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► **To cite this version:**

Jesús Mendoza-Avila, Denis Efimov, José Ángel Mercado Uribe, Johannes Schiffer. Fitting Characterizations of Integral ISS Notions for Multistable State Periodic Systems. *Automatica*, In press. <hal-05570587>

HAL Id: hal-05570587

<https://inria.hal.science/hal-05570587v1>

Submitted on 27 Mar 2026

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Fitting Characterizations of Integral ISS Notions for Multistable State Periodic Systems [★]

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Abstract

Novel characterizations of the integral Input-to-State Stability (iISS) property are provided for multistable systems with periodic dynamics in a part of the state. The concepts of *iISS-Leonov functions* and *output smooth dissipativity* are introduced, and their equivalence to the properties of *bounded-energy-bounded-state* and *global attractiveness* of unperturbed solutions are proven. Furthermore, the strong iISS property is examined in terms of the presented iISS-Leonov functions. The proposed approach represents a relaxed characterization of the iISS property for the considered class of systems, in contrast to conventional results, since iISS-Leonov functions are not necessarily positive definite. Finally, the usefulness of the theoretical results is illustrated by a robustness analysis of a synchronous generator with saturated control input.

Key words: Nonlinear systems, Robustness, Multistable system, Integral input-to-state stability, and Lyapunov function methods

1 Introduction

New developments in power electronics, biological and chemical systems involve the design of complex processes with highly nonlinear dynamics, whose analysis requires advanced approaches, for example, see [3,24]. In this context, the Lyapunov function method has been accepted as the main tool for the analysis of stability of dynamical systems [8,19,21,26]. This approach offers an elegant characterization of boundedness and convergence of the system's trajectories with respect to an invariant set via the existence of a continuously differentiable (or at least Lipschitz continuous) Lyapunov function, which is positive definite with respect to such an invariant set and whose time derivative is negative definite along the trajectories of the system under study.

Classical results on Lyapunov stability are mainly applicable to systems with a single equilibrium point at the origin. However, many engineering applications deal with systems with multiple equilibria or invariant sets, e.g., power or biological systems [3,11]. Hence, extensions of the Lyapunov stability approach and related notions have been developed for systems with several disjoint invariant sets [5,12,19,31]. Nevertheless, these extensions are far from being trivial, and require significant modifications and relaxations of the employed stability notions, see for example [1,4,12,34]. Therefore, the study of stability and robustness of strongly nonlinear systems continues attracting considerable attention in the control systems community.

The concept of Input-to-State Stability (ISS) [40] has been found very useful for the analysis of robust stability of nonlinear systems affected by external inputs, e.g., exogenous disturbances or measurement noises. Moreover, a weaker notion of ISS has been introduced with the concept of integral ISS (iISS) [6]. Qualitatively, we say that iISS is a relaxed property because it indicates robustness against disturbances with finite energy, while

[★] This paper was not presented at any IFAC meeting. Corresponding author Jesus Mendoza Avila.

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the former implies robustness against bounded disturbances. Although iISS is referred to as a “weak” concept, the results of [6] have shown that it is a natural property of well-behaved systems, so that, it is indeed a very useful notion. An intermediate property between ISS and iISS is called strong iISS [7]. This property is a combination of ISS with respect to small inputs and iISS, which means robustness against inputs with large amplitude but bounded energy and any input with a sufficiently small amplitude. Naturally, the connections between ISS notions and the Lyapunov function method have been widely studied, see for example the survey papers [9] and [41].

Recently, extensions of the ISS property in [2] and of the iISS property in [18] have been developed for systems with multiple invariant solutions constituting a decomposable set.¹ These works provides necessary and sufficient conditions for the global analysis of the corresponding property in terms of the existence of a nonnegative (taking zero values only at some elements of the decomposable invariant set) and continuously differentiable (in the manifold where the system evolves) ISS- or iISS-Lyapunov function, respectively, with a negative definite (with respect to such an invariant set) time-derivative along the solutions of the system under investigation.

A particular class of systems with multiple equilibria is formed by periodic dynamics with respect to a part of the state. Some examples of this class of systems include the forced nonlinear pendulum [17], phase-lock loops [24,25], microgrids [10,16,39], and power systems [11,33]. Unfortunately, the aforementioned Lyapunov-based approaches cannot be straightforwardly applied for the study of stability and robustness properties of this class of state periodic systems, then a more sophisticated solution is required. For instance, the implementation of the results in [2] or [18] leads to the construction of periodic Lyapunov functions with respect to the part of the state in which the system is periodic. This fact represents a strong limitation and in the most of the cases just local stability results can be obtained, see for example [11,33].

Recently, the concept of *ISS-Leonov functions* was introduced in [15] as a fitting characterization of the ISS property for multistable state periodic systems. Typically, ISS-Leonov functions are sign-indefinite, and their derivative is required to be non-positive only in the region of the state space where the respective function is nonnegative. Hence, they significantly relax the study of the ISS property for multistable state periodic systems, in contrast to the results of [2] based on ISS-Lyapunov functions in the multistable sense. The result of [15] is based on the so-called *cell structure* approach, which provides a fitting tool to establish global boundedness

of solutions for unforced periodic systems with a scalar periodic state variable [23] (see also [19,32,42]). Lately, the work [14] presents a multivariable version of the cell structure approach for systems that are periodic with respect to multiple state variables and introduces the concept of *Leonov functions*: sign-indefinite functions with respect to the periodic states, radially unbounded with respect to the non-periodic ones, and whose derivative is sign-definite. This approach has been successfully applied to establish almost global stability properties of power systems and microgrids in [36,37] and to ensure global synchronization of networks of heterogeneous Kuramoto oscillators in [29,30].

Then, motivated by [14,15] and [18], this paper is devoted to the development of a novel characterization of the iISS property for multistable system with periodic dynamics in a part of the state. For this purpose, we present the concepts of an *iISS-Leonov function* and *output smooth dissipativity*, and prove their equivalence to the properties of bounded-energy-bounded-state and global attractiveness of solutions in the absence of disturbances. Those notions were introduced by [18] as an appropriated definition of iISS for system with multiple equilibria (including periodic dynamics). Partial results of this research have been already published in [28]. The novelty of this paper is summarized as follows:

- Detailed proofs of the equivalence between the introduced properties and iISS notions in the multistable sense are presented.
- Refinements on the definition of iISS-Leonov functions are realized to strengthen the theoretical foundation of our results.
- An extension of our main results that establishes a connection between the proposed iISS-Leonov function framework and the original Leonov function approach of [14,35] is developed. This result may simplify the application of the proposed approach in practical cases, as it is illustrated in Section 5.
- The strong iISS property is examined in terms of iISS-Leonov functions for multistable state periodic systems, completing the characterization of ISS related properties within the Leonov function framework.
- The proposed approach is applied for analysis of robustness in a synchronous generator with saturated control input.

The remainder of the paper is structured as follows. In Section 2, the considered class of systems, useful definitions and preliminary results are presented. Section 3 contains the main results of our investigation. Section 4 is devoted to study the concept of strong iISS in the framework of iISS-Leonov functions. An analysis of robustness in a synchronous generator with saturated control input is developed in Section 5. The conclusions are given in Section 6. Finally, the proofs of some of our results are presented in Appendices A and B.

¹ In the sense of [31].

1.1 Notation

\mathbb{N} , \mathbb{Z} and \mathbb{R} stand for the sets of natural, integer and real numbers, respectively. Furthermore, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ represents the set of nonnegative real numbers.

Following [21], we say that a function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is of class \mathcal{K} if it is continuous, strictly increasing and $\alpha(0) = 0$. The function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is of class \mathcal{K}_∞ if $\alpha \in \mathcal{K}$ and it is unbounded. Besides, a function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is of class \mathcal{L} if it is continuous, decreasing and $\lim_{t \rightarrow \infty} \alpha(t) = 0$. Moreover, a continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to the class \mathcal{KL} if $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each fixed $t \in \mathbb{R}_+$, and $\beta(s, \cdot) \in \mathcal{L}$ for each fixed $s > 0$.

The notation $DV(\tilde{x})v$ stands for the directional (or Dini) derivative of a continuously differentiable (or locally Lipschitz continuous) function $V : \mathbb{R}^n \rightarrow \mathbb{R}$, in the direction of the vector $v \in \mathbb{R}^n$, and evaluated at the point $\tilde{x} \in \mathbb{R}^n$.

For a vector $\tilde{x} \in \mathbb{R}^n$, $|\tilde{x}|$ stands for the Euclidean norm and $|\tilde{x}|_\infty = \max_{1 \leq i \leq n} |\tilde{x}_i|$ represents the infinity norm, which satisfies $|\tilde{x}|_\infty \leq |\tilde{x}| \leq \sqrt{n}|\tilde{x}|_\infty$. Besides, for a locally essentially bounded and measurable signal $d : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ the essential supremum norm is defined as $\|d\|_\infty = \text{ess sup}_{t \geq 0} |d(t)|$, and the set of such signals with a finite norm is further denoted by \mathcal{L}_∞^m .

2 Preliminaries

Let a map $f : M \times \mathbb{R}^m \rightarrow T_x M$ be continuously differentiable with respect to its arguments, where M is a n -dimensional smooth connected orientable Riemannian manifold without boundary and $T_x M$ is its tangent space. Then, consider the system

$$\dot{x}(t) = f(x(t), d(t)), \quad t \geq 0, \quad (1)$$

where $x(t) \in M$ is the state and $d \in \mathcal{L}_\infty^m$ is an input signal. Moreover, for any $x_0 \in M$ and $d \in \mathcal{L}_\infty^m$, we denote by $X(t, x_0; d)$ the uniquely defined solution of (1) at time t fulfilling $X(0, x_0; d) = x_0$. Together with (1), consider its unperturbed version

$$\dot{x}(t) = f(x(t), 0). \quad (2)$$

In the sequel, the time variable t will be omitted for compactness, when the context allows it. Hence, a set $S \subset M$ is invariant for the unperturbed system (2) if $X(t, x_0; 0) \in S$ for all $t \in \mathbb{R}$ and for all $x_0 \in S$. For $x_0 \in M$, the point $y \in M$ belongs to its ω -limit (α -limit) set if there is a sequence t_i , $\lim_{i \rightarrow +\infty} t_i = +\infty$, such that $\lim_{i \rightarrow +\infty} X(t_i, x_0; 0) = y$ ($\lim_{i \rightarrow +\infty} X(-t_i, x_0; 0) = y$). For any $x_0 \in M$, the α - and ω -limit sets of the unperturbed system (2) are invariant [20]. Furthermore, the distance from a point $x \in M$ to a set $S \subset M$ is defined as $|x|_S = \inf_{a \in S} \varpi(x, a)$, where the symbol $\varpi(x_1, x_2)$ denotes the Riemannian distance between x_1 and x_2 in M , $|x| = |x|_0$ for $x \in M$ (0 is a point selected on M).

2.1 Decomposable sets

Let $\Lambda \subset M$ be a compact invariant set of the system (2).

Definition 1 [31] *A decomposition of Λ is a finite and disjoint family of compact invariant sets $\Lambda_1, \dots, \Lambda_k$ such that $\Lambda = \cup_{i=1}^k \Lambda_i$. Each Λ_i is called an atom.*

For an atom Λ_i , its attracting and repulsing sets are given by

$$\begin{aligned} \mathfrak{A}(\Lambda_i) &= \{x \in M : |X(t, x; 0)|_{\Lambda_i} \rightarrow 0 \text{ as } t \rightarrow +\infty\}, \\ \mathfrak{R}(\Lambda_i) &= \{x \in M : |X(t, x; 0)|_{\Lambda_i} \rightarrow 0 \text{ as } t \rightarrow -\infty\}. \end{aligned}$$

Definition 2 [31] *Let $\Lambda_1, \dots, \Lambda_k$ be a decomposition of Λ . For any $i \neq j$, the relation $\Lambda_i \prec \Lambda_j$ means $\mathfrak{A}(\Lambda_i) \cap \mathfrak{R}(\Lambda_j) \neq \emptyset$. Then,*

- (1) *An r -cycle ($r \geq 2$) is an ordered r -tuple of distinct indices i_1, \dots, i_r such that $\Lambda_{i_1} \prec \dots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$.*
- (2) *A 1-cycle is an index i so that $\mathfrak{A}(\Lambda_i) \cap \mathfrak{R}(\Lambda_i) \setminus \Lambda_i \neq \emptyset$.*
- (3) *A filtration ordering is a numbering of Λ_i so that $\Lambda_i \prec \Lambda_j \implies i \leq j$.*

Following [20], the existence of an r -cycle ($r \geq 2$) implies the existence of a heteroclinic orbit for the system (2). Likewise, the existence of a 1-cycle implies the existence of a homoclinic orbit for the system (2).

