $\sigma$  and  $\tau$  such that  $(p, \sigma, \tau)$  is a solution of

$$\begin{aligned} Fp + \tau p_{u^{(k-1)}} &= \gamma(x, p, p_x, p_{xx}) & Ep &= 0, \quad \sigma_x = 0, \\ Fp_x + \tau p_{x,u^{(k-1)}} &= \delta(x, p, p_x, p_{xx}) & p_{u^{(k-1)}} &\neq 0, \quad \tau_x \neq 0, \quad \sigma \neq 0 \end{aligned} \quad (20)$$

Indeed, (16) does imply the above relations with  $\sigma$  and  $\tau$  given by (17); in particular,  $\tau_x \neq 0$  is equivalent to (e) and  $\sigma \neq 0$  to (d); conversely, eliminating  $\sigma$  and  $\tau$  in (20), one recovers  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ . Note also that, with  $g$  and  $h$  related to  $\gamma$  and  $\delta$  by (6) and (7), any solution of the above equations and inequations satisfies

$$Dp_x = h(x, p, p_x, Dp - p_x \dot{x}) + g(x, p, p_x, Dp - p_x \dot{x}) \dot{x}. \quad (21)$$

The following will be used repeatedly in the paper :

**Lemma 3.5.** If  $p$  is a solution of system  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$  and

- (1) either it satisfies a relation of the type  $p_x = \alpha(x, p)$  with  $\alpha$  a function of two variables,
- (2) or it satisfies a relation of the type  $p_{xx} = \alpha(x, p, p_x)$  with  $\alpha$  a function of three variables,
- (3) or it satisfies two relations of the type  $p_{xxx} = \alpha(x, p, p_x, p_{xx})$  and  $Fp_{xx} + \tau p_{xxu^{(k-1)}} = \psi(x, p, p_x, p_{xx})$ , with  $\psi$  and  $\alpha$  two functions of four variables,

then it satisfies  $ED^i p = 0$  for all  $i \geq 0$  and hence is not a regular solution of  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ .

*Proof.* Point 1 implies point 2 because differentiating the relation  $p_x = \alpha(x, p)$  with respect to  $x$  yields  $p_{xx} = \alpha_x(x, p) + p_x \alpha_p(x, p)$ . Likewise, point 2 implies point 3 : differentiating the relation  $p_{x,x} = \alpha(x, p, p_x)$  with respect to  $x$  yields  $p_{xxx} = \alpha_x(x, p, p_x) + p_x \alpha_p(x, p, p_x) + p_{xx} \alpha_{p_x}(x, p, p_x)$  while differentiating it along the vector field  $F + \tau \partial/\partial u^{(k-1)}$  and using (20) yields  $Fp_{xx} + \tau p_{xxu^{(k-1)}} = \gamma(x, p, p_x, p_{xx}) \alpha_p(x, p, p_x) + \delta(x, p, p_x, p_{xx}) \alpha_{p_x}(x, p, p_x)$ .

Let us prove that point 3 implies  $ED^i p = ED^i p_x = ED^i p_{xx} = 0$  for all  $i \geq 0$ , hence the lemma. It is indeed true for  $i = 0$  and the following three relations

$$\begin{aligned} Dp &= \gamma(x, p, p_x, p_{xx}) + \dot{x} p_x, & Dp_x &= \delta(x, p, p_x, p_{xx}) + \dot{x} p_{xx}, \\ Dp_{xx} &= \psi(x, p, p_x, p_{xx}) + \dot{x} \alpha(x, p, p_x, p_{xx}), \end{aligned}$$

that are implied by (18), (20) and the two relations in point 3 allow one to go from  $i$  to  $i + 1$  ( $ED^i x = Ex^{(i)} = 0$  and  $ED^i \dot{x} = Ex^{(i+1)} = 0$  from the very definition of  $D$  and  $E$ ).  $\square$

**3.2. The relation with Monge parameterizations.** Let us now explain how a Monge parameterization for system (3) can be deduced from a regular solution  $p : O \rightarrow \mathbb{R}$  of  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ . This may seem anecdotic but it is not, for we shall prove (*cf.* sections 5 and 6) that all Monge parameterizations are of this type, except when the system (3) belongs to a special well-known class.

We saw in Remark 3.4 that (16-e) is equivalent to  $\tau_x \neq 0$ ; let  $(u_0 \dots v_0^{(\ell-1)})$  be a point in  $O$  such that

$$\tau_x(u_0, \dots, u_0^{(k-1)}, x_0, v_0, \dots, v_0^{(\ell-1)}) \neq 0. \quad (22)$$



Chose any pair  $(u_0^{(k)}, v_0^{(\ell)}) \in \mathbb{R}^2$  (for instance with  $v_0^{(\ell)} = 0$ ) such that

$$u_0^{(k)} - \sigma(u_0, \dots, u_0^{(k-1)}, v_0, \dots, v_0^{(\ell-1)}) v_0^{(\ell)} = \tau(u_0, \dots, u_0^{(k-1)}, x_0, v_0, \dots, v_0^{(\ell-1)}) . \quad (23)$$

Then, the implicit function theorem provides a neighborhood  $V$  of  $(u_0, \dots, u_0^{(k)}, v_0, \dots, v_0^{(\ell)})$  in  $\mathbb{R}^{k+\ell+2}$  and a real analytic map  $\varphi : V \rightarrow \mathbb{R}$  such that  $\varphi(u_0, \dots, u_0^{(k)}, v_0, \dots, v_0^{(\ell)}) = x_0$  and

$$\begin{aligned} \tau(u, \dots, u^{(k-1)}, \varphi(u \cdots v^{(\ell)}), v, \dots, v^{(\ell-1)}) \\ = u^{(k)} - \sigma(u, \dots, u^{(k-1)}, v, \dots, v^{(\ell-1)}) v^{(\ell)} \end{aligned} \quad (24)$$

identically on  $V$ . Two other maps  $V \rightarrow \mathbb{R}$  may be defined by

$$\psi(u, \dots, u^{(k)}, v, \dots, v^{(\ell)}) = p(u, \dots, u^{(k-1)}, \varphi(\cdots), v, \dots, v^{(\ell-1)}), \quad (25)$$

$$\chi(u, \dots, u^{(k)}, v, \dots, v^{(\ell)}) = p_x(u, \dots, u^{(k-1)}, \varphi(\cdots), v, \dots, v^{(\ell-1)}). \quad (26)$$

From these  $\varphi$ ,  $\psi$  and  $\chi$ , one can define a map  $\Gamma$  as in (10) that is a candidate for a parameterization. We prove below that, if  $p$  is a regular solution of  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ , then this  $\Gamma$  is indeed a parameterization, at least away from some singularities. The following lemma describes these singularities; it is proved in Appendix A.

**Lemma 3.6.** *Let  $O$  be an open connected subset of  $\mathbb{R}^{k+\ell+1}$  and  $p : O \rightarrow \mathbb{R}$  be a  $K$ -regular solution of system  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ , see (16). Define the map  $\pi : O \times \mathbb{R}^K \rightarrow \mathbb{R}^{K+2}$  by*

$$\pi(u \cdots u^{(k-1)}, x, v \cdots v^{(\ell-1)}, \dot{x} \cdots x^{(K)}) = \begin{pmatrix} p_x(u \cdots u^{(k-1)}, x, v \cdots v^{(\ell-1)}) \\ p(u \cdots u^{(k-1)}, x, v \cdots v^{(\ell-1)}) \\ Dp(u \cdots u^{(k-1)}, x, v \cdots v^{(\ell-1)}, \dot{x}) \\ \vdots \\ D^K p(u \cdots u^{(k-1)}, x, v \cdots v^{(\ell-1)}, \dot{x} \cdots x^{(K)}) \end{pmatrix}. \quad (27)$$

There exist two non-negative integers  $i_0 \leq k$  and  $j_0 \leq \ell$  such that  $i_0 + j_0 = K + 2$  and

$$\det \left( \frac{\partial \pi}{\partial u^{(k-i_0)}}, \dots, \frac{\partial \pi}{\partial u^{(k-1)}}, \frac{\partial \pi}{\partial v^{(\ell-j_0)}}, \dots, \frac{\partial \pi}{\partial v^{(\ell-1)}} \right) \quad (28)$$

is a nonzero real analytic function on  $O \times \mathbb{R}^K$ .

We can now state precisely the announced sufficient condition. Its interest is discussed in Remark 5.3.

**Theorem 3.7.** *Let  $p : O \rightarrow \mathbb{R}$ , with  $O \subset \mathbb{R}^{k+\ell+1}$  open, be a  $K$ -regular solution of system  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ , and  $i_0, j_0$  be given by Lemma 3.6. Then, the maps  $\varphi, \psi, \chi$  constructed above define a parameterization  $\Gamma$  of system (3) of order  $(k, \ell)$  (see Definition 2.2) at any jet of solutions  $(x_0, y_0, z_0, \dot{x}_0, \dots, x_0^{(K)}, \dot{y}_0, \dots, y_0^{(K)})$  such that, for some  $u_0, \dots, u_0^{(k-1)}, v_0, \dots, v_0^{(\ell-1)}$ ,*

$$\left. \begin{aligned} (u_0, \dots, u_0^{(k-1)}, x_0, v_0, \dots, v_0^{(\ell-1)}) &\in O, \\ z_0 &= p_x(u_0, \dots, u_0^{(k-1)}, v_0, \dots, v_0^{(\ell-1)}, x_0), \\ y_0^{(i)} &= D^i p(u_0, \dots, u_0^{(k-1)}, v_0, \dots, v_0^{(\ell-1)}, x_0, \dots, x_0^{(i)}) \quad 0 \leq i \leq K, \end{aligned} \right\} \quad (29)$$

the left-hand sides of (16-c,d,e) are all nonzero at  $(u_0, \dots, u_0^{(k-1)}, x_0, v_0, \dots, v_0^{(\ell-1)})$ , and the function  $ED^K p$  and the determinant (28) are nonzero at point  $(u_0, \dots, u_0^{(k-1)}, x_0, \dots, x_0^{(K)}, v_0, \dots, v_0^{(\ell-1)}) \in O \times \mathbb{R}^K$ .

*Proof.* Let us prove that  $\Gamma$  given by (10), with the maps  $\varphi, \psi, \chi$  constructed above, satisfies the three points of Definition 2.2. Differentiating (24) with respect to  $u^{(k)}$  and  $v^{(\ell)}$  yields  $\varphi_{u^{(k)}} \tau_x = 1$ ,  $\varphi_{v^{(\ell)}} \tau_x = -\sigma$ , hence the point 3 ( $\sigma \neq 0$  from (20)). To prove point 1, let  $u(\cdot), v(\cdot)$  be arbitrary and  $x(\cdot), y(\cdot), z(\cdot)$  be defined by (10). Differentiating (25) and (26), taking  $u^{(k)}$  from (24), one has, with  $F$  given by (14),

$$\dot{y}(t) = Fp + v^{(\ell)}(t)Ep + \dot{x}(t)z(t), \quad \dot{z}(t) = Fp_x + v^{(\ell)}(t)Ep_x + \dot{x}(t)p_{xx}$$

where the argument  $(u(t) \dots u^{(k-1)}(t), x(t), v(t) \dots v^{(\ell-1)}(t))$  for  $Fp, Fp_x, Ep, Ep_x$  and  $p_{xx}$  is omitted. Then, (20) implies, again omitting the arguments of  $p_{xx}$ , one has  $\dot{y}(t) = \gamma(x(t), y(t), z(t), p_{xx}) + z(t)\dot{x}(t)$ , and  $\dot{z}(t) = \delta(x(t), y(t), z(t), p_{xx}) + p_{xx}\dot{x}(t)$ . The first equation yields  $p_{xx} = g(x(t), y(t), z(t), \dot{y}(t) - z(t)\dot{x}(t))$  with  $g$  related to  $\gamma$  by (6), and then the second one yields (3), with  $h$  related to  $\delta$  by (7). This proves point 1. The rest of the proof is devoted to point 2.

Let  $t \mapsto (x(t), y(t), z(t))$  be a solution of (3). We may consider  $\Gamma(u, v) = (x, y, z)$  (see (10)) as a system of three ordinary differential equations in two unknown functions  $u, v$  :

$$u^{(k)} - \sigma(u, \dots, u^{(k-1)}, v, \dots, v^{(\ell-1)})v^{(\ell)} - \tau(u, \dots, u^{(k-1)}, x, v, \dots, v^{(\ell-1)}) = 0, \quad (30)$$

$$p(u, \dots, u^{(k-1)}, x, v, \dots, v^{(\ell-1)}) = y, \quad (31)$$

$$p_x(u, \dots, u^{(k-1)}, x, v, \dots, v^{(\ell-1)}) = z. \quad (32)$$

Differentiating (31)  $K+1$  times, substituting  $u^{(k)}$  from (30), and using the fact that  $ED^i p = 0$  for  $i \leq K$  (see Definition 3.1), we get

$$D^i p(u(t), \dots, u^{(k-1)}(t), v(t), \dots, v^{(\ell-1)}(t), x(t), \dots, x^{(i)}(t)) = \frac{d^i y}{dt^i}(t) \quad (33)$$

for  $i$  such that  $1 \leq i \leq K$ , and

$$v^{(\ell)}(t) ED^K p(u(t), \dots, u^{(k-1)}(t), v(t), \dots, v^{(\ell-1)}(t), x(t), \dots, x^{(K)}(t)) + D^{K+1} p(u(t), \dots, u^{(k-1)}(t), v(t), \dots, v^{(\ell-1)}(t), x(t), \dots, x^{(K+1)}(t)) = \frac{d^{K+1} y}{dt^{K+1}}(t). \quad (34)$$

Equations (31)-(32)-(33) can be written

$$\pi(u, \dots, u^{(k-1)}, x, v, \dots, v^{(\ell-1)}, \dot{x}, \dots, x^{(K)}) = \begin{pmatrix} z \\ y \\ \dot{y} \\ \vdots \\ y^{(K)} \end{pmatrix} \quad (35)$$

with  $\pi$  given by (27). Since the determinant (28) is nonzero, the implicit function theorem implies that (31)-(32)-(33) is, locally, equivalent to a system giving explicitly  $u^{(k-i_0)}, \dots, u^{(k-1)}, v^{(\ell-j_0)}, \dots, v^{(\ell-1)}$  as functions of  $u, \dots, u^{(k-i_0-1)}, v, \dots, v^{(\ell-j_0-1)}, x, \dots, x^{(K)}, y, \dots, y^{(K)}$  and  $z$ . Let us single out these giving the lowest order derivatives :

$$\begin{aligned} u^{(k-i_0)} &= f^1(u, \dots, u^{(k-1-i_0-1)}, v, \dots, v^{(\ell-j_0-1)}, x, \dots, x^{(K)}, z, y, \dots, y^{(K)}), \\ v^{(\ell-j_0)} &= f^2(u, \dots, u^{(k-1-i_0-1)}, v, \dots, v^{(\ell-j_0-1)}, x, \dots, x^{(K)}, z, y, \dots, y^{(K)}). \end{aligned} \quad (36)$$

