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A Global Robust Finite–Time Control for the Heisenberg System: Simultaneous Regulation and Tracking[★]

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Abstract

In this paper, a novel hybrid control strategy is proposed to ensure the global robust finite–time regulation and tracking of any admissible reference for the Heisenberg system, sometimes also known as the Brockett integrator. The proposed scheme is comprised of a local controller, ensuring the finite–time attractivity of the origin of the error vector; a global controller, providing finite–time convergence to the region of attraction of the local one; and a state–dependent switching condition, which combines both controllers into a single hybrid structure. The resulting strategy provides settling–time estimations and is straightforward to tune. Some simulations considering possible applications, such as the tracking of geodesic curves, are included to illustrate the implementability of the results.

Key words: Heisenberg System, Brockett’s Integrator, Finite–Time, Regulation, Tracking.

1 Introduction

The Heisenberg system, sometimes called the nonholonomic or Brockett’s integrator [1], takes its name from the relation of its vector fields with the Heisenberg algebra [2] and is a prototypical nonholonomic system diffeomorphic to many physical systems (*e.g.*, the Unicycle Mobile Robot [3] and the Induction Motor [4]) and to the well–known canonical chained form for $n = 3$ [5]. Additionally, the Heisenberg system belongs to a class of nonlinear systems for which its linearization around the origin is not controllable [6]. It is well–known that time–invariant continuous, and even some class of discontinuous [7], feedback controllers fail to globally stabilize its origin due to Brockett’s condition [8]. Besides, the Heisenberg system possesses a sub–Riemannian geometry. This means that the shortest path connecting

any two given points in its 3–dimensional manifold is not always a straight line, and its length does not necessarily coincide with the Riemannian metric (see, *e.g.*, [9] and [10]). For these reasons, the Heisenberg system is a popular benchmark for controller design and stability analysis of nonholonomic systems, and also it has been used as a classic example for design of nonlinear [11], time–varying [12], hybrid [13], and geometric [14] controls.

In the last decades, many control strategies have been developed with the aim to ensure the closed–loop attractivity, or even stability of the origin of nonholonomic systems, including the Heisenberg system (see, *e.g.*, [15], [16], and [17]). Most of these results only achieve their purpose with exponential convergence rates. There are other more recent results that do consider finite–time, fixed–time, and even prescribed–time convergence to the origin (see [18], [19], and [20]), and ensure the robustness of the closed–loop system. However, in these proposals, the control inputs are not uniformly bounded or the tuning requires selecting several parameters, constructing complex auxiliary functions, and some of these results are only local. In addition, the more general simultane-

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ous regulation and tracking problem, which for some applications is part of the so-called motion planning (see, e.g., [21] and [22]) is rarely taken into account. This task is challenging for nonholonomic systems, mainly because, in general, the regulation and tracking problems are not equivalent for this class of systems.

Particularly for the Unicycle Mobile Robot (UMR), which is an important and popular application for the Heisenberg system, some interesting results ensuring simultaneous stabilization and tracking have been published. In [23], a particular change of coordinate is used to design a backstepping controller, ensuring the asymptotic convergence to the origin of the error vector. The results presented in [24] extend previous results of tracking and regulation of the UMR in a unifying simultaneous scheme. The authors of [25] consider limited amplitude control inputs but only guarantee asymptotic convergence of the error vector to the origin or to some compact neighborhood of it if perturbations are taken into account. Nonetheless, none of these results obtain faster than asymptotic convergence rates.

Motivated by the issues mentioned above, in this paper, we propose a uniformly bounded hybrid controller, that is straightforward to tune and guarantees the exact global finite-time convergence to any given point of the state space and any admissible trajectory, despite the presence of time-varying or constant matched perturbations. The design also let us estimate the settling time of the tracking error. The hybrid nature of the resulting design is obtained using the framework of [26]. The hybrid systems framework allows the design of robust controllers for complex nonlinear, piece-wise smooth and switched systems (see some examples in [26] and [27]), and help us to analyze the closed-loop system properties through a structured formal analysis, which is relevant for this particular result considering the nonholonomic constraints of the Heisenberg system.

In order to illustrate the workability of the proposed algorithm, a simulation of the tracking of geodesics connecting the origin to any arbitrary point in the state space is included. The controller design proposed in the paper can be seen as an extension of the results previously introduced in [28]. However, some important differences and additional contributions of the present paper should be noted:

- The global finite-time convergence to the origin of the tracking error is ensured by designing a complementary controller and a hybrid system structure. The global property is relevant since many controller designs for nonholonomic systems require the initial conditions to be selected outside of a zero-measure set.
- The resulting hybrid controller solves both the regulation and tracking problems, which lets us modify any initial state to any arbitrary desired reference

through any desired admissible trajectory without any additional modification to the control structure. This, in turn, extends its application range to a broader class of tasks, e.g., motion planning with almost no inconvenience.

- The global uniform boundedness of the control law and the simplicity of the tuning (the selection of only 1 additional parameter is required) are preserved.

The structure of the rest of this paper is as follows: some relevant definitions and important theoretical notions are included in Section 2. The problem statement can be found in Section 3. In Section 4, the proposed control design is given together with the robustness analysis. In Section 5, some simulations illustrating possible implementations of the proposed control are presented. Finally, in Section 6, the concluding remarks can be found.

Notation. Denote $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$, $\mathbb{R}_- = \{x \in \mathbb{R} : x < 0\}$ and $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$, where \mathbb{R} is the set of all real numbers; $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^n . Define the function $[a]^\gamma = |a|^\gamma \text{sign}(a)$, for any $\gamma \in \mathbb{R}_{\geq 0}$ and any $a \in \mathbb{R}$. For a set $\mathcal{A} \subset U$ with the boundary $\partial\mathcal{A}$, \mathcal{A}^c denotes the complement of the set \mathcal{A} in U , i.e., $\mathcal{A}^c = U \setminus \mathcal{A}$, while A° denotes its interior, i.e., $A^\circ = A \setminus \partial A$, and $cl(\mathcal{A})$ its closure, i.e., $cl(\mathcal{A}) = \mathcal{A} \cup \partial\mathcal{A}$. The notation S^1 represents the 1-sphere and $SO(2)$ denotes the special orthogonal group in \mathbb{R}^2 . A rotation by an angle $\theta \in S^1$ of a vector $v \in \mathbb{R}^2$, can be computed by the multiplication of v by the rotation matrix $\mathbf{R}(\theta) \in SO(2)$, as $\mathbf{R}(\theta)v$, with

$$\mathbf{R}(\theta) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

2 Preliminaries

2.1 Finite-Time Attractivity

Consider the system

$$\dot{x} = f(t, x), \quad t \in \mathbb{R}_{\geq 0}, \quad x(0) = x_0, \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector. The function $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is assumed to be locally bounded uniformly in t . For f locally measurable but discontinuous with respect to x , the solutions are understood in the sense of Filippov [29]. That is, $x(t, x_0)$ is a solution to (1) if it is absolutely continuous, and if it satisfies the differential inclusion

$$\dot{x} \in K[f](t, x) = \bar{co} \bigcap_{\varepsilon > 0} \bigcap_{\mu N=0} f(t, B(x, \varepsilon) \setminus N),$$

where $\bar{\text{co}}M$ represents the convex closure of a set M , $B(x, \varepsilon)$ represents the ball centered at x with radius ε , μN is the Lebesgue measure of a set N . Note that the intersections are taken over all sets N of measure zero, for all $\varepsilon > 0$.