Definition 3 [2] *The set Λ is called decomposable if it admits a finite decomposition without cycles, $\Lambda = \cup_{i=1}^k \Lambda_i$, for some nonempty disjoint compact sets Λ_i , which form a filtration ordering of Λ .*

Then, we establish the following assumption for an invariant set $\mathcal{W} \subset M$ of the system (2).

Assumption 4 *Let a set $\mathcal{W} \subset M$ containing all α - and ω -limit sets of (2) be compact and decomposable.*

2.2 Robustness notions for multistable systems

The following notions of robustness have been introduced by [2] and [18] for multistable systems like (1), with an invariant set \mathcal{W} satisfying Assumption 4 (for details on the standard ISS and iISS frameworks see [6,9]).

Definition 5 [2] *The system (1) has the practical asymptotic gain (pAG) property if there exist $\nu \in \mathcal{K}_\infty$ and $c_1 \geq 0$, such that for all $x_0 \in M$ and all $d \in \mathcal{L}_\infty^m$ the solutions are defined for all $t \geq 0$ and*

$$\limsup_{t \rightarrow \infty} |X(t, x_0; d)|_{\mathcal{W}} \leq \nu(\|d\|_\infty) + c_1.$$

If $c_1 = 0$, then we say that the asymptotic gain (AG) property holds. Moreover, if $d \equiv 0$, then the AG property implies the so-called zero-global attraction (0-GATT) property for the unperturbed system (2) to the set \mathcal{W} .

Definition 6 [18] *The system (1) is said to have the uniform bounded-energy bounded-state (UBEBS) property if, for some class \mathcal{K}_∞ functions χ, ς and η , and a positive constant c_2 , the following estimate holds for all $t \geq 0$, all $x_0 \in M$ and all $d \in \mathcal{L}_\infty^m$:*

$$\chi(|X(t, x_0; d)|_{\mathcal{W}}) \leq \varsigma(|x_0|_{\mathcal{W}}) + \int_0^t \eta(|d(s)|) ds + c_2. \quad (3)$$

Definition 7 [2] *Suppose that the system (1) satisfies Assumption 4 and possesses the pAG property. Then, it is said to be ISS in the multistable sense with respect to the set \mathcal{W} and the input $d \in \mathcal{L}_\infty^m$.*

Definition 8 [18] *Under Assumption 4, the system (1) is said to be iISS in the multistable sense with respect to \mathcal{W} if it possesses the UBEBS and 0-GATT properties.*

The lack of Lyapunov stability in the general case of multistable systems is the main difference between the classical notions of ISS and iISS [6,40] and the ones presented above. The importance of Assumption 4 to deal with this issue can be appreciated in the following example. Consider a system with a single equilibrium point at the origin. If such a system is practically ISS but the origin is locally unstable, there may exist a limit cycle or a chaotic trajectory. We cannot say that the system is ISS in the classical sense but it could be ISS in the multistable sense, under Assumption 4.

Definition 9 [2,18] *A \mathcal{C}^1 function $L : M \rightarrow \mathbb{R}_+$ is said to be a practical ISS-Lyapunov function for the system (1) if there exist functions $\alpha_1, \alpha_2, \alpha_3, \gamma \in \mathcal{K}_\infty$, and scalars $\ell, r \geq 0$, such that, the inequalities*

$$\alpha_1(|x|_{\mathcal{W}}) \leq L(x) \leq \alpha_2(|x|_{\mathcal{W}} + \ell) \quad (4)$$

holds for all $x \in M$, and the dissipation inequality

$$DL(x)f(x, d) \leq -\alpha_3(|x|_{\mathcal{W}}) + \gamma(|d|) + r \quad (5)$$

is satisfied for all $(x, d) \in M \times \mathbb{R}^m$. If $r = 0$ in (5), then L is said to be an ISS-Lyapunov function. Moreover, if α_3 is just a positive definite function and $r = 0$, then L is said to be an iISS-Lyapunov function.

Next, we present an extract of the main theorems of [2,18], which establish the equivalence between the aforementioned robustness notions and the existence of practical ISS- and iISS-Lyapunov functions as in Definition 9.

Theorem 10 [2] *Consider a nonlinear system as in (1) and let \mathcal{W} be as in Assumption 4. Then, the following properties are equivalent:*

- (1) pAG;
- (2) AG;
- (3) existence of a (practical) ISS-Lyapunov function;
- (4) existence of an ISS-Lyapunov function L , such that $DL(x) = 0$ for all $x \in \mathcal{W}$.

Theorem 11 [18] *Consider a nonlinear system as in (1) and let \mathcal{W} be as in Assumption 4. Then, the following properties are equivalent:*

- (1) 0-GATT and UBEBS;
- (2) existence of a smooth iISS-Lyapunov function L such that $DL(x) = 0$ for all $x \in \mathcal{W}$;
- (3) existence of an \mathcal{C}^1 iISS-Lyapunov function L .

Remark 12 *In Theorems 10 and 11, only compactness of the set \mathcal{W} is necessary from Assumption 4 to show that the existence of an ISS- or iISS-Lyapunov function L , such that $DL(x) = 0$ for all $x \in \mathcal{W}$, implies the rest of the properties, respectively. Indeed, decomposability of the set \mathcal{W} can be proven with this kind of ISS- or iISS-Lyapunov function (see the proof of [2, Claim 3]). However, the converse results require all conditions of Assumption 4. Particularly, the equivalence of the pAG and AG properties is based on the decomposability of \mathcal{W} (see the proof of [2, Claim 4]).*

2.3 Boundedness of solutions of state periodic systems

In this subsection, we start by assuming that $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ in (1) and (2), and we will denote its argument by $\tilde{x} \in \mathbb{R}^n$. Consider $k, q \in \mathbb{N}$, such that $n = k + q$. Let $\tilde{z} \in \mathbb{R}^k$ and $\tilde{\theta} \in \mathbb{R}^q$ be two subsets of the state \tilde{x} , i.e., $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$. Then, let us establish the following special feature over the system (2):

Assumption 13 *The vector field f in (2) is 2π -periodic with respect to $\tilde{\theta}$, i.e., for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$*

$$f(\tilde{x}, 0) = f(\tilde{x} + \xi_j, 0), \quad (6)$$

where $\xi_j = [0_k^\top \ 2\pi j^\top]^\top \in \mathbb{R}^n$, with 0_k being a zero vector of dimension k and $j \in \mathbb{Z}^q$.

The systems that satisfy Assumption 13 are called state periodic systems. Under this assumption, we can define a manifold $M = \mathbb{R}^k \times \mathbb{S}^q$, where \mathbb{S} is the unit sphere, and a projection $\mathcal{P} : \mathbb{R}^n \rightarrow M$, that is just a modulus of the $\tilde{\theta}$ coordinates over 2π . Therefore, for any $\tilde{x} \in \mathbb{R}^n$ we have $\mathcal{P}(\tilde{x}) = x$, where $x = (z, \theta) \in M$, with $z \in \mathbb{R}^k$ and $\theta \in \mathbb{S}^q$. Hence, the system (2) can be embedded in a manifold M , i.e., its solutions $\tilde{X}(t, \tilde{x}_0; 0) \in \mathbb{R}^n$, with $\tilde{x}_0 \in \mathbb{R}^n$, can be projected to the trajectories $X(t, x_0; 0) \in M$, with $x_0 = \mathcal{P}(\tilde{x}_0) \in M$, by applying $X(t, x_0; 0) = \mathcal{P}(\tilde{X}(t, \tilde{x}_0; 0))$, such that both solutions are defined on the same time interval. Similarly, the set $\tilde{\mathcal{W}} \subset \mathbb{R}^n$, containing all α - and ω -limit sets of the system (2), can be projected to $\mathcal{W} \subset M$ by using the periodicity of the last q variables. Note that \mathcal{W} can be compact in M but the set $\tilde{\mathcal{W}}$ becomes unbounded in \mathbb{R}^n . Moreover, $|\tilde{x}|_{\tilde{\mathcal{W}}} = \inf_{y \in \tilde{\mathcal{W}}} |\tilde{x} - y|$ represents a distance from a point $\tilde{x} \in \mathbb{R}^n$ to the set $\tilde{\mathcal{W}}$, where $|\tilde{x}| = |\tilde{x}|_{\{0\}}$

is the usual Euclidean norm for a vector $\tilde{x} \in \mathbb{R}^n$. Since the elements of the set \mathcal{W} are uniformly distributed in the whole \mathbb{R}^n due to the periodicity of (2), then we have $|x|_{\mathcal{W}} = |\tilde{x}|_{\tilde{\mathcal{W}}}$ provided that $\mathcal{P}(\tilde{x}) = x$.

Remark 14 *In Assumption 13, we can consider any interval of periodicity. For a very general case, we can define a matrix $\Delta = \text{diag}(\Delta_1, \Delta_2, \dots, \Delta_q)$, with $\Delta_i > 0$ ($i = 1, \dots, q$), such that the system (2) is assumed to be Δ -periodic with respect to $\tilde{\theta}$, i.e., for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$:*

$$f(\tilde{x}, 0) = f(\tilde{x} + \xi_j, 0), \quad (7)$$

where $\xi_j = [0_k^\top (\Delta_j)^\top]^\top \in \mathbb{R}^n$ with $j \in \mathbb{Z}^q$. Then, the vector field f can exhibit different kinds of periodicity in each component of the vector $\tilde{\theta} \in \mathbb{R}^q$.

Now, consider a change of coordinates $\tilde{y} = S\tilde{x}$, where $S = \text{diag}(\mathbf{I}_k, 2\pi\Delta^{-1})$, with \mathbf{I}_k being the identity matrix of dimension k . The system (2) in coordinates \tilde{y} becomes

$$\dot{\tilde{y}} = Sf(S^{-1}\tilde{y}, 0) = Sf(S^{-1}(\tilde{y} + \tilde{\xi}_j), 0),$$

where $\tilde{\xi}_j = S\xi_j = [0_k^\top 2\pi j^\top]^\top$. Thus, the vector field $f(\cdot, 0)$ is Δ -periodic in coordinates \tilde{x} , and 2π -periodic in coordinates \tilde{y} . Hence, without losing generality, we only consider 2π -periodicity in Assumption 13.

The cell structure approach allows us to establish boundedness of solutions for state periodic systems satisfying Assumption 13 for $q = 1$. It was introduced by [23] and [32], independently, and further developed by [19,42]. This approach can be seen as a fine handling of attractiveness and periodicity, which leads to boundedness of trajectories. That is, the existence of a sign-indefinite function $U_0(\tilde{x})$, whose time derivative satisfies $\dot{U}_0(\tilde{x}) \leq -2\rho U_0(\tilde{x})$ for all $t \geq 0$, where $\rho > 0$, implies attractiveness and forward invariance of the set $\Omega_0 = \{\tilde{x} \in \mathbb{R}^n : U_0(\tilde{x}) \leq 0\}$, i.e., $\tilde{X}(t, \tilde{x}_0; 0) \in \Omega_0$ for all $t \in [0, +\infty]$ provided that $\tilde{x}_0 \in \Omega_0$. Then, the functions $U_j(\tilde{x}) = U_j((\tilde{z}, \tilde{\theta})) = U_0((\tilde{z}, \tilde{\theta} + 2\pi j))$ are defined for all $j \in \mathbb{Z}$, and by the periodicity of (2), it is proven that the sets $\Omega_j = \{\tilde{x} \in \mathbb{R}^n : U_j(\tilde{x}) \leq 0\}$ are also attractive and forward invariant. Finally, it is shown that the intersections of the sets Ω_j form a kind of cell cover of the domain of the variable $\tilde{\theta}$, where each cell is isolated, attractive and forward invariant. Hence, for any $\tilde{x}_0 \in \mathbb{R}^n$, the solution $\tilde{X}(t, \tilde{x}_0)$ of (2) asymptotically enters and remains in one of those cells.

Recently, the works in [13,14] introduced the concept of Leonov functions to develop a generalization of the cell structure approach for systems whose dynamics are periodic with respect to multiple state variables, i.e., for the case when the system (2) satisfies Assumption 13 for $q > 1$. These results have been further extended in [35].

Consider the sets: $\mathcal{U} = \cup_{r \in \mathbb{Z}^+} \mathcal{U}_r$, where $\mathcal{U}_r = \{\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n : |\tilde{\theta}|_\infty = 2r\pi, f(\tilde{x}, 0) \equiv 0\}$, and $\mathcal{V} \supseteq \{\tilde{x} \in \mathbb{R}^n : \underline{c} \leq |\tilde{\theta}|_\infty \leq \bar{c}\}$ for some $\underline{c} \in (0, 2\pi)$ and $\bar{c} \in (\pi, 2\pi)$, with $\underline{c} < \bar{c}$. Additionally, the equilibrium set of (2) is given by $\mathcal{D} = \{\tilde{x} \in \mathbb{R}^n : f(\tilde{x}, 0) \equiv 0\}$, and consequently, $\mathcal{U} \subseteq \mathcal{D} \subseteq \tilde{\mathcal{W}}$. Then, let us present the following corollary from [35].