Let us prove that, provided that  $(x, y, z)$  is a solution of (3), system (36) is equivalent to (30)-(31)-(32), *i.e.* to  $\Gamma(u, v) = (x, y, z)$ . It is obvious that any  $t \mapsto (u(t), v(t), x(t), y(t), z(t))$  that satisfies (3), (30), (31) and (32) also satisfies (36), because these equations were obtained from consequences of those. Conversely, let  $t \mapsto (u(t), v(t), x(t), y(t), z(t))$  be such that (3) and (36) are satisfied; differentiating

(36) and substituting each time  $\dot{z}$  from (3) and  $(u^{(k-i_0)}, v^{(\ell-j_0)})$  from (36), one obtains, for all  $i, i$  in  $\mathbb{N}$ ,

$$\begin{aligned} u^{(k-i_0+i)} &= f^{1,i}(u, \dots, u^{(k-1-i_0-1)}, v, \dots, v^{(\ell-j_0-1)}, x, \dots, x^{(K+i)}, z, y, \dots, y^{(K+i)}), \\ v^{(\ell-j_0+j)} &= f^{2,j}(u, \dots, u^{(k-1-i_0-1)}, v, \dots, v^{(\ell-j_0-1)}, x, \dots, x^{(K+j)}, z, y, \dots, y^{(K+j)}). \end{aligned} \quad (37)$$

Now, substitute the values of  $u^{(k-i_0)}, \dots, u^{(k)}, v^{(\ell-j_0)}, \dots, v^{(\ell)}$  from (37) into (30), (31) and (32); either the obtained relations are identically satisfied, and hence it is true that any solution of (3) and (36) also satisfies (30)-(31)-(32), or one obtains at least one relation of the form (we assume  $k \leq \ell$ ; if not put  $K+k$  instead of  $K+\ell$ ):

$$R(u, \dots, u^{(k-1-i_0-1)}, v, \dots, v^{(\ell-j_0-1)}, x, \dots, x^{(K+\ell)}, z, y, \dots, y^{(K+\ell)}) = 0.$$

This relation has been obtained (indirectly) by differentiating and combining (3)-(30)-(31)-(32). This is absurd because (30)-(31)-(32)-(33)-(34) are the only independent relations of order  $k, \ell$  obtained by differentiating and combining<sup>1</sup> (30)-(31)-(32) because, on the one hand, since  $D^K p \neq 0$ , differentiating more (34) and (30) will produce higher order differential equations in which higher order derivatives cannot be eliminated, and on the other hand, differentiating (32) and substituting  $\dot{z}$  from (3),  $u^{(k)}$  from (30) and  $\dot{y}$  from (33) for  $i = 1$  yields the trivial  $0 = 0$  because  $p$  is a solution of  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ , see the proof of point 1 above.

We have now established that, for  $(x, y, z)$  a solution of (3),  $\Gamma(u, v) = (x, y, z)$  is equivalent to (36). Using Cauchy Lipschitz theorem with continuous dependence on the parameters, one can define a continuous map  $s : \mathcal{V} \rightarrow \mathcal{U}$  mapping a germ  $(x, y, z)$  to the unique germ of solution of (36) with fixed initial condition  $(u, \dots, u^{(k-i_0-1)}, v, \dots, v^{(\ell-j_0-1)}) = (u_0, \dots, u_0^{(k-i_0-1)}, v_0, \dots, v_0^{(\ell-j_0-1)})$ . Then  $s$  is a continuous right inverse of  $\Gamma$ , *i.e.*  $\Gamma \circ s = Id$ . This proves point 2.  $\square$

### 3.3. On (non-)existence of regular solutions of system $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ .

**Conjecture 3.8.** For any real analytic functions  $\gamma$  and  $\delta$  (with  $\gamma_4 \neq 0$ ), and any integers  $k, \ell$ , the partial differential system  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$  (see (16)) does not admit any regular solution  $p$ .

Note that Theorem 3.7 is pointless if this conjecture is true; see remark 5.3 for more comments. In this section, we comment the conjecture itself and prove it for small enough values of the order  $(k, \ell)$ .

An equivalent way of stating this conjecture is that the equations  $ED^i p = 0$ , for  $1 \leq i \leq k + \ell - 2$ , are consequences of (16) (note that “ $ED^i p = 0$ ” in fact encodes several partial differential relations on  $p$ ; see Remark 3.2). If  $\gamma$  and  $\delta$  are polynomials, then this can be easily phrased in terms of the differential ideals in the set of polynomials with respect to the variables  $u, \dots, u^{(k-1)}, x, v, \dots, v^{(\ell-1)}$  with  $k + \ell + 1$  commuting derivatives (all the partial derivatives with respect to these variables).

This conjecture is still a conjecture for general integers  $k$  and  $\ell$ , but we were able to prove it for “small enough”  $k, \ell$ , namely if one of them is smaller than 3 or if  $k = \ell = 3$ . As usual, all the statements assume  $k \leq \ell$  (see remark 2.3).

<sup>1</sup>In other words, (30)-(31)-(32)-(33)-(34), as a system of ODEs in  $u$  and  $v$ , is formally integrable (see e.g. [3, Chapter IX]). This means, for a systems of ODEs with independent variable  $t$ , that no new independent equation of the same orders ( $k$  with respect to  $u$  and  $\ell$  with respect to  $v$ ) can be obtained by differentiating with respect to  $t$  (an arbitrary number of times) and combining these equations. It is known [3, Chapter IX] that a sufficient condition is that this is true when differentiating only once and the system allows one to express the highest order derivatives as functions of the others. Formal integrability also means that, given any initial condition  $(u(0), \dots, u^{(k)}(0), v(0), \dots, v^{(\ell)}(0))$  that satisfies these relations, there is a solution of the system of ODEs with these initial conditions.

**Proposition 3.9.** *If system  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ , with  $k \leq \ell$ , admits a regular solution, then necessarily  $k \geq 3$ ,  $\ell \geq 4$  and the determinant*

$$\begin{vmatrix} p_{u^{(k-1)}} & p_{u^{(k-2)}} & p_{u^{(k-3)}} \\ p_{xu^{(k-1)}} & p_{xu^{(k-2)}} & p_{xu^{(k-3)}} \\ p_{xxu^{(k-1)}} & p_{xxu^{(k-2)}} & p_{xxu^{(k-3)}} \end{vmatrix} \quad (38)$$

is a nonzero real analytic function.

*Proof.* This is an straightforward consequence of Lemma 3.5 and the three following lemmas, proved in appendix B.  $\square$

**Lemma 3.10.** *If  $p$  is a solution of system  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$  and either  $k = 1$  or*

*$\begin{vmatrix} p_{u^{(k-1)}} & p_{u^{(k-2)}} \\ p_{xu^{(k-1)}} & p_{xu^{(k-2)}} \end{vmatrix} = 0$ , then around each point such that  $p_{u^{(k-1)}} \neq 0$ , there exists a function  $\alpha$  of two variables such that a relation  $p_x = \alpha(x, p)$  holds identically on a neighborhood of that point.*

**Lemma 3.11.** *Suppose that  $p$  is a solution of  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$  with*

$$\ell \geq k \geq 2, \quad p_{u^{(k-1)}} \neq 0, \quad \begin{vmatrix} p_{u^{(k-1)}} & p_{u^{(k-2)}} \\ p_{xu^{(k-1)}} & p_{xu^{(k-2)}} \end{vmatrix} \neq 0. \quad (39)$$

*If either  $k = 2$  or the determinant (38) is identically zero, then, around any point where the two quantities in (39) are nonzero, there exists a function  $\alpha$  of three variables such that a relation  $p_{x,x} = \alpha(x, p, p_x)$  holds identically on a neighborhood of that point.*

**Lemma 3.12.** *Let  $k = \ell = 3$ . For any solution  $p$  of  $(\mathcal{E}_{3,3}^{\gamma,\delta})$ , in a neighborhood of any point where the determinant (38) is nonzero, there exists two functions  $\alpha$  and  $\psi$  of four variables such that  $p_{xxx} = \alpha(x, p, p_x, p_{xx})$  and  $Fp_{xx} + \tau p_{xxu^{(k-1)}} = \psi(x, p, p_x, p_{xx})$  identically on a neighborhood of that point.*

#### 4. REMARKS ON THE CASE WHERE $S = T = 0$ .

**Lemma 4.1.** *Consider an open subset  $\Omega$  of  $\mathbb{R}^4$ , and  $g$  and  $h$  two smooth functions  $\Omega \rightarrow \mathbb{R}$ , and assume that  $S$  and  $T$ , given by (13), are identically zero. Then, for any point  $(x_0, y_0, z_0, \lambda_0) \in \Omega$  such that  $g_4(x_0, y_0, z_0, \lambda_0) \neq 0$ , there exists*

- an open set  $W \subset \mathbb{R}^3$  and an open interval  $I \subset \mathbb{R}$  such that  $(x_0, y_0, z_0, \lambda_0) \in \Omega' = W \times I \subset \Omega$ ,

- and seven smooth functions  $W \rightarrow \mathbb{R}$  denoted by  $\hat{a}^0, \hat{a}^1, \hat{a}^2, \hat{b}^0, \hat{b}^1, \hat{c}^0, \hat{c}^1$  such that

$$\text{Rank} \begin{pmatrix} \hat{a}^0 & (g_{4,4} h_4 - h_{4,4} g_4) \lambda^2 + (2 h_4 g_4 - h_{4,4} g_4) \lambda - 2 h g_4 \\ \hat{a}^1 & -2 (g_{4,4} h_4 - h_{4,4} g_4) \lambda + h g_{4,4} - 2 h_4 g_4 \\ \hat{a}^2 & g_{4,4} h_4 - h_{4,4} g_4 \\ \hat{b}^0 & (-g g_{4,4} + 2 g_4^2) \lambda - 2 g g_4 \\ \hat{b}^1 & g g_{4,4} - 2 g_4^2 \\ \hat{c}^0 & -g_{4,4} \lambda - 2 g_4 \\ \hat{c}^1 & g_{4,4} \end{pmatrix} = 1 \quad (40)$$

(this means that the first column is proportional to the second one, that cannot be zero for  $g_4 \neq 0$ ). Then,  $\hat{c}^0 + \hat{c}^1 \lambda \neq 0$  on  $\Omega$ ,  $\hat{c}^1 \hat{b}^0 - \hat{b}^1 \hat{c}^0 \neq 0$  on  $W$ , and, for all

$x, y, z, \lambda$  in  $\Omega$ ,

$$\begin{aligned} g(x, y, z, \lambda) &= \frac{\hat{b}^0(x, y, z) + \hat{b}^1(x, y, z)\lambda}{\hat{c}^0(x, y, z) + \hat{c}^1(x, y, z)\lambda}, \\ h(x, y, z, \lambda) &= \frac{\hat{a}^0(x, y, z) + \hat{a}^1(x, y, z)\lambda + \hat{a}^2(x, y, z)\lambda^2}{\hat{c}^0(x, y, z) + \hat{c}^1(x, y, z)\lambda}. \end{aligned} \quad (41)$$

*Proof.* By differentiating, one checks that the ratios between the elements of the second column in (40) do not depend on  $\lambda$ . Chosen as follows, the functions  $\hat{c}^i, \hat{b}^i, \hat{a}^i$  therefore do not depend on  $\lambda$ :

$$\begin{aligned} \text{if } g_{4,4} \neq 0, & \begin{cases} \hat{a}^0 = (h_4 - g_4 h_{4,4}/g_{4,4})\lambda^2 + (2h_4 g_4/g_{4,4} - h)\lambda - 2h g_4/g_{4,4}, \\ \hat{a}^1 = -2(h_4 - g_4 h_{4,4}/g_{4,4})\lambda + h - 2h_4 g_4/g_{4,4}, \\ \hat{a}^2 = h_4 - g_4 h_{4,4}/g_{4,4}, \\ \hat{b}^0 = (-g + 2g_4^2/g_{4,4})\lambda - 2g g_4/g_{4,4}, & \hat{b}^1 = g - 2g_4^2/g_{4,4}, \\ \hat{c}^0 = -\lambda - 2g_4/g_{4,4}, & \hat{c}^1 = 1; \end{cases} \\ \text{if } g_{4,4} = 0, & \begin{cases} \hat{a}^0 = \frac{1}{2}h_{4,4}\lambda^2 - h_4\lambda + h, & \hat{a}^1 = -h_{4,4}\lambda + h_4, & \hat{a}^2 = \frac{1}{2}h_{4,4}, \\ \hat{b}^0 = -g_4\lambda + g, & \hat{b}^1 = g_4, & \hat{c}^0 = 1, & \hat{c}^1 = 0. \end{cases} \end{aligned} \quad (42)$$

The relation (41) is a simple consequence of (40) or (42).  $\square$

To analyze further the case of systems for which  $S = T = 0$ , let us assume that  $g$  and  $h$  take the form (41). Then define the differential forms in the three variables  $x, y, z$ :

$$\hat{\omega}^1 = dy - zdx, \quad \hat{\omega}^2 = \hat{b}^1 dx + \hat{a}^2 \hat{\omega}^1 - \hat{c}^1 dz, \quad \hat{\omega}^3 = \hat{b}^0 dx + \hat{a}^1 \hat{\omega}^1 - \hat{c}^0 dz. \quad (43)$$

Denoting by  $\xi$  the point of coordinates  $x, y, z$ , equation (3) can be written

$$\langle \hat{\omega}^1, \dot{\xi} \rangle \langle \hat{\omega}^2, \dot{\xi} \rangle + \langle \hat{\omega}^3, \dot{\xi} \rangle + \hat{a}^0 = 0. \quad (44)$$

Note that  $\hat{\omega}^1, \hat{\omega}^2$  and  $\hat{\omega}^3$  are differential forms in the variables  $x, y, z$  only, but, from (40),  $\hat{\omega}^2$  is proportional to  $\omega$  defined in (11), *i.e.* there is a functions  $k \neq 0$ , of the four variables  $x, y, z, \lambda$ , such that<sup>2</sup>  $\omega = k \hat{\omega}^2$ . Hence, from (12), assuming  $S = T = 0$  one has

$$J = 0 \Leftrightarrow d\omega \wedge \omega = 0 \Leftrightarrow d\hat{\omega}^2 \wedge \hat{\omega}^2 = 0. \quad (45)$$

**Theorem 4.2.** *If  $S = T = J = 0$ , then system (3) admits a parameterization of order (1, 2) at any  $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \ddot{x}_0, \ddot{y}_0) \in (\hat{\Omega} \times \mathbb{R}^2) \setminus F$ , where  $F \subset \hat{\Omega} \times \mathbb{R}^2$  is closed with empty interior.*

This is a consequence of [18, Proposition 4.5], although orders of parameterizations are not given in [18]. For the sake of self-containedness, let us however give a short proof.

