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# A finite-time observer for time-varying systems with applications to epidemiological compartment systems

Radosław Patelski, Rosane Ushirobira, Denis Efimov

**Abstract**—In this paper, we present the design of a finite-time observer for estimating the spread of an epidemic in a population. The dynamics of the epidemic is represented by a nonlinear time-varying compartment model, and the system is written in a polytopic form. By using the implicit Lyapunov function approach, conditions for the observer’s convergence are given in terms of linear matrix inequalities (LMI). We provide an analytical proof of robustness to a certain extent of structural uncertainty. In addition, we show the input-to-state stability (ISS) with respect to exogenous inputs and measurement noise. A numerical simulation confirms the validity of the proposed solution.

## I. INTRODUCTION

Studies on modeling and analysis of the spread of infectious diseases are crucial to ensure public health, especially in densely populated urban areas. In order to investigate the evolution of a disease, compartmental models were introduced in [1], the population is then represented by homogeneous compartments grouped by selected characteristics. The change in a compartment volume corresponds to the flow of individuals passing through different stages of infection or treatment. An overview of selected models focused on the representation of actively commuting populations is given in [2].

In epidemiological research, estimating the momentary state of the epidemic from available measurements arises as a significant issue. While the use of control theory methods to tackle the problem of epidemiological systems estimation is not widespread, in the literature certain solutions utilizing linear [3] and nonlinear [4] observers, as well as Kalman filtering [5] can be found. Due to the nonlinear character of the dynamics of population flows between different compartments, the design of state observers for these systems poses a nontrivial challenge, with solutions available for selected classes of systems only. Assuming the measurability of certain nonlinear terms in the model dynamics, standard observers for linear time-varying (LTV) systems [6] can also be efficiently used for state estimation of compartment models. Nevertheless, these solutions offer asymptotic convergence of estimation errors, which may not be acceptable for some systems. In particular, selected compartment models are characterized by vanishing observability in equilibrium configurations, leading to a decrease in the convergence rate, an increase in noise or disturbance sensitivity, and

an inability to correctly recover the information about the system’s state.

A finite-time observer is proposed in this paper to address these problems. The epidemiological system under consideration is first rewritten in an exact convex form [7], leading to a semi-linear model structure. To ensure a suitable form of the system dynamics, parts of the model are treated as external disturbances, which nonetheless do not deteriorate the convergence properties of the algorithm. Using the Implicit Lyapunov Function approach, the observer is then designed to have a kind of homogeneity of the estimation error dynamics and finite-time convergence of the estimates [8]. As the transformed system is described by matrices, which may not conform to any canonical form, the observer design is based on linear generalized homogeneity [9] to obtain non-asymptotic state estimates. In particular, our solution extends the results of [9] by applying the method to time-varying systems and explicitly establishing the qualitative estimate of the extent of rejectable disturbances.

The structure of the paper is as follows. Section II describes the epidemiological compartment model under consideration. In Section III, some key results on finite-time stability are recalled. Section IV presents the proposed observer design and stability analysis, while Section V contains the numerical verification of the method. Section VI concludes the paper.

The following notation is used in the paper. We denote  $\mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$  and  $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} : x \geq 0\}$  with  $\mathbb{R}$  being the field of real numbers. Without loss of generality,  $\|\cdot\|$  stands for the Euclidean norm of a vector in  $\mathbb{R}^n$  or the induced norm of a matrix in  $\mathbb{R}^{n \times m}$ . The identity matrix of dimension  $n \times n$  is denoted by  $\mathbf{I}_n$ . The notation  $\exp(\mathbf{A})$  stands for the matrix exponential for  $\mathbf{A} \in \mathbb{R}^{n \times n}$  or the ordinary exponential for scalar arguments. For  $\mathbf{A} \in \mathbb{R}^{n \times n}$ , we denote  $[\mathbf{A}] := \inf_{\mathbf{x} \in \mathbb{R}^n} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|}$ , while  $\lambda_{\max}(\mathbf{A})$ ,  $\lambda_{\min}(\mathbf{A})$  denote the maximum and minimum eigenvalues of a symmetric matrix, respectively. A continuous function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  belongs to class  $\mathcal{K}$  if it is strictly increasing and  $\alpha(0) = 0$ . For  $p, n \in \mathbb{N}$  with  $\mathbb{N}$  being the field of natural numbers and  $p \leq n$ , the set  $\{p, p+1, \dots, n\}$  is represented by  $\bar{p}, \bar{n}$ . The symbol  $(*)$  denotes transposition of the previous term in the formula, *i.e.*,  $\mathbf{A} + (*) = \mathbf{A} + \mathbf{A}^\top$ .

## II. EPIDEMIOLOGICAL COMPARTMENT SYSTEM

To motivate our studies, we consider a dynamic model of tuberculosis spread with active treatment procedures, as given in [2]. The considered model comprises four compartments, representing susceptible  $S$ , exposed  $E$ , infectious  $I$

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and treated  $T$  individuals. The transmission of the disease to susceptible and treated individuals is characterized by different rates  $\beta_1, \beta_2 \in \mathbb{R}_{\geq 0}$  respectively. Flow of individuals from the exposed to the infectious compartment happens at rate  $v \in \mathbb{R}_{\geq 0}$ . The treatment rates of exposed and infectious individuals are denoted by  $r_1, r_2 \in \mathbb{R}_{\geq 0}$ , and  $q \in \mathbb{R}_{\geq 0}$  is a fraction ratio of successful treatments. The death and birth rates are assumed to be equal and denoted by  $b$ , leading to a constant population size, see Fig. 1. Denoting

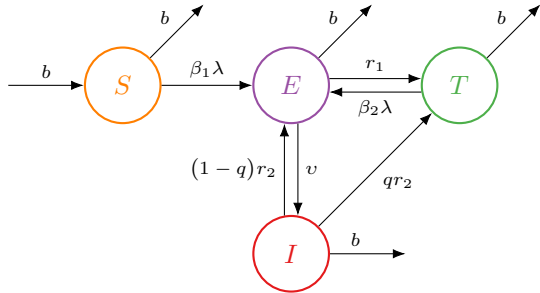


Fig. 1. Treatment model diagram

$\mathbf{x} = [E \ I \ S \ T]^\top \in \mathbb{R}^4$  and  $\lambda(\mathbf{x}) = \frac{I}{N}$ , where  $N = S + E + I + T$ , yields the dynamics

$$\dot{\mathbf{x}} = \mathbf{A}_0(\lambda(\mathbf{x}))\mathbf{x} \quad (1)$$

with

$$\mathbf{A}_0(\lambda) = \begin{bmatrix} -b - v - r_1 & (1-q)r_2 & \beta_1\lambda & \beta_2\lambda \\ v & -b - r_2 & 0 & 0 \\ b & b & -\beta_1\lambda & b \\ r_1 & qr_2 & 0 & -b - \beta_2\lambda \end{bmatrix}.$$