Corollary 15 [35] *If for the system (2), under Assumptions 4 and 13, there exists a \mathcal{C}^1 function $U : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying*

$$U(0) \leq 0, \quad \inf_{\tilde{x} \in \mathcal{U} \setminus \mathcal{U}_0} U(\tilde{x}) \leq 0, \quad \inf_{\tilde{x} \in \mathcal{V}} U(\tilde{x}) > 0,$$

and a dissipation inequality

$$DU(\tilde{x})f(\tilde{x}, 0) < 0, \quad \forall \tilde{x} \in \mathbb{R}^n \setminus \{\mathcal{D} \cup \{\tilde{x} \in \mathbb{R}^n : U(\tilde{x}) \leq 0\}\},$$

then for all $\tilde{x}_0 \in \mathbb{R}^n$ the corresponding trajectories $\tilde{X}(t, \tilde{x}_0)$ are bounded for all $t \geq 0$.

2.4 ISS-Leonov functions

The concept of practical ISS-Leonov functions was originally introduced by [15] as a fitting characterization of the pAG property for multistable state periodic systems. In general, practical ISS-Leonov functions are sign-indefinite and non-periodic permitting to address global analysis of the ISS property for the system (1) with respect to \mathcal{W} , under Assumptions 4 and 13. Here, we present a refined definition of practical ISS-Leonov functions. To this end, let us introduce a set $\mathcal{W}' \subseteq \mathcal{W}$ (then $\tilde{\mathcal{W}}'$ and $\tilde{\mathcal{W}}$ are their projections in \mathbb{R}^n , respectively).

Definition 16 *A \mathcal{C}^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a practical ISS-Leonov function for the system (1) if there exist functions $\vartheta_1, \vartheta_2, \sigma_1, \gamma_1, \lambda_1 \in \mathcal{K}_\infty$, and scalars $g_1, \ell_1, r_1 \geq 0$, such that the inequalities*

$$\vartheta_1(|\tilde{x}|_{\tilde{\mathcal{W}}'}) - \sigma_1(|\tilde{\theta}|) - g_1 \leq V(\tilde{x}) \leq \vartheta_2(|\tilde{x}|_{\tilde{\mathcal{W}}'}) + \ell_1 \quad (8)$$

hold for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$, and the dissipation inequality

$$DV(\tilde{x})f(\tilde{x}, d) \leq -\lambda_1(V(\tilde{x})) + \gamma_1(|d|) + r_1 \quad (9)$$

holds for all $\tilde{x} \in \{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) \geq 0\}$ and all $d \in \mathbb{R}^m$. If the latter inequality holds for $r_1 = 0$, then V is said to be an ISS-Leonov function.

The definition of practical ISS-Leonov functions in [15] considers $g_1 = 0$, $\mathcal{W}' = \mathcal{W}$, and the dissipation inequality (9) holding for all $\tilde{x} \in \mathbb{R}^n$, with $\lambda_1 \in \mathcal{K}_\infty$ only for nonnegative arguments. Thus, Definition 16 enlarges the family of practical ISS-Leonov functions. The next Lemma validates the effectiveness of ISS-Leonov function as in Definition 16 to characterize the ISS property in the multistable sense.

Lemma 17 [15] *Let Assumptions 4 and 13 be satisfied by the system (1) and its invariant set \mathcal{W} . Then, the existence of a practical ISS-Leonov function implies the pAG property with respect to \mathcal{W} .*

PROOF. The proof of Lemma 17 was originally given in [15] by using a more restrictive class of practical ISS-Leonov functions. Now, let us show that the same proof can be repeated with a practical ISS-Leonov function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ as described in Definition 16.

Take $\tilde{x}_0 \in \mathbb{R}^n$ and consider a solution $\tilde{X}(t, \tilde{x}_0; d)$ of the system (1), which is defined at least locally in time for $t \in [0, T)$, with some $T > 0$. Then, we split this interval of time as $[0, T) = \mathcal{T}_+ \cup \mathcal{T}_-$, where

$$\begin{aligned}\mathcal{T}_+ &= \{t \in [0, T) : V(\tilde{X}(t, \tilde{x}_0; d)) \geq 0\}, \\ \mathcal{T}_- &= \{t \in [0, T) : V(\tilde{X}(t, \tilde{x}_0; d)) < 0\}.\end{aligned}$$

Notice that for any $t \in \mathcal{T}_-$ and any isolated $t \in \mathcal{T}_+$ we have $V(\tilde{X}(t, \tilde{x}_0; d)) \leq 0$. Hence, from (8) we obtain

$$\vartheta_1(|\tilde{X}(t, \tilde{x}_0; d)|_{\tilde{\mathcal{W}}'}) \leq \sigma_1(|\tilde{\theta}(t)|) + g_1. \quad (10)$$

If $t \in \mathcal{T}_+$ is not an isolated element of \mathcal{T}_+ , then there is an interval $[t_s, t_e] \subset \mathcal{T}_+$ with $V(\tilde{X}(t, \tilde{x}_0; d)) \geq 0$ for all $t \in [t_s, t_e]$. Therefore, on $[t_s, t_e]$ the function V satisfies the definition of practical ISS-Lyapunov functions and we can apply the standard results of the ISS theory [41]. Accordingly, there exists a function $\beta \in \mathcal{KL}$ such that for all $t \in [t_s, t_e]$:

$$\begin{aligned}V(\tilde{X}(t, \tilde{x}_0; d)) &\leq \beta(V(\tilde{X}(t_s, \tilde{x}_0; d)), t - t_s) \\ &\quad + \lambda_1^{-1}(\gamma_1(\|d\|_\infty) + r_1).\end{aligned} \quad (11)$$

By definition, if $t_s = 0$, then $V(\tilde{X}(t_s, \tilde{x}_0; d)) \geq 0$, while if $t_s > 0$ then $V(\tilde{X}(t_s, \tilde{x}_0; d)) = 0$. Hence, combining (8), (10) and (11), we have for all $t \in [0, T)$ that

$$\begin{aligned}\vartheta_1(|\tilde{X}(t, \tilde{x}_0; d)|_{\tilde{\mathcal{W}}'}) &\leq \beta(V(\tilde{x}_0), t) + \sigma_1(|\tilde{\theta}(t)|) \\ &\quad + g_1 + \lambda_1^{-1}(\gamma_1(\|d\|_\infty) + r_1),\end{aligned}$$

Then, consider the projections of the trajectory $\tilde{X}(t, \tilde{x}_0; d) \in \mathbb{R}^n$ to $X(t, x_0; d) \in M$, and the set $\tilde{\mathcal{W}}' \subset \mathbb{R}^n$ to $\mathcal{W}' \subset M$. Therefore, $|\tilde{X}(t, \tilde{x}_0; d)|_{\tilde{\mathcal{W}}'} = |X(t, x_0; d)|_{\mathcal{W}'}$, and the latter inequality is also satisfied in M . Moreover, since $\mathcal{W}' \subseteq \mathcal{W}$, then $|X(t, x_0; d)|_{\mathcal{W}} \leq |X(t, x_0; d)|_{\mathcal{W}'}$. Besides, due to compactness of the set \mathcal{W} , there exists a constant $b \geq 0$, such that $|x_0|_{\mathcal{W}'} \leq |x_0|_{\mathcal{W}} + b$. Thus, since $\vartheta_1 \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$, we obtain

$$\begin{aligned}\vartheta_1(|X(t, \tilde{x}_0; d)|_{\mathcal{W}}) &\leq \beta(\vartheta_2(|x_0|_{\mathcal{W}} + b + \ell_1), t) + g_1 \\ &\quad + \sigma_1(|\tilde{\theta}(t)|) + \lambda_1^{-1}(\gamma_1(\|d\|_\infty) + r_1).\end{aligned}$$

After this step, the proof of [15, Lemma 1] can be repeated just as it was originally developed. \square

3 Main results: iISS-Leonov functions and equivalent properties

Inspired by [14,15] and [18], we present the following two definitions as new characterizations of the iISS property in the multistable sense for periodic systems evolving in \mathbb{R}^n . Recall the set $\mathcal{W}' \subseteq \mathcal{W}$.

Definition 18 *A \mathcal{C}^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be an iISS-Leonov function for the system (1) if there exist functions $\vartheta_3, \vartheta_4, \sigma_2, \gamma_2 \in \mathcal{K}_\infty$, a continuous positive definite function $\lambda_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and scalars $g_2, \ell_2 \geq 0$, such that for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$:*

$$\vartheta_3(|\tilde{x}|_{\tilde{\mathcal{W}}'}) - \sigma_2(|\tilde{\theta}|) - g_2 \leq V(\tilde{x}) \leq \vartheta_4(|\tilde{x}|_{\tilde{\mathcal{W}}'} + \ell_2), \quad (12)$$

and a dissipation inequality

$$DV(\tilde{x})f(\tilde{x}, d) \leq -\lambda_2(V(\tilde{x})) + \gamma_2(\|d\|) \quad (13)$$

holds for all $\tilde{x} \in \{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) \geq 0\}$ and $d \in \mathbb{R}^m$.

Definition 19 *The system (1) is said to be output smoothly dissipative (OSD) if there exist a \mathcal{C}^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}$, functions $\vartheta_3, \vartheta_4, \sigma_2, \gamma_3 \in \mathcal{K}_\infty$, a continuous nonnegative function $\lambda_3 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, constants $g_2, \ell_2 \geq 0$, and a continuous output map $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$, such that V satisfies inequality (12) for all $\tilde{x} \in \mathbb{R}^n$, and a dissipation inequality*

$$DV(\tilde{x})f(\tilde{x}, d) \leq -\lambda_3(|h(\tilde{x})|) + \gamma_3(\|d\|) \quad (14)$$

holds for all $\tilde{x} \in \{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) \geq 0\}$ and $d \in \mathbb{R}^m$.

These two definitions have been introduced in [28] for the case $\mathcal{W}' = \mathcal{W}$. However, due to Assumption 4 we have that $|\tilde{x}|_{\tilde{\mathcal{W}}'} \leq |\tilde{x}|_{\tilde{\mathcal{W}}} \leq |\tilde{x}|_{\tilde{\mathcal{W}}'} + b$ for some $b \geq 0$. Hence, the present versions are more general.

The next two lemmas represent the key contributions of this work. Their proofs are presented in Appendix A.

Lemma 20 *Let Assumptions 4 and 13 be satisfied by the system (1) and its invariant set \mathcal{W} . Then, the existence of an iISS-Leonov function for (1) implies the 0-GATT and UBEBS properties with respect to \mathcal{W} .*

Lemma 21 *Let Assumptions 4 and 13 be satisfied by the system (1) and its invariant set \mathcal{W} . If the system (1) is endowed with the OSD property, then it also possesses the 0-GATT and UBEBS properties with respect to \mathcal{W} .*

Our main result about the characterization of the iISS property for periodic system by iISS-Leonov functions is established in the next theorem.

Theorem 22 *Let Assumptions 4 and 13 be satisfied by the system (1) and its invariant set \mathcal{W} . Then, for the system (1) the following properties are equivalent:*

- (a) *iISS property in the multistable sense with respect to the set \mathcal{W} ;*
- (b) *existence of an iISS-Leonov function;*
- (c) *existence of an output function that makes the system OSD.*

PROOF. By Definition 8, the system (1) is said to have the iISS property in the multistable sense if it satisfies the UBEBS and 0-GATT properties. By Theorem 11, we have that if the system (1) possesses the UBEBS and 0-GATT properties with respect to the set \mathcal{W} , then there exists a \mathcal{C}^1 iISS-Lyapunov function $L : M \rightarrow \mathbb{R}_+$ as in Definition 9 where $M = \mathbb{R}^k \times \mathbb{S}^q$. Due to periodicity of the vector field f in (1), L is also 2π -periodic with respect to $\tilde{\theta} \in \mathbb{R}^q$, i.e., $L((z, \theta - 2\pi j)) = L((\tilde{z}, \tilde{\theta})) = L(\tilde{x})$ where $j = [j_1, \dots, j_q] \in \mathbb{Z}^q$. Hence, from Definition 9 we have that for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$:

$$\alpha_1(|\tilde{x}|_{\tilde{\mathcal{W}}}) \leq L(\tilde{x}) \leq \alpha_2(|\tilde{x}|_{\tilde{\mathcal{W}}} + \ell), \quad (15)$$

and

$$DL(\tilde{x})f(\tilde{x}, d) \leq -\alpha_3(|\tilde{x}|_{\tilde{\mathcal{W}}}) + \gamma(|d|), \quad (16)$$

with $\tilde{\mathcal{W}}$ being extended from $\mathcal{W} \subset M$ to \mathbb{R}^n by using the periodicity of (1) in θ , and considering $|x|_{\mathcal{W}} = |\tilde{x}|_{\tilde{\mathcal{W}}}$. Clearly, L satisfies all conditions of Definition 19 with $\mathcal{W}' = \mathcal{W}$, $h(x) := \alpha_3(|\tilde{x}|_{\tilde{\mathcal{W}}})$, and λ_3 equal to the identity function. Thus, we conclude that (a) implies (c).