*Proof.* From Lemma 4.1, system (3) can be written as (44) where, from (45),  $d\hat{\omega}^2 \wedge \hat{\omega}^2 = 0$ . Since, in addition,  $\hat{\omega}^1$  and  $\hat{\omega}^2$  are linearly independent there is a (local) change of coordinates  $(\tilde{x}, \tilde{y}, \tilde{z}) = P(x, y, z)$  such that  $\hat{\omega}^2 = k' d\tilde{x}$  and  $\hat{\omega}^1 = k''(d\tilde{y} - \tilde{z}d\tilde{x})$  with  $k' \neq 0$  and  $k'' \neq 0$ ; in these coordinates, with  $\kappa, a, b, c$  some smooth maps,  $\kappa \neq 0$ , system (44) reads

$$\dot{\tilde{z}} = \kappa(\tilde{x}, \tilde{y}, \tilde{z}) \tilde{x} \tilde{\lambda} + a(\tilde{x}, \tilde{y}, \tilde{z}) \tilde{\lambda} + b(\tilde{x}, \tilde{y}, \tilde{z}) \tilde{\dot{x}} + c(\tilde{x}, \tilde{y}, \tilde{z}) \quad \text{with} \quad \tilde{\lambda} = \dot{\tilde{y}} - \tilde{z} \dot{\tilde{x}}.$$

<sup>2</sup>In fact, from (12),  $S = T = 0$  means that the Lie derivative of  $\omega$  along  $\partial/\partial\lambda$  is co-linear to  $\omega$ , and this classically implies that  $\omega$  itself is collinear to a form in three variables  $x, y, z$ , the first integrals of  $\partial/\partial\lambda$ .

In  $\mathbb{R}^4$  with coordinates  $\tilde{x}, \tilde{y}, \tilde{z}, \dot{\tilde{x}}$ , define the vector fields  $X, Y, \Xi$  as follows, and compute the bracket  $[Y, \Xi]$  :

$$X = \dot{\tilde{x}} \left( \frac{\partial}{\partial \tilde{x}} + \tilde{z} \frac{\partial}{\partial \tilde{y}} \right) + (b\tilde{x} + c) \frac{\partial}{\partial \tilde{z}}, \quad Y = \frac{\partial}{\partial \tilde{y}} + (a + \dot{\tilde{x}}\kappa) \frac{\partial}{\partial \tilde{z}}, \quad \Xi = \frac{\partial}{\partial \dot{\tilde{x}}}, \quad [Y, \Xi] = -\kappa \frac{\partial}{\partial \dot{\tilde{z}}}$$

Since  $\kappa \neq 0$ ,  $[Y, \Xi]$  and  $Y$  are linearly independent. Let  $\varphi$  be a function of four variables  $(\tilde{x}, \tilde{y}, \tilde{z}, \dot{\tilde{x}})$  such that  $Y\varphi = 0$  and  $[Y, \Xi]\varphi \neq 0$ ; since  $Y\varphi = 0$ , one has  $[Y, \Xi]\varphi = Y\Xi\varphi$ . Define

$$\varphi'(\tilde{x}, \tilde{y}, \tilde{z}, \dot{\tilde{x}}, \ddot{\tilde{x}}) = X\varphi(\tilde{x}, \tilde{y}, \tilde{z}, \dot{\tilde{x}}) + \ddot{\tilde{x}}\Xi\varphi(\tilde{x}, \tilde{y}, \tilde{z}, \dot{\tilde{x}}).$$

Since  $Y\Xi\varphi$  is nonzero real analytic, this is a closed subset of  $W \times \mathbb{R}^2$  with empty interior :

$$\tilde{F} = \{(\tilde{x}, \tilde{y}, \tilde{z}, \dot{\tilde{x}}, \ddot{\tilde{x}}) \in W \times \mathbb{R}^2, YX\varphi(\tilde{x}, \tilde{y}, \tilde{z}, \dot{\tilde{x}}) + \ddot{\tilde{x}}Y\Xi\varphi(\tilde{x}, \tilde{y}, \tilde{z}, \dot{\tilde{x}}) = 0\}.$$

At any point  $(\tilde{x}, \tilde{y}, \tilde{z}, \dot{\tilde{x}}, \ddot{\tilde{x}}) \in (W \times \mathbb{R}^2) \setminus \tilde{F}$ , the map

$$(\tilde{x}, \tilde{y}, \tilde{z}, \dot{\tilde{x}}, \ddot{\tilde{x}}) \mapsto (\tilde{x}, \varphi(\tilde{x}, \tilde{y}, \tilde{z}, \dot{\tilde{x}}), \varphi'(\tilde{x}, \tilde{y}, \tilde{z}, \dot{\tilde{x}}, \ddot{\tilde{x}}), \dot{\tilde{x}}, \ddot{\tilde{x}})$$

is a local diffeomorphism ; let its inverse be

$$(\tilde{x}, u, \dot{u}, \dot{\tilde{x}}, \ddot{\tilde{x}}) \mapsto (\tilde{x}, \tilde{\psi}(\tilde{x}, u, \dot{u}, \dot{\tilde{x}}, \ddot{\tilde{x}}), \tilde{\chi}(\tilde{x}, u, \dot{u}, \dot{\tilde{x}}, \ddot{\tilde{x}}), \dot{\tilde{x}}, \ddot{\tilde{x}}).$$

Let then  $F$  be the set of  $(x, y, z, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}) \in \hat{\Omega} \times \mathbb{R}^2$  such that

$$(P^1(x, y, z), P^2(x, y, z), P^3(x, y, z), \dot{P}^1(x, y, z, \dot{x}, \dot{y}), \ddot{P}^1(x, y, z, \dot{x}, \dot{y}, \ddot{x}, \ddot{y})) \in \tilde{F}$$

where  $\dot{P}^1$  and  $\ddot{P}^1$  are computed according to (3). Then, in a neighborhood of any  $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \ddot{x}_0, \ddot{y}_0) \in (\hat{\Omega} \times \mathbb{R}^2) \setminus F$ , a parameterization (10) of order (1,2) is provided by  $(x, y, z) = P^{-1}(v, \tilde{\psi}(v, u, \dot{u}, \dot{v}, \ddot{v}), \tilde{\chi}(v, u, \dot{u}, \dot{v}, \ddot{v}))$ .  $\square$

Now, let us turn to the case where  $J$  is nonzero.

**Proposition 4.3.** *Assume that the functions  $g$  and  $h$  defining system (3) are such that  $S$  and  $T$  defined by (12) or (13) are identically zero on  $\Omega$ , but  $J$  is not, and let  $(x_0, y_0, z_0, \lambda_0) \in \Omega$  be such that  $J(x_0, y_0, z_0, \lambda_0) \neq 0$ .*

*There exists an open set  $W \subset \mathbb{R}^3$  and an open interval  $I \subset \mathbb{R}$  such that  $(x_0, y_0, z_0, \lambda_0) \in \Omega' = W \times I \subset \Omega$ , a smooth diffeomorphism  $P$  from  $W$  to  $P(W) \subset \mathbb{R}^3$  and six smooth functions  $P(W) \rightarrow \mathbb{R}$  denoted  $\kappa, \alpha, \beta, a, b, c$  such that, with the change of coordinates  $(\tilde{x}, \tilde{y}, \tilde{z}) = P(x, y, z)$ , system (3) reads*

$$\dot{\tilde{z}} = \kappa(\tilde{x}, \tilde{y}, \tilde{z}) (\dot{\tilde{y}} - \alpha(\tilde{x}, \tilde{y}, \tilde{z}) \dot{\tilde{x}}) (\dot{\tilde{y}} - \beta(\tilde{x}, \tilde{y}, \tilde{z}) \dot{\tilde{x}}) + a(\tilde{x}, \tilde{y}, \tilde{z}) \dot{\tilde{x}} + b(\tilde{x}, \tilde{y}, \tilde{z}) \dot{\tilde{y}} + c(\tilde{x}, \tilde{y}, \tilde{z}) \quad (46)$$

*and none of the functions  $\kappa, \alpha - \beta, \alpha_3$  and  $\beta_3$  vanish on  $W$ .*

Note that if no denominator appears in (41) (*i.e.*  $\hat{c}^0 = 1$  and  $\hat{c}^1 = 0$ ), then (41) is already in the form (46) with  $\kappa = -\hat{a}^2$ ,  $\alpha = z$ ,  $\beta = z + \hat{b}^1/\hat{a}^2$ ,  $a = \hat{b}^0 - \hat{a}^1 z$ ,  $b = \hat{a}^1$ ,  $c = \hat{a}^0$ .

*Proof.* From Lemma 4.1,  $g$  and  $h$  are given by (41). Let  $P^1, P^2$  be a pair of independent first integrals of the vector field  $\hat{c}^1 \left( \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} \right) + \hat{b}^1 \frac{\partial}{\partial z}$  in the coordinates  $x, y, z$ ; from (43),  $\hat{\omega}^1, \hat{\omega}^2$  span the annihilator of this vector field, and hence are linear combinations of  $dP^1$  and  $dP^2$ . Possibly modifying  $P^1, P^2$ , the component on  $dP^1$  of  $\hat{\omega}^1$  and  $\hat{\omega}^2$  are nonzero, *i.e.* there exists smooth function  $k^1, k^2, f^1, f^2$  such that  $\hat{\omega}^i = k^i (dP^2 - f^i dP^1)$ ,  $k^i \neq 0$ ,  $i = 1, 2$ . Now, take for  $P^3$  any function such that  $dP^1 \wedge dP^2 \wedge dP^3 \neq 0$ . Decomposing  $\hat{\omega}^3$ , we get three smooth functions  $p^0, p^1, p^2$  such that  $\hat{\omega}^3 = p^0 (-dP^3 + p^1 dP^1 + p^2 dP^2)$ .

From (43),  $\hat{\omega}^1 \wedge \hat{\omega}^2 \wedge \hat{\omega}^3 = (\hat{b}^1 \hat{c}^0 - \hat{b}^0 \hat{c}^1) dx \wedge dy \wedge dz \neq 0$  while, from the above expressions,  $\hat{\omega}^1 \wedge \hat{\omega}^2 \wedge \hat{\omega}^3 = p^0 (f^1 - f^2) dP^1 \wedge dP^2 \wedge dP^3$ ; hence  $p^0 \neq 0$  and  $f^1 - f^2 \neq 0$ . Also, (43) implies  $d\hat{\omega}^1 \wedge \hat{\omega}^1 \neq 0$ , (45) implies  $d\hat{\omega}^2 \wedge \hat{\omega}^2 \neq 0$ , while, from the above expressions,  $d\hat{\omega}^i \wedge \hat{\omega}^i = -(k^i)^2 dP^1 \wedge dP^2 \wedge df^i$  for  $i = 1, 2$ ; hence, recalling the above inequality,

$$p^0 \neq 0, \quad f^1 - f^2 \neq 0, \quad dP^1 \wedge dP^2 \wedge df^i, \quad i = 1, 2. \quad (47)$$

Defining a change of coordinates  $\tilde{x} = P^1(x, y, z)$ ,  $\tilde{y} = P^2(x, y, z)$ ,  $\tilde{z} = P^3(x, y, z)$ , with inverse  $(x, y, z) = \psi(\tilde{x}, \tilde{y}, \tilde{z})$ , and six functions  $\kappa, \alpha, \beta, a, b, c$  by :

$$\kappa = \frac{k^1 k^2}{p^0} \circ \psi, \quad \alpha = f^1 \circ \psi, \quad \beta = f^2 \circ \psi, \quad a = p^1 \circ \psi, \quad b = p^2 \circ \psi, \quad c = \frac{\hat{a}^0}{p^0} \circ \psi,$$

equation (44) translates into (46) and (47) implies, since  $k^1 \neq 0$  and  $k^2 \neq 0$ , that  $\kappa, \alpha - \beta, \alpha_3$  and  $\beta_3$  are nonzero.  $\square$

Equation (46), where  $\alpha$  and  $\beta$  play exactly the same role, is not really a particular case of (3), but it would be with  $z$  instead of  $\alpha$  or  $\beta$ . Since  $\alpha_3 \neq 0$  and  $\beta_3 \neq 0$ ,

$$(\tilde{x}, \tilde{y}, \tilde{z}) \mapsto A(\tilde{x}, \tilde{y}, \tilde{z}) = (\tilde{x}, \tilde{y}, \alpha(\tilde{x}, \tilde{y}, \tilde{z})) \quad \text{and} \quad (\tilde{x}, \tilde{y}, \tilde{z}) \mapsto B(\tilde{x}, \tilde{y}, \tilde{z}) = (\tilde{x}, \tilde{y}, \beta(\tilde{x}, \tilde{y}, \tilde{z})) \quad (48)$$

define two local diffeomorphisms, both providing coordinates that turn (46) to a system of the form (3) with  $h$  and  $g$  as in (41) and  $\hat{c}^1 = 0, \hat{c}^0 = 1$ , but with different functions  $\hat{a}^i$  and  $\hat{b}^i$ ; in both cases,  $\hat{c}^1 = 0, \hat{c}^0 = 1$ ). These two different possibilities for  $g$  and  $h$  yield two possible sets of functions  $\gamma$  and  $\delta$ —to be used in the differential system  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ , see (16)—computed according to (6) and (7); let us give their explicit expression :

$$\gamma^i(x, y, z, w) = \frac{w - m^{i,0}(x, y, z)}{m^{i,1}(x, y, z)}, \quad \delta^i = n^{i,0} + n^{i,1}\gamma + n^{i,2}\gamma^2, \quad i \in \{1, 2\} \quad (49)$$

with (these are obtained from each other by interchanging  $\alpha$  and  $\beta$ ) :

$$\begin{aligned} m^{1,0} &= (\alpha_1 + \alpha \alpha_2 + (a + b \alpha) \alpha_3) \circ A^{-1}, & m^{1,1} &= (\kappa \alpha_3 (\alpha - \beta)) \circ A^{-1}, \\ n^{1,0} &= \alpha_3 \circ A^{-1}, & n^{1,1} &= (\alpha_2 + b \alpha_3) \circ A^{-1}, & n^{1,2} &= (\kappa \alpha_3) \circ A^{-1}, \\ m^{2,0} &= (\beta_1 + \beta \beta_2 + (a + b \beta) \beta_3) \circ B^{-1}, & m^{2,1} &= (\kappa \beta_3 (\beta - \alpha)) \circ B^{-1}, \\ n^{2,0} &= \beta_3 \circ B^{-1}, & n^{2,1} &= (\beta_2 + b \beta_3) \circ B^{-1}, & n^{2,2} &= (\kappa \beta_3) \circ B^{-1}. \end{aligned} \quad (50)$$

## 5. MAIN RESULTS

The following theorem is central in this paper. The proof is very short because it relies on local necessary conditions and sufficient conditions that were already given or are stated in section 6. Since the singularities are not the same for necessary and sufficient conditions, we do not give necessary and sufficient conditions at a precise point; but we use the “somewhere” as in Definitions 3.3 and 2.6.