For brevity, we will further skip the argument  $\mathbf{x}$  in  $\lambda(\mathbf{x})$ . Henceforth, the compartments of infectious and susceptible individuals are considered to be measurable outputs of the system, and thus

$$\mathbf{y} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{C}\mathbf{x}. \quad (2)$$

We also assume that the total population is known and normalized, without loss of generality, so  $N = 1$  and  $\lambda \in [0, 1]$ . Consequentially, function  $\lambda$  is fully known and can be utilized in the observer synthesis.

We first decompose the system to treat all terms involving  $r_2$  and  $\beta_2$  as disturbances, which will not be directly utilized in the core observer design procedure. This decomposition will enable the fulfillment of the LMI conditions required for the proposed observer. In particular, due to the structure of the measurable output, the LMIs to be formulated are feasible if the nonmeasurable state variables have no time-varying coefficients in the system dynamics. While this property is not a necessary condition of LMIs feasibility, such decomposition greatly simplifies the observer design. Nonetheless, the robustness properties of the observer will ensure that the system remains finite-time stable in the presence of these disturbing dynamics. Denote  $\mathbf{A}(\lambda) = \mathbf{A}_0(\lambda)|_{r_2, \beta_2=0}$  and  $\mathbf{G}(\lambda) = \mathbf{A}_0(\lambda) - \mathbf{A}(\lambda)$ . The resultant system can now be expressed in an equivalent convex form, inspired by [7]:

$$\dot{\mathbf{x}} = \mathbf{A}_0(\lambda)\mathbf{x} = \sum_{i=1}^2 h_i(\lambda)\mathbf{A}_i\mathbf{x} + \mathbf{G}(\lambda)\mathbf{x}, \quad (3)$$

with  $h_1(\lambda) = \lambda, h_2(\lambda) = 1 - \lambda$ , and constant known matrices  $\mathbf{A}_1 = \mathbf{A}(1) = \mathbf{A}(\lambda)|_{\lambda=1}, \mathbf{A}_2 = \mathbf{A}(0) = \mathbf{A}(\lambda)|_{\lambda=0}$ .

### III. PRELIMINARIES

Some definitions and properties of homogeneous systems following [10, 11, 9, 12] can be found in the sequel.

*Definition 1:* The map  $\mathbf{d} : \mathbb{R} \rightarrow \mathbb{R}^n$  is called a *linear dilation* in  $\mathbb{R}^n$  if:

- 1)  $\mathbf{d}$  is a continuous map with  $\mathbf{d}(0) = \mathbf{I}_n$  and  $\mathbf{d}(s+t) = \mathbf{d}(s)\mathbf{d}(t) = \mathbf{d}(t)\mathbf{d}(s)$ , for all  $t, s \in \mathbb{R}$ ; ( $\mathbf{d}$  is a one-parameter subgroup);
- 2)  $\lim_{s \rightarrow -\infty} \|\mathbf{d}(s)\mathbf{x}\| = 0$  and  $\lim_{s \rightarrow +\infty} \|\mathbf{d}(s)\mathbf{x}\| = +\infty$  uniformly on the unit sphere  $S = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| = 1\}$ .

In particular, the dilation  $\mathbf{d}$  can be written as:

$$\mathbf{d}(s) = \exp(\mathbf{G}_d s) = \sum_{i=0}^{\infty} \frac{s^i \mathbf{G}_d^i}{i!} \quad (4)$$

where  $\mathbf{G}_d \in \mathbb{R}^{n \times n}$  is the constant *generator matrix*.

*Theorem 1:* If  $\mathbf{d}$  is a dilation in  $\mathbb{R}^n$ , then

- 1) the matrix generator  $\mathbf{G}_d$  is anti-Hurwitz<sup>1</sup> and there exists a matrix  $\mathbf{P} \in \mathbb{R}^{n \times n}$  such that

$$\mathbf{P}\mathbf{G}_d + \mathbf{G}_d^\top \mathbf{P} \succ 0, \mathbf{P} \succ 0; \quad (5)$$

- 2) the dilation  $\mathbf{d}$  satisfies

$$\begin{aligned} e^{\alpha s} &\leq \lfloor \mathbf{d}(s) \rfloor \leq \|\mathbf{d}(s)\| \leq e^{\beta s}, \quad s \leq 0 \\ e^{\beta s} &\leq \lfloor \mathbf{d}(s) \rfloor \leq \|\mathbf{d}(s)\| \leq e^{\alpha s}, \quad s > 0 \end{aligned} \quad (6)$$

where  $\alpha = \frac{1}{2} \lambda_{\max}(\mathbf{P}^{\frac{1}{2}} \mathbf{G}_d \mathbf{P}^{-\frac{1}{2}} + \mathbf{P}^{-\frac{1}{2}} \mathbf{G}_d^\top \mathbf{P}^{\frac{1}{2}})$  and  $\beta = \frac{1}{2} \lambda_{\min}(\mathbf{P}^{\frac{1}{2}} \mathbf{G}_d \mathbf{P}^{-\frac{1}{2}} + \mathbf{P}^{-\frac{1}{2}} \mathbf{G}_d^\top \mathbf{P}^{\frac{1}{2}})$ .

*Definition 2:* The scalar function  $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$  and the vector field  $\mathbf{f}_1 : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are  *$\mathbf{d}$ -homogeneous of degree  $v \in \mathbb{R}$*  if

$$\begin{aligned} f_0(\mathbf{d}(s)\mathbf{x}) &= e^{vs} f_0(\mathbf{x}), \\ \mathbf{f}_1(\mathbf{d}(s)\mathbf{x}) &= e^{vs} \mathbf{d}(s) \mathbf{f}_1(\mathbf{x}), \end{aligned} \quad (7)$$

for all  $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}, s \in \mathbb{R}$ .

*Definition 3:* The *canonical homogeneous norm*  $\|\cdot\|_d : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is defined as a  $\mathbf{d}$ -homogeneous function of degree 1 satisfying

$$\|\mathbf{d}(-\ln(\|\mathbf{x}\|_d))\mathbf{x}\| = 1 \quad (8)$$

for  $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$ , and  $\|0\|_d = 0$ .