From (15), we can see that $L(\tilde{x}) \leq \alpha_2(\ell)$ for all $\tilde{x} \in \tilde{\mathcal{W}}$. Hence, consider a constant $g_2 \geq \alpha_2(2\ell)$ and the function $V(\tilde{x}) = L(\tilde{x}) - g_2$, such that $V(\tilde{x}) \leq 0$ for all $\tilde{x} \in \tilde{\mathcal{W}}$, and

$$\alpha_1(|\tilde{x}|_{\tilde{\mathcal{W}}}) - g_2 \leq V(\tilde{x}) \leq \alpha_2(2|\tilde{x}|_{\tilde{\mathcal{W}}}), \quad (17)$$

for all $\tilde{x} \in \mathbb{R}^n$. Moreover, by invoking [6, Lemma IV.1], we have that for the positive definite function α_3 in (16) there exists $\alpha_{31} \in \mathcal{K}_\infty$ and $\alpha_{32} \in \mathcal{L}$, such that

$$\alpha_3(|\tilde{x}|_{\tilde{\mathcal{W}}}) \geq \alpha_{31}(|\tilde{x}|_{\tilde{\mathcal{W}}})\alpha_{32}(|\tilde{x}|_{\tilde{\mathcal{W}}}).$$

Then, considering (17), we define the function

$$\hat{\alpha}_3(s) = \alpha_{31}(0.5\alpha_2^{-1}(s))\alpha_{32}(\alpha_1^{-1}(2 \max\{s, g_2\})),$$

which is positive definite and satisfies the condition $\hat{\alpha}_3(V(\tilde{x})) \leq \alpha_3(|\tilde{x}|_{\tilde{\mathcal{W}}})$ for all $\tilde{x} \in \{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) \geq 0\}$. Therefore, from inequality (16) we obtain

$$DV(\tilde{x})f(\tilde{x}, d) \leq -\hat{\alpha}_3(V(\tilde{x})) + \gamma(|d|), \quad (18)$$

for all $\tilde{x} \in \{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) \geq 0\}$. Due to (17) and (18), we can say that V satisfies Definition 18. Thus, it is an iISS-Leonov function for (1). Then, (a) implies (b).

From Lemma 20, we have that the existence of an iISS-Leonov function for the system (1) implies that it admits the UBEBS and 0-GATT properties with respect to \mathcal{W} . Then, by Definition 8 and Assumption 4, we can say that the system (1) has the iISS property in the multistable sense. Therefore, (b) implies (a). Similarly, by Lemma 21 we can conclude that (c) implies (a). Finally, all equivalencies have been justified, which completes the proof. \square

Remark 23 *Note that Definition 19 allows to set $\lambda_3 \equiv 0$. This condition is equivalent to the so-called zero-output smooth dissipativity (0-OSD) property [6]. In general, 0-OSD is not a characterization of the classical iISS property but we can see in the proof of Lemma 21 that it is enough to conclude the UBEBS property for the system (1), under Assumption 13.*

Remark 24 *In results based on iISS-Lyapunov functions, the 0-GATT property is concluded by using a weak zero-detectability condition [18]. Differently, the conclusion of 0-GATT in the previous results based on iISS-Leonov functions is obtained by exploiting Assumption 4.*

Now, let us introduce the following assumption.

Assumption 25 *For the system (1), there exists a function $\varepsilon \in \mathcal{K}_\infty$, such that $|f_\theta(\tilde{x}, d) - f_\theta(\tilde{x}, 0)| \leq \varepsilon(|d|)$ for all $\tilde{x} \in \mathbb{R}^n$ and $d \in \mathbb{R}^m$, where f_θ denotes the components of the vector field f describing the dynamics of the periodic states $\tilde{\theta}$.*

Then, by combining Corollary 15 with Definition 19 and Lemma 21, we present the following theorem, which relaxes the conditions for the function V in Definition 19, and may simplify the verification of the OSD property in practice, as we can see in the case study of Section 5. Recall the sets \mathcal{U} , \mathcal{V} and \mathcal{D} defined before Corollary 15.

Theorem 26 *Let Assumptions 4, 13 and 25 be satisfied by the system (1) and its invariant set \mathcal{W} . If there exist a \mathcal{C}^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}$, functions $\vartheta_5, \vartheta_6, \sigma_3, \gamma_4 \in \mathcal{K}_\infty$, a continuous positive definite function $\lambda_4 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, constants $g_4, \ell_3 \geq 0$, and a continuous output map $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$, with $h(\tilde{x}) \equiv 0$ if and only if $|\tilde{x}|_{\mathcal{D}} = 0$, such that the conditions*

$$\begin{aligned} \vartheta_5(|\tilde{z}|) - \sigma_3(|\tilde{\theta}|) - g_4 \leq V(\tilde{x}) \leq \vartheta_6(|\tilde{x}| + \ell_3), \\ V(0) \leq 0, \quad \inf_{\tilde{x} \in \mathcal{U} \setminus \mathcal{U}_0} V(\tilde{x}) \leq 0, \quad \inf_{\tilde{x} \in \mathcal{V}} V(\tilde{x}) > 0 \end{aligned} \quad (19)$$

hold for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$, and the dissipation inequality

$$DV(\tilde{x})f(\tilde{x}, d) \leq -\lambda_4(|h(\tilde{x})|) + \gamma_4(|d|) \quad (20)$$

holds for all $\tilde{x} \in \{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) \geq 0\}$ and $d \in \mathbb{R}^m$, then the system (1) possesses the 0-GATT and UBEBS properties with respect to \mathcal{W} .

The proof of Theorem 26 is given in Appendix B. Note that the function $V(\tilde{x})$ in Theorem 26 satisfies all conditions of Corollary 15 for $d \equiv 0$, therefore it allows us to ensure that the trajectories $\tilde{X}(t, \tilde{x}_0; 0)$ of the system (2) are bounded for all $\tilde{x}_0 \in \mathbb{R}^n$ and $t \geq 0$.

4 Relation with the strong iISS property

The most typical scenario in practice is that processes are affected by both signals with large amplitude but bounded energy (such as impulse modeled signals), and persistently exciting signals of small amplitude (such as noise). For the study of these cases, reference [7] introduces the strong iISS property as the combination of ISS with respect to small inputs and iISS. Let us emphasize that ISS with respect to small inputs is a stronger condition than local ISS since the former implies global asymptotic stability in the absence of perturbations and the latter is a local property in both the state and the input. The extension of the strong iISS property for multistable systems is given in [18] as the fusion of the iISS property in the multistable sense (Definition 8) and the AG property for small inputs (i.e., when Definition 5 holds only if $\|d\|_\infty \leq R$ for some $R > 0$):

Definition 27 *The system (1) is said to be strongly iISS in the multistable sense if it has the UBEBS property with respect to \mathcal{W} , and there exists $\kappa \in \mathcal{K}$ and a constant $R > 0$, such that*

$$\|d\|_\infty \leq R \implies \limsup_{t \rightarrow \infty} |X(t, x_0; d)|_{\mathcal{W}} \leq \kappa(\|d\|_\infty), \quad (21)$$

for all $x_0 \in M$ and all $d \in \mathcal{L}_\infty^m$.

Remark 28 *Since the 0-GATT property is a consequence of inequality (21), Definition 27 is equivalent to the original definition of strong iISS introduced by [18].*

The following result provides sufficient conditions for verifying the strong iISS property by means of iISS-Leonov functions.

Lemma 29 *Let Assumptions 4 and 13 be satisfied by the system (1) and its invariant set \mathcal{W} . If there exists an iISS-Leonov function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ as in Definition 18 with the dissipation inequality (13) admitting a function λ_2 of class \mathcal{K} , then the system (1) has the strong iISS property with respect to \mathcal{W} .*

PROOF. From Definition 18, recall the functions $\vartheta_3, \vartheta_4, \sigma_2, \gamma_2 \in \mathcal{K}_\infty$, $\lambda_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, and the scalars $g_2, \ell_2 \geq 0$. Note that the condition $\lambda_2 \in \mathcal{K}$ represents a particular case of Definition 18. Therefore, the 0-GATT and UBEBS properties are established by Lemma 20. Now, we only have to show that inequality (21) is satisfied by the system (1), which is equivalent to the AG property for small inputs.

To this end, let us select a constant $R > 0$ such that $\gamma_2(R) \leq \frac{1}{2}\lambda_2(+\infty)$. Accordingly, λ_2 is invertible in $[0, 2\gamma_2(R)]$. Next, consider $\|d\|_\infty \leq R$, and from the inequality (13) we can see that if $V(\tilde{x}) \geq \lambda_2^{-1}(2\gamma_2(\|d\|))$, then we get

$$DV(\tilde{x})f(\tilde{x}, d) \leq -\frac{1}{2}\lambda_2(V(\tilde{x})). \quad (22)$$

Consequently, the set $\{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) \leq \lambda_2^{-1}(2\gamma_2(\|d\|))\}$ is globally attractive and forward invariant for the trajectories of the system (1).

Now, recall that q is the dimension of the periodic state variable θ . So, denote by $j \in \mathbb{Z}^q$ a multi-index vector, and let us check the behavior of the functions $V_j(\tilde{x}) = V((\tilde{z}, \tilde{\theta} - 2\pi j))$ for any $j \in \mathbb{Z}^q$ (i.e., $V_0(\tilde{x}) = V(\tilde{x})$). First, note that $|\tilde{x}|_{\mathcal{W}'} = |(\tilde{z}, \tilde{\theta})|_{\mathcal{W}'} = |(\tilde{z}, \tilde{\theta} - 2\pi j)|_{\mathcal{W}'}$ and $f(\tilde{x}, d) = f((\tilde{z}, \tilde{\theta}), d) = f((\tilde{z}, \tilde{\theta} - 2\pi j), d)$ due to Assumption 13. Hence, from (12) and (22) we have

$$\begin{aligned} \vartheta_3(|\tilde{x}|_{\mathcal{W}'}) - \sigma_2(|\tilde{\theta} - 2\pi j|) - g_2 \\ \leq V_j(\tilde{x}) \leq \vartheta_4(|\tilde{x}|_{\mathcal{W}'} + \ell_2), \end{aligned} \quad (23)$$

for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$, and for $\|d\|_\infty \leq R$, the condition $V_j(\tilde{x}) \geq \lambda_2^{-1}(2\gamma_2(\|d\|))$ implies

$$DV_j(\tilde{x})f(\tilde{x}, d) \leq -\frac{1}{2}\lambda_2(V_j(\tilde{x})),$$

for any $j \in \mathbb{Z}^q$. In consequence, we can say that the set $\Omega_d = \bigcap_{j \in \mathbb{Z}^q} \{\tilde{x} \in \mathbb{R}^n : V_j(\tilde{x}) \leq \lambda_2^{-1}(2\gamma_2(\|d\|))\}$, with $\|d\|_\infty \leq R$, is globally attractive and forward invariant for the system (1). Hence, for any $\tilde{x}_0 \in \mathbb{R}^n \setminus \Omega_d$, the trajectory $\tilde{X}(t, \tilde{x}_0; d)$ enters to Ω_d in a finite time T_d , and remains there for all future time. Next, from (23), we get

$$\begin{aligned} \vartheta_3(|\tilde{X}(t, \tilde{x}_0; d)|_{\mathcal{W}'}) - \sigma_2(|\tilde{\theta}(t) - 2\pi j|) - g_2 \\ \leq V_j(\tilde{X}(t, \tilde{x}_0; d)) \leq \lambda_2^{-1}(2\gamma_2(\|d(t)\|)), \end{aligned}$$

for all $\tilde{X}(t, \tilde{x}_0; d) \in \Omega_d$, such that

$$\begin{aligned} \vartheta_3(|\tilde{X}(t, \tilde{x}_0; d)|_{\mathcal{W}'}) \leq \lambda_2^{-1}(2\gamma_2(\|d(t)\|) \\ + \sigma_2(|\tilde{\theta}(t) - 2\pi j|) + g_2. \end{aligned}$$

Since this inequality is satisfied for all possible j , then we can take the minimum with respect to j of its right-hand side. Note that it is composed of three independent terms, which can be minimized separately, and only one of them depends on j . Since by construction $\min_{j \in \mathbb{Z}^q} |\tilde{\theta}(t) - 2\pi j| \leq \sqrt{q}\pi$, we have

$$\begin{aligned} \vartheta_3(|\tilde{X}(t, \tilde{x}_0; d)|_{\mathcal{W}'}) \leq \lambda_2^{-1}(2\gamma_2(\|d(t)\|) \\ + g_1 + \sigma_2(\sqrt{q}\pi)), \end{aligned} \quad (24)$$

for all $t \geq T_d$, $\tilde{x}_0 \in \mathbb{R}^n$ and $\|d\|_\infty \leq R$.