**Theorem 5.1.** *System (3) admits a parameterization of order  $(k, \ell)$  somewhere in  $\Omega$  if and only if*

- (1) *either  $S = T = J = 0$  on  $\Omega$  (in this case, one can take  $(k, \ell) = (1, 2)$ ),*
- (2) *or  $S = T = 0$  on  $\Omega$  and one of the two systems  $(\mathcal{E}_{k,\ell}^{\gamma^1, \delta^1})$  or  $(\mathcal{E}_{k,\ell}^{\gamma^2, \delta^2})$  with  $\gamma^i, \delta^i$  given by (49)-(50), admits a regular solution somewhere in  $\hat{\Omega}$ .*
- (3) *or  $S$  and  $T$  are not both identically zero, and the system  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$  with  $\gamma$  and  $\delta$  defined from  $g$  and  $h$  according to (6) and (7) admits a regular solution somewhere in  $\hat{\Omega}$ .*

*Proof.* Sufficiency : the parameterization is provided, away from an explicitly described set of singularities, by Theorem 4.2 if point 1 holds, and by Theorem 4.2 if one of the two other points holds. For necessity, assume that there is a parameterization of order  $(k, \ell)$  at a point  $(x, y, z, \dot{x}, \dot{y}, \dots, x^{(L)}, y^{(L)})$  in  $(\widehat{\Omega} \times \mathbb{R}^{2L-2}) \setminus F$ . From Theorems 6.2 and 6.5, it implies that one of the three points holds.  $\square$

This theorem clearly gives some importance to the system of PDEs  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ . Our conjecture (Conjecture 3.8) is that this system never has “regular solutions”, and this would imply the

**Conjecture 5.2.** If  $d\omega \wedge \omega$  (or  $(S, T, J)$ ) is not identically zero on  $\Omega$ , then system (3) does not admit a parameterization of any order at any point (jet of any order).

**Remark 5.3.** If our conjecture 3.8 is correct, the systems  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$  never have any regular solutions, and the sufficiency part of Theorem 5.1 (apart from case 1) is essentially void, and so is Theorem 3.7. However, Conjecture 3.8 is still a conjecture, and the interest of these sufficient conditions is that they make the two conjectures—the above one on parameterizations of system (3) and Conjecture 3.8 that only deals with a set of partial differential equalities and inequalities—equivalent. For instance, if one comes up with a regular solution of some of these systems  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ , this will yield a new class of systems that admit a parameterization.

We do have the following result, weaker than the conjecture because it is restricted to “small” values of  $(k, \ell)$  (small in the sense of the order in remark 2.3, but still, the conjecture is proved if  $k$  is 1 or 2 and  $\ell$  arbitrary).

**Theorem 5.4.** *If system (3) admits a parameterization of order  $(k, \ell)$ , with  $k \leq \ell$ , at some jet, then either  $S = T = J = 0$  or  $k \geq 3$  and  $\ell \geq 4$ .*

*Proof.* This is a simple consequence of Theorem 5.1 and Proposition 3.9.  $\square$

Finally, note that the results of the present paper are sufficient to recover the results contained in the very technical paper [18]. The main result in that reference can be phrased :

“ (1) is  $(x, u)$ -dynamic linearizable (*i.e.*  $(x, u)$ -flat) if and only if  $S = T = J = 0$  ”,

and the difficult part is to prove that  $S = T = J = 0$  is necessary; in fact that proof is very technical, and relies on some simplifications performed via computer algebra. From our Proposition 7.4,  $(x, u)$ -flatness implies existence of a parameterization of some order  $(k, \ell)$  with  $k \leq 3$  and  $\ell \leq 3$ . Hence Theorem 5.1 does imply the above statement (provided that a parameterization of order (1,2) for (3) implies  $(x, u)$ -flatness for (1); this is elementary seen the construction in the proof of Theorem 4.2).

## 6. NECESSARY CONDITIONS

**6.1. The case where  $S$  and  $T$  are not both zero.** The following lemma is needed to state the theorem.

**Lemma 6.1.** *If  $(S, T, J) \neq (0, 0, 0)$  and system (3) admits a parameterization  $(\varphi, \psi, \chi)$  of order  $(k, \ell)$  at point  $(x_0, y_0, z_0, \dots, x_0^{(L)}, y_0^{(L)}) \in \mathbb{R}^{2L+3}$ , then  $\varphi_{u^{(k)}}$  is a nonzero real analytic function.*

*Proof.* Assume a parameterization where  $\varphi$  does not depend on  $u^{(k)}$ . Substituting in (3) yields

$$\dot{\chi} = h(\varphi, \psi, \chi, \dot{\psi} - \chi\dot{\varphi}) + g(\varphi, \psi, \chi, \dot{\psi} - \chi\dot{\varphi})\dot{\varphi}.$$



Since  $\dot{\varphi}$  does not depend on  $u^{(k+1)}$ , differentiating twice with respect to  $u^{(k+1)}$  yields

$$\chi_{u^{(k)}} = \psi_{u^{(k)}}(h_4 + g_4\dot{\varphi}), \quad 0 = \psi_{u^{(k)}}^2(h_{4,4} + g_{4,4}\dot{\varphi}).$$

If  $\psi_{u^{(k)}}$  was zero, then, from the first relation,  $\chi_{u^{(k)}}$  would too, and this would contradict point 3 in Definition 2.2; hence the second relation implies that  $h_{4,4} + g_{4,4}\dot{\varphi}$  is identically zero. From point 2 in the same definition, it implies that *all* solutions of (3) satisfy the relation :  $h_{4,4}(x, y, z, \dot{y} - z\dot{x}) + g_{4,4}(x, y, z, \dot{y} - z\dot{x})\dot{x} = 0$ . From Lemma 2.1, this implies that  $h_{4,4}$  and  $g_{4,4}$  are the zero function of four variables, and hence  $S = T = J = 0$ . This proves the lemma.  $\square$

**Theorem 6.2.** *Assume that either  $S$  or  $T$  is not identically zero on  $\Omega$ , and that system (3) admits a parameterization of order  $(k, \ell)$  at  $\mathcal{X} = (x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dots, x_0^{(L)}, y_0^{(L)}) \in \widehat{\Omega} \times \mathbb{R}^{2L-2}$ , with  $k, \ell, L$  some integers and  $\varphi, \psi, \chi$  defined on  $U \subset \mathbb{R}^{k+\ell+2}$ .*

*Then  $k \geq 1, \ell \geq 1$  and, for any point  $(u_0, \dots, u_0^{(k)}, v_0, \dots, v_0^{(\ell)}) \in U$  (not necessarily sent to  $\mathcal{X}$  by the parameterization) such that*

$$\varphi_{u^{(k)}}(u_0, \dots, u_0^{(k)}, v_0, \dots, v_0^{(\ell)}) \neq 0,$$

*there exists a neighborhood  $O$  of  $(u_0, \dots, u_0^{(k-1)}, \varphi(u_0 \cdots v_0^{(\ell)}), v_0, \dots, v_0^{(\ell-1)})$  in  $\mathbb{R}^{k+\ell+1}$  and a regular solution  $p : O \rightarrow \mathbb{R}$  of  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ , related to  $\varphi, \psi, \chi$  by (24), (25) and (26), the functions  $\gamma$  and  $\delta$  being related to  $g$  and  $h$  by (6) and (7).*

**Remark 6.3.** From theorem 3.7, if  $p$  is a  $K$ -regular solution ( $K \leq k + \ell - 2$ ), one can always take  $L \geq K$  in the jet at which the parameterization is valid. In other word, the open set  $\mathcal{V}$  of germs of solutions in Definition 2.2 derives from  $V \subset \widehat{\Omega} \times \mathbb{R}^{2K-2}$ .

*Proof.* Assume that system (3) admits a parameterization  $(\varphi, \psi, \chi)$  of order  $(k, \ell)$  at  $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dots, x_0^{(L)}, y_0^{(L)})$ . Since  $\varphi_{u^{(k)}}$  does not vanish, one can apply the inverse function theorem to the map

$$(u, \dot{u}, \dots, u^{(k)}, v, \dot{v}, \dots, v^{(\ell)}) \mapsto (u, \dots, u^{(k-1)}, \varphi(u, \dots, u^{(k)}, v, \dots, v^{(\ell)}), v, \dots, v^{(\ell)})$$

and define locally a function  $r$  of  $k + \ell + 2$  variables such that

$$\varphi(u, \dot{u}, \dots, u^{(k)}, v, \dot{v}, \dots, v^{(\ell)}) = x \Leftrightarrow r(u, \dot{u}, \dots, u^{(k-1)}, x, v, \dot{v}, \dots, v^{(\ell)}) = u^{(k)}. \quad (51)$$

Defining two functions  $p, q$  by substitution of  $u^{(k)}$  in  $\psi, \chi$ , the parameterization can be re-written implicitly as

$$\begin{cases} y = p(u, \dot{u}, \dots, u^{(k-1)}, x, v, \dot{v}, \dots, v^{(\ell)}), \\ z = q(u, \dot{u}, \dots, u^{(k-1)}, x, v, \dot{v}, \dots, v^{(\ell)}), \\ u^{(k)} = r(u, \dot{u}, \dots, u^{(k-1)}, x, v, \dot{v}, \dots, v^{(\ell)}). \end{cases} \quad (52)$$

We now work with this form of the parameterization and  $u, \dot{u}, \dots, u^{(k-1)}, x, \dot{x}, \ddot{x}, \dots, v, \dot{v}, \dots, v^{(\ell)}, v^{(\ell+1)}, \dots$  instead of  $u, \dot{u}, \dots, u^{(k-1)}, u^{(k)}, u^{(k+1)}, \dots, v, \dot{v}, \dots, v^{(\ell)}, v^{(\ell+1)}, \dots$ .

In order to simplify notations, we define (the vector field  $F$  is defined in (14))

$$\mathcal{P} = Fp + rp_{u^{(k-1)}} + v^{(\ell)}p_{v^{(\ell-1)}} + v^{(\ell+1)}p_{v^{(\ell)}}, \quad \mathcal{Q} = Fq + rq_{u^{(k-1)}} + v^{(\ell)}q_{v^{(\ell-1)}} + v^{(\ell+1)}q_{v^{(\ell)}}.$$

We shall prove later that  $\ell \geq 1$  and  $k \geq 1$ ; until then, all terms involving  $\ell - 1$  or  $k - 1$  are assumed to be zero by convention.  $\mathcal{P}$  and  $\mathcal{Q}$  depend on  $u, \dot{u}, \dots, u^{(k-1)}, x, v, \dot{v}, \dots, v^{(\ell)}, v^{(\ell+1)}$  but not on  $\dot{x}$ ;  $Fp$  and  $Fq$  depend neither on  $\dot{x}$  nor on  $v^{(\ell+1)}$ . When substituting (52) in (3), using  $\dot{y} = \mathcal{P} + \dot{x}p_x$  and  $\dot{z} = \mathcal{Q} + \dot{x}q_x$ , one obtains :

$$\mathcal{Q} + \dot{x}q_x = h(x, p, q, \lambda) + g(x, p, q, \lambda)\dot{x} \quad \text{with } \lambda = \mathcal{P} + \dot{x}(p_x - q). \quad (53)$$

Differentiating each side three times with respect to  $\dot{x}$ , one obtains :

$$q_x = (h_4(x, p, q, \lambda) + g_4(x, p, q, \lambda)\dot{x})(p_x - q) + g(x, p, q, \lambda), \quad (54)$$

$$0 = (h_{4,4}(x, p, q, \lambda) + g_{4,4}(x, p, q, \lambda)\dot{x})(p_x - q)^2 + 2g_4(x, p, q, \lambda)(p_x - q), \quad (55)$$

$$0 = (h_{4,4,4}(x, p, q, \lambda) + g_{4,4,4}(x, p, q, \lambda)\dot{x})(p_x - q)^3 + 3g_{4,4}(x, p, q, \lambda)(p_x - q)^2. \quad (56)$$

Combining (55) and (56) to cancel the first term in each equation, one obtains (see  $S$  and  $T$  in (13)) :

$$\left( T(x, p, q, \lambda) + S(x, p, q, \lambda)\dot{x} \right) (p_x - q)^2 = 0. \quad (57)$$

If  $T + S\dot{x}$  was identically zero as a function of  $u, \dots, u^{(k-1)}, x, v, \dots, v^{(\ell-1)}$ , this would imply, by Definition 2.2 (point 2), that *all* solutions  $(x(t), y(t), z(t))$  of (3) satisfy  $T(x, y, z, \dot{y} - z\dot{x}) + \dot{x}S(x, y, z, \dot{y} - z\dot{x}) = 0$  identically; this would imply that  $S$  and  $T$  are identically zero functions of 4 variables, but we supposed the contrary. Hence  $q = p_x$ .

We first notice that this implies

$$\lambda = \mathcal{P} = Fp + rp_{u^{(k-1)}} + v^{(\ell)}p_{v^{(\ell-1)}} + v^{(\ell+1)}p_{v^{(\ell)}} \quad (58)$$

and that (54) yields  $p_{xx} = g(x, p, p_x, \lambda)$ , or, with  $\gamma$  defined by (6),

$$\lambda = \gamma(x, p, p_x, p_{xx}). \quad (59)$$

Since neither  $p$  nor  $Fp$  nor  $r$  depend on  $v^{(\ell+1)}$ , these two equations yield  $p_{v^{(\ell)}} = 0$ , *i.e.*  $p$  is a function of  $u, \dots, u^{(k-1)}, x, v, \dots, v^{(\ell-1)}$  only. Then (58) and (59) imply (112) with  $f = \gamma$ . Furthermore,  $r_{v^{(\ell)}} \neq 0$  because, from (51),  $r_{v^{(\ell)}} = 0$  would imply  $\varphi_{v^{(\ell)}} = 0$  and hence contradict point 3 of Definition 2.2. We may then apply lemma C.1 (see appendix). If  $p$  was a function of  $x$  only, then all solutions of (62) should satisfy a relation  $y(t) = p(x(t))$ , which is absurd from Lemma 2.1; hence the (113) holds, this proves (16-c,d).

From (59), the left-hand side of (58) does not depend on  $v^{(\ell)}$ ; hence, differentiating both sides twice with respect to  $v^{(\ell)}$  yields  $r_{v^{(\ell)}v^{(\ell)}}p_{u^{(k-1)}} = 0$ ; since  $p_{u^{(k-1)}} \neq 0$ ,  $r$  must be affine with respect to  $v^{(\ell)}$ . Let us call  $\sigma$  and  $\tau$  the functions of  $u, \dots, u^{(k-1)}, x, v, \dots, v^{(\ell-1)}$  such that

$$r = \tau + \sigma v^{(\ell)}. \quad (60)$$

Since  $p, q = p_x, \lambda$  and  $q_x = p_{xx}$  do not depend on  $v^{(\ell)}$ , (53) implies that  $\mathcal{Q}$  does not depend on  $v^{(\ell)}$  either; with  $p_x = q$ , and  $r$  given by (60), the expression of  $\mathcal{Q}_{v^{(\ell)}}$  is  $\sigma p_{xu^{(k-1)}} + p_{xv^{(\ell-1)}}$  while, from (58), the expression of  $\mathcal{P}_{v^{(\ell)}}$ . Collecting this, one gets

$$\sigma p_{xu^{(k-1)}} + p_{xv^{(\ell-1)}} = \sigma p_{xu^{(k-1)}} + p_{xv^{(\ell-1)}} = 0. \quad (61)$$

Then, the expressions defining  $\mathcal{P}$  and  $\mathcal{Q}$  before (53) yield, with  $r$  given by (60),  $\mathcal{P} = \lambda = Fp + \tau p_{u^{(k-1)}}$  and  $\mathcal{Q} = Fp_x + \tau p_{xu^{(k-1)}}$ . Both sides of (53) are affine with respect to  $\dot{x}$ , hence it yields two relations : the coefficient of  $\dot{x}$  yields (59) and the constant term yields  $\mathcal{Q} = h(x, p, p_x, \lambda)$ . Collecting this, we get

$$Fp - \gamma(x, p, p_x, p_{xx}) + p_{u^{(k-1)}}\tau = 0, \quad Fp_x - \delta(x, p, p_x, p_{xx}) + p_{xu^{(k-1)}}\tau = 0,$$

with  $\delta$  defined by (7); eliminating  $\tau$ , one gets (16-a), while (16-b) is obtained by eliminating  $\sigma$  in (61).