*Lemma 1:* Let  $\mathbf{d}$  be a dilation with generator  $\mathbf{G}_d$ . Then, the vector field  $\mathbf{f}_0(\mathbf{x}) = \mathbf{A}\mathbf{x}$  with  $\mathbf{A} \in \mathbb{R}^{n \times n}$  is  $\mathbf{d}$ -homogeneous of degree  $v \in \mathbb{R}$  if and only if

$$\mathbf{A}\mathbf{G}_d - \mathbf{G}_d \mathbf{A} = v\mathbf{A}. \quad (9)$$

Recall now some properties of finite-time and input-to-state stable systems (see, for instance, [13]). In particular, consider a general nonlinear dynamic system in the form of

$$\dot{\mathbf{x}}(t) = \mathbf{f}_0(t, \mathbf{x}(t), \mathbf{r}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad (10)$$

<sup>1</sup>A matrix  $A$  is *anti-Hurwitz* if  $-A$  is a Hurwitz matrix.

where  $\mathbf{x}(t) \in \mathbb{R}^n$  is the state of the system,  $\mathbf{f}_0 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a continuous vector field that guarantees existence of solutions at least locally in time, and  $\mathbf{r}(t) \in \mathbb{R}^m$  is an essentially bounded disturbance input.

*Definition 4:* The system (10) is *finite-time input-to-state stable (FT-ISS)* if for any initial time  $t_0 \in \mathbb{R}_{\geq 0}$ , any initial state  $\mathbf{x}_0$ , any essentially bounded input  $\mathbf{r}$ , and for any  $t \geq t_0$ , the trajectory of the system exists and satisfies

$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(t_0)\|, t - t_0) + \gamma(\text{ess sup}_{s \in [t_0, t]} \|\mathbf{r}(s)\|),$$

where  $\gamma$  is a function of class  $\mathcal{K}$ , and  $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is such that  $\beta(\cdot, 0)$  is a function of class  $\mathcal{K}$  and for each fixed  $s$ , function  $\beta(s, t)$  is continuous, decreasing to zero and  $\beta(s, t) = 0$  for any  $t \geq T(s)$  with  $T \in \mathcal{K}$ .

*Theorem 2 (Implicit FT-ISS Lyapunov function):* The system (10) is globally FT-ISS if there exists a continuous function  $Q : \mathbb{R}_{> 0} \times \mathbb{R}^n \rightarrow \mathbb{R}$  that satisfies the following conditions:

- (C1)  $Q$  is continuously differentiable in  $\mathbb{R}_{> 0} \times \mathbb{R}^n \setminus \{0\}$ ;
- (C2) for any  $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$  there exists  $V \in \mathbb{R}_{> 0}$  such that  $Q(V, \mathbf{x}) = 0$ ;
- (C3) let  $\Omega = \{(V, \mathbf{x}) \in \mathbb{R}_{> 0} \times \mathbb{R}^n : Q(V, \mathbf{x}) = 0\}$  then

$$\lim_{\substack{\mathbf{x} \rightarrow 0 \\ (V, \mathbf{x}) \in \Omega}} V = 0^+, \quad \lim_{\substack{V \rightarrow 0^+ \\ (V, \mathbf{x}) \in \Omega}} \|\mathbf{x}\| = 0, \quad \lim_{\substack{\|\mathbf{x}\| \rightarrow 0 \\ (V, \mathbf{x}) \in \Omega}} V = \infty;$$

- (C4)  $\frac{\partial Q(V, \mathbf{x})}{\partial V} < 0$  holds for all  $V \in \mathbb{R}_{> 0}$  and  $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$ ;
- (C5) there exist constant  $b \in \mathbb{R}_{> 0}$  and  $\mu \in (0, 1]$  such that

$$\sup_{t \in \mathbb{R}_{\geq 0}} \frac{\partial Q(V, \mathbf{x})}{\partial \mathbf{x}} \mathbf{f}_0(t, \mathbf{x}, \mathbf{r}) \leq bV^{1-\mu} \frac{\partial Q(V, \mathbf{x})}{\partial V}$$

for all  $(V, \mathbf{x}) \in \Omega$  and all  $\mathbf{x}$  such that  $\|\mathbf{x}\| \geq \chi(\|\mathbf{r}\|)$  with  $\chi$  being  $\mathcal{K}$  function.

The settling time estimate is given by  $T(\mathbf{x}_0) \leq \frac{V_0^\mu}{b\mu}$  where  $V_0 \in \mathbb{R}_{> 0}$  is such that  $Q(V_0, \mathbf{x}_0) = 0$ .

*Theorem 3 (Explicit FT-ISS Lyapunov function):* The system (10) is FT-ISS if there exists positive definite, radially unbounded function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  such that

$$\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}_0(t, \mathbf{x}, \mathbf{r}) \leq -b_1 V^{1-\mu_1}(\mathbf{x})$$

for all  $\mathbf{x}$  such that  $\|\mathbf{x}\| \geq \chi(\|\mathbf{r}\|)$  with  $\chi \in \mathcal{K}$ ,  $b_1 \in \mathbb{R}_{> 0}$ , and  $\mu_1 \in (0, 1]$ .

#### IV. OBSERVER DESIGN

To facilitate the finite-time observer design, recall the system in the convex form representation (3) under unknown disturbances  $\mathbf{r} \in \mathbb{R}^n$  and measurement noise  $\mathbf{v} \in \mathbb{R}^n$ :

$$\dot{\mathbf{x}} = \sum_{i=1}^m h_i(\lambda) \mathbf{A}_i \mathbf{x} + \mathbf{G}(\lambda) \mathbf{x} + \mathbf{r}, \quad \mathbf{y} = \mathbf{C}(\mathbf{x} + \mathbf{v}), \quad (11)$$

where  $m \in \mathbb{R}_{> 0}$  is a number of terms obtained through transformation of the nominal system into a convex form, and

$\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{y} \in \mathbb{R}^z$ . The observer is proposed in the following form:<sup>2</sup>