Definition 1 [30]. *Let the origin be an equilibrium of (1) i.e., let $\mathbf{0} \in K[f](t, \mathbf{0})$. The origin is said to be finite-time attractive, with a region of attraction $\Omega \subseteq \mathbb{R}^n$, if there exists a locally bounded function $T : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that $x(t, x_0) = \mathbf{0}$, for any $x_0 \in \Omega$, and all $t \geq T(x_0)$. Such a function T is called a settling-time function.*

2.2 Hybrid Dynamical Systems

A hybrid dynamical system, comprised of continuous and discrete dynamics can be formally described, as proposed in [26], by $\mathcal{H} = (\mathcal{C}, \mathcal{D}, F, G)$, where $\mathcal{C} \subseteq \mathbb{R}^n$ is the flow set, $\mathcal{D} \subseteq \mathbb{R}^n$ is the jump set, $F : \mathcal{C} \rightarrow \mathbb{R}^n$ is the flow map, and $G : \mathcal{D} \rightarrow \mathbb{R}^n$ is the jump map. Then, the explicit expressions describing the evolution of the hybrid state $\xi \in \mathbb{R}^n$ are given as

$$\dot{\xi} = F(\xi), \quad \xi \in \mathcal{C}, \quad (2a)$$

$$\xi^+ = G(\xi), \quad \xi \in \mathcal{D}. \quad (2b)$$

Let a hybrid arc $\phi(t, k)$ with the initial condition $\phi_0 = \phi(0, 0)$, be a solution of (2) parametrized by the continuous time parameter $t \in \mathbb{R}_{\geq 0}$ and the discrete time parameter $k \in \mathbb{N}$, with $\text{dom } \phi$ denoting the corresponding hybrid time domain of ϕ .

Definition 2 *The set $\Omega \subset \mathbb{R}^n$ is said to be globally finite-time attractive for (2), if there exists a locally bounded function $T : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that for each $\phi_0 \in \mathbb{R}^n$, all respective solutions $\phi(t, k) \in \Omega$ for all $t + k \geq T(\phi_0)$, for any $(t, k) \in \text{dom } \phi$.*

3 Problem Statement

The Heisenberg system, affected by non-vanishing but bounded matched perturbations, is described by the equations

$$\dot{z}(t) = Y^\top(t) \mathbf{J} X(t) + \Delta^\top(t) \mathbf{J} X(t), \quad (3)$$

$$\dot{X}(t) = Y(t) + \Delta(t), \quad (4)$$

with the vector state $q(t) = [z(t), X^\top(t)]^\top \in \mathbb{R}^3$, the initial conditions $q(0) = q_0$, and the perturbation $\Delta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^2$, bounded as $\|\Delta(t)\| \leq \bar{\Delta}$, for all $t \geq 0$, and for a known real scalar $\bar{\Delta} > 0$. The function $Y : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^2$ is the control, and $\mathbf{J} \in \mathbb{R}^{2 \times 2}$ is the skew-symmetric matrix

$$\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Given an initial point $r_0 = [z_{r_0}, X_{r_0}^\top]^\top \in \mathbb{R}^3$ and a feasible desired trajectory $r : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^3$ such that $r(t) = [z_r(t), X_r(t)]^\top \in \mathbb{R}^3$, $r(0) = r_0$, where $z_r(t) = z_{r_0} + \int_0^t Y_r^\top(\tau) \mathbf{J} X_r(\tau) d\tau$, and $X_r(t) = X_{r_0} + \int_0^t Y_r(\tau) d\tau$, for a given function $Y_r : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^2$; the problem is to design a control law $Y(t)$ such that the origin of the tracking error $e(t) = q(t) - r(t)$

$$e(t) = \begin{bmatrix} \tilde{z}(t) \\ \tilde{X}(t) \end{bmatrix} = \begin{bmatrix} z(t) - z_r(t) \\ X(t) - X_r(t) \end{bmatrix}, \quad (5)$$

is globally finite-time attractive, i.e., $e(t) = 0$, for all $t \geq T_e(e_0) \geq 0$, for any $e_0 = e(0) = [\tilde{z}_0, \tilde{X}_0^\top]^\top$.

Notice that no other constraint, besides the integrability of the terms Y_r and $Y_r^\top \mathbf{J} X_r$, is imposed on the desired trajectory r . This means that the controller should be able to track piece-wise smooth reference trajectories r . Moreover, due to this, the controller should also be able to solve the regulation case, i.e., to track $r(t) = r_d$, for all $t \geq 0$, for a given arbitrary constant vector $r_d \in \mathbb{R}^3$, and also to track asymptotic, or even finite-time, convergent trajectories i.e., $r(t) \rightarrow r_d$ as $t \rightarrow \infty$, or $r(t) = r_d$, for $t \geq T > 0$.

In the remainder of the paper, the argument t is omitted whenever it is clear and does not lead to any ambiguity.

4 Controller Design

The controller design is carried out in three stages. First, a local controller ensuring the robust finite-time attractivity of $e = \mathbf{0}$ from a known region of attraction is obtained. Then, the design of a complementary controller, guaranteeing that any solution starting outside of the region of attraction of the first controller reaches its interior in finite-time, is carried out. Finally, a switching condition is added in order to combine both controllers, and to obtain the robust globally attractivity property of $e = \mathbf{0}$.

4.1 Local Finite-Time Attractive Controller Design

Taking into account system (3)–(4) and the tracking error (5), let us introduce the auxiliary variable $\Gamma = \tilde{z} - \tilde{X}^\top \mathbf{J} X_r$, with its time derivative given by $\dot{\Gamma} = (Y - Y_r)^\top \mathbf{J} \tilde{X} + \Delta^\top \mathbf{J} \tilde{X}$, and initial condition $\Gamma_0 = \Gamma(e_0)$. Now, defining $\tilde{Y} = Y - Y_r$, the dynamics for Γ and \tilde{X} can be rewritten as

$$\dot{\Gamma} = \tilde{Y}^\top \mathbf{J} \tilde{X} + \Delta^\top \mathbf{J} \tilde{X}, \quad (6)$$

$$\dot{\tilde{X}} = \tilde{Y} + \Delta. \quad (7)$$

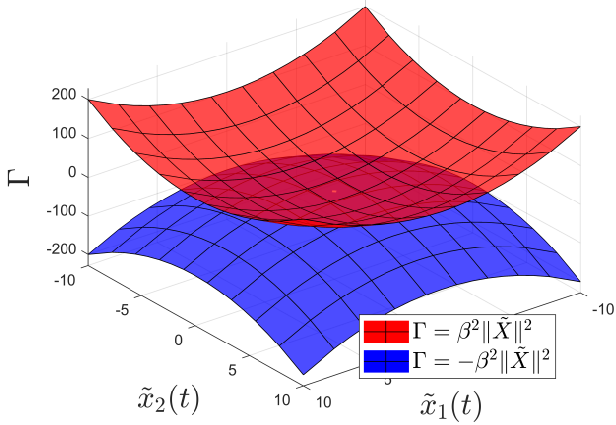


Fig. 1. Representation of the set \mathcal{D}_β for $\beta = 1$. The set \mathcal{D}_β is comprised of the red and blue manifolds and the region between them.

Note that if $\tilde{X} = \mathbf{0}$ and $\Gamma = 0$ then $\tilde{z} = 0$. So, the finite-time tracking problem for system (3)–(4) can be equivalently solved by designing a control law \tilde{Y} for the the system (6)–(7), that ensures the finite-time convergence to the origin of the state vector $\tilde{q} = [\Gamma, \tilde{X}^\top]^\top \in \mathbb{R}^3$, verifying that $\tilde{q} = \mathbf{0}$ implies $e = \mathbf{0}$. Before introducing the proposed controller, some additional definitions and previous control design results for the local attractivity of the origin of the Heisenberg system are required.