Now, to prove by contradiction  $K$ -regularity for some  $K \leq k + \ell + 1$ , assume that  $ED^i p = 0$  for  $0 \leq i \leq k + \ell$ . Then  $p_x, p, \dots, D^{(k+\ell-1)}p, x, \dots, x^{(k+\ell-1)}$  are  $2k+2\ell+1$  functions in the  $2k+2\ell$  variables  $u, \dots, u^{(k-1)}, v, \dots, v^{(\ell-1)}, x, \dots, x^{(k+\ell-1)}$ . At points where the Jacobian matrix has constant rank, there is at least one nontrivial

relation between them. From point 2 of Definition 2.2, this would imply that all solutions of system (3) satisfy this relation, say  $R(z(t), y(t), \dots, y^{(k+\ell-1)}(t), x(t), \dots, x^{(k+\ell-1)}(t)) = 0$ , which is absurd from Lemma 2.1.  $\square$

**6.2. The case where  $S$  and  $T$  are zero.** Again, we establish that, if  $J$  is not zero, any parameterization “derives from” a solution of the system of PDEs (16). However, the situation is slightly more complicated: there are two distinct (non equivalent) choices for  $\gamma$  and  $\delta$ .

If  $J \neq 0$ , we saw, in section 4, that possibly after a change of coordinates, system (3) can be written as (46), then we re-write here without the tildes:

$$\dot{z} = \kappa(x, y, z) (\dot{y} - \alpha(x, y, z) \dot{x}) (\dot{y} - \beta(x, y, z) \dot{x}) + a(x, y, z) \dot{x} + b(x, y, z) \dot{y} + c(x, y, z), \quad (62)$$

where  $\kappa, \alpha, \beta, a, b, c$  are real analytic functions of three variables and  $\kappa \neq 0$ ,  $\alpha - \beta \neq 0$ ,  $\partial\alpha/\partial x \neq 0$ ,  $\partial\beta/\partial x \neq 0$ . We state the theorem for this class of systems, because it is simpler to describe the two possible choices for  $\gamma$  and  $\delta$  than with (3), knowing that  $S = T = 0$ .

**Lemma 6.4.** *If system (62) admits a parameterization  $(\varphi, \psi, \chi)$  of order  $(k, \ell)$  at a point, then  $\varphi_{u^{(k)}}$  is a nonzero real analytic function.*

*Proof.* After a change of coordinates (48), use Lemma 6.1.  $\square$

**Theorem 6.5.** *Let  $(x_0, y_0, z_0)$  be a point where  $\kappa, \alpha - \beta, \alpha_3$  and  $\beta_3$  are nonzero, and  $k, \ell, L$  three integers. If system (62) has a parameterization of order  $(k, \ell)$  at  $\mathcal{X} = (x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dots, x_0^{(L)}, y_0^{(L)})$  with  $\varphi, \psi, \chi$  defined on  $U \subset \mathbb{R}^{k+\ell+2}$ , then  $k \geq 1, \ell \geq 1$  and, for any point  $(u_0, \dots, u_0^{(k)}, v_0, \dots, v_0^{(\ell)}) \in U$  (not necessarily sent to  $\mathcal{X}$  by the parameterization) such that*

$$\varphi_{u^{(k)}}(u_0, \dots, u_0^{(k)}, v_0, \dots, v_0^{(\ell)}) \neq 0,$$

*there exists a neighborhood  $O$  of  $(u_0, \dots, u_0^{(k-1)}, \varphi(u_0 \cdots v_0^{(\ell)}), v_0, \dots, v_0^{(\ell-1)})$  in  $\mathbb{R}^{k+\ell+1}$  and a regular solution  $p : O \rightarrow \mathbb{R}$  of one of the two systems  $(\mathcal{E}_{k,\ell}^{\gamma^1, \delta^1})$  or  $(\mathcal{E}_{k,\ell}^{\gamma^2, \delta^2})$  with  $\gamma^i, \delta^i$  given by (49)-(50), such that  $p, \varphi, \psi, \chi$  are related by (24), (25) and (26).*

Remark 6.3 applies to this theorem in the same way as theorem 6.2.

*Proof.* Like in the beginning of the proof of Theorem 6.2, a parameterization  $(\varphi, \psi, \chi)$  of order  $(k, \ell)$  with  $\varphi_{u^{(k)}} \neq 0$  yields an implicit form (52). Substituting in (62), one obtains an identity between two polynomials in  $v^{(\ell+1)}$  and  $\dot{x}$ . The coefficient of  $(v^{(\ell+1)})^2$  in the right-hand side must be zero and this yields that  $p$  cannot depend on  $v^{(\ell)}$ ; the linear term in  $v^{(\ell+1)}$  then implies that  $q$  does not depend on  $v^{(\ell)}$  either. To go further, let us define, as in the proof of Theorem 6.2,

$$\mathcal{P} = Fp + rp_{u^{(k-1)}} + v^{(\ell)} p_{v^{(\ell-1)}}, \quad \mathcal{Q} = Fq + rq_{u^{(k-1)}} + v^{(\ell)} q_{v^{(\ell-1)}}, \quad (63)$$

with  $F$  is defined in (14). The terms of degree 0, 1 and 2 with respect to  $\dot{x}$  then yield

$$\begin{aligned} \mathcal{Q} &= \kappa(x, p, q) \mathcal{P}^2 + b(x, p, q) \mathcal{P} + c(x, p, q) \\ q_x &= \kappa(x, p, q) (2p_x - \alpha(x, p, q) - \beta(x, p, q)) \mathcal{P} + a(x, p, q) + b(x, p, q) p_x \\ 0 &= (p_x - \alpha(x, p, q)) (p_x - \beta(x, p, q)) \end{aligned}$$

From (62), the factors in the third equation cannot both be zero. Let us assume

$$p_x - \alpha(x, p, q) = 0, \quad p_x - \beta(x, p, q) \neq 0; \quad (64)$$

for the other alternative, just interchange the roles of  $\alpha$  and  $\beta$  in the sequel. The first and second equations then yield

$$\mathcal{P} = \frac{q_x - a(x, p, q) - b(x, p, q)p_x}{\kappa(x, p, q)(p_x - \beta(x, p, q))}, \quad \mathcal{Q} = \kappa(x, p, q)\mathcal{P}^2 + b(x, p, q)\mathcal{P} + c(x, p, q). \quad (65)$$

Since  $\alpha_3 \neq 0$ , one can obtain from (64), by the inverse function theorem, an expression of  $q$  as a function of  $x$ ,  $p$  and  $p_x$  and hence of  $q_x$  as a function of  $x$ ,  $p$ ,  $p_x$  and  $p_{xx}$ . Substituting in (65) yields  $\mathcal{P} = f(x, p, p_x, p_{xx})$  for some smooth function  $f$ . Since  $\mathcal{P}$  is given by (63), one may apply Lemma C.1. Like in the proof of Theorem 6.2 we note that  $r_{v^{(\ell)}} = 0$  would imply  $\varphi_{v^{(\ell)}} = 0$  and contradict point 3 of Definition 2.2, and also that it would be absurd that  $p$  depend on  $x$  only because all solutions of (62) would satisfy a relation  $y(t) = p(x(t))$  (see Lemma 2.1); hence (113) holds.

From (65),  $\mathcal{P}$  does not depend on  $v^{(\ell)}$ . Since  $p_{u^{(k-1)}} \neq 0$ , (63) then implies that  $r$  assumes the form (60) with  $\sigma$  and  $\tau$  some functions of the  $k + \ell + 1$  variables  $u, \dot{u}, \dots, u^{(k-1)}, x, v, \dot{v}, \dots, v^{(\ell-1)}$  only. Since  $\mathcal{P}$  and  $\mathcal{Q}$  do not depend on  $v^{(\ell)}$ , we must have (we added a third equation is a consequence of (64) and the other two) :

$$\sigma p_{u^{(k-1)}} + p_{v^{(\ell-1)}} = 0, \quad \sigma q_{u^{(k-1)}} + q_{v^{(\ell-1)}} = 0, \quad \sigma p_{xu^{(k-1)}} + p_{xv^{(\ell-1)}} = 0, \quad (66)$$

and then  $\mathcal{P} = Fp + \tau p_{u^{(k-1)}}$  and  $\mathcal{Q} = Fq + \tau q_{u^{(k-1)}}$ . Differentiating (64), we get

$$p_{xx} = \alpha_1 + \alpha_2 p_x + \alpha_3 q_x, \quad Fp_x + \tau p_{xu^{(k-1)}} = \alpha_2 \mathcal{P} + \alpha_3 \mathcal{Q}.$$

Substituting this in (65) yields, with  $\gamma$  and  $\delta$  given by (49),

$$Fp + \tau p_{u^{(k-1)}} = \gamma(x, p, p_x, p_{xx}), \quad Fp_x + \tau p_{xu^{(k-1)}} = \delta(x, p, p_x, p_{xx}). \quad (67)$$

Eliminating  $\tau$  between these two equations and then  $\sigma$  between the first and last equations in (66) yields the two relations (16-a,b).

For the proof of  $K$ -regularity with  $K \leq k + \ell + 1$  and  $K \leq L + 1$ , see the end of the proof of Theorem 6.2 above.  $\square$

## 7. FLAT OUTPUTS AND DIFFERENTIAL FLATNESS

**Definition 7.1** (flat outputs, flatness, endogenous parameterization [7]). Let  $j$  be a non-negative integer. A pair  $A = (a, b)$  of real analytic functions defined on a connected open subset  $\widehat{U}$  of  $\widehat{\Omega} \times \mathbb{R}^{2j-2} \subset \mathbb{R}^{2j+3}$  is a *flat output of order  $j$*  for system (3) at point  $\mathcal{X} = (x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dots, x_0^{(L)}, y_0^{(L)})$ , with  $L \geq j$  if and only if, for some integers  $k, \ell$ , there exists a Monge parameterization of order  $(k, \ell)$  at this point  $\mathcal{X}$  and, for  $U$  and  $V$  possibly smaller than these given in Definition 2.2, one has the following property : if  $(x(\cdot), y(\cdot), z(\cdot)) \in \mathcal{V}$  is a germ of solution of (3), then the germ  $(x(\cdot), y(\cdot), z(\cdot), u(\cdot), v(\cdot)) \in \mathcal{V} \times \mathcal{U}$  satisfies (68) if and only if it satisfies (69) :

$$\left. \begin{aligned} \varphi(u(t), \dot{u}(t), \dots, u^{(k)}(t), v(t), \dot{v}(t), \dots, v^{(\ell)}(t)) &= x(t) \\ \psi(u(t), \dot{u}(t), \dots, u^{(k)}(t), v(t), \dot{v}(t), \dots, v^{(\ell)}(t)) &= y(t) \\ \chi(u(t), \dot{u}(t), \dots, u^{(k)}(t), v(t), \dot{v}(t), \dots, v^{(\ell)}(t)) &= z(t) \end{aligned} \right\} \quad (68)$$

$$\left. \begin{aligned} u(t) &= a(x(t), y(t), z(t), \dot{x}(t), \dot{y}(t), \ddot{x}(t), \ddot{y}(t), \dots, x^{(j)}(t), y^{(j)}(t)) \\ v(t) &= b(x(t), y(t), z(t), \dot{x}(t), \dot{y}(t), \ddot{x}(t), \ddot{y}(t), \dots, x^{(j)}(t), y^{(j)}(t)) \end{aligned} \right\} \quad (69)$$

System (3) is called *flat* if and only if it admits a flat output of order  $j$  for some  $j \in \mathbb{N}$ . We call a Monge parameterization *endogenous*<sup>3</sup> if and only if there exists a flat output associated to this parameterization as above.

<sup>3</sup>This terminology (endogenous vs. exogenous) is borrowed from the authors of [7, 14]; it usually qualifies feedbacks rather than parameterization, but the notion is exactly the same.

Although the focus of this paper is on Monge parameterization rather than flatness, it is interesting to tell when the parameterizations derived from a regular solution of  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$  according to Theorem 3.7 (from Section 5, these are the only possible parameterizations unless  $S = T = J = 0$ ) are endogenous, i.e. allow the system to be flat. The following proposition is given for the sake of completeness, but is not used anywhere in the paper.

**Proposition 7.2.** *Let  $p : O \rightarrow \mathbb{R}$ , with  $O \subset \mathbb{R}^{k+\ell+1}$  open, be a regular solution of system  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$ . The parameterization of order  $(k, \ell)$  of system (3) associated to  $p$  according to Theorem 3.7 is endogenous if and only if  $p$  is exactly  $(k+\ell-2)$ -regular; then, the associated flat output is of order  $j \leq k + \ell - 2$ .*

*Proof.* In the end of the proof of Theorem 3.7, it was established that  $\Gamma(u, v) = (x, y, z)$  (which is nothing but (68)) is equivalent to (36) if  $(x, y, z)$  is a solution of (3). If on the other hand it is equivalent to (69), then necessarily  $i_0 = k$  and  $j_0 = \ell$  because if either  $i_0 < k$  or  $j_0 < \ell$  then there are, for fixed  $x(\cdot), y(\cdot), z(\cdot)$ , infinitely many solutions  $u(\cdot), v(\cdot)$  of (36) while there is a unique one for (69). Then  $K = i_0 + j_0 - 2 = k + \ell - 2$  so that  $p$  is  $(k + \ell - 2)$ -regular and (36) (where  $u$  and  $v$  do not appear in the right-hand side) is of the form (69) with  $j = K = k + \ell - 2$ .  $\square$

In order to compare the results in the present paper with the results from [18], let us define what is called  $(x, u)$ -dynamic linearizability of system (1) in that reference ( $(x, u)$ -flatness would be more appropriate, or “endogenous” should be added). This notion means existence of a flat output whose components are functions of  $\xi^1, \xi^2, \xi^3, \xi^4, w^1, w^2$ . For system (3), this means existence of a flat output depending on  $x, y, z, \lambda, \dot{x}, \dot{\lambda}$ :

**Definition 7.3.** System (1) is “ $(x, u)$ -dynamic linearizable” if and only if system (3) admits a flat output of order 2 of a special kind :  $A(x, y, z, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}) = \mathbf{a}(x, y, z, \lambda, \dot{x}, \dot{\lambda})$  for some smooth  $\mathbf{a}$ .

In the following result, the functions  $\gamma$  and  $\delta$  in (1) are supposed to be related to the functions  $g$  and  $h$  in (3) according to (6) and (7).