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \sum_{i=1}^m h_i (\mathbf{A}_i \hat{\mathbf{x}} - \mathbf{L}_i (\mathbf{y} - \mathbf{C} \hat{\mathbf{x}}) - \mathbf{g}_i (\mathbf{y} - \mathbf{C} \hat{\mathbf{x}})) + \mathbf{G} \hat{\mathbf{x}}, \\ \text{with } \mathbf{g}_i(\mathbf{s}) &= \left\| \tilde{\mathbf{P}} \mathbf{s} \right\|^{\frac{v-\varepsilon}{\varepsilon}} \mathbf{d} \left( \ln \left( \left\| \tilde{\mathbf{P}} \mathbf{s} \right\|^{\frac{1}{\varepsilon}} \right) \right) \mathbf{M}_i \mathbf{s}, \end{aligned} \quad (12)$$

where  $\mathbf{L}_i, \mathbf{M}_i \in \mathbb{R}^{n \times z}$ , for  $i \in \overline{1, m}$ , are the observer gains,  $v \in [-1, 0)$  corresponds to the desired degree of homogeneity of the resultant closed loop error system, while  $\varepsilon \in \mathbb{R}_{> 0}$  and  $\tilde{\mathbf{P}} \in \mathbb{R}^{z \times z}$  are the design parameters to be specified. The selection of  $\mathbf{L}_i, \mathbf{M}_i$  is derived in the theorem given below. Denote now  $\tilde{\mathbf{x}} = \mathbf{x} - \hat{\mathbf{x}}$  with the dynamics expressed by

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \sum_{i=1}^m h_i ((\mathbf{A}_i + \mathbf{L}_i \mathbf{C}) \tilde{\mathbf{x}} + \mathbf{g}_i (\mathbf{C} \tilde{\mathbf{x}} + \mathbf{C} \mathbf{v})) \\ &+ \sum_{i=1}^m h_i \mathbf{L}_i \mathbf{C} \mathbf{v} + \mathbf{G} \tilde{\mathbf{x}} + \mathbf{r} = \mathbf{f}(t, \tilde{\mathbf{x}}, \mathbf{v}, \tilde{\mathbf{r}}), \end{aligned} \quad (13)$$

with  $\tilde{\mathbf{r}} := \sum_{i=1}^m h_i \mathbf{L}_i \mathbf{C} \mathbf{v} + \mathbf{G} \tilde{\mathbf{x}} + \mathbf{r}$ .

*Theorem 4:* Assume  $\mathbf{v} \equiv 0$ . For  $\mathbf{H} \in \mathbb{R}^{n \times n}$ ,  $\mathbf{X} \in \mathbb{R}^{z \times z}$  with  $\mathbf{X} \succ 0$ ,  $v \in [-1, 0)$ , and  $\varepsilon \in \mathbb{R}_{\geq 0}$ , consider  $\mathbf{G}_d = v\mathbf{H} + \varepsilon \mathbf{I}_n$  and  $\tilde{\mathbf{P}} = \mathbf{X}^{\frac{1}{2}}$ . Assume all of the following are satisfied:

- 1) for some  $\mathbf{Y}_i \in \mathbb{R}^{n \times z}$  ( $i \in \overline{1, m}$ ),

$$\mathbf{A}_i \mathbf{H} - \mathbf{H} \mathbf{A}_i - \mathbf{A}_i - \mathbf{Y}_i \mathbf{C} = 0, \quad \forall i \in \overline{1, m}, \quad (14)$$

$$\mathbf{C} \mathbf{H} = 0, \quad (15)$$

$$\mathbf{H} + \mathbf{H}^\top + 2\mathbf{I}_n \succ 0; \quad (16)$$

- 2) for some  $\mathbf{P} \in \mathbb{R}^{n \times n}$  with  $\mathbf{P} \succ 0$ ,  $\xi_i, \theta_i, \tau_i \in \mathbb{R}_{\geq 0}$  with  $\xi_i > \tau_i$ ,  $\mathbf{Z}_i \in \mathbb{R}^{n \times n}$  with  $\mathbf{Z}_i \succ 0$ ,  $\mathbf{Y}_i \in \mathbb{R}^{n \times z}$ ,  $\mathbf{L}_i = (\mathbf{H} + \mathbf{I}_n)^{-1} \mathbf{Y}_i$ ,  $\mathbf{M}_i = \mathbf{P}^{-1} \mathbf{Y}_i$  ( $i \in \overline{1, m}$ ),

$$\begin{bmatrix} \mathbf{P}(\mathbf{A}_i + \mathbf{L}_i \mathbf{C} + \mathbf{M}_i \mathbf{C}) + (*) + \xi_i \mathbf{P} & \mathbf{P} \\ \mathbf{P} & -\mathbf{Z}_i \end{bmatrix} \preceq 0, \quad (17)$$

$$\begin{bmatrix} \tau_i \mathbf{X} & \mathbf{Y}_i^\top \\ \mathbf{Y}_i & \theta_i \mathbf{P} \end{bmatrix} \succeq 0, \quad \forall i \in \overline{1, m}, \quad (18)$$

$$\mathbf{P} - \mathbf{C}^\top \mathbf{X} \mathbf{C} \succeq 0, \quad (19)$$

$$v\mathbf{P} \mathbf{H} + v\mathbf{H}^\top \mathbf{P} + 2\varepsilon \mathbf{P} \succ 0 \quad (20)$$

- 3) for all  $\lambda \in (0, 1]$ , set  $\Xi(\lambda) = \lambda^{v-\varepsilon} \mathbf{d}(\ln(\lambda)) - \mathbf{I}_n$ ,

$$\lambda^{2\varepsilon} \Xi^\top(\lambda) \mathbf{Z}_i \Xi(\lambda) - \frac{1}{\theta_i} \mathbf{P} \preceq 0, \quad \forall i \in \overline{1, m}. \quad (21)$$

Then the system (13) is globally FT-ISS with respect to the input  $\tilde{\mathbf{r}}$ . The asymptotic gain estimate is so that

$$\|\tilde{\mathbf{r}}\|^2 \leq \delta \min \left\{ (\lambda_{\min}(\mathbf{P}) \|\tilde{\mathbf{x}}\|^2)^{\frac{v+\alpha}{\beta}}, (\lambda_{\min}(\mathbf{P}) \|\tilde{\mathbf{x}}\|^2)^{\frac{v+\beta}{\alpha}} \right\} \quad (22)$$

for some  $\delta \in \mathbb{R}_{\geq 0}$  satisfying

$$\mathbf{I}_n - \delta^{-1} \mathbf{P} \succeq 0, \quad (23)$$

<sup>2</sup>We omit  $\lambda$  in the arguments to simplify the notation.