First, we introduce the functions

$$\omega(\tilde{q}) = \text{atan}(\Phi(\tilde{q})),$$

$$\Phi(\tilde{q}) = \begin{cases} \frac{2\beta|\Gamma|^{1/2}}{\|\tilde{X}\|}, & \text{if } \tilde{q} \neq \mathbf{0}, \\ 0, & \text{if } \tilde{q} = \mathbf{0}, \end{cases}$$

with $\beta > 0$. Then, define the sets $\mathcal{M}_{\tilde{x}} = \{\tilde{q} \in \mathbb{R}^3 : \tilde{X} = \mathbf{0}\}$ and $\mathcal{D}_\beta = \{\tilde{q} \in \mathbb{R}^3 : |\Phi(\tilde{q})| \leq 2\beta^2\}$, or equivalently, as shown in Figure 1, $\mathcal{D}_\beta = \{\tilde{q} \in \mathbb{R}^3 : |\Gamma| \leq \beta^2 \|\tilde{X}\|^2\}$. Evidently $\mathcal{M}_{\tilde{x}} \cap \mathcal{D}_\beta = \{\mathbf{0}\}$. The following theorem, which summarizes the results presented in [28], gives a controller design and sufficient conditions on its parameters, ensuring the invariance of the set \mathcal{D}_β and the finite-time attractivity of the origin from it.

Theorem 3 [28]. Consider the system (6)–(7) with the control input

$$\tilde{Y} = \tilde{Y}_1 = -\gamma \mathbf{R}(\omega(\tilde{q})) \frac{\tilde{X}}{\|\tilde{X}\|}, \quad (8)$$

where $\mathbf{R} \in SO(2)$, and by definition $\tilde{X}/\|\tilde{X}\| \in \{v \in \mathbb{R}^2 : \|v\| \leq 1\}$ for $\tilde{X} = \mathbf{0}$; with the parameters $\beta > |\Gamma_0|^{1/2}/\|\tilde{X}_0\|$ and $\gamma > \max\{\bar{\Delta}/C_\beta, \gamma_0\}$; and the

constants $\omega_\beta = \text{atan}(2\beta^2)$, $C_\beta = \cos(\omega_\beta)$, and

$$\gamma_0 = \frac{\sqrt{\|\tilde{X}_0\|^2 + 4\beta^2|\Gamma_0|} \left(\|\tilde{X}_0\|^2 + 2|\Gamma_0| \right)}{2|\Gamma_0|^{1/2} \|\tilde{X}_0\| \left(\beta \|\tilde{X}_0\| - |\Gamma_0|^{1/2} \right)} \bar{\Delta}. \quad (9)$$

Then, the origin of the closed-loop system is finite-time attractive, with the region of attraction \mathcal{D}_β . Moreover, for any $\tilde{q}_0 = \tilde{q}(0) \in \mathcal{D}_\beta$ there exists $T_{\tilde{q}}(\tilde{q}_0) \leq \|\tilde{X}_0\|/(\gamma C_\beta - \bar{\Delta})$, such that $\tilde{q}(t) = \mathbf{0}$, for all $t \geq T_{\tilde{q}}(\tilde{q}_0)$. Also, this parameter selection establishes that $|\Phi(\tilde{q}(t))|$ is strictly decreasing for $t > 0$, implying that \mathcal{D}_β , and \mathcal{D}_β^c are invariant sets for the solutions of the closed-loop system.

Note that, if $\tilde{X}_0 \notin \mathcal{M}_{\tilde{x}}$, there always exists $\beta > 0$ satisfying the conditions of Theorem 3. This condition ensures that $\tilde{q}_0 \in \mathcal{D}_\beta$. Also, from the definition of Φ and the invariance of \mathcal{D}_β , it holds that $|\Gamma(t)| \leq \beta^2 \|\tilde{X}(t)\|^2$ for all $t > 0$, for any trajectory initiated in \mathcal{D}_β . Thus, for any $T > 0$, $\tilde{X}(T) = \mathbf{0}$ implies that $\Gamma(T) = 0$, and hence, $\tilde{z}(T) = 0$. However, if $\tilde{X}_0 \in \mathcal{M}_{\tilde{x}}$, there does not exist β such that $\tilde{q}_0 \in \mathcal{D}_\beta$.

4.2 Complementary Controller Design

The main drawback of (8) is that, the finite-time attractivity of the origin is only achieved locally on \mathcal{D}_β . Moreover, if $\tilde{X}_0 \in \mathcal{M}_{\tilde{x}}$, (8) cannot drive \tilde{q} to the origin. In order to extend the finite-time attractivity property to \mathcal{D}_β^c , and consequently obtain a globally attractive origin of (6)–(7) in closed-loop, the following complementary controller is proposed.

Theorem 4 Consider the set \mathcal{D}_β with a given $\beta > 0$, and system (6)–(7), with initial conditions $\tilde{q}_0 \in \mathcal{D}_\beta^c$ and the control input

$$\tilde{Y} = \tilde{Y}_2 = \begin{cases} \gamma \hat{p}, & \text{if } \tilde{q} \in \mathcal{M}_{\tilde{x}} \\ \gamma \frac{\tilde{X}}{\|\tilde{X}\|}, & \text{if } \tilde{q} \notin \mathcal{M}_{\tilde{x}} \end{cases} \quad (10)$$

with $\hat{p} = [1, 0]^\top$, and $\gamma > (1 + 2\beta^2)\bar{\Delta}/2\beta^2$. Then, the trajectories of the tracking error satisfy $\tilde{q}(t) \in \mathcal{D}_\beta$ for all $t \geq T_\beta(\tilde{q}_0)$ with $T_\beta(\tilde{q}_0) \leq (\frac{1}{\beta}|\Gamma_0|^{1/2} - \|\tilde{X}_0\|)/(\bar{\gamma} - \bar{\Delta}/2\beta^2)$, where $\bar{\gamma} = \gamma - \bar{\Delta}$.

All of the proofs are postponed to the Appendix.

Despite that the control input (10) ensures the finite-time attractivity of \mathcal{D}_β , for any given $\beta > 0$, the problem is only partially solved, since it is necessary to switch the control input to (8), once the set \mathcal{D}_β has been reached. The subsequent section addresses this issue.

4.3 Global Finite-Time Attractive Hybrid Controller Design

For the purpose of ensuring the global finite-time attractivity of the closed-loop system origin, a hybrid controller design, obtained by combining (8) and (10), is proposed.

Consider the discrete state variable $\sigma \in \{1, 2\}$ and the hybrid state vector $\eta = [\tilde{q}^\top, \sigma]^\top \in \mathbb{R}^3 \times \{1, 2\}$. Then, the corresponding dynamics are given as

$$\begin{aligned} \dot{\eta} &= F(\eta) = \begin{bmatrix} \tilde{Y}_\sigma^\top \mathbf{J} \tilde{X} + \Delta^\top \mathbf{J} \tilde{X} \\ \tilde{Y}_\sigma + \Delta \\ 0 \end{bmatrix}, \quad \eta \in \mathcal{C}, \\ \eta^+ &= G(\eta) = \begin{bmatrix} \tilde{q} \\ 3 - \sigma \end{bmatrix}, \quad \eta \in \mathcal{D}, \end{aligned}$$

where the closed flow and jump sets are given as

$$\mathcal{C} = \bigcup_{i=1}^2 \mathcal{C}_i \times \{i\}, \quad \mathcal{D} = \bigcup_{i=1}^2 \mathcal{D}_i \times \{i\},$$