**Proposition 7.4.** *If system (1) is “ $(x, u)$ -dynamic linearizable” in the sense of [18], then (3) admits a parameterization of order  $(k, \ell)$  with  $k \leq 3$  and  $\ell \leq 3$ .*

*Proof.* Consider the map

$$(x, y, z, \lambda, \dot{x}, \dot{\lambda}, \dots, x^{(4)}, \lambda^{(4)}) \mapsto \begin{pmatrix} \mathbf{a}(x, y, z, \lambda, \dot{x}, \dot{\lambda}) \\ \dot{\mathbf{a}}(x, y, z, \lambda, \dot{x}, \dot{\lambda}, \ddot{x}, \ddot{\lambda}) \\ \ddot{\mathbf{a}}(x, y, z, \lambda, \dot{x}, \dot{\lambda}, \dots, x^{(3)}, \lambda^{(3)}) \\ \mathbf{a}^{(3)}(x, y, z, \lambda, \dot{x}, \dot{\lambda}, \dots, x^{(4)}, \lambda^{(4)}) \end{pmatrix}$$

Its Jacobian is  $8 \times 12$ , and has rank 8, but the  $8 \times 8$  sub-matrix corresponding to derivatives with respect to  $\dot{x}, \dot{\lambda}, \dots, x^{(4)}, \lambda^{(4)}$  has rank 4 only. Hence  $x, y, z$ , and  $\lambda$  can be expressed as functions of the components of  $\mathbf{a}, \dot{\mathbf{a}}, \ddot{\mathbf{a}}, \mathbf{a}^{(3)}$ , yielding a Monge parameterization of order at most  $(3, 3)$ .  $\square$

#### APPENDIX A.

*Proof of Lemma 3.6.* For this proof only, the notation  $\mathcal{F}_{i,j}$  ( $0 \leq i \leq k, 0 \leq j \leq \ell$ ) stands either for the following family of  $i+j$  vectors in  $\mathbb{R}^{K+2}$  or for the corresponding  $(K+2) \times (i+j)$  matrix :

$$\mathcal{F}_{i,j} = \left( \frac{\partial \pi}{\partial u^{(k-i)}}, \dots, \frac{\partial \pi}{\partial u^{(k-1)}}, \frac{\partial \pi}{\partial v^{(\ell-j)}}, \dots, \frac{\partial \pi}{\partial v^{(\ell-1)}} \right)$$

with the convention that if  $i$  or  $j$  is zero the corresponding list is empty;  $\mathcal{F}_{i,j}$  depends on  $u, \dots, u^{(k-1)}, v, \dots, v^{(\ell-1)}, x, \dots, x^{(K)}$ . Let us first prove that, at least outside a closed subset of empty interior,

$$\text{Rank } \mathcal{F}_{k,\ell} = K + 2. \quad (70)$$

Indeed, if it is smaller at all points of  $O \times \mathbb{R}^K$ , then, around points (they form an open dense set) where it is locally constant, there is at least one function  $R$  such that a non-trivial identity  $R(p_x, p, \dots, D^K p, x, \dots, x^{(K)}) = 0$  holds and the partial derivative of  $R$  with respect to at least one of its  $K + 2$  first arguments is nonzero. Since  $p$  is  $K$ -regular, applying  $E$  to this relation, shows that  $R$  does not depend on  $D^K p$ , and hence does not depend on  $x^{(K)}$  either. Then, applying  $ED$ ,  $ED^2$  and so on, and using the fact that, according to (21),  $Dp_x$  is a function of  $p_x, p, Dp, x, \dot{x}$ , we get finally a relation  $R(p_x, p, x) = 0$  with  $(R_{p_x}, R_p) \neq (0, 0)$ . Differentiating with respect to  $u^{(k-1)}$ , one obtains  $R_{p_x} p_{x u^{(k-1)}} + R_p p_{u^{(k-1)}} = 0$ ; hence, from the first relation in (16-c),  $R_{p_x} \neq 0$ , and the relation  $R(p_x, p, x) = 0$  implies, in a neighborhood of almost any point,  $p_x = f(p, x)$  for some smooth function  $f$ . From Lemma 3.5, this would contradict the fact that the solution  $p$  is  $K$ -regular. This proves (70).

Let  $W_s$  ( $1 \leq s \leq K + 2$ ) be the set of pairs  $(i, j)$  such that  $i + j = s$  and the rank of  $\mathcal{F}_{i,j}$  is  $s$  at least at one point in  $O \times \mathbb{R}^K$ , *i.e.* one of the  $s \times s$  minors of  $\mathcal{F}_{i,j}$  is a nonzero real analytic function on  $O \times \mathbb{R}^K$ . The lemma states that  $W_{K+2}$  is nonempty; in order to prove it by contradiction, suppose that  $W_{K+2} = \emptyset$  and let  $\bar{s}$  be the smallest  $s$  such that  $W_s = \emptyset$ . From (16-c),  $W_1$  contains  $(1, 0)$ , hence  $2 \leq \bar{s} \leq K + 2 < k + \ell + 1$ . Take  $(i', j')$  in  $W_{\bar{s}-1}$ ;  $\mathcal{F}_{i',j'}$  has rank  $i' + j'$  (*i.e.* is made of  $i' + j'$  linearly independent vectors) on an open dense set  $A \subset O \times \mathbb{R}^K$ . Let the  $i_1 \leq k$  and  $j_1 \leq \ell$  be the largest such that  $\mathcal{F}_{i_1, j'}$  and  $\mathcal{F}_{i', j_1}$  have rank  $\bar{s} - 1$  on  $A$ . On the one hand, since  $i' + j' = \bar{s} - 1 < k + \ell$ , one has either  $i' < k$  or  $j' < \ell$ . On the other hand since  $W_{\bar{s}}$  is empty, it contains neither  $(i' + 1, j')$  nor  $(i', j' + 1)$ ; hence the rank of  $\mathcal{F}_{i'+1, j'}$  is less than  $i' + j' + 1$  if  $i' < k$ , and so is the rank of  $\mathcal{F}_{i', j'+1}$  if  $j' < \ell$ . To sum up,

$$i' < k \Rightarrow i_1 \geq i' + 1, \quad j' < \ell \Rightarrow j_1 \geq j' + 1.$$

From (70), one has either  $i_1 < k$  or  $j_1 < \ell$ . Possibly exchanging  $u$  and  $v$ , assume  $i_1 < k$ ; all the vectors  $\partial\pi/\partial u^{(k-i_1)}, \dots, \partial\pi/\partial u^{(k-i'+1)}, \partial\pi/\partial u^{(\ell-j_1)}, \dots, \partial\pi/\partial u^{(\ell-j'+1)}$  are then linear combinations of the vectors in  $\mathcal{F}_{i', j'}$ , while  $\partial\pi/\partial u^{(k-i_1-1)}$  is not :

$$\text{Rank } \mathcal{F}_{i', j'} = i' + j', \quad (71)$$

$$\text{Rank } \mathcal{F}_{i_1, j_1} = i' + j', \quad (72)$$

$$\text{Rank} \left( \frac{\partial\pi}{\partial u^{(k-i_1-1)}}, \mathcal{F}_{i', j'} \right) = i' + j' + 1 \quad (73)$$

on an open dense subset of  $O \times \mathbb{R}^K$ , that we still call  $A$  although it could be smaller. In a neighborhood of any point in this set, one can, from (73), apply the inverse function theorem and obtain, for an open  $\Omega \subset \mathbb{R}^{k+\ell+K+1}$ , a map  $\Omega \rightarrow \mathbb{R}^{i'+j'+1}$  that expresses  $u^{(k-i')}, \dots, u^{(k-1)}, v^{(\ell-j')}, \dots, v^{(\ell-1)}$  and  $u^{(k-i_1-1)}$  as functions of  $u, \dots, u^{(k-i_1-2)}, u^{(k-i_1)}, \dots, u^{(k-i'-1)}, v, \dots, v^{(\ell-j'-1)}, x, \dots, x^{(K)}$  and  $i' + j' + 1$  functions chosen among  $p_x, p, Dp, \dots, D^K p$  ( $i' + j' + 1$  columns defining an invertible minor in  $(\partial\pi/\partial u^{(k-i_1-1)}, \mathcal{F}_{i', j'})$ ). Focusing on  $u^{(k-i_1-1)}$ , one has

$$u^{(k-i_1-1)} = B \left( u, \dots, u^{(k-i_1-2)}, u^{(k-i_1)}, \dots, u^{(k-i'-1)}, \right. \\ \left. v, \dots, v^{(\ell-j'-1)}, x, \dots, x^{(K)}, p_x, p, \dots, D^K p \right) \quad (74)$$

where  $B$  is some smooth function of  $k + \ell + 2K + 2 - i' - j'$  variables and we have written all the functions  $p_x, p, Dp, \dots, D^{K-1}p$  although  $B$  really depends only on  $i' + j' + 1$  of them.

Differentiating (74) with respect to  $u^{(k-i')}, \dots, u^{(k-1)}, v^{(\ell-j')}, \dots, v^{(\ell-1)}$ , one has, with obvious matrix notation,  $\left( \frac{\partial B}{\partial p_x} \frac{\partial B}{\partial p} \cdots \frac{\partial B}{\partial D^{K-1}p} \right) \mathcal{F}_{i',j'} = 0$ , where the right-hand side is a line-vector of dimension  $i' + j'$ ; from (71) and (72), this implies that one also has

$$\left( \frac{\partial B}{\partial p_x} \frac{\partial B}{\partial p} \cdots \frac{\partial B}{\partial D^{K-1}p} \right) \mathcal{F}_{i_1, j_1} = 0, \quad (75)$$

where the right-hand side is now a bigger line-vector of dimension  $i_1 + j_1$ . Differentiating (74) with respect to  $u^{(k-i_1)}, \dots, u^{(k-i_1-1)}, v^{(\ell-j_1)}, \dots, v^{(\ell-j_1-1)}$  and using (75) yields that  $B$  does not depend on its arguments  $u^{(k-i_1)}, \dots, u^{(k-i_1-1)}$  and  $v^{(\ell-j_1)}, \dots, v^{(\ell-j_1-1)}$ .  $B$  cannot depend on  $D^K p$  either because  $ED^K p \neq 0$  and all the other arguments of  $B$  are constant along  $E$ ; then it cannot depend on  $x^{(K)}$  either because  $x^{(K)}$  appears in no other argument; (74) becomes

$$u^{(k-i_1-1)} = B\left(u, \dots, u^{(k-i_1-2)}, v, \dots, v^{(\ell-j_1-1)}, x, \dots, x^{(K-1)}, p_x, p, \dots, D^{K-1}p\right).$$

Applying  $D$ , using (21) and substituting  $u^{(k-i_1-1)}$  from above, one gets, from some smooth  $C$ ,

$$u^{(k-i_1)} = C\left(u, \dots, u^{(k-i_1-2)}, \underbrace{v, \dots, v^{(\ell-j_1)}}_{\text{empty if } j_1 = \ell}, x, \dots, x^{(K)}, p_x, p, \dots, D^K p\right). \quad (76)$$

Differentiating with respect to  $u^{(k-i')}, \dots, u^{(k-1)}, v^{(\ell-j')}, \dots, v^{(\ell-1)}$  yields  $\left( \frac{\partial C}{\partial p_x} \frac{\partial C}{\partial p} \cdots \frac{\partial C}{\partial D^{K-1}p} \right) \mathcal{F}_{i',j'} = 0$ , the right-hand side being a line-vector of dimension  $i' + j'$ . From (71) and (72),  $\partial\pi/\partial u^{(k-i_1-1)}$  is a linear combination of the columns of  $\mathcal{F}_{i',j'}$ , hence one also has

$$\left( \frac{\partial C}{\partial p_x} \frac{\partial C}{\partial p} \cdots \frac{\partial C}{\partial D^{K-1}p} \right) \frac{\partial\pi}{\partial u^{(k-i_1-1)}} = 0.$$

This implies that the derivative of the right-hand side of (76) with respect to  $u^{(k-i_1-1)}$  is zero. This is absurd.  $\square$

## APPENDIX B. PROOF OF LEMMAS 3.10, 3.11 AND 3.12

We need some notations and preliminaries. With  $F, E$  and  $\tau$  defined in (14) and (17), define the vector fields

$$X = \frac{\partial}{\partial x}, \quad Y = F + \tau \frac{\partial}{\partial u^{(k-1)}} \quad (77)$$

$$X_1 = [X, Y], \quad X_2 = [X_1, Y], \quad E_2 = [E, Y], \quad E_3 = [E_2, Y]. \quad (78)$$

Then (20) obviously implies

$$Yp = \gamma(x, p, p_x, p_{x,x}), \quad Yp_x = \delta(x, p, p_x, p_{x,x}), \quad X\sigma = 0, \quad (79)$$

and a simple computation yields (we recall  $E$  from (17)) :

$$\begin{aligned} X_1 &= \tau_x \frac{\partial}{\partial u^{(k-1)}}, \quad X_2 = \tau_x \frac{\partial}{\partial u^{(k-2)}} + (\cdots) \frac{\partial}{\partial u^{(k-1)}}, \\ E &= \frac{\partial}{\partial v^{(\ell-1)}} + \sigma \frac{\partial}{\partial u^{(k-1)}}, \quad E_2 = \frac{\partial}{\partial v^{(\ell-2)}} + \sigma \frac{\partial}{\partial u^{(k-2)}} + (\cdots) \frac{\partial}{\partial u^{(k-1)}}, \\ E_3 &= \frac{\partial}{\partial v^{(\ell-3)}} + \sigma \frac{\partial}{\partial u^{(k-3)}} + (\cdots) \frac{\partial}{\partial u^{(k-2)}} + (\cdots) \frac{\partial}{\partial u^{(k-1)}}. \end{aligned} \quad (80)$$

The vector field  $X_1$  and  $X_2$  are linearly independent because  $\tau_x \neq 0$ , see (20). Computing the following brackets and decomposing on  $X_1$  and  $X_2$ , one gets

$$[X, X_1] = \lambda X_1, \quad [X_1, X_2] = \lambda' X_1 + \lambda'' X_2, \quad (81)$$

$$\left. \begin{aligned} [X, E] = 0, \quad [X, E_2] = \mu X_1, \\ [X, E_3] = \mu' X_1 + \mu'' X_2, \quad [E_2, X_2] = \nu' X_1 + \nu'' X_2. \end{aligned} \right\} \quad (82)$$

for some functions  $\lambda, \lambda', \lambda'', \mu, \mu', \mu'', \nu', \nu''$ .

*Proof of Lemma 3.10.* From (16-c),  $y = p(u, \dots, u^{(k-1)}, x, v, \dots, v^{(\ell-1)})$  defines local coordinates  $u, \dots, u^{(k-2)}, y, x, v, \dots, v^{(\ell-1)}$ . Composing  $p_x$  by the inverse of this change of coordinates, there is a function  $\alpha$  of  $k + \ell + 1$  variables such that  $p_x = \alpha(u, \dots, u^{(k-2)}, p, x, \dots, v^{(\ell-1)})$  identically. Since  $Ep = Ep_x = 0$  (see (20)), applying  $E$  to both sides of this identity yields that  $\alpha$  does not depend on its argument  $v^{(\ell-1)}$ . Similarly, if  $k \geq 2$ , differentiating both sides of the same identity with respect to  $u^{(k-1)}$  and  $u^{(k-2)}$ , the fact that the determinant in the lemma is zero implies that  $\alpha$  does not depend on its argument  $u^{(k-2)}$ . To sum up,  $p$  and  $p_x$  satisfy an identity

$$p_x = \alpha(u, \dots, u^{(k-3)}, p, x, v, \dots, v^{(\ell-2)}),$$

where the first list is empty if  $k = 1$  or  $k = 2$ . Now define two integers  $m \leq k-3$  and  $n \leq \ell-2$  as the smallest such that  $\alpha$  depends on  $u, \dots, u^{(m)}, x, y, v, \dots, v^{(n)}$ , with the convention that  $m < 0$  if  $k = 1, k = 2$ , or  $\alpha$  depends on none of the variables  $u, \dots, u^{(k-3)}$  and  $n < 0$  if  $\alpha$  depends on none of the variables  $v, \dots, v^{(\ell-2)}$ .