$2\delta < \sup_{i \in \overline{1, m}} \{\xi_i - \tau_i\}$ , and  $v + \beta \geq 0$ , with  $\alpha, \beta$  as in Theorem 1.

*Proof:* Consider the implicit Lyapunov function candidate given by

$$Q(V, \tilde{\mathbf{x}}) = \tilde{\mathbf{x}}^\top \mathbf{d}^\top (-\ln(V)) \mathbf{P} \mathbf{d} (-\ln(V)) \tilde{\mathbf{x}} - 1. \quad (24)$$

To prove the theorem, (24) should satisfy the conditions of Theorem 2. The function  $Q$  is continuously differentiable and clearly satisfies condition (C1). Due to properties of homogeneous norm (8), condition (C2) is satisfied. Condition (C3) holds also by design. Consider now

$$\partial_V Q = -V^{-1} \tilde{\mathbf{x}}^\top \mathbf{d}^\top (-\ln(V)) (v \mathbf{P} \mathbf{H} + v \mathbf{H}^\top \mathbf{P} + 2\varepsilon \mathbf{P}) \times \mathbf{d} (-\ln(V)) \tilde{\mathbf{x}}. \quad (25)$$

Using (20), we can conclude that condition (C4) also holds. Finally, recall that  $\mathbf{v} \equiv 0$  and consider

$$\begin{aligned} \partial_{\tilde{\mathbf{x}}} Q \dot{\tilde{\mathbf{x}}} &= 2\tilde{\mathbf{x}}^\top \mathbf{d}^\top (-\ln(V)) \mathbf{P} \sum_{i=1}^m h_i(\mathbf{d}(-\ln(V)) (\mathbf{A}_i + \mathbf{L}_i \mathbf{C})) \tilde{\mathbf{x}} \\ &\quad + \mathbf{d}(-\ln(V)) \mathbf{g}_i(\mathbf{C}\tilde{\mathbf{x}}) + \epsilon, \end{aligned} \quad (26)$$

where  $\epsilon = 2\tilde{\mathbf{x}}^\top \mathbf{d}^\top (-\ln(V)) \mathbf{P} \mathbf{d} (-\ln(V)) \tilde{\mathbf{r}}$  is a non-zero disturbance term. By substituting  $\mathbf{G}_d = v \mathbf{H} + \varepsilon \mathbf{I}_n$  into (15) and recalling (4), one can show that  $\mathbf{C} \exp(-\varepsilon s \mathbf{I}_n) = \mathbf{C} \mathbf{d}(-s)$ . Denoting  $\lambda = V^{-1} \|\tilde{\mathbf{P}} \mathbf{C} \tilde{\mathbf{x}}\|_{\frac{1}{\varepsilon}}$  yields

$$\begin{aligned} \partial_{\tilde{\mathbf{x}}} Q \dot{\tilde{\mathbf{x}}} &= 2\tilde{\mathbf{x}}^\top \mathbf{d}^\top (-\ln(V)) \mathbf{P} \sum_{i=1}^m h_i(\mathbf{d}(-\ln(V)) (\mathbf{A}_i + \mathbf{L}_i \mathbf{C})) \\ &\quad + \lambda^{v-\varepsilon} \mathbf{d}(\ln(\lambda)) V^v \mathbf{M}_i \mathbf{C} \mathbf{d}(-\ln(V)) \tilde{\mathbf{x}} + \epsilon. \end{aligned} \quad (27)$$

By substituting  $\mathbf{L}_i$  into (14), adding and subtracting the term  $\frac{\varepsilon}{v} (\mathbf{A}_i + \mathbf{L}_i \mathbf{C})$ , and recalling the definition of  $\mathbf{G}_d$ , one shows that the vector fields defined by  $\mathbf{A}_i + \mathbf{L}_i \mathbf{C}$  satisfies the conditions of Lemma 1 and thus (9) holds. Consequently

$$\begin{aligned} \partial_{\tilde{\mathbf{x}}} Q \dot{\tilde{\mathbf{x}}} &= 2V^v \sum_{i=1}^m h_i(\mathbf{z}_0^\top \mathbf{P} (\mathbf{A}_i + \mathbf{L}_i \mathbf{C} + \mathbf{M}_i \mathbf{C}) \mathbf{z}_0 \\ &\quad + \mathbf{z}_0^\top \mathbf{P} \mathbf{z}_i) + \epsilon, \end{aligned} \quad (28)$$

where  $\mathbf{z}_i = \Xi(\lambda) \mathbf{M}_i \mathbf{C} \mathbf{d}(-\ln(V)) \tilde{\mathbf{x}}$ , for  $i \in \overline{1, m}$ , and  $\mathbf{z}_0 = \mathbf{d}(-\ln(V)) \tilde{\mathbf{x}}$ . Due to (17), it follows that

$$\partial_{\tilde{\mathbf{x}}} Q \dot{\tilde{\mathbf{x}}} \leq V^v \sum_{i=1}^m h_i(-\mathbf{z}_0^\top \xi_i \mathbf{P} \mathbf{z}_0 + \mathbf{z}_i^\top \mathbf{Z}_i \mathbf{z}_i) + \|\epsilon\|.$$

Notice now that  $\mathbf{z}_0^\top \mathbf{P} \mathbf{z}_0 = Q + 1$  and consider only  $(V, \tilde{\mathbf{x}}) \in \Omega$ , which implies  $Q = 0$ , and thus

$$\partial_{\tilde{\mathbf{x}}} Q \dot{\tilde{\mathbf{x}}} \leq V^v \sum_{i=1}^m h_i(-\xi_i + \mathbf{z}_i^\top \mathbf{Z}_i \mathbf{z}_i) + \|\epsilon\|.$$

Substitute back  $\mathbf{z}_i$ , then due to (21) and the properties implied by (15) it holds that

$$\begin{aligned} \partial_{\tilde{\mathbf{x}}} Q \dot{\tilde{\mathbf{x}}} &\leq V^v \sum_{i=1}^m h_i(\lambda^{-2\varepsilon} V^{-2\varepsilon} \theta_i^{-1} \tilde{\mathbf{x}}^\top \mathbf{C}^\top \mathbf{M}_i^\top \mathbf{P} \mathbf{M}_i \mathbf{C} \tilde{\mathbf{x}} \\ &\quad - \xi_i) + \|\epsilon\|. \end{aligned}$$