Now, by Assumption 13, we have that for any $\tilde{x}_0 \in \mathbb{R}^n$ the solution $\tilde{X}(t, \tilde{x}_0; d) \in \mathbb{R}^n$ can be projected to the solution $X(t, x_0; d) \in M$, such that, both solutions are defined on the same time interval. Also, the set $\tilde{\mathcal{W}}' \subset \mathbb{R}^n$ can be projected to $\mathcal{W}' \subset M$. Hence, $|\tilde{X}(t, \tilde{x}_0; d)|_{\tilde{\mathcal{W}}'} = |X(t, x_0; d)|_{\mathcal{W}'}$, and we can say that inequality (24) is also satisfied for the trajectories in M . Next, recall that $\mathcal{W}' \subseteq \mathcal{W}$, therefore we have $|X(t, x_0; d)|_{\mathcal{W}} \leq |X(t, x_0; d)|_{\mathcal{W}'}$. Thus, (24) implies that Definition 5 is verified with $\nu(\cdot) = \vartheta_3^{-1}(2\lambda_1^{-1}(2\gamma_2(\cdot)))$ and $c_1 = \vartheta_3^{-1}(2(g_2 + \sigma_2(\sqrt{q}\pi)))$. Hence, we conclude that the system (1) has the pAG property with respect to \mathcal{W} , provided that $\|d\|_\infty \leq R$. Moreover, the result of [2, Claim 4] establishes the equivalence between the pAG and AG properties for systems evolving in M , with the set \mathcal{W} satisfying Assumption 4. Then, we can say that the system (1) has the AG property with respect to \mathcal{W} for all $x_0 \in M$ and $\|d\|_\infty \leq R$. \square

The function λ_2 in inequality (13) is called the dissipation rate. The proof of Lemma 29 shows that if the dissipation rate of an iISS-Leonov function is of class \mathcal{K} , then it is possible to prove the AG property for small inputs for the system (1). However, the converse may not be true, i.e., if the system (1) is strong iISS, the corresponding iISS-Leonov function does not necessarily have a class \mathcal{K} dissipation rate. Recall that the converse results in Theorem 22 are proven by using iISS-Lyapunov functions, and the previous assertion has been supported through a counter-example based on iISS-Lyapunov functions in [7].

The hierarchy of ISS-related notions establishes that the iISS property is a special case of the ISS property, and the strong iISS is an intermediate property between them. Thus, we can unify Definitions 16 and 18, and Lemmas 17, 20 and 29, such that the existence of a function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ as in Definition 16 with a dissipation rate:

- of class \mathcal{K}_∞ implies ISS for the system (1),
- of class \mathcal{K} implies strong iISS for the system (1),
- positive definite implies iISS for the system (1).

5 Applying the iISS-Leonov function approach to assess the iISS properties of a synchronous generator with saturated control input

Consider the classical representation of a regulated synchronous generator by a constant voltage behind transient reactance, and whose turbine is connected to a governing system equipped with droop control, which allows to manage its mechanical power output P_M and speed ω_M . As usual, we assume ideal governor dynamics, and a linear relationship between the valve position and the mechanical power. Recall also that the mechanical speed ω_M is related to the electrical speed ω via $\omega = 0.5p_l\omega_M$,

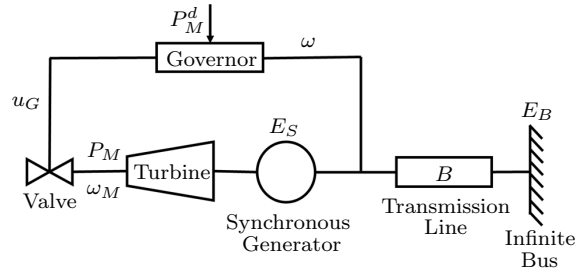


Fig. 1. Block diagram of a synchronous generator connected to an infinite bus.

with p_l being the number of machine poles. Moreover, the synchronous generator is connected to an infinite bus through a transmission line, whose impedance is purely inductive with a susceptance $B < 0$. Figure 1 shows a block diagram of the described system. The active power flow is given by

$$P(t) = G_S E_S^2 + |B| E_S E_B \sin(\hat{\delta}(t)),$$

where $G_S > 0$ represents the self-admittance, and $E_S, E_B > 0$ denote the voltage amplitudes of the synchronous generator and the infinite bus, respectively. Besides, $\hat{\delta} = \delta - \delta_B$ represents the difference between the phase angle of the synchronous generator δ and the one of the infinite bus δ_B . Thus, $\hat{\omega} = \dot{\hat{\delta}}$ represents the difference of their frequencies (recall that the infinite bus has a constant frequency). Under the aforementioned conditions, the droop control takes the form

$$u_G = P_M(t) = P_M^d - k\varphi(\hat{\omega}(t)),$$

where $k > 0$ is a proportional gain, and $P_M^d > 0$ is the reference setpoint for the mechanical power. Furthermore, $\varphi : \mathbb{R} \rightarrow [-1, 1]$ is a continuously differentiable and non-decreasing function, satisfying $\varphi(0) = 0$ and $s\varphi(s) > 0$, for all $s \neq 0$, e.g., $\varphi(s) = \tanh(s)$ or $\varphi(s) = \frac{s}{1+|s|}$. This means that the control input u_G is saturated. Finally, we denote by $M > 0$ the inertia coefficient, and consider a zero damping term. Then, a simplified mathematical model for the described system is given by

$$\begin{aligned} \dot{\hat{\delta}}(t) &= \hat{\omega}(t), \\ M\dot{\hat{\omega}}(t) &= -A \sin(\hat{\delta}(t)) - k\varphi(\hat{\omega}(t)) + c + \frac{d(t)}{1+|\hat{\delta}(t)|}, \end{aligned} \quad (25)$$

where $A = |B|E_S E_B$, $c = P_M^d + G_S E_S^2$, with $|c| < k$, and $d \in \mathcal{L}_\infty$ is a perturbation that becomes negligible if the generator phase angle significantly deviates from the bus one. In the sequel, the time argument of all variables will be omitted for compactness. The system (25) and the steps of modeling were adapted from those presented in [38] (see also [22,27] for further details about modeling and control of synchronous generators).

Now, for $|c| < A$ and $d = 0$, the system (25) has equilibria in the manifold $\{[\hat{\delta}, \hat{\omega}] \in \mathbb{R}^2 : -\pi \leq \hat{\delta} \leq \pi\}$ at

the points $[\hat{\delta}_0, \hat{\omega}_0]^\top = [\text{asin}(cA^{-1}), 0]^\top$ and $[\hat{\delta}'_0, \hat{\omega}'_0]^\top = [\pi - \text{asin}(cA^{-1}), 0]^\top$. Then, consider the coordinates $\zeta = [\zeta_1, \zeta_2]^\top \in \mathbb{R}^2$, where $\zeta_1 = \hat{\delta} - \hat{\delta}_0$ and $\zeta_2 = \hat{\omega}$. The system (25) in coordinates ζ becomes

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2, \\ M\dot{\zeta}_2 &= -A\phi(\zeta_1) - k\varphi(\zeta_2) + \frac{d}{1+|\zeta_1+\hat{\delta}_0|}, \end{aligned} \quad (26)$$

where $\phi(\zeta_1) = \sin(\zeta_1 + \hat{\delta}_0) - \sin(\hat{\delta}_0)$. Let us define the set $\tilde{\mathcal{W}}_{SG} = \{[2j\pi, 0]^\top \cup [(1+2j)\pi - 2\hat{\delta}_0, 0]^\top : j \in \mathbb{Z}\}$, which contains all equilibria of the system (26). Hence, for this case we have $\mathcal{D} = \tilde{\mathcal{W}}_{SG}$.

We define $|\zeta|_{\tilde{\mathcal{W}}_{SG}}^2 = \zeta_2^2 + \phi^2(\zeta_1)$ as the distance from the trajectories of (26) to the set $\tilde{\mathcal{W}}_{SG}$, and $|\zeta|^2 = \zeta_1^2 + \zeta_2^2$ as the distance from the trajectories of (26) to the origin.

Now, we apply the methodology develop in Section 3 to verify the iISS properties of the system (26) with respect to the set \mathcal{W}_{SG} . To this end, consider a iISS-Leonov function candidate of the form

$$\begin{aligned} V(\zeta) &= \frac{1}{2}\zeta_2^2 + \epsilon\phi(\zeta_1)\varphi(\zeta_2) - \frac{c}{M}\zeta_1 \\ &\quad + \frac{A}{M}(\cos(\hat{\delta}_0) - \cos(\zeta_1 + \hat{\delta}_0)), \end{aligned}$$

where $\epsilon > 0$ is a constant parameter. Next, we define $\varrho(\zeta_1) = A(\cos(\hat{\delta}_0) - \cos(\zeta_1 + \hat{\delta}_0)) - c\zeta_1 - \frac{1}{2}vA\zeta_1^2$, then

$$\begin{aligned} \varrho'(\zeta_1) &= A\sin(\zeta_1 + \hat{\delta}_0) - c - vA\zeta_1, \\ \varrho''(\zeta_1) &= A\cos(\zeta_1 + \hat{\delta}_0) - vA. \end{aligned}$$

Note that the equation $\varrho'(\zeta_1) = 0$ has a solution at $\zeta_1 = 0$. Moreover, the equation $\varrho''(\zeta_1) = 0$ has no solutions for $v > 1$. Hence, $\varrho'(\zeta_1)$ is strictly decreasing in such a case. Thus, $\zeta_1 = 0$ is the unique solution to the equation $\varrho'(\zeta_1) = 0$ for $v > 1$. Since the inequality $\varrho''(0) < 0$ holds for $v > 1$, then we can conclude that $\zeta_1 = 0$ is a global maximum of the function $\varrho(\zeta_1)$, and $\varrho(0) = 0$. Therefore, we have that $\varrho(\zeta_1)$ is negative definite for $v > 1$, which implies that $A(\cos(\hat{\delta}_0) - \cos(\zeta_1 + \hat{\delta}_0)) - c\zeta_1 \leq \frac{1}{2}vA\zeta_1^2$ under such a condition. By using similar arguments, we can obtain $A(\cos(\hat{\delta}_0) - \cos(\zeta_1 + \hat{\delta}_0)) - c\zeta_1 \geq -\frac{1}{2}vA\zeta_1^2$ for $v > 1$ as well. Therefore, we get

$$\frac{1}{2}\zeta_2^2 - \frac{vA}{2M}\zeta_1^2 - 2\epsilon \leq V(\zeta) \leq \frac{1}{2}\zeta_2^2 + \frac{vA}{2M}\zeta_1^2 + 2\epsilon, \quad (27)$$

for all $\zeta \in \mathbb{R}^2$, where $v > 1$.