Applying  $Y$  to both sides of the above identity yields

$$Yp_x = \alpha_y Yp + \sum_{i=0}^m u^{(i+1)} \alpha_{u^{(i)}} + \sum_{i=0}^n v^{(i+1)} \alpha_{v^{(i)}},$$

where, if  $m < 0$  or  $n < 0$ , the corresponding sum is empty. Using (79), since  $p_{xx} = \alpha_x + \alpha\alpha_y$ , one can replace  $Yp$  with  $\gamma(x, y, \alpha, \alpha_x + \alpha\alpha_y)$  and  $Yp_x$  with  $\delta(x, y, \alpha, \alpha_x + \alpha\alpha_y)$  in the above equation, where all terms except the last one of each non-empty sum therefore depend on  $u, \dots, u^{(m)}, x, y, v, \dots, v^{(n)}$  only. Differentiating with respect to  $u^{(m+1)}$  and  $v^{(n+1)}$  yields  $\alpha_{u^{(m)}} = \alpha_{v^{(n)}} = 0$ , which is possible only if  $m < 0$  and  $n < 0$ , hence the lemma.  $\square$

*Proof of Lemma 3.11.* From (39), setting

$$y = p(u, \dots, u^{(k-1)}, x, v, \dots, v^{(\ell-1)}), \quad z = p_x(u, \dots, u^{(k-1)}, x, v, \dots, v^{(\ell-1)}), \quad (83)$$

one gets some local coordinates  $(u, \dots, u^{(k-3)}, x, y, z, v, \dots, v^{(\ell-1)})$ . In these coordinates, the vector fields  $X$  and  $Y$  defined by (77) have the following expressions, where  $\chi$  and  $\alpha$  are some functions, to be studied further :

$$X = \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + \alpha \frac{\partial}{\partial z}, \quad (84)$$

$$Y = \gamma \frac{\partial}{\partial y} + \delta \frac{\partial}{\partial z} + \chi \frac{\partial}{\partial u^{(k-3)}} + \sum_{i=0}^{k-4} u^{(i+1)} \frac{\partial}{\partial u^{(i)}} + \sum_{i=0}^{\ell-1} v^{(i+1)} \frac{\partial}{\partial v^{(i)}}. \quad (85)$$

In the expression of  $Y$ , the third term is zero if  $k = 2$ , the fourth term ( $\sum_{i=0}^{k-4} \dots$ ) is zero if  $k = 2$  or  $k = 3$ , and the notations  $\gamma$  and  $\delta$  are slightly abusive :  $\gamma$  stands for the function

$$(u, \dots, u^{(k-3)}, x, y, z, v, \dots, v^{(\ell-1)}) \mapsto \gamma(x, y, z, \alpha(u, \dots, u^{(k-3)}, x, y, z, v, \dots, v^{(\ell-1)})),$$

and the same for  $\delta$ . With the same abuse of notations, (16-e) reads

$$X\gamma - \delta \neq 0. \quad (86)$$



The equalities  $(\sigma \frac{\partial}{\partial u^{(k-1)}} + \frac{\partial}{\partial v^{(\ell-1)}})u^{(k-2)} = \frac{\partial}{\partial x}u^{(k-2)} = \frac{\partial}{\partial u^{(k-1)}}u^{(k-2)} = 0$  are obvious in the original coordinates. Since the inverse of the change of coordinates (83) is given by

$$\begin{aligned} u^{(k-2)} &= \chi(u, \dots, u^{(k-3)}, x, y, z, v, \dots, v^{(\ell-1)}), \\ u^{(k-1)} &= Y\chi(u, \dots, u^{(k-3)}, x, y, z, v, \dots, v^{(\ell-1)}), \end{aligned}$$

and  $E$ ,  $X$  and  $X_1$  are given by (80), those equalities imply

$$E\chi = X\chi = X_1\chi = 0. \quad (87)$$

Then, from (78), (84) and (85),

$$X_1 = (X\gamma - \delta) \frac{\partial}{\partial y} + (X\delta - Y\alpha) \frac{\partial}{\partial z}, \quad (88)$$

$$[X, X_1] = (X^2\gamma - 2X\delta - Y\alpha) \frac{\partial}{\partial y} + (X^2\delta - XY\alpha - X_1\alpha) \frac{\partial}{\partial z}. \quad (89)$$

With these expressions of  $X$  and  $X_1$ , the first relation in (81) implies :

$$\begin{vmatrix} X\gamma - \delta & X^2\gamma - 2X\delta + Y\alpha \\ X\delta - Y\alpha & X^2\delta - XY\alpha - X_1\alpha \end{vmatrix} = 0. \quad (90)$$

The definition of  $\alpha$  implies  $Xz = \alpha$ . In the original coordinates, this translates into the identity  $p_{xx} = \alpha(u, \dots, u^{(k-3)}, x, p, p_x, v, \dots, v^{(\ell-1)})$ . Since  $Ep = Ep_x = Ep_{xx} = 0$  (see (20)), applying  $E$  to both sides of this identity yields that  $\alpha$  does not depend on its argument  $v^{(\ell-1)}$ . Also, if  $k \geq 3$ , differentiating both sides with respect to  $u^{(k-1)}$ ,  $u^{(k-2)}$  and  $u^{(k-3)}$ , we obtain that the determinant (38) is zero if and only if  $\alpha$  does not depend on its argument  $u^{(k-3)}$ . To sum up, under the assumptions of the lemma,

$$\alpha \text{ depends on } u, \dots, u^{(k-4)}, x, y, z, v, \dots, v^{(\ell-2)} \text{ only} \quad (91)$$

with the convention that the first list is empty if  $k = 2$  or  $k = 3$ . Now define two integers  $m \leq k - 4$  and  $n \leq \ell - 2$  as the smallest such that  $\alpha$  depends on  $u, \dots, u^{(m)}, x, y, v, \dots, v^{(n)}$ , with the convention that  $m < 0$  if  $k = 2$ ,  $k = 3$ , or  $\alpha$  depends on none of the variables  $u, \dots, u^{(k-4)}$ , and  $n < 0$  if  $\alpha$  depends on none of the variables  $v, \dots, v^{(\ell-2)}$ . We have

$$m \geq 0 \Rightarrow \alpha_{u^{(m)}} \neq 0, \quad n \geq 0 \Rightarrow \alpha_{v^{(n)}} \neq 0. \quad (92)$$

Since  $m$  is no larger than  $k - 4$ ,  $\chi$  does not appear in the expression of  $Y\alpha$  :

$$Y\alpha = \gamma\alpha_y + \delta\alpha_z + \sum_{i=0}^m u^{(i+1)}\alpha_{u^{(i)}} + \sum_{i=0}^n v^{(i+1)}\alpha_{v^{(i)}} \quad (93)$$

where the first (second) sum is empty if  $m$  ( $n$ ) is negative.

Returning to (90), we note that all the terms depend only on  $u, \dots, u^{(m)}, x, y, z, v, \dots, v^{(n)}$  except  $Y\alpha$ ,  $XY\alpha$  and  $X_1\alpha$  that depend on  $u^{(m+1)}$  if  $m \geq 0$  or on  $v^{(n+1)}$  if  $n \geq 0$ , in an affine manner according to the above formula; hence the determinant in (90) is a polynomial of degree two with respect to  $u^{(m+1)}$  and  $v^{(n+1)}$  with coefficients depending on  $u, \dots, u^{(m)}, x, y, z, v, \dots, v^{(n)}$  only, and the term of degree two comes from  $(Y\alpha)^2$ ; it is :

$$\left( \alpha_{u^{(m)}} u^{(m+1)} + \alpha_{v^{(n)}} v^{(n+1)} \right)^2.$$

Then (90) implies that  $\alpha_{u^{(m)}}$  and  $\alpha_{v^{(n)}}$  are identically zero and (92) then implies that  $m$  and  $n$  are negative, and by definition of these integers, this implies that  $\alpha$  depends on  $(x, y, z)$  only, yielding, in the original coordinates, an identity  $p_{xx} = \alpha(x, p, p_x)$ .  $\square$

Before proving Lemma 3.12, we need to extract more information from the previous proof :

**Lemma B.1.** *Assume, as in Lemma 3.11, that  $p$  is a solution of  $(\mathcal{E}_{k,\ell}^{\gamma,\delta})$  satisfying (39), but assume also that  $\ell \geq k \geq 3$  and the determinant (38) is nonzero. Then  $[X, E_2] = [X, E_3] = 0$ .*

*Proof.* Starting as in the proof of Lemma 3.11, one does not obtain (91) but, since (38) is nonzero,

$$\alpha \text{ depends on } u, \dots, u^{(k-4)}, x, y, z, v, \dots, v^{(\ell-2)} \text{ and } \alpha_{u^{(k-3)}} \neq 0. \quad (94)$$

Since  $Ep = Ep_x = 0$ , one has  $E = \partial/\partial v^{(\ell-1)}$  in these coordinates. The first equation in (87) then reads  $\chi_{v^{(\ell-1)}} = 0$ , and (78) and (85) yield

$$E_2 = \frac{\partial}{\partial v^{(\ell-2)}}, \quad [X, E_2] = -\alpha_{v^{(\ell-2)}} \frac{\partial}{\partial z}.$$

Since  $[X, E_2] = \mu X_1$  (see (82)), relations (88) and (86) imply that  $\alpha_{v^{(\ell-2)}}$ ,  $\mu$ , and the bracket  $[X, E_2]$  are zero, and prove the first part of the lemma. Let us turn to  $[X, E_3]$ : from (78) and (85), one gets, since  $E_2$  and  $X$  commute, and  $X\chi = 0$ ,

$$E_3 = \chi_{v^{(\ell-2)}} \frac{\partial}{\partial u^{(k-3)}} + \frac{\partial}{\partial v^{(\ell-3)}}, \quad [X, E_3] = -(E_3\alpha) \frac{\partial}{\partial z}. \quad (95)$$

In order to prove that  $E_3\alpha = 0$ , let us examine equation (90). For short, we use the symbol  $\mathcal{O}$  to denote *any function* that depends on  $u, \dots, u^{(k-3)}, x, y, z, v, \dots, v^{(\ell-3)}$  only. For instance,  $X\gamma - \delta = \mathcal{O}$ , and all terms in the determinant are of this nature, except the following three :

$$\begin{aligned} Y\alpha &= \chi \alpha_{u^{(k-3)}} + v^{(\ell-2)} \alpha_{v^{(\ell-3)}} + \mathcal{O}, \\ XY\alpha &= \chi X\alpha_{u^{(k-3)}} + v^{(\ell-2)} X\alpha_{v^{(\ell-3)}} + \mathcal{O}, \\ X_1\alpha &= -\alpha_z \left( \chi \alpha_{u^{(k-3)}} + v^{(\ell-2)} \alpha_{v^{(\ell-3)}} \right) + \mathcal{O} \end{aligned}$$

(we used  $X\chi = 0$ ). Setting  $\zeta = \chi \alpha_{u^{(k-3)}} + v^{(\ell-2)} \alpha_{v^{(\ell-3)}}$ , one has

$$X\zeta = \frac{X\alpha_{u^{(k-3)}}}{\alpha_{u^{(k-3)}}} \zeta + \mathbf{b} v^{(\ell-2)} \quad \text{with} \quad \mathbf{b} = X\alpha_{v^{(\ell-3)}} - \alpha_{v^{(\ell-3)}} \frac{X\alpha_{u^{(k-3)}}}{\alpha_{u^{(k-3)}}}, \quad (96)$$

and equation (90) reads

$$\zeta^2 + \mathcal{O}\zeta - (X\gamma - \delta) \mathbf{b} v^{(\ell-2)} + \mathcal{O} = 0. \quad (97)$$

Differentiating with respect to  $X$  and using (96) yields

$$2 \frac{X\alpha_{u^{(k-3)}}}{\alpha_{u^{(k-3)}}} \zeta^2 + \left( 2\mathbf{b} v^{(\ell-2)} + \mathcal{O} \right) \zeta + \mathcal{O} v^{(\ell-2)} + \mathcal{O} = 0.$$

Then, eliminating  $\zeta$  between these two polynomials yields the resultant

$$\begin{vmatrix} 1 & \mathcal{O} & -(X\gamma - \delta) \mathbf{b} v^{(\ell-2)} + \mathcal{O} & 0 \\ 0 & 1 & \mathcal{O} & -(X\gamma - \delta) \mathbf{b} v^{(\ell-2)} + \mathcal{O} \\ 2 \frac{X\alpha_{u^{(k-3)}}}{\alpha_{u^{(k-3)}}} & 2\mathbf{b} v^{(\ell-2)} + \mathcal{O} & \mathcal{O} v^{(\ell-2)} + \mathcal{O} & 0 \\ 0 & 2 \frac{X\alpha_{u^{(k-3)}}}{\alpha_{u^{(k-3)}}} & 2\mathbf{b} v^{(\ell-2)} + \mathcal{O} & \mathcal{O} v^{(\ell-2)} + \mathcal{O} \end{vmatrix} = 0.$$

This is a polynomial of degree at most three with respect to  $v^{(\ell-2)}$ , the coefficient of  $(v^{(\ell-2)})^3$  being  $-4\mathbf{b}^3(X\gamma - \delta)$ . Hence  $\mathbf{b} = 0$  and, from (97),  $\zeta$  does not depend on  $v^{(\ell-2)}$ . This implies  $E_3\alpha = 0$  because, from (95) and the definition of  $\zeta$ , one has  $\zeta_{v^{(\ell-2)}} = E_3\alpha$ .  $\square$

*Proof of Lemma 3.12.* The independent variables in  $(\mathcal{E}_{3,3}^{\gamma,\delta})$  are  $u, \dot{u}, \ddot{u}, x, v, \dot{v}, \ddot{v}$ . Since the determinant (38) is nonzero, one defines local coordinates  $(x, y, z, w, v, \dot{v}, \ddot{v})$  by

$$y = p(u, \dot{u}, \ddot{u}, x, v, \dot{v}, \ddot{v}), \quad z = p_x(u, \dot{u}, \ddot{u}, x, v, \dot{v}, \ddot{v}), \quad w = p_{xx}(u, \dot{u}, \ddot{u}, x, v, \dot{v}, \ddot{v}). \quad (98)$$