By the definitions of  $\mathbf{M}_i, \tilde{\mathbf{P}}$ , and following (18), it yields

$$\partial_{\tilde{\mathbf{x}}} Q \dot{\tilde{\mathbf{x}}} \leq V^v \sum_{i=1}^m h_i(\lambda^{-2\varepsilon} V^{-2\varepsilon} \tau_i \tilde{\mathbf{x}}^\top \mathbf{C}^\top \tilde{\mathbf{P}} \mathbf{P} \mathbf{C} \tilde{\mathbf{x}} - \xi_i) + \|\epsilon\|.$$

One can now consider the term involving  $\|\tilde{\mathbf{P}} \mathbf{C} \tilde{\mathbf{x}}\|^2$ . By recalling Definition 1, one may use (15) and the definition of  $\tilde{\mathbf{P}}$  to show that due to (19), for all  $(V, \tilde{\mathbf{x}}) \in \Omega$ , it holds that  $\|\tilde{\mathbf{P}} \mathbf{C} \tilde{\mathbf{x}}\|^2 \leq V^{2\varepsilon}$  and thus  $\lambda \leq 1$ . Hence

$$\partial_{\tilde{\mathbf{x}}} Q \dot{\tilde{\mathbf{x}}} \leq V^v \sum_{i=1}^m h_i(-\xi_i + \tau_i) + \|\epsilon\|. \quad (29)$$

By Theorem 1, we have  $2\alpha \mathbf{P} - \mathbf{P} \mathbf{G}_d - \mathbf{G}_d^\top \mathbf{P} \succeq 0$ , and from (25), it can be shown that for  $(V, \tilde{\mathbf{x}}) \in \Omega$ , it holds that  $\partial_V Q \geq -2\alpha V^{-1}$ . Thus, it follows that

$$\begin{aligned} \partial_{\tilde{\mathbf{x}}} Q \dot{\tilde{\mathbf{x}}} &\leq V^{v+1} \sum_{i=1}^m h_i(-\xi_i + \tau_i) \frac{\partial_V Q}{2\alpha} + \|\epsilon\| \\ &\leq V^{v+1} \sup_{i \in \overline{1, m}} \{-\xi_i + \tau_i\} \frac{\partial_V Q}{2\alpha} + \|\epsilon\|, \end{aligned} \quad (30)$$

which satisfies condition (C5) of Theorem 2 if  $\epsilon \equiv 0$  due to condition 2).

To consider a more general case of non-zero disturbance, assume that  $\tilde{\mathbf{r}}$  satisfies

$$\begin{aligned} \tilde{\mathbf{r}}^\top \mathbf{d}^\top (-\ln(V)) \mathbf{d} (-\ln(V)) \tilde{\mathbf{r}} &\leq \\ &V^{2v} \delta \tilde{\mathbf{x}}^\top \mathbf{d}^\top (-\ln(V)) \mathbf{P} \mathbf{d} (-\ln(V)) \tilde{\mathbf{x}} \end{aligned} \quad (31)$$

with  $\delta$  satisfying (23). From (29) it thus follows that

$$\partial_{\tilde{\mathbf{x}}} Q \dot{\tilde{\mathbf{x}}} \leq V^v \sum_{i=1}^m h_i(-\xi_i + \tau_i) + 2V^v \delta$$

and, following the analysis of (30), one can show that

$$\partial_{\tilde{\mathbf{x}}} Q \dot{\tilde{\mathbf{x}}} \leq V^{v+1} \sup_{i \in \overline{1, m}} \{-\xi_i + \tau_i + 2\delta\} \frac{\partial_V Q}{2\alpha}$$

and the system is finite-time stable if  $2\delta < \sup_{i \in \overline{1, m}} \{\xi_i - \tau_i\}$ . To finalize the proof, we must show that (31) is satisfied if (22) holds. Note, that for  $(V, \tilde{\mathbf{x}}) \in \Omega$ , (22) holds if

$$\|\mathbf{d}(-\ln(V))\|^2 \|\tilde{\mathbf{r}}\|^2 \leq V^{2v} \delta. \quad (32)$$

Recalling (6) consider first case when  $V > 1$ . Thus,  $\|\mathbf{d}(-\ln(V))\|^2 \|\tilde{\mathbf{r}}\|^2 \leq \|\tilde{\mathbf{r}}\|^2 e^{-2\beta \ln V}$ . Taking now  $\|\tilde{\mathbf{r}}\|^2 \leq \delta (\lambda_{\min}(\mathbf{P}) \|\tilde{\mathbf{x}}\|^2)^{(v+\beta)/\alpha}$ , condition (32) is equivalent to

$$\left( \frac{\lambda_{\min}(\mathbf{P}) \|\tilde{\mathbf{x}}\|^2}{V^{2\alpha}} \right)^{\frac{v+\beta}{\alpha}} \leq 1.$$

From (24) and (6), it follows also that  $V^{-2\alpha} \lambda_{\min}(\mathbf{P}) \|\tilde{\mathbf{x}}\|^2 \leq \lambda_{\min}(\mathbf{P}) [\mathbf{d}(-\ln(V))]^2 \|\tilde{\mathbf{x}}\|^2 \leq Q + 1$  and thus for  $(V, \tilde{\mathbf{x}}) \in \Omega$ , (32) is satisfied if  $v + \beta \geq 0$ . By similar analysis, one can show that for  $V \leq 1$ , (31) holds if  $\|\tilde{\mathbf{r}}\|^2 \leq \delta (\lambda_{\min}(\mathbf{P}) \|\tilde{\mathbf{x}}\|^2)^{\frac{v+\alpha}{\beta}}$  and  $v + \alpha \geq 0$ . As  $\alpha \geq \beta$ , an explicit condition on only smaller of these is necessary, then the presented conditions holds if (22) is satisfied. ■

*Remark 1:* To verify the conditions of Theorem 4, it is necessary to solve the nonlinear inequality (21). In [11], an equivalent condition is given in the form of linear matrix inequality enabling more accessible validation.

*Theorem 5:* If the conditions of Theorem 4 are satisfied, then system (13) is FT-ISS with respect to  $\mathbf{v}$ .