Now, the derivative of the function V along the trajectories of the system (26) is given by

$$\begin{aligned} \dot{V}(\zeta) &= -\left(\frac{k}{M} - \epsilon\cos(\zeta_1 + \hat{\delta}_0)\right)\zeta_2\varphi(\zeta_2) - \frac{\epsilon A}{M}\varphi'(\zeta_2)\phi^2(\zeta_1) \\ &\quad - \frac{\epsilon k}{M}\varphi'(\zeta_2)\phi(\zeta_1)\varphi(\zeta_2) + \frac{\epsilon\varphi'(\zeta_2)\phi(\zeta_1)+\zeta_2}{M(1+|\zeta_1+\hat{\delta}_0|)}d. \end{aligned}$$

Since the function $\varphi(\zeta_2)$ is non-decreasing and bounded, we have $\varphi'(\zeta_2) \geq 0$ and $\varphi'(\zeta_2)$ vanishes as $|\zeta_2|$ grows up. Moreover, by using Young's inequality, we obtain $2\phi(\zeta_1)\varphi(\zeta_2) \leq \phi^2(\zeta_1) + \varphi^2(\zeta_2)$. Therefore, we have

$$\begin{aligned} \dot{V}(\zeta) &\leq -\left(\frac{k}{M} - \epsilon\right)\zeta_2\varphi(\zeta_2) + \frac{\epsilon k}{2M}\varphi'(\zeta_2)\varphi^2(\zeta_2) \\ &\quad - \frac{\epsilon}{2M}\varphi'(\zeta_2)(2A - k)\phi^2(\zeta_1) + \frac{\epsilon\varphi'(\zeta_2)\phi(\zeta_1)+\zeta_2}{M(1+|\zeta_1+\hat{\delta}_0|)}d. \end{aligned}$$

Due to the function $\varphi(\zeta_2)$ is continuously differentiable and bounded, we can find two constants $\Delta > 0$ and $\varphi'_{\max} > 0$, such that $|\varphi(\zeta_2)| \leq \Delta|\zeta_2|$ and $|\varphi'(\zeta_2)| \leq \varphi'_{\max}$ for all $\zeta_2 \in \mathbb{R}$. Hence, we obtain

$$\begin{aligned} \dot{V}(\zeta) &\leq -\left(\frac{k}{M} - \epsilon - \frac{\epsilon k \Delta \varphi'_{\max}}{2M}\right)\zeta_2\varphi(\zeta_2) \\ &\quad - \frac{\epsilon}{2M}\varphi'(\zeta_2)(2A - k)\phi^2(\zeta_1) + \frac{\epsilon\varphi'(\zeta_2)\phi(\zeta_1)+\zeta_2}{M(1+|\zeta_1+\hat{\delta}_0|)}d. \end{aligned}$$

Now, consider the set $\Omega = \{\zeta \in \mathbb{R}^2 : V(\zeta) \geq 0\}$ and the auxiliary function

$$W(\zeta) = \begin{cases} \ln(1 + V(\zeta)) & \text{for } \zeta \in \Omega, \\ V(\zeta) & \text{for } \zeta \in \mathbb{R}^2 \setminus \Omega, \end{cases}$$

which is a continuous function since $W(\zeta) = V(\zeta) = \ln(1 + V(\zeta)) = 0$ for all $\zeta \in \partial\Omega$, where $\partial\Omega$ denotes the boundary of Ω . Note that the term $\ln(1 + V(\zeta))$ is well-defined for all $\zeta \in \Omega$, and by properties of the natural logarithm, we have $0 \leq \ln(1 + V(\zeta)) \leq V(\zeta)$ for all $\zeta \in \Omega$. Besides, since $\ln(1 + |s|)$ is a \mathcal{K}_∞ -function, we can deduce the inequality

$$\ln(1 + \frac{1}{4}\zeta_2^2) \leq \ln(1 + \frac{1}{2}\zeta_2^2 - \frac{vA}{2M}\zeta_1^2 - 2\epsilon) + \ln(1 + \frac{vA}{2M}\zeta_1^2 + 2\epsilon),$$

where the argument of the first logarithm in the right-hand side is nonnegative on the set Ω . Accordingly, we obtain from (27) that for all $\zeta \in \mathbb{R}^2$:

$$\begin{aligned} \ln(1 + \frac{1}{4}\zeta_2^2) - \frac{vA}{2M}\zeta_1^2 - 2\epsilon &\leq W(\zeta) \\ &\leq \frac{1}{2}\max\{1, \frac{vA}{M}\}|\zeta|^2 + 2\epsilon. \end{aligned} \quad (28)$$

Define the set $\mathcal{U} = \{[2j\pi, 0]^\top : j \in \mathbb{Z}\}$ for the system (26). Being that $V([0, 0]) = 0$, and $V([2j\pi, 0]) \leq 0$ for a sufficiently large positive value of $j \in \mathbb{Z}$, we have that $W([0, 0]) = 0$, and

$$\inf_{\zeta \in \mathcal{U} \setminus \{[0, 0]\}} W(\zeta) \leq 0.$$

Also, $V([\pi, 0]) = -\frac{c}{M}\pi + \frac{2A}{M}\cos(\hat{\delta}_0) > 0$ provided that $|c|\pi < 2A\cos(\hat{\delta}_0)$, which is ensured for a sufficiently large value of $A > 0$. By continuity of V , we can claim the existence of constants $\underline{c} \in (0, \pi)$ and $\bar{c} \in (\pi, 2\pi)$, such that $V(\zeta) > 0$ for all $\zeta \in \mathcal{V} \supseteq \{\zeta \in \mathbb{R}^2 : \underline{c} \leq |\zeta_1|_\infty \leq \bar{c}\}$.

Consequently, we obtain that

$$\inf_{\zeta \in \mathcal{V}} W(\zeta) > 0.$$

Thus, we can conclude that the function W satisfies (19).

Now, let us check the derivative of W along the trajectories of the system (26) for all $\zeta \in \Omega$, which is given by

$$\dot{W}(\zeta) = \frac{\dot{V}(\zeta)}{1+V(\zeta)} \leq -h(\zeta) + \frac{\epsilon \varphi'(\zeta_2) \phi(\zeta_1) + \zeta_2}{M(1+V(\zeta))(1+|\zeta_1 + \delta_0|)} |d|,$$

where

$$h(\zeta) = \frac{1}{(1+V(\zeta))} \left(\left(\frac{k}{M} - \epsilon - \frac{\epsilon k \Delta \varphi'_{\max}}{2M} \right) \zeta_2 \varphi(\zeta_2) - \frac{\epsilon}{2M} \varphi'(\zeta_2) (2A - k) \phi^2(\zeta_1) \right).$$

By setting $2A > k$, $k > \frac{2\epsilon M}{2 - \epsilon \Delta \varphi'_{\max}}$ and $2 > \epsilon \Delta \varphi'_{\max}$, the output $h(\zeta)$ is nonnegative for all $\zeta \in \Omega$. In addition, $h(\zeta) = 0$ if and only if $\zeta \in \tilde{\mathcal{W}}_{SG}$. Besides, since the quotient $\frac{\zeta_2}{(1+V(\zeta))(1+|\zeta_1 + \delta_0|)}$ is bounded for all $\zeta \in \Omega$, there is a constant $a = \sup_{\zeta \in \Omega} \left\{ \frac{\zeta_2}{(1+V(\zeta))(1+|\zeta_1 + \delta_0|)} \right\}$. Then, we get

$$\dot{W}(\zeta) \leq -|h(\zeta)| + \frac{a}{M} |d|, \quad (29)$$

for all $\zeta \in \Omega$. Thus, we conclude that the function W and the output h satisfies all conditions of Theorem 26. Hence, we can confirm that the system (26) has the UBES and 0-GATT properties with respect to $\tilde{\mathcal{W}}_{SG}$, provided that $2A > \max \left\{ k, \frac{|c| \pi}{\cos(\hat{\delta}_0)} \right\}$, $k > \frac{2\epsilon M}{2 - \epsilon \Delta \varphi'_{\max}}$ and $2 > \epsilon \Delta \varphi'_{\max}$.

Moreover, from inequality (B.4) in the proof of Theorem 26, we can see how the parameters of inequalities (28) and (29) are related to those in Definition 6. So, similarly to (B.4), we can obtain the following inequality for the solutions of the system (26) for all $\zeta(0) \in \mathbb{R}^2$:

$$\varrho(|\zeta(t)|_{\mathcal{W}_{SG}}) \leq \bar{\vartheta}(|\zeta(0)|_{\mathcal{W}_{SG}}) + c_3 + \frac{1}{M} \sigma_6 \left(\int_0^t |d(s)| ds \right), \quad (30)$$

where $\bar{\vartheta}(\cdot) = 18 \max \left\{ 1, \frac{vA}{M} \right\} ((\cdot)^2 + \rho_1^2(\cdot))$, $\varrho(\cdot) = \ln \left(1 + \frac{1}{64} (\cdot)^2 \right)$, $\sigma_6(\cdot) = 2 \max \{ 8vA(\cdot)^2, a(\cdot) \}$ and $c_3 = 4\epsilon + 18 \max \left\{ 1, \frac{vA}{M} \right\} (\pi - 2\hat{\delta}_0)^2 + \frac{32vA}{M} \rho_0^2 + \frac{9}{2} \max \left\{ 1, \frac{vA}{M} \right\} \pi^2 + \frac{2vA}{M} \pi^2$, with parameters $\rho_0 > 0$ and $\rho_1 \in \mathcal{K}_\infty$ as in the proof of Theorem 26. Note that the functions ε , γ_4 , and consequently γ_5 in (B.4), are linear for this case. We can see from inequality (30) that the inertia parameter M is the main actor in the attenuation of perturbation's effects. Basically, higher

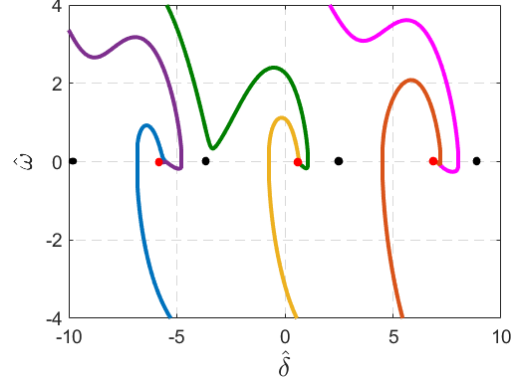


Fig. 2. Behavior of the system (25), with $d = 3(1 - \tanh^2(0.2t - 4))$. Each curve represent a trajectory of the system (25) from an arbitrarily chosen initial condition. Red and black points represent stable focus and saddle points of the system (25), respectively.

the inertia, lower the influence of perturbation on system's behavior. In this case, the damping coefficient k is not considered in the attenuation of perturbations because of the saturation. We omitted the step of applying $\sigma_6^{-1}(\frac{1}{3}(\cdot))$ in both sides of inequality (30) for simplicity, and because it does not reveal further insights about the behavior of the trajectories of the system (26).

Finally, by restricting the system (26) to the manifold $\{\zeta \in \mathbb{R}^2 : -\pi \leq \zeta_1 < \pi\}$, all points of the set $\tilde{\mathcal{W}}_{SG}$ are projected to the set $\mathcal{W}_{SG} = \{[0, 0]^\top, [\pi - 2\hat{\delta}_0, 0]^\top\}$. Therefore, we have $|\zeta| \leq |\zeta|_{\mathcal{W}_{SG}} + \pi - 2\hat{\delta}_0$. This fact together with inequality (30) implies that the trajectories of the system (26), with $d(t) = 0$, are bounded in the manifold $\{\zeta \in \mathbb{R}^2 : -\pi \leq \zeta_1 < \pi\}$ for all $\zeta(0) \in \mathbb{R}^2$ and all $t \geq 0$. Local stability properties of the system (26) have been extensively studied in the literature. For instance, we can invoke [38, Proposition 4.7] to claim that the set \mathcal{W}_{SG} is attractive for the trajectories of the system (26). Thus, the set \mathcal{W}_{SG} is decomposable in the sense of Definition 3. Finally, we have that the system (26) satisfies Assumptions 4 and 13, and by Theorem 26 and Definition 8, we conclude that it has the iISS property in the multistable sense with respect to the set \mathcal{W}_{SG} .

5.1 Numerical example

For the system (25), we select $\varphi(\hat{\omega}_2) = \frac{\hat{\omega}_2}{1+|\hat{\omega}_2|}$, and the parameters $c = 1.5$, $A = 3$, $M = 1$ and $k = 5$. Hence, the system (25) has equilibrium points in the set $\tilde{\mathcal{W}}_{SG} = \{[\frac{\pi}{6} + 2j\pi, 0]^\top \cup [(1+2j)\pi - \frac{\pi}{6}, 0]^\top\}$ where $j \in \mathbb{Z}$. Considering $d = 2(1 - \tanh(0.15t - 4))$, some trajectories are depicted in Figure 2. We can see that the trajectories approach the stable equilibria in spite of the effect of the bounded-energy perturbation. Hence, we can observe that the system (25) has the iISS property with respect to the set \mathcal{W}_{SG} .

6 Conclusions

Novel characterizations of the iISS property has been developed for multistable systems whose dynamics are periodic with respect to a part of the state. The concepts of iISS-Leonov functions and OSD have been presented, and their equivalence with the UBEBS and 0-GATT properties have been proven for the considered class of systems. Also, a new result that establishes a connection between the proposed iISS-Leonov function framework and the original Leonov function approach was developed, such that robustness against inputs with bounded energy and boundedness of solutions in absence of perturbations are ensured for the system under study. Additionally, the relation of iISS-Leonov functions and the concept of strong iISS was studied. Finally, our results were applied for robustness analysis of a synchronous generator with saturated control input. Future work includes the development of numerical methods based on machine learning and physics-inspired neural networks for the design of ISS- and iISS-Leonov functions.