In these coordinates,  $X$  and  $Y$ , defined in (77), have the following expressions, with  $\psi$  and  $\alpha$  some functions to be studied further :

$$X = \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} + \alpha \frac{\partial}{\partial w}, \quad (99)$$

$$Y = \gamma \frac{\partial}{\partial y} + \delta \frac{\partial}{\partial z} + \psi \frac{\partial}{\partial w} + \dot{v} \frac{\partial}{\partial v} + \ddot{v} \frac{\partial}{\partial \dot{v}}. \quad (100)$$

Then, using, for short, the following notation  $\Gamma$  :

$$\Gamma = X\gamma - \delta \neq 0, \quad (101)$$

one has

$$X_1 = \Gamma \frac{\partial}{\partial y} + (X\delta - \psi) \frac{\partial}{\partial z} + (X\psi - Y\alpha) \frac{\partial}{\partial w}, \quad (102)$$

$$\begin{aligned} [X, X_1] &= (X\Gamma - X\delta + \psi) \frac{\partial}{\partial y} + (X^2\delta - 2X\psi + Y\alpha) \frac{\partial}{\partial z} \\ &\quad + (X^2\psi - XY\alpha - X_1\alpha) \frac{\partial}{\partial w}. \end{aligned} \quad (103)$$

Also,

$$E = \frac{\partial}{\partial \ddot{v}}, \quad E_2 = [E_1, Y] = \psi_{\ddot{v}} \frac{\partial}{\partial w} + \frac{\partial}{\partial \dot{v}}, \quad [X, E_2] = \psi_{\ddot{v}} \frac{\partial}{\partial z} + (X\psi_{u^{(k-1)}} - E_2\alpha) \frac{\partial}{\partial w}$$

but, from Lemma B.1, one has  $[X, E_2] = 0$ , hence  $\psi_{\ddot{v}} = 0$ ,  $E_2 = \partial/\partial \dot{v}$  and  $\alpha_{\dot{v}} = 0$ . Then

$$E_3 = [\frac{\partial}{\partial \dot{v}}, Y] = \psi_{\dot{v}} \frac{\partial}{\partial w} + \frac{\partial}{\partial v}, \quad [X, E_3] = \psi_{\dot{v}} \frac{\partial}{\partial z} + (X\psi_{\dot{v}} - E_3\alpha) \frac{\partial}{\partial w},$$

but, from Lemma B.1, one has  $[X, E_3] = 0$ , hence  $\psi_{\dot{v}} = 0$ ,  $E_3 = \partial/\partial v$  and  $\alpha_v = 0$ . To sum up,

$$E = \frac{\partial}{\partial \ddot{v}}, \quad E_2 = \frac{\partial}{\partial \dot{v}}, \quad E_3 = \frac{\partial}{\partial v}, \quad (104)$$

$\alpha$  depends at most on  $(x, y, z, w)$  only and  $\psi$  on  $(x, y, z, w, v)$ .

**Notation:** until the end of this proof,  $\mathcal{O}$  stands for *any* function of  $x, y, z, w$  only. For instance,  $\alpha = \mathcal{O}$ ,  $\gamma = \mathcal{O}$ ,  $\delta = \mathcal{O}$ ,  $\Gamma = \mathcal{O}$ ,  $X\Gamma = \mathcal{O}$ ,  $X\delta = \mathcal{O}$  and  $X^2\delta = \mathcal{O}$ .

From (81), (102) and (103), one has  $\begin{vmatrix} \Gamma & X\delta - \psi \\ X\Gamma - X\delta + \psi & X^2\delta - 2X\psi + Y\alpha \end{vmatrix} = 0$ .

Hence

$$X\psi = \frac{1}{2\Gamma} \psi^2 + \mathcal{O}\psi + \mathcal{O}. \quad (105)$$

We now write the expression (102) of  $X_1$  as

$$X_1 = X_1^0 + \psi X_1^1 + \psi^2 X_1^2 \quad (106)$$

$$\text{with } X_1^0 = \Gamma \frac{\partial}{\partial y} + \mathcal{O} \frac{\partial}{\partial z} + \mathcal{O} \frac{\partial}{\partial w}, \quad X_1^1 = -\frac{\partial}{\partial z} + \mathcal{O} \frac{\partial}{\partial w}, \quad X_1^2 = \frac{1}{2\Gamma} \frac{\partial}{\partial w}. \quad (107)$$

Note that  $X_1^0$ ,  $X_1^1$  and  $X_1^2$  are vector fields in the variables  $x, y, z, w$  only. Now define :

$$U = -X_1^1 - \frac{\psi}{\Gamma} \frac{\partial}{\partial w} = \frac{\partial}{\partial z} + \left( \mathcal{O} - \frac{\psi}{\Gamma} \right) \frac{\partial}{\partial w}, \quad (108)$$

$$V = X_1^0 - \psi^2 X_1^2 = \Gamma \frac{\partial}{\partial y} + \mathcal{O} \frac{\partial}{\partial z} + \left( \mathcal{O} - \frac{\psi^2}{2\Gamma} \right) \frac{\partial}{\partial w}, \quad (109)$$

so that

$$X_1 = V - \psi U \quad (110)$$

and, from (100) and (106) one deduces the following expression of  $X_2 = [X_1, Y]$  :

$$X_2 = (Y\psi)U + (X_1\psi) \frac{\partial}{\partial w} + \psi^3 \frac{\Gamma_w}{2\Gamma^2} \frac{\partial}{\partial w} + \psi^2 \left( \frac{\gamma_w}{2\Gamma} \frac{\partial}{\partial y} + \mathcal{O} \frac{\partial}{\partial z} + \mathcal{O} \frac{\partial}{\partial w} \right) + \psi X_2^1 + X_2^0 \quad (111)$$

where  $X_2^1$  and  $X_2^0$  are two vector fields in the variables  $x, y, z, w$  only.

This formula and (104) imply  $[E_2, X_2] = (Y\psi)_v U = \psi_v U$ ; hence, from the last relation in (82), either  $\psi_v$  is identically zero or  $U$  is a linear combination of  $X_1$  and  $X_2$ . We assume, until the end of the proof, that  $U$  is a linear combination of  $X_1$  and  $X_2$ . This implies, using (110), that  $X_2$  and  $X_1$  are linear combinations of  $U$  and  $V$ ; hence  $U, V$  is another basis for  $X_1, X_2$ . Also, from (81)  $[U, V]$  must be a linear combination of  $U$  and  $V$ . From (108) and (109),

$$[U, V] = \frac{X_1\psi}{\Gamma} \frac{\partial}{\partial w} - \psi^2 \mathcal{O} \frac{\partial}{\partial w} + \psi W^1 + W^0$$

where  $W^1$  and  $W^0$  are two vector fields in the variables  $x, y, z, w$  only, and, finally, with  $Z^1$  and  $Z^0$  two other vector fields in the variables  $x, y, z, w$  only, one has, from (111)

$$X_2 - (Y\psi)U - \Gamma[U, V] = \psi^3 \frac{\Gamma_w}{2\Gamma^2} \frac{\partial}{\partial w} + \psi^2 \left( \frac{\gamma_w}{2\Gamma} \frac{\partial}{\partial y} + \mathcal{O} \frac{\partial}{\partial z} + \mathcal{O} \frac{\partial}{\partial w} \right) + \psi Z^1 + Z^0.$$

This vector field is also a linear combination of  $U$  and  $V$ . Computing the determinant in the basis  $\partial/\partial y, \partial/\partial z, \partial/\partial w$ , one has, using (108) and (109),

$$\det(U, V, X_2 - (Y\psi)U - \Gamma[U, V]) = \frac{\gamma_w}{\Gamma^3} \psi^4 + \mathcal{O}\psi^3 + \mathcal{O}\psi^2 + \mathcal{O}\psi + \mathcal{O} = 0.$$

This polynomial of degree 4 with respect to  $\psi$  is nontrivial because, from (15),  $\gamma_w \neq 0$ ; its coefficients depend on  $x, y, z, w$  only, hence  $\psi$  cannot depend on  $v$ .

We have proved that, in any case, both  $\alpha$  and  $\psi$  depend on  $x, y, z, w$  only, and this yields the desired identities in the lemma.  $\square$

### APPENDIX C.

**Lemma C.1.** *Let  $p$  be a smooth function of  $u, \dots, u^{(k-1)}, x, v, \dots, v^{(\ell-1)}$ ,  $r$  a smooth function of  $u, \dots, u^{(k-1)}, x, v, \dots, v^{(\ell)}$ , with  $r_{v^{(\ell)}} \neq 0$ , and  $f$  a smooth function of four variables such that*

$$\sum_{i=0}^{k-2} u^{(i+1)} p_{u^{(i)}} + r p_{u^{(k-1)}} + \sum_{i=0}^{\ell-1} v^{(i+1)} p_{v^{(i)}} = f(x, p, p_x, p_{xx}) \quad (112)$$

where, by convention,  $r p_{u^{(k-1)}}$  is zero if  $k = 0$  and the first (resp. last) sum is zero if  $k \leq 1$  (resp.  $\ell = 0$ ). Then either  $p$  depends on  $x$  only or

$$k \geq 1, \quad \ell \geq 1, \quad p_{u^{(k-1)}} \neq 0, \quad p_{v^{(\ell-1)}} \neq 0. \quad (113)$$

*Proof.* Let  $m \leq k-1$  and  $n \leq \ell-1$  be the smallest integers such that  $p$  depends on  $u, \dots, u^{(m)}, x, v, \dots, v^{(n)}$ ; if  $p$  depends on none of the variables  $u, \dots, u^{(k-1)}$  (or  $v, \dots, v^{(\ell-1)}$ ), take  $m < 0$  (or  $n < 0$ ). Then  $p_{u^{(m)}} \neq 0$  if  $m \geq 0$  and  $p_{v^{(n)}} \neq 0$  if  $n \geq 0$ .

The lemma states that either  $m < 0$  and  $n < 0$  or  $k \geq 1, \ell \geq 1$  and  $(m, n) = (k-1, \ell-1)$ . This is indeed true :

- if  $m = k-1$  and  $k \geq 1$  then  $n = \ell-1$  and  $\ell \geq 1$  because if not, differentiating both sides in (112) with respect to  $v^{(\ell)}$  would yield  $r_{v^{(\ell)}} p_{u^{(k-1)}} = 0$ , but the lemma assumes that  $r_{v^{(\ell)}} \neq 0$ ,

- if  $m < k-1$  or  $m = 0$ , (112) becomes :  $\sum_{i=0}^m u^{(i+1)} p_{u^{(i)}} + \sum_{i=0}^n v^{(i+1)} p_{v^{(i)}} = f(x, p, p_x, p_{xx})$ ; if  $m \geq 0$ , differentiating with respect to  $u^{(m+1)}$  yields  $p_{u^{(m)}} = 0$

and if  $n \geq 0$ , differentiating with respect to  $u^{(m+1)}$  yields  $p_{v^{(n)}} = 0$ ; hence  $m$  and  $n$  must both be negative.  $\square$

**Acknowledgements.** The authors are very grateful to their colleague *José Grimm* at INRIA Sophia Antipolis for an extremely careful reading of the manuscript that led to many improvements.

#### REFERENCES

- [1] E. Aranda-Bricaire, C. H. Moog, and J.-B. Pomet. An infinitesimal Brunovsky form for nonlinear systems with applications to dynamic linearization. In *Geometry in Nonlinear Control and Differential Inclusions*, volume 32 of *Banach Center Publications*, pages 19–33. Polish Acad. Sci., Warsaw, 1995.
- [2] D. Avanesoff. –, PhD thesis, INRIA Sophia Antipolis, 2005. under preparation.
- [3] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmitt, and P. A. Griffiths. *Exterior Differential Systems*, volume 18 of *Mathematical Sciences Research Institute Publications*. Springer-Verlag, 1991.
- [4] É. Cartan. Sur l'intégration de certains systèmes indéterminés d'équations différentielles. *J. für reine und angew. Math.*, 145:86–91, 1915.
- [5] B. Charlet, J. Lévine, and R. Marino. On dynamic feedback linearization. *Syst. & Control Lett.*, 13:143–151, 1989.
- [6] B. Charlet, J. Lévine, and R. Marino. Sufficient conditions for dynamic state feedback linearization. *SIAM J. on Control and Optim.*, 29:38–57, 1991.
- [7] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Sur les systèmes non linéaires différentiellement plats. *C. R. Acad. Sci. Paris, Série I*, 315:619–624, 1992.
- [8] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Flatness and defect of nonlinear systems: Introductory theory and examples. *Int. J. Control*, 61(6):1327–1361, 1995.
- [9] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems. *IEEE Trans. Automat. Control*, 44(5):922–937, 1999.
- [10] M. Fliess, J. Lévine, P. Martin, and P. Rouchon. Some open questions related to flat nonlinear systems. In *Open problems in mathematical systems and control theory*, pages 99–103. Springer, London, 1999.
- [11] M. Golubitsky and V. Guillemin. *Stable mappings and their singularities*. Springer-Verlag, New York, 1973. Graduate Texts in Mathematics, Vol. 14.
- [12] D. Hilbert. Über den Begriff der Klasse von Differentialgleichungen. *Math. Annalen*, 73:95–108, 1912.
- [13] A. Isidori, C. H. Moog, and A. de Luca. A sufficient condition for full linearization via dynamic state feedback. In *Proc. 25th IEEE Conf. on Decision & Control, Athens*, pages 203–207, 1986.
- [14] P. Martin. *Contribution à l'étude des systèmes différentiellement plats*. PhD thesis, L'École Nationale Supérieure des Mines de Paris, 1992.
- [15] P. Martin, R. M. Murray, and P. Rouchon. Flat systems. In *Mathematical control theory, Part 1, 2 (Trieste, 2001)*, ICTP Lect. Notes, VIII, pages 705–768 (electronic). Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002.
- [16] P. Martin and P. Rouchon. Feedback linearization and driftless systems. *Math. of Control, Signals & Systems*, 7:235–254, 1994.
- [17] J.-B. Pomet. A differential geometric setting for dynamic equivalence and dynamic linearization. In *Geometry in Nonlinear Control and Differential Inclusions*, volume 32 of *Banach Center Publications*, pages 319–339. Polish Acad. Sci., Warsaw, 1995.
- [18] J.-B. Pomet. On dynamic feedback linearization of four-dimensional affine control systems with two inputs. *ESAIM: Control, Optim. & C. of Variations*, 2:151–230, June 1997. URL: <http://www.edpsciences.com/cocv/>.
- [19] P. Rouchon. Flatness and oscillatory control: some theoretical results and case studies. Tech. report PR412, CAS, Ecole des Mines, Paris, Sept. 1992.
- [20] P. Rouchon. Necessary condition and genericity of dynamic feedback linearization. *J. of Math. Systems, Estimation, and Control*, 4:1–14, 1994.
- [21] W. M. Sluis. A necessary condition for dynamic feedback linearization. *Syst. & Control Lett.*, 21:277–283, 1993.
- [22] M. van Nieuwstadt, M. Rathinam, and R. Murray. Differential flatness and absolute equivalence of nonlinear control systems. *SIAM J. on Control and Optim.*, 36(4):1225–1239, 1998. <http://epubs.siam.org:80/sam-bin/dbq/article/27402>.
- [23] P. Zervos. Le problème de Monge. *Mémoires des Sciences Mathématiques*, LIII, 1932.