*Proof:* If  $\mathbf{v}, \tilde{\mathbf{r}} \equiv 0$ , then by Theorem 4, (2) implicitly defines a continuously differentiable (out of the origin), positive definite and radially unbounded Lyapunov function  $V$  such that

$$\frac{\partial V(\tilde{\mathbf{x}})}{\partial \tilde{\mathbf{x}}} \mathbf{f}(t, \tilde{\mathbf{x}}, 0, 0) < -bV^{1+v}(\tilde{\mathbf{x}}) \quad (33)$$

for  $\tilde{\mathbf{x}} \in \mathbb{R} \setminus \{0\}$ . By (24) and comparing the resultant Lyapunov function on trajectories of  $\tilde{\mathbf{x}}$  and  $\mathbf{d}(s)\tilde{\mathbf{x}}$  it can be shown, that  $V(\mathbf{d}(s)\tilde{\mathbf{x}}) = e^s V(\tilde{\mathbf{x}})$ . Denoting  $\mathbf{q} \in S$  such that  $\tilde{\mathbf{x}} = \mathbf{d}(\ln \|\tilde{\mathbf{x}}\|_{\mathbf{d}}) \mathbf{q}$  and recalling positive-definiteness and homogeneity of the Lyapunov function, it follows also that  $V(\tilde{\mathbf{x}}) = \|\tilde{\mathbf{x}}\|_{\mathbf{d}} V(\mathbf{q})$ . Thus there exists  $a_0 \in \mathbb{R}_{>0}$  such that  $V(\tilde{\mathbf{x}}) \geq a_0 \|\tilde{\mathbf{x}}\|_{\mathbf{d}}$ . From (33) it directly follows then that  $\partial_{\tilde{\mathbf{x}}} V(\tilde{\mathbf{x}}) \mathbf{f}(t, \tilde{\mathbf{x}}, 0, 0) \leq -ba_0^{1+v} \|\tilde{\mathbf{x}}\|_{\mathbf{d}}^{1+v}$ . Finally, it can be shown that dynamics (13) satisfy

$$\mathbf{f}(t, \mathbf{d}(s)\tilde{\mathbf{x}}, \mathbf{d}(s)\mathbf{v}, \tilde{\mathbf{d}}(s)\tilde{\mathbf{r}}) = e^{vs} \mathbf{d}(s) \mathbf{f}(t, \tilde{\mathbf{x}}, \mathbf{v}, \tilde{\mathbf{r}}) \quad (34)$$

with  $\tilde{\mathbf{d}}(s) = e^{vs} \mathbf{d}(s)$ . One can thus recall  $\tilde{\mathbf{x}} = \mathbf{d}(\ln \|\tilde{\mathbf{x}}\|_{\mathbf{d}}) \mathbf{q}$  to show that  $\partial_{\tilde{\mathbf{x}}} V(\tilde{\mathbf{x}}) \mathbf{f}(t, \tilde{\mathbf{x}}, 0, 0) = \|\tilde{\mathbf{x}}\|_{\mathbf{d}}^{v+1} \partial_{\mathbf{q}} V(\mathbf{q}) \mathbf{f}(t, \mathbf{q}, 0, 0)$ . It follows, that there exists  $c_0 \in \mathbb{R}_{>0}$  such that  $\|\partial_{\mathbf{q}} V(\mathbf{q})\| \leq c_0$ . Consider now the derivative of  $V(\tilde{\mathbf{x}})$  with nonzero  $\mathbf{v}, \tilde{\mathbf{r}}$ , expressed by

$$\dot{V}(\tilde{\mathbf{x}}) = \partial_{\tilde{\mathbf{x}}} V(\tilde{\mathbf{x}}) \mathbf{f}(t, \tilde{\mathbf{x}}, \mathbf{v}, \tilde{\mathbf{r}}).$$

Due to (34), it follows that

$$\begin{aligned} \dot{V}(\tilde{\mathbf{x}}) &= \partial_{\tilde{\mathbf{x}}} V(\tilde{\mathbf{x}}) \mathbf{f}(t, \tilde{\mathbf{x}}, 0, 0) - \|\tilde{\mathbf{x}}\|_{\mathbf{d}}^{v+1} \partial_{\mathbf{q}} V(\mathbf{q}) (\mathbf{f}(t, \mathbf{q}, 0, 0) \\ &\quad - \mathbf{f}(t, \mathbf{q}, \mathbf{d}(-\ln(\|\tilde{\mathbf{x}}\|_{\mathbf{d}})) \mathbf{v}, \tilde{\mathbf{d}}(-\ln(\|\tilde{\mathbf{x}}\|_{\mathbf{d}})) \tilde{\mathbf{r}})). \end{aligned}$$

Since  $\mathbf{f}$  is continuous, then there exist  $v_0, r_0 \in \mathbb{R}_{\geq 0}$  that for any  $s \in \mathbb{R}$  if  $\|\tilde{\mathbf{d}}(s)\tilde{\mathbf{r}}\| \leq r_0$  and  $\|\mathbf{d}(s)\mathbf{v}\| \leq v_0$  then  $\|\mathbf{f}(t, \mathbf{q}, 0, 0) - \mathbf{f}(t, \mathbf{q}, \mathbf{d}(s)\mathbf{v}, \tilde{\mathbf{d}}(s)\tilde{\mathbf{r}})\| \leq f_0$  with  $f_0 \in \mathbb{R}_{\geq 0}$  being any constant. Substituting the norm with a homogeneous ones, utilizing their homogeneity properties and setting  $f_0 = \frac{ba_0^{1+v}}{2c_0}$ ,  $s = -\ln(\|\tilde{\mathbf{x}}\|_{\mathbf{d}})$ , yields that if  $\|\tilde{\mathbf{x}}\|_{\mathbf{d}} \geq \max\{v_0^{-1} \|\mathbf{v}\|_{\mathbf{d}}, r_0^{-1} \|\tilde{\mathbf{r}}\|_{\tilde{\mathbf{d}}}\}$  then

$$\dot{V}(\tilde{\mathbf{x}}) \leq -\frac{1}{2} ba_0^{1+v} \|\tilde{\mathbf{x}}\|_{\mathbf{d}}^{1+v} \leq -\frac{1}{2} bV^{1+v}(\tilde{\mathbf{x}}),$$

what satisfies the conditions of Theorem 3.  $\blacksquare$

## V. NUMERICAL VALIDATION

The validation of the proposed method is carried out in simulation. Namely, we consider system (3), with constant parameters chosen as  $\beta_1 = 0.7$ ,  $\beta_2 = 0.4$ ,  $v = 0.5$ ,  $r_1 = \frac{1}{3}\beta_1$ ,  $r_2 = \frac{1}{2}\beta_2$ ,  $q = 0.1$ ,  $b = 0.2$ . The initial state of the system is taken as  $\mathbf{x}_0 = [0.3 \ 0.15 \ 0.3 \ 0.25]^\top$  resulting in the normalized population of  $N = 1$ . The observer is tuned with  $\varepsilon = 7.677$  and