with

$$\begin{aligned} \mathcal{C}_1 &= \left\{ \tilde{q} \in \mathbb{R}^3 : |\Gamma| \leq \beta^2 \|\tilde{X}\|^2 \right\}, \quad \mathcal{D}_1 = \text{cl}(\mathcal{C}_1^c) \setminus \{\mathbf{0}\}, \\ \mathcal{D}_2 &= \left\{ \tilde{q} \in \mathbb{R}^3 : |\Gamma| \leq \beta_\varepsilon^2 \|\tilde{X}\|^2 \right\}, \quad \mathcal{C}_2 = \text{cl}(\mathcal{D}_2^c) \setminus \{\mathbf{0}\}, \end{aligned}$$

for some positive scalars β and $\beta_\varepsilon < \beta$. Now, the closed-loop system can be defined as a hybrid system $\mathcal{H}_c = (\mathcal{C}, \mathcal{D}, F, G)$, with the given maps and sets¹. Note that, if the parameters γ, β , and β_ε are adequately selected, the definition of the sets \mathcal{C} and \mathcal{D} generates a hysteresis-like switching mechanism [26] for the control signal \tilde{Y}_σ , ensuring that only one switch occurs at the same time instant, thus avoiding any Zeno behaviour.

The mechanism leaves \tilde{Y}_2 active if the variable $\sigma = 2$ and $\tilde{q} \in \mathcal{C}_2$. Thus, for any initial condition $\tilde{q}_0 \in \mathcal{C}_2 \supset \mathcal{D}_\beta^c$, \tilde{q} is driven to $\mathcal{D}_2 \subset \mathcal{C}_1^o$ while \tilde{Y}_2 is active. On the other hand, the mechanism leaves \tilde{Y}_1 if the variable $\sigma = 1$ and $\tilde{q} \in \mathcal{C}_1$. So, once $\tilde{q} \in \mathcal{C}_1 = \mathcal{D}_\beta$, \tilde{q} is driven to the origin in a finite time while \tilde{Y}_1 is active. The variable σ switches while $\tilde{q} \in \mathcal{D}_i$, for $i \in \{1, 2\}$. Notice that $\mathcal{D}_2 \subset \mathcal{C}_1^o$, then if $\sigma = 2$ and $\tilde{q} \in \mathcal{D}_2$, the switch to $\sigma = 1$ only occurs on \mathcal{C}_1^o . Additionally, because $\mathcal{C}_1^o = \mathcal{D}_\beta^o$ is an invariant set for the continuous closed-loop system with \tilde{Y}_1 , and $\mathcal{D}_1 \cap \mathcal{C}_1^o = \emptyset$, then \tilde{q} cannot reach \mathcal{D}_1 once it is in \mathcal{C}_1^o , and it is impossible to switch back to $\sigma = 2$. In the worst

¹ Formally, the solutions of \mathcal{H}_c are understood in the Krasovskii sense (see [26, Chapter 4]).

case, *i.e.*, when the initial conditions are such that $\sigma = i$ while $\tilde{q}_0 \in \mathcal{C}_j \cap \mathcal{D}_i$, for $i \neq j$, only two switches can happen at most. In this case, an instantaneous switch occurs according to $\sigma^+ = 3 - \sigma$ at the initial continuous time $t = 0$, setting $\sigma = j$, and leaving the subsequent evolution of the state continuously on \mathcal{C}_j exactly as any of the cases explained above. Therefore, only one additional switch happens with respect to any other cases, giving a maximum of two switches and a minimum of zero.

Remark 5 *The nominal system, *i.e.*, \mathcal{H}_c with $\Delta \equiv \mathbf{0}$, fulfill the hybrid basic conditions given in [26, Assumption 6.5], these are sufficient conditions for the nominal well-posedness of the system.*

Remark 5 also ensures the well-posedness of the system \mathcal{H}_c in the presence of continuous vanishing perturbations [26, Proposition 6.8], *e.g.*, some possible unmodeled dynamics. Although this class of perturbations are not formally considered in the present analysis, the well-posedness of the closed-loop system rules out any possible finite-time escape.

The following Theorem gives the conditions on the parameter selection ensuring the desired global finite-time attractivity property of the point $[\mathbf{0}^\top, 1]^\top$ for \mathcal{H}_c .

Theorem 6 *Consider $\tilde{Y} = \tilde{Y}_\sigma$ given by (8) and (10), respectively, for $\sigma \in \{1, 2\}$ and*

$$\begin{aligned} \gamma &> \max \left\{ \bar{\Delta}/C_\beta, \gamma_\beta, \frac{(1 + \beta_\varepsilon^2) \bar{\Delta}}{\beta_\varepsilon^2} \right\}, \\ \gamma_\beta &= \frac{\sqrt{1 + 4\beta_\varepsilon} (\beta + (2\beta_\varepsilon^4/\beta))}{2\beta_\varepsilon^2 (\beta - (\beta_\varepsilon^2/\beta))} \bar{\Delta}, \end{aligned}$$

with $0 < \beta_\varepsilon < \beta$. Then, $\eta = [\mathbf{0}^\top, 1]^\top$ is globally finite-time attractive for the hybrid system \mathcal{H}_c , with the settling-time $T(\tilde{q}_0) \leq m_c \bar{T}_1(\tilde{q}_0) + \bar{T}_2(\tilde{q}_0)$, where $m_c = 1 + \frac{\bar{\Delta}}{2\beta_\varepsilon^2(\gamma C_\beta - \bar{\Delta})}$,

$$\begin{aligned} \bar{T}_1(\tilde{q}_0) &= \max \left\{ 0, \frac{\frac{1}{\beta_\varepsilon} |\Gamma_0|^{\frac{1}{2}} - \|\tilde{X}_0\|}{\gamma - (\bar{\Delta}/2\beta_\varepsilon^2)} \right\}, \\ \bar{T}_2(\tilde{q}_0) &= \max \left\{ \frac{|\Gamma_0|^{\frac{1}{2}}}{\beta_\varepsilon (\gamma C_\beta - \bar{\Delta})}, \frac{\|\tilde{X}_0\|}{(\gamma C_\beta - \bar{\Delta})} \right\}. \end{aligned}$$

The next result, which considers the unperturbed case, *i.e.*, $\Delta \equiv \mathbf{0}$, follows directly from Theorem 6.

Corollary 7 *If $\Delta \equiv \mathbf{0}$, then $\eta = [\mathbf{0}^\top, 1]^\top$ is globally finite-time attractive, for the hybrid system \mathcal{H}_c , with $0 < \beta_\varepsilon < \beta$, and the control inputs \tilde{Y}_σ given by (8) and (10), respectively, for $\sigma \in \{1, 2\}$, with $\gamma > 0$. Moreover,*

the settling-time can be exactly calculated as

$$T(\tilde{q}_0) = \max \left\{ 0, \frac{1}{\gamma} \left(\frac{1}{\beta_\varepsilon} |\Gamma_0|^{\frac{1}{2}} - \|\tilde{X}_0\| \right) \right\} + \frac{|\Gamma_0|^{\frac{1}{2}}}{\gamma \beta_\varepsilon C_\beta}.$$

4.4 On the Implementation

Note that the actual control input applied to the system (3)–(4) is $Y = \tilde{Y}_\sigma + Y_r$, and although the integrability is the only formal assumption on Y_r , in order to physically implement the proposed controller it is necessary that Y_r is uniformly bounded, *i.e.*, $\|Y_r(t)\| \leq \bar{Y}_r, \forall t \geq 0$, for some $0 < \bar{Y}_r < \infty$. In this case, Y_r could be considered as part of the matched perturbation as $\Delta_y(t) = \Delta(t) + Y_r(t)$, with the upper bound $\bar{\Delta}_y = \bar{\Delta} + \bar{Y}_r$. Therefore, we can directly design the control law Y as in Theorem 6, without taking into account Y_r (*i.e.*, $Y = \tilde{Y}_\sigma$), but considering $\bar{\Delta}_y$ instead of $\bar{\Delta}$.