A Proofs of Lemmas 20 and 21

Consider a \mathcal{C}^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying inequality (12). Moreover, take any $\tilde{x}_0 \in \mathbb{R}^n$ and consider a solution $\tilde{X}(t, \tilde{x}_0; d)$ of the system (1), which is defined at least locally in time for $t \in [0, T)$ for some $T > 0$. Then, we can introduce two subsets of instants of time

$$\begin{aligned} \mathcal{T}_+ &= \{t \in [0, T) : V(\tilde{X}(t, \tilde{x}_0; d)) \geq 0\}, \\ \mathcal{T}_- &= \{t \in [0, T) : V(\tilde{X}(t, \tilde{x}_0; d)) < 0\}, \end{aligned} \quad (\text{A.1})$$

such that $[0, T) = \mathcal{T}_+ \cup \mathcal{T}_-$. Therefore, for any $t \in \mathcal{T}_-$ we have from inequality (12) that

$$\vartheta_3(|\tilde{X}(t, \tilde{x}_0; d)|_{\mathcal{W}'}) - \sigma_2(|\tilde{\theta}(t)|) - g_2 \leq V(\tilde{X}(t, \tilde{x}_0; d)) < 0,$$

then

$$\vartheta_3(|\tilde{X}(t, \tilde{x}_0; d)|_{\mathcal{W}'}) < \sigma_2(|\tilde{\theta}(t)|) + g_2.$$

for all these instants of time. Moreover, by continuity of V , for any isolated element $t \in \mathcal{T}_+$ we have $V(\tilde{X}(t, \tilde{x}_0; d)) = 0$, then as previously

$$\vartheta_3(|\tilde{X}(t, \tilde{x}_0; d)|_{\mathcal{W}'}) \leq \sigma_2(|\tilde{\theta}(t)|) + g_2. \quad (\text{A.2})$$

If $t \in \mathcal{T}_+$ is not an isolated element of \mathcal{T}_+ , then there is an interval $[t_s, t_e] \subset \mathcal{T}_+$ with $V(\tilde{X}(t, \tilde{x}_0; d)) \geq 0$ for all $t \in [t_s, t_e]$. Therefore, on the interval $[t_s, t_e]$ both inequalities, (13) and (14), become

$$DV(\tilde{X}(t, \tilde{x}_0; d))f(\tilde{X}(t, \tilde{x}_0; d), d) \leq \gamma_2(|d|), \quad (\text{A.3})$$

and by integration it follows that

$$V(\tilde{X}(t, \tilde{x}_0; d)) - V(\tilde{X}(t_s, \tilde{x}_0; d)) \leq \int_{t_s}^t \gamma_2(|d(s)|) ds.$$

By definition of t_s we have $V(\tilde{X}(t_s, \tilde{x}_0; d)) \geq 0$ if $t_s = 0$, and $V(\tilde{X}(t_s, \tilde{x}_0; d)) = 0$ if $t_s > 0$. Hence, we have

$$V(\tilde{X}(t, \tilde{x}_0; d)) \leq V(\tilde{x}_0) + \int_{t_s}^t \gamma_2(|d(s)|) ds,$$

for all $t \in [t_s, t_e]$. Now, considering inequality (12) and estimate (A.2) we obtain that

$$\begin{aligned} \vartheta_3(|\tilde{X}(t, \tilde{x}_0; d)|_{\mathcal{W}'}) &\leq \vartheta_4(|\tilde{x}_0|_{\mathcal{W}'} + \ell_2) + g_2 \\ &\quad + \sigma_2(|\tilde{\theta}(t)|) + \int_0^t \gamma_2(|d(s)|) ds, \end{aligned}$$

for any $t \in [0, T)$. Recall that q is the dimension of the periodic state variable θ . Denote by $j \in \mathbb{Z}^q$ a multi-index vector, and define the functions $V_j(\tilde{x}) = V((\tilde{z}, \tilde{\theta} - 2\pi j))$ for any $j \in \mathbb{Z}^q$ (then $V_0(\tilde{x}) = V(\tilde{x})$). From (12), and either (13) or (14), we have for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$:

$$\vartheta_3(|\tilde{x}|_{\mathcal{W}'}) - \sigma_2(|\tilde{\theta} - 2\pi j|) - g_2 \leq V_j(\tilde{x}) \leq \vartheta_4(|\tilde{x}|_{\mathcal{W}'} + \ell_2),$$

and for all $\tilde{x} \in \{\tilde{x} \in \mathbb{R}^n : V_j(\tilde{x}) \geq 0\}$ and all $d \in \mathbb{R}^m$:

$$DV_j(\tilde{x})f(\tilde{x}, d) \leq \gamma_2(|d|).$$

Since $|\tilde{x}|_{\mathcal{W}'} = |(\tilde{z}, \tilde{\theta})|_{\mathcal{W}'} = |(\tilde{z}, \tilde{\theta} - 2\pi j)|_{\mathcal{W}'}$ and $f(\tilde{x}, d) = f((\tilde{z}, \tilde{\theta}), d) = f((\tilde{z}, \tilde{\theta} - 2\pi j), d)$. Then, by repeating the same calculations as above for the function V_j for any multi-index vector j , we have for all $t \in [0, T)$:

$$\begin{aligned} \vartheta_3(|\tilde{X}(t, \tilde{x}_0; d)|_{\mathcal{W}'}) &\leq \vartheta_4(|\tilde{x}_0|_{\mathcal{W}'} + \ell_2) + \sigma_2(|\tilde{\theta}(t) - 2\pi j|) \\ &\quad + g_2 + \int_0^t \gamma_2(|d(s)|) ds. \end{aligned}$$

Since this inequality is satisfied for all possible j , then the minimum with respect to j of its right-hand side can be calculated. Note that it is composed of five independent nonnegative terms that can be minimized independently, and only one of them is dependent on j . Furthermore, by construction $\min_{j \in \mathbb{Z}^q} |\tilde{\theta}(t) - 2\pi j| \leq \sqrt{q}\pi$. Thus, we get

$$\begin{aligned} \vartheta_3(|\tilde{X}(t, \tilde{x}_0; d)|_{\mathcal{W}'}) &\leq \vartheta_4(|\tilde{x}_0|_{\mathcal{W}'} + \ell_2) + \sigma_2(\sqrt{q}\pi) \\ &\quad + g_2 + \int_0^t \gamma_2(|d(s)|) ds, \end{aligned} \quad (\text{A.4})$$

for all $t \in [0, T)$. Now, due to the periodicity of the system (2), we have that for any $\tilde{x}_0 \in \mathbb{R}^n$ the solution $\tilde{X}(t, \tilde{x}_0; d) \in \mathbb{R}^n$ can be projected to the solution $X(t, x_0; d) \in M$ by using the periodicity of the variables $\tilde{\theta} \in \mathbb{R}^q$, hence both solutions are defined on the same time interval. Also, the set $\mathcal{W}' \subset \mathbb{R}^n$ can be projected to $\mathcal{W} \subset M$. Then, $|\tilde{X}(t, \tilde{x}_0; d)|_{\mathcal{W}'} = |X(t, x_0; d)|_{\mathcal{W}'}$, and consequently, the property (A.4) is also satisfied

in M , with $T = +\infty$. Next, recall that $\mathcal{W}' \subseteq \mathcal{W}$, thus $|X(t, x_0; d)|_{\mathcal{W}} \leq |X(t, x_0; d)|_{\mathcal{W}'}$. Moreover, since the set \mathcal{W} is assumed to be compact, then there exists a constant $b \geq 0$, such that $|x_0|_{\mathcal{W}'} \leq |x_0|_{\mathcal{W}} + b$. Therefore, considering that ϑ_3, ϑ_4 are class \mathcal{K}_∞ functions, we obtain

$$\vartheta_3(|X(t, x_0; d)|_{\mathcal{W}}) \leq \vartheta_4(2|x_0|_{\mathcal{W}}) + c + \int_0^t \gamma_2(|d(s)|)ds,$$

where γ_2 is a class \mathcal{K}_∞ function, and $c = \vartheta_4(2(\ell_2 + b)) + g_2 + \sigma_2(\sqrt{q}\pi)$ is a positive constant. Thus, we can say that the system (1) has the UBEBS property with respect to \mathcal{W} .

Moreover, from the latter inequality with $d \equiv 0$, we have that the distance of the trajectories $X(t, x_0; 0)$ to the set \mathcal{W} is bounded for all $x_0 \in M$ and all $t \geq 0$. By Assumption 4, the set \mathcal{W} contains all α - and ω -limit sets of (2), and is compact and decomposable. Due to compactness of \mathcal{W} , for any $x \in M$ there is a constant $b' \geq 0$ such that $|x| \leq |x|_{\mathcal{W}} + b'$. Hence, the trajectory $X(t, x_0; 0)$ is bounded in M for any $x_0 \in M$. Therefore, by [21, Lemma 4.1] we have

$$\lim_{t \rightarrow +\infty} |X(t, x_0; 0)|_{\mathcal{W}} = 0,$$

for all $x_0 \in M$. Thus, we can conclude that the system (1) has the 0-GATT property in M .

Note that this proof can be developed by using a \mathcal{C}^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying either Definition 18 or 19. Thus, this proof is valid for both Lemmas 20 and 21.

B Proof of Theorem 26

From Theorem 26, there exists a \mathcal{C}^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying condition (19) for all $\tilde{x} \in \mathbb{R}^n$, and (20) for all $\tilde{x} \in \{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) \geq 0\}$ and $d \in \mathbb{R}^m$.

Recall the property $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for any $a, b \geq 0$. Thus, we obtain $|\tilde{x}| \leq |\tilde{z}| + |\tilde{\theta}|$ for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$. Then, by properties of \mathcal{K}_∞ -functions we have

$$\vartheta_5\left(\frac{1}{2}(|\tilde{x}|)\right) \leq \vartheta_5\left(\frac{1}{2}(|\tilde{z}| + |\tilde{\theta}|)\right) \leq \vartheta_5(|\tilde{z}|) + \vartheta_5(|\tilde{\theta}|).$$

Now, recall the set $\tilde{\mathcal{W}} \subset \mathbb{R}^n$. Due to Assumptions 4 and 13, we have that $\tilde{\mathcal{W}}$ is compact with respect to the subspace \mathbb{R}^k of coordinates \tilde{z} , and uniformly distributed in the whole subspace \mathbb{R}^q of coordinates $\tilde{\theta}$. Hence, there exists a constant $g_3 = \min_{\tilde{x} \in \tilde{\mathcal{W}}} |\tilde{x}|$, such that $|\tilde{x}|_{\tilde{\mathcal{W}}} \leq |\tilde{x}| + g_3$. So, we can rewrite the first inequality in (19) as

$$\vartheta_5\left(\frac{1}{4}|\tilde{x}|_{\tilde{\mathcal{W}}}\right) - \sigma_4(|\tilde{\theta}|) - g_5 \leq V(\tilde{x}) \leq \vartheta_6(|\tilde{x}| + \ell_3), \quad (\text{B.1})$$

where $\sigma_4(|\tilde{\theta}|) = \sigma_3(|\tilde{\theta}|) + \vartheta_5(|\tilde{\theta}|)$ and $g_5 = g_4 + \vartheta_5(\frac{1}{2}g_3)$.

Take any $\tilde{x}_0 \in \mathbb{R}^n$ and consider a solution $\tilde{X}(t, \tilde{x}_0; d)$ of the system (1), which is defined at least locally in time for $t \in [0, T)$ for some $T > 0$. Recall the sets of instants of time \mathcal{T}_+ and \mathcal{T}_- in (A.1), such that $[0, T) = \mathcal{T}_+ \cup \mathcal{T}_-$. Thus, for any $t \in \mathcal{T}_-$ we have $V(\tilde{X}(t, \tilde{x}_0; d)) < 0$, and by continuity of V , for any isolated element $t \in \mathcal{T}_+$ we have $V(\tilde{X}(t, \tilde{x}_0; d)) = 0$. Accordingly, from inequality (B.1) we have that $V(\tilde{X}(t, \tilde{x}_0; d)) \leq 0$, implies

$$\vartheta_5\left(\frac{1}{4}|\tilde{X}(t, \tilde{x}_0; d)|_{\tilde{\mathcal{W}}}\right) \leq \sigma_4(|\tilde{\theta}(t)|) + g_5, \quad (\text{B.2})$$

for all these instants of time. Besides, if $t \in \mathcal{T}_+$ is not an isolated element, then there is an interval $[t_s, t_e] \subset \mathcal{T}_+$ with $V(\tilde{X}(t, \tilde{x}_0; d)) \geq 0$ for all $t \in [t_s, t_e]$. Moreover, since λ_4 in (20) is a nonnegative function, we obtain

$$DV(\tilde{X}(t, \tilde{x}_0; d))f(\tilde{X}(t, \tilde{x}_0; d), d) \leq \gamma_4(|d|),$$

for all $t \in [t_s, t_e]$. By integration it follows that

$$V(\tilde{X}(t, \tilde{x}_0; d)) \leq V(\tilde{x}_0) + \int_0^t \gamma_4(|d(s)|)ds,$$

for all $t \in [t_s, t_e]$. Now, by using (B.1) in the latter inequality, we can establish

$$\begin{aligned} \vartheta_5\left(\frac{1}{4}|\tilde{X}(t, \tilde{x}_0; d)|_{\tilde{\mathcal{W}}}\right) &\leq \vartheta_6(|\tilde{x}_0| + \ell_3) + \sigma_4(|\tilde{\theta}(t)|) \\ &\quad + g_5 + \int_0^t \gamma_4(|d(s)|)ds, \end{aligned}$$

for all $t \in [t_s, t_e]$. Notice that considering (B.2) we can extend the latter inequality to the whole interval $[0, T)$.