$$\tilde{\mathbf{P}} = \begin{bmatrix} 2.915 & -0.186 \\ -0.186 & 2.956 \end{bmatrix}, \mathbf{L}_1 = \begin{bmatrix} 0 & -1.742 \\ 0 & 1.133 \\ 0.2 & 0 \\ -0.2 & 0.609 \end{bmatrix},$$

$$\mathbf{L}_2 = \begin{bmatrix} -0.7 & -1.742 \\ 0 & 1.133 \\ 0.9 & 0 \\ -0.2 & 0.609 \end{bmatrix}, \mathbf{M}_{1,2} = \begin{bmatrix} -0.056 & 0.530 \\ -0.001 & -1.718 \\ -1.717 & -0.002 \\ -5.409 & 0.969 \end{bmatrix},$$

that enables conditions of Theorem 4 to be satisfied. The initial state estimates are selected as  $\hat{\mathbf{x}}_0 = [0 \ 0 \ 0 \ 0]^\top$ . Fig. 2 shows the simulation results for different values of the parameter  $v$ . Specifically, the evolution of the state variables of the original system is given and compared with the estimates produced by the observer. Absolute values of the estimation errors are also presented in the logarithmic scale. The plots also show the extent of disturbance acting upon the system (in solid lines) and the feasible disturbance bounds (in dashed lines) according to (22).

Nonasymptotic convergence of the estimation errors is visible on the plots in logarithmic scale, with a settling time of about 7 days for the observer tuned with  $v = -1$ . The considered system approaches equilibrium after

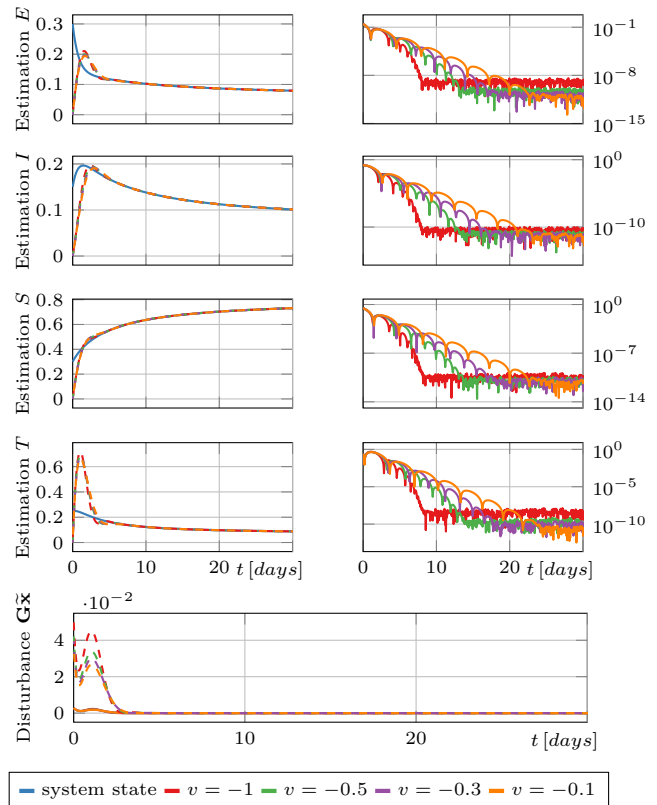


Fig. 2. Simulation results of the proposed observer

approximately 50 days. Thus, employing our solution enables efficient estimation of the disease spread long before the epidemic ceases to evolve. While the convergence rate in the initial period is not significantly affected by the selection of  $v$  parameter, it can be changed by choosing tuning parameters leading to maximization of  $\xi_i - \tau_i$  terms. Notably, the finite-time convergence is achieved despite the presence of the disturbance term  $\mathbf{G}\tilde{\mathbf{x}}$  which is not directly addressed by the synthesis rules of Theorem 4, which highlights the robustness of the proposed design.

The simulations are repeated in the noisy environment to verify the conclusions of Theorem 5 empirically. Fig. 3 presents the results of the numerical trials with additional measurement noise in the form of uniformly distributed values in the range of  $[-0.05, 0.05]$  (i.e., an uncertainty of 10% of the total population). The results of the trial confirm the ISS property of the observer, and a proper estimation of the states of the system is exhibited despite the presence of non-vanishing measurement noises.

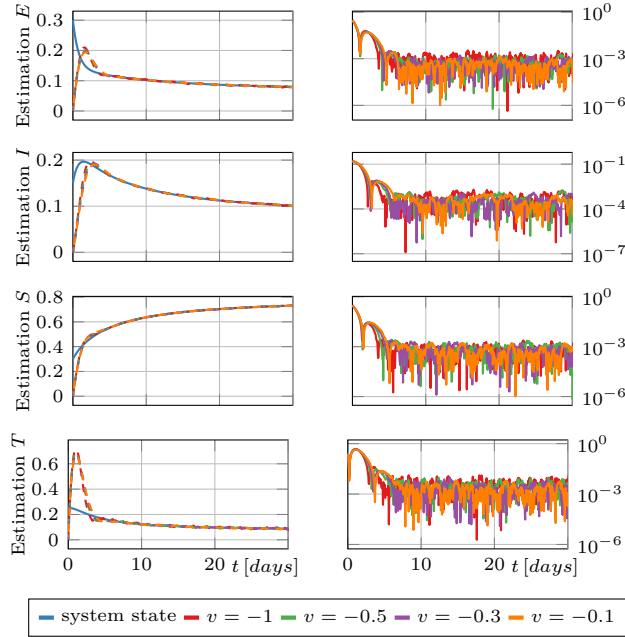


Fig. 3. Simulation results with noised measurement

## VI. CONCLUSION

In the paper a finite-time observer for a class of time-varying systems with known and bounded nonlinearities was proposed. We used a convex representation of the nominal system to synthesize the observer and formulate conditions for its convergence in the form of linear matrix inequalities. It was shown analytically that, under proper tuning, the observer ensures the convergence of the estimation errors in finite time and exhibits robustness to structural uncertainties and measurement noises. Additionally, we discussed the applicability of the observer to certain classes of epidemiological systems and validated through numerical trials. While the provided theorems establish the method of selection of feasible set of parameters of the observer, its fine tuning to achieve desired performance goal is not fully intuitive and constitutes a topic of future research.

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