Implementing the control considering this additional assumption has two important consequences. In first place, it is not necessary to feed Y_r as a part of Y in order to have global finite-time convergence to the origin of the tracking error, as long as the gain γ is selected adequately, by considering the upper bound of the perturbation as $\bar{\Delta}_y$. And secondly, because no other additional condition is imposed in the perturbation Δ_y , it is possible to track piece-wise smooth trajectories references r , generated by discontinuous but bounded functions Y_r , *e.g.*, a bounded upper semi-continuous function.

For the tuning, the selection of β is arbitrary and does not depend on the initial conditions. The only requirement on β is that $\beta > \beta_\varepsilon > 0$, but β_ε can always be selected as $\beta_\varepsilon = \varepsilon\beta$ with $\varepsilon \in (0, 1)$. Once β and β_ε are fixed, it is simply required to select γ according to the conditions given in Theorem 6, which can be directly calculated without any additional information because these conditions only depend on $\bar{\Delta}$, β , and β_ε ; or if it is assumed that $\Delta \equiv \mathbf{0}$, simply select $\gamma > 0$, as in Corollary 7.

5 Simulations: Tracking of Geodesics

It is well-known that due to the sub-Riemannian geometry of the Heisenberg system, the geodesics, *i.e.*, admissible curves of minimal length connecting any two given points of the manifold, are of particular interest [9]. These curves also have the particularity that their time derivatives go to zero in a finite time. Thus, one illustrative and interesting application of the proposed controller is the tracking of geodesic trajectories, showcasing that the controller achieves the simultaneous regulation and tracking objective.

For simplicity, in these simulations we only consider geodesics connecting the origin and any given point $r_d \in$

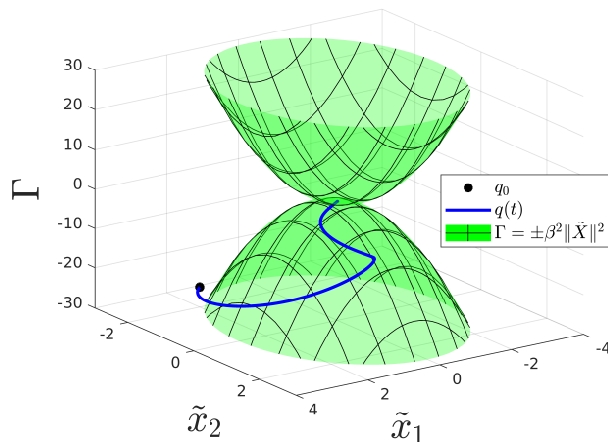


Fig. 2. Trajectories of the Tracking Error in \mathbb{R}^3 – Case 1

\mathbb{R}^3 . The explicit expressions for a geodesic of the Heisenberg system from the origin to an arbitrary point $r_d \neq \mathbf{0}$ are the following [10]

$$X_r(t) = 1/h_0 \begin{bmatrix} \sin(\theta_0 + h_0 t) - \sin(\theta_0) \\ -\cos(\theta_0 + h_0 t) + \cos(\theta_0) \end{bmatrix},$$

$$z_r(t) = \int_0^t \dot{X}_r(s) \mathbf{J} X_r(s) ds = 1/h_0^2 (h_0 t - \sin(h_0 t)),$$

for $0 \leq t \leq T$ and $r(t) = r_d$, for $t > T$. Therefore, a geodesic connecting the origin and $r_d = [57.0789, -10, -10]^\top$, in $T = 15.7080$ [s], can be obtained by selecting $h_0 = 0.1$ and $\theta_0 = \pi$. Additionally, it is assumed that the system is affected by the perturbation $\Delta(t) = (0.05 \cos(0.01t) + 0.01, 0.05 \sin(0.05t) - 0.01)^\top$, and hence, we have that $\bar{\Delta} = 0.0849$. Considering this, the constants $\beta = 2$ and $\gamma = 2.5$ were chosen as parameters for the controllers (8) and (10). Also, $\beta_\varepsilon = 1.99$ was chosen.

Case 1: $\tilde{q}_0 \in \mathcal{D}_\beta$. For this case, the initial conditions for (3)–(4) are selected as $q_0 = [-25, 2, -2]^\top$, and $\sigma(0) = 1$ ensuring that $\tilde{q}_0 \in \mathcal{C}_1^o = \mathcal{D}_\beta^o$ as shown in Figure 2. It is clear that no switching occurs due to the invariance of \mathcal{C}_1^o for $\sigma = 1$ and $\mathcal{C}_1^o \cap \mathcal{D}_1 = \emptyset$, so only Y_1 is active. The upper bound for the settling-time is $T_{\tilde{q}} \leq 10.0522$. The tracking motion of q related to the geodesic r is shown in Figure 3, where it is clear that q converges to r , and then, both r and q reach r_d , remaining on this point afterwards. The control inputs are depicted in Figure 4, where it is shown that these are only discontinuous at $\tilde{q} = \mathbf{0}$, which occurs after 5 seconds.

Case 2: $\tilde{q} \in \mathcal{D}_\beta^c$. For the second case, the initial conditions for (3)–(4) are selected as $q_0 = [-25, 0, 0]^\top$, and $\sigma(0) = 2$, this implies that $\tilde{q}_0 \in \mathcal{M}_{\tilde{x}} \cap \mathcal{C}_2 \subset \mathcal{D}_\beta^c$, as shown in Figure 5. In this case, the controller Y_2 is active initially and remains active until $\tilde{q}(t) \in \partial \mathcal{D}_2 \subset \mathcal{D}_\beta$. At this