Let $j \in \mathbb{Z}^q$ be a multi-index vector, and 0_k be a zero vector of dimension k . Moreover, define the functions $V_j(\tilde{x}) = V((\tilde{z}, \tilde{\theta} - 2\pi j))$ for any $j \in \mathbb{Z}^q$. Hence, from (B.1), we have for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$:

$$\begin{aligned} \vartheta_5\left(\frac{1}{4}|\tilde{x}|_{\tilde{\mathcal{W}}}\right) - \sigma_4(|\tilde{\theta} - 2\pi j|) - g_5 &\leq V_j(\tilde{x}) \\ &\leq \vartheta_6(|\tilde{x}|_{[0_k, 2\pi j]} + \ell_3), \end{aligned}$$

and for all $\tilde{x} \in \{\tilde{x} \in \mathbb{R}^n : V_j(\tilde{x}) \geq 0\}$ and all $d \in \mathbb{R}^m$:

$$DV_j(\tilde{x})f(\tilde{x}, d) \leq \gamma_4(|d|),$$

because $|\tilde{x}|_{\tilde{\mathcal{W}}'} = |(\tilde{z}, \tilde{\theta})|_{\tilde{\mathcal{W}}'} = |(\tilde{z}, \tilde{\theta} - 2\pi j)|_{\tilde{\mathcal{W}}'}$, and $f(\tilde{x}, d) = f((\tilde{z}, \tilde{\theta}), d) = f((\tilde{z}, \tilde{\theta} - 2\pi j), d)$.

By repeating the same calculations as above for the function V_j for any $j \in \mathbb{Z}^q$, we have for all $t \in [0, T)$:

$$\begin{aligned} \vartheta_5\left(\frac{1}{4}|\tilde{X}(t, \tilde{x}_0; d)|_{\tilde{\mathcal{W}}}\right) &\leq \vartheta_6(|\tilde{x}_0|_{[0_k, 2\pi j]} + \ell_3) + g_5 \\ &\quad + \sigma_4(|\tilde{\theta}(t) - 2\pi j|) + \int_0^t \gamma_4(|d(s)|)ds. \end{aligned}$$

Moreover, we have that $|\tilde{x}_0|_{[0, 2\pi j]} \leq |\tilde{z}_0| + |\tilde{\theta}_0 - 2\pi j|$, and $|\tilde{\theta}(t) - 2\pi j| \leq |\tilde{\theta}(t) - \tilde{\theta}_0| + |\tilde{\theta}_0 - 2\pi j|$. Thus, we can rewrite the latter inequality as

$$\begin{aligned} \vartheta_5 \left(\frac{1}{4} |\tilde{X}(t, \tilde{x}_0; d)|_{\tilde{\mathcal{W}}} \right) &\leq \vartheta_6(3|\tilde{z}_0|) + \vartheta_6(3\ell_3) + g_5 \\ &+ \sigma_5(|\tilde{\theta}_0 - 2\pi j|) + \sigma_4(2|\tilde{\theta}(t) - \tilde{\theta}_0|) + \int_0^t \gamma_4(|d(s)|) ds, \end{aligned}$$

where $\sigma_5(r) = \vartheta_6(3r) + \sigma_4(2r)$. Since this inequality is valid for all $j \in \mathbb{Z}^q$, the minimum with respect to j of its right-hand side can be calculated. Note that it is composed of six independent nonnegative terms that can be minimized independently, and only one of them is dependent on j . Furthermore, by construction $\min_{j \in \mathbb{Z}^q} |\tilde{\theta}_0 - 2\pi j| \leq \sqrt{q}\pi$. Thus, we obtain

$$\begin{aligned} \vartheta_5 \left(\frac{1}{4} |\tilde{X}(t, \tilde{x}_0; d)|_{\tilde{\mathcal{W}}} \right) &\leq \vartheta_6(3|\tilde{z}_0|) + \vartheta_6(3\ell_3) + \sigma_5(\sqrt{q}\pi) \\ &+ g_5 + \sigma_4(2|\tilde{\theta}(t) - \tilde{\theta}_0|) + \int_0^t \gamma_4(|d(s)|) ds, \end{aligned}$$

for all $t \in [0, T)$. Now, let us denote by $\tilde{\theta}^u(t)$ the periodic components of the solution $\tilde{X}(t, \tilde{x}_0; 0)$ of the unperturbed system (2). Accordingly, we get

$$\begin{aligned} \vartheta_5 \left(\frac{1}{4} |\tilde{X}(t, \tilde{x}_0; d)|_{\tilde{\mathcal{W}}} \right) &\leq \vartheta_6(3|\tilde{z}_0|) + \vartheta_6(3\ell_3) + \sigma_5(\sqrt{q}\pi) \\ &+ g_5 + \sigma_4(4(|\tilde{\theta}(t) - \tilde{\theta}_0| - |\tilde{\theta}^u(t) - \tilde{\theta}_0|)) \\ &+ \sigma_4(4|\tilde{\theta}^u(t) - \tilde{\theta}_0|) + \int_0^t \gamma_4(|d(s)|) ds. \end{aligned}$$

Moreover, by the reverse triangle inequality we have $||\tilde{\theta}(t) - \tilde{\theta}_0| - |\tilde{\theta}^u(t) - \tilde{\theta}_0|| \leq |\tilde{\theta}(t) - \tilde{\theta}^u(t)|$. Furthermore, we can write $\tilde{\theta}(t) = \tilde{\theta}(0) + \int_0^t f_\theta(\tilde{x}(s), d(s)) ds$ and $\tilde{\theta}^u(t) = \tilde{\theta}(0) + \int_0^t f_\theta(\tilde{x}(s), 0) ds$, where f_θ denotes the components of the vector field f describing the dynamics of the periodic states $\tilde{\theta}$. Consequently, we get $|\tilde{\theta}(t) - \tilde{\theta}^u(t)| \leq \int_0^t |f_\theta(\tilde{x}(s), d(s)) - f_\theta(\tilde{x}(s), 0)| ds$, and due to Assumption 25, we obtain for all $t \in [0, T)$:

$$\begin{aligned} \vartheta_5 \left(\frac{1}{4} |\tilde{X}(t, \tilde{x}_0; d)|_{\tilde{\mathcal{W}}} \right) &\leq \vartheta_6(3|\tilde{z}_0|) + \vartheta_6(3\ell_3) + \sigma_5(\sqrt{q}\pi) \\ &+ g_5 + \sigma_4(4|\tilde{\theta}^u(t) - \tilde{\theta}_0|) + \int_0^t \gamma_4(|d(s)|) ds \\ &+ \sigma_4 \left(4 \int_0^t \varepsilon(|d(s)|) ds \right). \end{aligned}$$

Consider the case $d \equiv 0$. Since Theorem 26 establishes that λ_4 is a positive definite function and $h(\tilde{x}) \equiv 0$ if and only if $|\tilde{x}|_{\mathcal{D}} = 0$, we can rewrite the inequality (20) as

$$DV(\tilde{x})f(\tilde{x}, d) < 0, \quad \forall \tilde{x} \in \{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) \geq 0\} \setminus \mathcal{D}.$$

Therefore, we have that the function $V(\tilde{x})$ in Theorem 26 satisfies also all conditions of Corollary 15. Hence, we

can say that the solutions of the system (1), with $d \equiv 0$, are bounded for all $\tilde{x} \in \mathbb{R}^n$ and all $t \geq 0$. Consequently, we can claim that there exist a constant $\rho_0 > 0$ and a function $\rho_1 \in \mathcal{K}_\infty$, such that $|\tilde{\theta}^u(t) - \tilde{\theta}_0| \leq \rho_0 + \rho_1(|\tilde{\theta}_0|)$ for all $t \geq 0$.

Additionally, let us define the class \mathcal{K}_∞ functions $\gamma_5(r) = \max\{4\varepsilon(r), \gamma_4(r)\}$, and $\sigma_6(r) = 2 \max\{r, \sigma_4(r)\}$. Then, we obtain for all $t \in [0, T)$:

$$\begin{aligned} \vartheta_5 \left(\frac{1}{4} |\tilde{X}(t, \tilde{x}_0; d)|_{\tilde{\mathcal{W}}} \right) &\leq \vartheta_6(3|\tilde{z}_0|) + \vartheta_6(3\ell_3) + \sigma_5(\sqrt{q}\pi) \\ &+ g_5 + \sigma_4(4(\rho_0 + \rho_1(|\tilde{\theta}_0|))) + \sigma_6 \left(\int_0^t \gamma_5(|d(s)|) ds \right). \end{aligned}$$

Finally, note that $|\tilde{z}| \leq |\tilde{x}|$ as well as $|\tilde{\theta}| \leq |\tilde{x}|$, then by applying $\sigma_6^{-1}(\frac{1}{3}(\cdot))$ in both sides of the latter inequality, we obtain

$$\vartheta_7 \left(\frac{1}{4} |\tilde{X}(t, \tilde{x}_0; d)|_{\tilde{\mathcal{W}}} \right) \leq \vartheta_8(3|\tilde{x}_0|) + c + \int_0^t \gamma_5(|d(s)|) ds, \quad (\text{B.3})$$

where $\vartheta_7(\cdot) = \sigma_6^{-1}(\frac{1}{3}\vartheta_5(\cdot))$, $\vartheta_8(\cdot) = \sigma_6^{-1}(\vartheta_6(3(\cdot)) + \sigma_4(8\rho_1(\cdot)))$, and $c = \sigma_6^{-1}(\vartheta_6(3\ell_3) + \sigma_5(\sqrt{q}\pi) + g_5 + \sigma_4(8\rho_0))$.

Now, by Assumption 13 we have that for any $\tilde{x}_0 \in \mathbb{R}^n$ the solution $\tilde{X}(t, \tilde{x}_0; d) \in \mathbb{R}^n$ can be projected to the solution $X(t, x_0; d) \in M$, hence both solutions are defined on the same time interval. Furthermore, the set $\tilde{\mathcal{W}}' \subset \mathbb{R}^n$ can be projected to $\mathcal{W}' \subset M$. Besides, we have that $|\tilde{x}| = |x|$ provided that $x = \mathcal{P}(\tilde{x})$. Thus, the property (B.3) is also satisfied in M . Hence, $T = +\infty$.

Note that due to compactness of $\mathcal{W} \subset M$, there exists a constant $b' = \max_{x \in \mathcal{W}} |x|$. Then, by triangle inequality we can establish that $|x| \leq |x|_{\mathcal{W}} + b'$ for all $x \in M$. Moreover, since ϑ_8 is a class \mathcal{K}_∞ function, we have that $\vartheta_8(3|x_0|) \leq \vartheta_8(6|x_0|_{\mathcal{W}}) + \vartheta_8(6b')$. Hence, we obtain

$$\vartheta_7 \left(\frac{1}{4} |X(t, x_0; d)|_{\mathcal{W}} \right) \leq \vartheta_8(6|x_0|_{\mathcal{W}}) + c' + \int_0^t \gamma_5(|d(s)|) ds. \quad (\text{B.4})$$

where $c' = c + \vartheta_8(6b')$. Thus, we can say that the system (1) has the UBEBS property with respect to \mathcal{W} .

Finally, for $d \equiv 0$, we have already established boundedness of solutions $\tilde{X}(t, \tilde{x}_0; 0)$ of the system (2) in \mathbb{R}^n , and this property is straightforwardly extended to the projected solutions $X(t, x_0; 0)$ in M . This fact together with Assumption 4 and [21, Lemma 4.1] confirm the 0-GATT property with respect to \mathcal{W} for the trajectories of the system (1) in M .

Acknowledgements

This work was partially supported by ANR-DFG projects: SyNPiD - project number 446182476, and SyNNuM - project number 545632618.

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