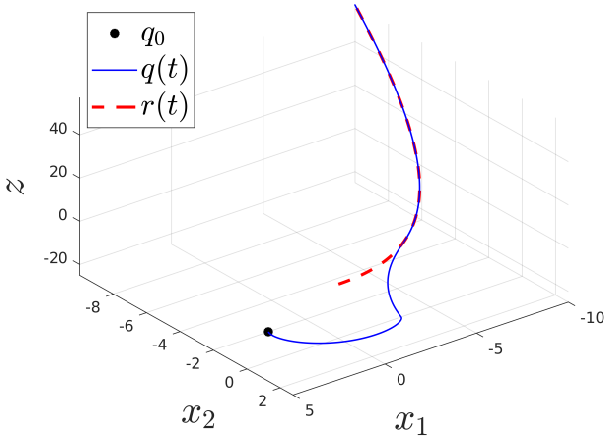


Fig. 3. Trajectories of the System in \mathbb{R}^3 – Case 1

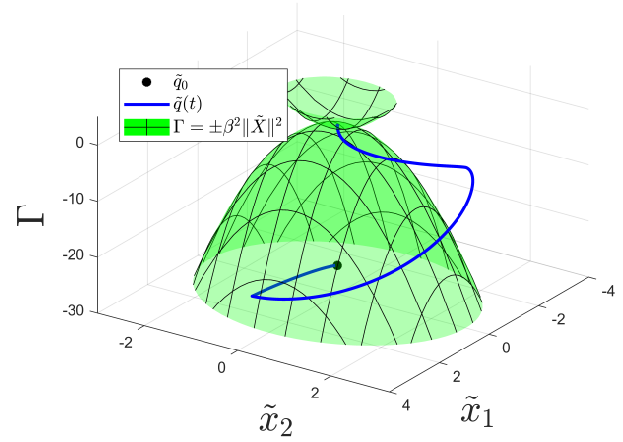


Fig. 5. Trajectories of the Tracking Error in \mathbb{R}^3 – Case 2

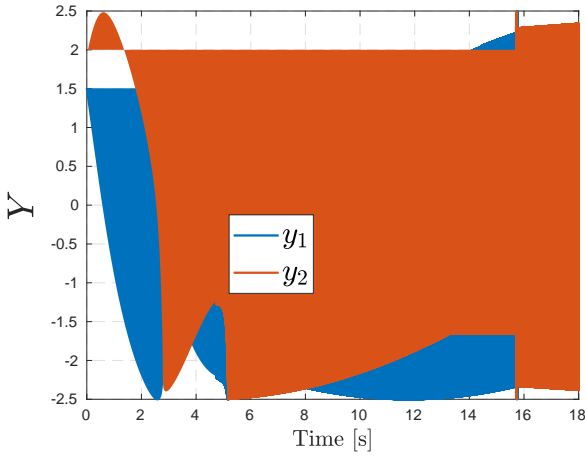


Fig. 4. Time evolution of the control input – Case 1

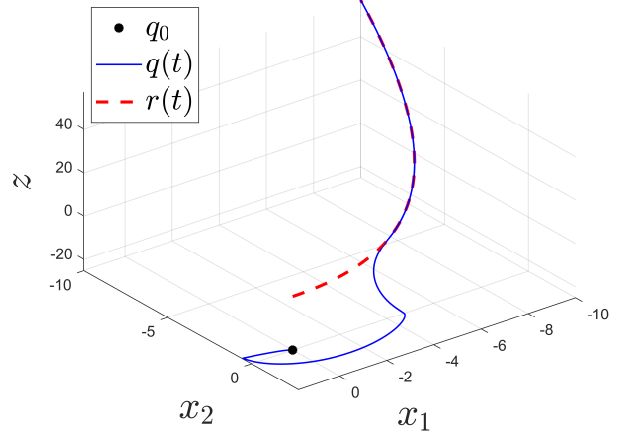


Fig. 6. Trajectories in \mathbb{R}^3 – Case 2

moment, the controller switches to Y_1 , and then the solution reaches the origin with $T_{\bar{q}} \leq 11.8951$. In Figure 6, the finite-time tracking of the geodesic r is presented. Finally, the control signals are depicted in Figure 7, where it is possible to notice a jump around the 1 second mark, this jump is evidently caused by the hybrid nature of the controller, and the trajectories reaching the jump set \mathcal{D}_2 , effectively switching $\sigma = 2$ to $\sigma = 1$ at this instant. No other switching of this nature happens, until the control input becomes discontinuous again only at the origin after around 5 seconds.

6 Conclusions

A simple to tune globally uniformly bounded finite-time controller, that solves the simultaneous regulation and tracking, is proposed for the Heisenberg system. The main contributions of this paper can be summarized as:

- An extension of the semi-global results from [28]

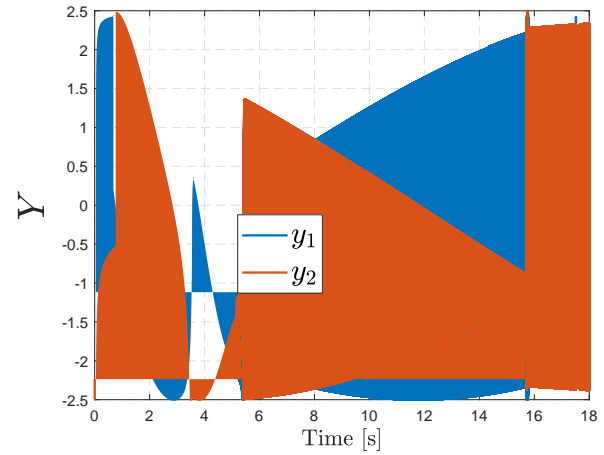


Fig. 7. Time evolution of the control input – Case 2

to the simultaneous tracking and regulation case is proposed.

- b) A new complementary controller, which ensures the global attractivity of the domain of convergence of the closed-loop error dynamics with the controller obtained in item a), is designed.
- c) A hybrid controller, ensuring the global attractivity of the origin of the closed-loop error dynamics, comprised of the controllers mentioned in items a) and b), and a state-dependent hysteresis-like supervisor are developed.

Finally, in order to illustrate the implementation, a simulation of the tracking of geodesics connecting the origin to any arbitrary point in \mathbb{R}^n is included.

Appendix

Proof of Theorem 4: Notice that $\mathcal{M}_{\bar{x}}$ is not an invariant set for the closed-loop system in \mathcal{D}_β^c , due to the definition of \tilde{Y}_2 in $\mathcal{M}_{\bar{x}} \cap \mathcal{D}_\beta^c$, and $\gamma > \bar{\Delta}$. This implies that, for any $\tilde{q}_0 \in \mathcal{M}_{\bar{x}} \cap \mathcal{D}_\beta^c$, there exists an arbitrarily small $t_\varepsilon > 0$, such that $\tilde{q}(t_\varepsilon) \in \mathcal{D}_\beta^c \setminus \mathcal{M}_{\bar{x}}$. Then, assuming $\tilde{q}(t_\varepsilon) \in \mathcal{D}_\beta^c \setminus \mathcal{M}_{\bar{x}}$ and applying controller (10) to system (6)–(7), the closed-loop tracking error dynamics satisfies $\dot{\Gamma} = \Delta^\top \mathbf{J} \tilde{X}$, and $\dot{\tilde{X}} = \gamma \tilde{X} / \|\tilde{X}\| + \Delta$. Now, define the function $C_{\bar{x}} = \frac{1}{2} \|\tilde{X}\|^2$, with the time derivative lower bounded as

$$\begin{aligned} \dot{C}_{\bar{x}} &= \gamma \|\tilde{X}\| + \tilde{X}^\top \Delta \\ &\geq \gamma \|\tilde{X}\| - \bar{\Delta} \|\tilde{X}\| = \sqrt{2} \gamma C_{\bar{x}}^{\frac{1}{2}} \end{aligned} \quad (11)$$

with $\bar{\gamma} = \gamma - \bar{\Delta}$, this implies that for any $t \geq 0$,

$$t \leq \frac{1}{\bar{\gamma}} \left(\|\tilde{X}(t)\| - \|\tilde{X}_0\| \right). \quad (12)$$

Inequality (11), establishes that $\mathcal{M}_{\bar{x}}$ is completely unstable on \mathcal{D}_β^c (see [31, Theorem 2.13]). Thus, if $\tilde{q}(t_\varepsilon) \in \mathcal{D}_\beta^c \setminus \mathcal{M}_{\bar{x}}$, then $\tilde{q}(t) \notin \mathcal{M}_{\bar{x}} \cap \mathcal{D}_\beta^c$ for all $t > t_\varepsilon$. In a similar manner, define $C_\Gamma = |\Gamma|$, with the time derivative given by $\dot{C}_\Gamma = [\Gamma]^0 \Delta \mathbf{J} \tilde{X}$, using the fact that as long as $\tilde{q} \in \mathcal{D}_\beta^c$, $|\Gamma|^{\frac{1}{2}} > \beta \|\tilde{X}\|$, we can upper-bound \dot{C}_Γ as $\dot{C}_\Gamma \leq |\Delta \mathbf{J} \tilde{X}| \leq \bar{\Delta} \|\tilde{X}\| \leq \frac{\bar{\Delta}}{\beta} C_\Gamma^{\frac{1}{2}}$, which in turn implies that

$$|\Gamma(t)|^{\frac{1}{2}} \leq |\Gamma_0|^{\frac{1}{2}} + \frac{\bar{\Delta}}{2\beta} t. \quad (13)$$

We are looking for an upper estimate of the time instant T_β , such that $\tilde{q}(T_\beta) \in \mathcal{D}_\beta$, this is the first instant T_β that

$|\Gamma(T_\beta)|^{\frac{1}{2}} = \beta \|\tilde{X}(T_\beta)\|$. From (12) and (13)

$$\begin{aligned} T_\beta &\leq \frac{1}{\bar{\gamma}} \left(\|\tilde{X}(T_\beta)\| - \|\tilde{X}_0\| \right) = \frac{1}{\bar{\gamma} \beta} |\Gamma(T_\beta)|^{\frac{1}{2}} - \frac{1}{\bar{\gamma}} \|\tilde{X}_0\| \\ &\leq \frac{1}{\bar{\gamma} \beta} \left(|\Gamma_0|^{\frac{1}{2}} + \frac{\bar{\Delta}}{2\beta} T_\beta \right) - \frac{1}{\bar{\gamma}} \|\tilde{X}_0\| \\ &= \frac{1}{\bar{\gamma}} \left(\frac{1}{\beta} |\Gamma_0|^{\frac{1}{2}} - \|\tilde{X}_0\| + \frac{\bar{\Delta}}{2\beta^2} T_\beta \right). \end{aligned}$$

Then, it is straightforward to deduce that

$$T_\beta \leq \frac{\frac{1}{\beta} |\Gamma_0|^{\frac{1}{2}} - \|\tilde{X}_0\|}{\bar{\gamma} - \bar{\Delta}/2\beta^2}. \quad (14)$$

The numerator of (14) is always positive due to the selection of β . Then, to ensure the positivity of the right-hand side of (14) it is required, from the denominator and the definition $\bar{\gamma} = \gamma - \bar{\Delta}$, that $\gamma - \bar{\Delta} - (\bar{\Delta}/2\beta^2) > 0$. Thus, $\gamma > (1 + 2\beta^2)\bar{\Delta}/2\beta^2 > \bar{\Delta}$ should be satisfied. ■

Proof of Theorem 6: Recall that a jump (switch) can only occur if $\tilde{q} \in \mathcal{D}_i$ while $\sigma = i$, and the jump maps dictates that $\sigma^+ = 3 - i$. Otherwise, the time evolution of the system is given by the corresponding i -esim flow map such that $\tilde{q} \in \mathcal{C}_i$. Note that $\mathcal{C}_1 = \mathcal{D}_\beta$, $\mathcal{C}_1 \cup \mathcal{C}_2 = \mathbb{R}^3$, $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$, $\mathcal{C}_j^o \cap \mathcal{D}_j = \emptyset$, but $\mathcal{C}_j \cap \mathcal{D}_j = \partial \mathcal{C}_j \setminus \{\mathbf{0}\} \subset \mathcal{C}_j^o \cap \mathcal{D}_j$, for $i \neq j$. Also, the definition of the sets \mathcal{C} and \mathcal{D} , as it will be shown, ensures that there does not exist the possibility of more than one switch at the same time instant for \mathcal{H}_c , and no more than 2 switches in total, thus avoiding any possible Zeno behaviour or sliding motion on the boundary. Additionally, it is possible to verify that the singleton $[\mathbf{0}^\top, 1]^\top$ is strongly invariant²: flow invariant, in Filippov sense, due to $[\mathbf{0}^\top, 1]^\top \in K[F]([\mathbf{0}^\top, 1]^\top) \cap \mathcal{C}_1$, and jump invariant because $\mathbf{0} \notin \mathcal{D}_1$.

The convergence analysis is carried out considering the following 4 cases:

1) For initial conditions $\eta_a(0, 0) \in \mathcal{C}_2^o \times \{2\}$, *i.e.*, $\tilde{q}_0 \in \mathcal{C}_2^o \cap \mathcal{D}_\beta^c$ and $\sigma(0) = 2$, initially \tilde{Y}_2 is acting on (6)–(7). From the definition of \mathcal{D}_2 , the fact that $\mathcal{C}_2 \cap \mathcal{D}_2 = \emptyset$ and Theorem 4, this control input drives \tilde{q} towards \mathcal{D}_2 , and remains active until the time instant T_1 , such that $\tilde{q}(T_1) \in \partial \mathcal{D}_2 \subset \mathcal{D}_\beta$. At this instant

$$|\Gamma(T_1)|^{\frac{1}{2}} = \beta_\varepsilon^2 \|\tilde{X}(T_1)\| < \beta^2 \|\tilde{X}(T_1)\|. \quad (15)$$

Using a similar argument as in the proof of Theorem 4 it is possible to upper-bound T_1 as $T_1(\tilde{q}_0) \leq (\frac{1}{\beta_\varepsilon} |\Gamma_0|^{\frac{1}{2}} - \|\tilde{X}_0\|) / (\bar{\gamma} - \bar{\Delta}/2\beta_\varepsilon^2)$. Furthermore, at T_1 due to $\tilde{q} \in$

² A set S is strongly invariant for a system \mathcal{H} , if for each $\xi_0 \in S$ all the hybrid arcs (solutions) of \mathcal{H} , $\phi(\xi_0) \subset S$.

$\mathcal{D}_2 \cap \mathcal{C}_1^o$, this is $|\Gamma(T_1)| < \beta^2 \|\tilde{X}(T_1)\|^2$, a switch occurs and $\sigma^+(T_1) = 1$. From this moment on, \tilde{Y}_1 starts acting on system (6)–(7). The invariance of $\mathcal{D}_\beta^o = \mathcal{C}_1^o$ with controller \tilde{Y}_1 , is ensured if³

$$\begin{aligned} \frac{d|\Phi(t)|}{dt} \Big|_{t=T_1} &\leq -\gamma \frac{2\beta \left(\beta - \frac{1}{2\beta} |\Phi(T_1)| \right)}{\|\tilde{X}(T_1)\| \sqrt{1 + \Phi^2(T_1)}} \\ &+ |\Gamma(T_1)|^{-\frac{1}{2}} \left(\beta + \frac{1}{2\beta} |\Phi(T_1)|^2 \right) \bar{\Delta} < 0. \end{aligned}$$

The fulfillment of previous inequality requires that, $\beta > |\Gamma(T_1)|^{\frac{1}{2}} / \|\tilde{X}(T_1)\|$, which is fulfilled at this instant, due to (15). Additionally, it is required that

$$\gamma > \frac{\sqrt{1 + \Phi^2(T_1)} \left(\beta + \frac{1}{2\beta} |\Phi(T_1)|^2 \right)}{|\Phi(T_1)| \left(\beta - \frac{1}{2\beta} |\Phi(T_1)| \right)} \bar{\Delta},$$

such a condition is equivalent to $\gamma > \gamma_\beta$, taking into account the definition of Φ and (15). Once the invariance of \mathcal{C}_1^o for (6)–(7), with \tilde{Y}_1 is ensured, it is guaranteed that no further switches would occur due to $\mathcal{D}_1 \cap \mathcal{C}_1^o = \emptyset$; and it is concluded that the set $\mathcal{C}_1^o \times \{1\}$ is strongly invariant for \mathcal{H}_c . Moreover, from Theorem 3, we may conclude that, on this set, $\tilde{q}(t)$ converges to the origin in a finite time T_2 upper-bounded as $T_2(\tilde{q}(T_1)) \leq \|\tilde{X}(T_1)\| / (\gamma C_\beta - \bar{\Delta})$. Using (13) and (15), an upper-bound for the total settling-time can be computed as

$$\begin{aligned} T(\tilde{q}_0) &= T_1(\tilde{q}_0) + T_2(\tilde{q}(T_1)) \\ &\leq T_1(\tilde{q}_0) + \frac{\|\tilde{X}(T_1)\|}{\gamma C_\beta - \bar{\Delta}} = T_1(\tilde{q}_0) + \frac{|\Gamma(T_1)|^{\frac{1}{2}}}{\beta_\varepsilon (\gamma C_\beta - \bar{\Delta})} \\ &\leq T_1(\tilde{q}_0) + \frac{|\Gamma_0|^{\frac{1}{2}} + \frac{\bar{\Delta}}{2\beta_\varepsilon} T_1(\tilde{q}_0)}{\beta_\varepsilon (\gamma C_\beta - \bar{\Delta})} \\ &= m_c T_1(\tilde{q}_0) + \frac{|\Gamma_0|^{\frac{1}{2}}}{\beta_\varepsilon (\gamma C_\beta - \bar{\Delta})}. \end{aligned}$$

Then, using the previously obtained upper-bound for T_1 , we have that $T(\tilde{q}_0) \leq m_c \left(\frac{1}{\beta_\varepsilon} |\Gamma_0|^{\frac{1}{2}} - \|\tilde{X}_0\| \right) / (\bar{\gamma} - \bar{\Delta} / 2\beta_\varepsilon^2) + T_1(\tilde{q}_0)$. Formally, any $\eta_a(0, 0) \in \mathcal{C}_2^o \times \{2\}$, with the hybrid time domain $\text{dom } \eta_a = [0, T_1] \times \{0\} \cup [T_1, \infty) \times$

$\{1\}$, is given by

$$\begin{aligned} \eta_a(t, 0) &= \begin{bmatrix} \Gamma_0 + \int_0^t \tilde{Y}_2^\top(\tau) \mathbf{J} \tilde{X}(\tau) d\tau \\ \tilde{X}_0 + \int_0^t \tilde{Y}_2(\tau) d\tau \\ 2 \end{bmatrix}, \\ \eta_a(t, 1) &= \begin{bmatrix} \Gamma(T_1) + \int_{T_1}^t \tilde{Y}_1^\top(\tau) \mathbf{J} \tilde{X}(\tau) d\tau \\ \tilde{X}(T_1) + \int_{T_1}^t \tilde{Y}_1(\tau) d\tau \\ 1 \end{bmatrix}. \end{aligned}$$

From the previously stated arguments, we can conclude that $\eta_a(t, k) = [\mathbf{0}^\top, 1]^\top$, for all $t \geq T(\tilde{q}_0)$ and $k = 1$.

2) Assume initial conditions such that $\eta_b(0, 0) \in \mathcal{C}_1^o \times \{1\}$, *i.e.*, $\tilde{q}_0 \in \mathcal{C}_1^o = \mathcal{D}_\beta^o$, while $\sigma(0) = 1$, so at this instant \tilde{Y}_1 is acting on (6)–(7), and due to the previously arguments, the set $\mathcal{C}_1^o \times \{1\}$ is strongly invariant for \mathcal{H}_c . Also, from Theorem 3, on this set $\tilde{q}(t)$ converges to the origin in a finite time upper bounded as $T(\tilde{q}_0) \leq \|\tilde{X}_0\| / (\gamma C_\beta - \bar{\Delta})$. So, any solution $\eta_b(0, 0) \in \mathcal{C}_1^o \times \{1\}$, with the hybrid time domain $\text{dom } \eta_b = [0, \infty) \times \{0\}$, is given by

$$\eta_b(t, 0) = \begin{bmatrix} \Gamma_0 + \int_0^t \tilde{Y}_1^\top(\tau) \mathbf{J} \tilde{X}(\tau) d\tau \\ \tilde{X}_0 + \int_0^t \tilde{Y}_1(\tau) d\tau \\ 1 \end{bmatrix}.$$

Once again, we may conclude that $\eta_b(t, k) = [\mathbf{0}^\top, 1]^\top$, for all $t \geq T(\tilde{q}_0)$ and $k = 0$.

3) For $\eta_c(0, 0) = [\mathbf{0}^\top, 2]^\top \in \mathcal{C}_1 \times \mathcal{D}_2$, an instantaneous switch happens, and we have $\eta_c(0, 1) = [\mathbf{0}^\top, 1]^\top$. Thus, for $\eta_c(0, 0) = [\mathbf{0}^\top, 2]^\top$, with $\text{dom } \eta_c = \{0\} \times \{0\} \cup [0, \infty) \times \{1\}$, then $\eta_c(t, k) = [\mathbf{0}^\top, 1]^\top$ for $t \geq 0$ and $k = 1$.

4) Recalling that $\partial \mathcal{C}_j \setminus \{\mathbf{0}\} \subset \mathcal{C}_i^o \cap \mathcal{D}_j$. For any other initial condition *i.e.*, $\eta_d(0, 0) \in \mathcal{C}_i^o \cap \mathcal{D}_j \times \{j\}$ with $i \neq j$, an instantaneous switch occurs, bringing the solution to $\eta_d(0, 1) \in \mathcal{C}_i^o \times \{i\}$, which can be analyzed analogously as case 1) or 2), respectively, considering the discrete time parameter k increased by 1. And, it is possible to conclude that $\eta_d(t, k) = [\mathbf{0}^\top, 1]^\top$ for all $t \geq T(\tilde{q}_0)$ and $k \leq 2$.

Notice that the sets described in items 1) to 4) generate a partition of $\mathbb{R}^3 \times \{1, 2\}$. Thus, for any initial condition $\eta(0, 0) \in \mathbb{R}^3 \times \{1, 2\}$, there exists a hybrid arc $\eta(t, k)$, and a locally bounded positive $T_h(\tilde{q}_0) = T(\tilde{q}_0) + 2$ such that $\eta(t, k) = [\mathbf{0}^\top, 1]^\top$, for all $t + k \geq T_h$, and $(t, k) \in \text{dom } \eta$.

Due to the local boundedness of F and G , the hybrid arc can be extended continuously from any initial condition $\eta(0, 0) \in \mathcal{C} \setminus [\mathbf{0}^\top, 1]^\top$ locally. Additionally, jumps

³ For the detailed explanation on how this upper bound is obtained we refer the reader to [28].

are bounded as $|\sigma^+ - \sigma| \leq 2$. Combining this with the global finite-time attractivity of $[\mathbf{0}^\top, 1]^\top$ it is possible to conclude that the hybrid arc is globally bounded and no finite escape time exists, also η converges to $[\mathbf{0}^\top, 1]^\top$, in a continuous finite time $T^* \leq T(\tilde{q}_0)$, with $T(\tilde{q}_0) < \infty$. Furthermore, no Zeno behavior exists because it is proven that the discrete time is globally uniformly bounded. Specifically, $k \leq 2$ for any $\eta(0, 0) \in \mathbb{R}^3 \times \{1, 2\}$. Thus, $\text{dom } \eta$ is always bounded in k . Finally, the convexity of $F([\mathbf{0}^\top, 1]^\top)$ and the fact $[\mathbf{0}^\top, 1]^\top \in K[F]([\mathbf{0}^\top, 1]^\top)$, ensure that, in Filippov sense, the hybrid arc can be extended indefinitely and invariantly as $\eta(t, \cdot) = [\mathbf{0}^\top, 1]^\top$ for all $t > T^*$. Consequently, $\text{dom } \eta$ is unbounded in t , and the hybrid arc η is complete and either continuous ($k = 0$), or eventually continuous ($1 \leq k \leq 2$). This concludes the proof. ■

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