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On delay robustness of ILF-based hyperexponential control*

K. Zimenko¹, D. Efimov², A. Polyakov² and A. Kremlev¹

Abstract—The stability of hyperexponential control based on the implicit Lyapunov function method under input delay is analyzed. Using the theory of homogeneous systems, a nonlinear feedback is proposed that ensures global asymptotic stability with respect to a compact set containing the origin for any bounded input delay. The tuning of control parameters and estimation of the convergence set in the presence of input delay are based on solving linear matrix inequalities. For this control a parameter adjustment method is proposed to ensure stability with respect to an arbitrarily small set containing the origin. In the absence of input delay, the proposed feedback guarantees hyperexponential stability at the origin.

I. INTRODUCTION

Input delays are unavoidable in many practical control applications involving remote or networked regulation, sampled-data control, scenarios with communication, measurement delays, and other similar factors. Therefore, systems with delayed control have remained an active area of research for many years (see, e.g., [1], [2], [3], [4], [5], [6], [7], [8], [9]). In the case of linear systems, there are numerous constructive conditions (e.g., in the form of Linear Matrix Inequalities (LMIs)) for analyzing and designing control for linear systems with input (constant or time-varying) delay [1], [2]. However, significantly fewer results are available for stability analysis when a nonlinear control is applied.

In [8], [4], sliding mode and homogeneous control methods for systems with input delay were studied, where the conditions of ultimate boundedness of solutions of the delayed system are obtained using LMI formulations. For example, [4] shows that homogeneous control guarantees global asymptotic stability with respect to a compact set containing the origin for any bounded delay, and this set can be estimated based on LMI parameters. In addition to robustness against input delay, these control methods are attractive because, in the absence of delay, they ensure accelerated (finite-time) convergence along with several useful robust properties. Accelerated convergence is relevant for various control applications (see, e.g., [10], [11], [12], [13]). Besides the mentioned finite-time control methods, there also exist hyperexponential methods, which guarantee convergence

faster than any exponential rate [14], [15], [16], [17]. In [14], a hyperexponential control method for linear systems is proposed based on the implicit Lyapunov function approach. Although hyperexponential stability is weaker than finite-time one, numerical simulations and robustness analysis (see [14], [18]) have demonstrated its competitive performance in the presence of noise and additive disturbances. Furthermore, numerical simulations in [14] showed that this control method is significantly less sensitive to input delay, ensuring convergence to a much smaller set compared to its finite-time counterpart.

This paper focuses on analyzing the delay robustness of Implicit Lyapunov Function (ILF)-based hyperexponential control proposed in [14]. It is shown that, similar to fixed-time control, hyperexponential control ensures local stability with respect to a compact set containing the origin in the presence of input delay. The paper proposes a combination of hyperexponential control at the origin with homogeneous control, enabling global stability with respect to a set under any bounded input delay. Compared to the homogeneous control from [4], the proposed control is significantly less sensitive to input delay. A parameter tuning approach based on LMIs is also proposed, which, given the maximum input delay, ensures stability with respect to an arbitrarily small set.

The structure of the paper is as follows. Section II covers the preliminary results essential for the study. Section III formulates the problem statement. Section IV presents the analysis of input delay robustness with illustrative simulations. Finally, Section V summarizes the conclusions.

Notation:

- \mathbb{R} is the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.
- \mathbb{R}^n represents the n -dimensional Euclidean space with the vector norm $|\cdot|$.
- \mathbb{N} is the set of natural numbers.
- $P > 0$ for $P \in \mathbb{R}^{n \times n}$ indicates that P is symmetric and positive definite. The smallest and largest eigenvalues of P are represented by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively.
- For a scalar function $\sigma(\cdot)$, the notation $\sigma^{-1}(\cdot)$ denotes the power -1 and does not refer to an inverse function.
- The Banach space of continuous functions $\phi : [-\tau, 0] \rightarrow \mathbb{R}^n$ with the uniform norm $\|\phi\| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|$ is denoted by $\mathbb{C}_{[-\tau, 0]}$.
- $1, m$ is used to denote a sequence of integers $1, \dots, m$.

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II. PRELIMINARIES

A. Hyperexponential stability

Consider the nonlinear system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad t \geq 0, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous vector field, $f(0) = 0$. Assume that solutions $\Phi(t, x_0)$ of (1) exist and unique for all $x_0 \in \mathbb{R}^n$ at least locally in time.

For $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_r)^\top \in \mathbb{R}_+^{r+1}$ and $r \in \mathbb{N}$, define the function of nested exponentials $\rho_{r,\alpha} : \mathbb{R} \rightarrow \mathbb{R}_+$ using the following recursive formula:

$$\begin{aligned} \rho_{0,\alpha}(z) &= \alpha_0 z, \\ \rho_{i,\alpha}(z) &= \alpha_i \left(e^{\rho_{i-1,\alpha}(z)} - 1 \right), \quad i = \overline{1, r}. \end{aligned} \quad (2)$$

Definition 1 [14] *The origin of the system (1) is hyperexponentially stable of degree $r \in \mathbb{N}$ if there exists $\varphi \in \mathcal{K}_\infty$ and $\alpha \in \mathbb{R}_+^{r+1}$ such that the inequality*

$$|\Phi(t, x_0)| \leq \varphi(|x_0|) e^{-\rho_{r,\alpha}(t)} \quad (3)$$

is satisfied for all $t \geq 0$ and $x_0 \in D$, where $D \subset \mathbb{R}^n$ is an open neighbourhood of the origin. If $D = \mathbb{R}^n$, then the origin of the system (1) is said to be globally hyperexponentially stable of degree r .

This definition implies that convergence is hyperexponential both near the origin and away from it. Hyperexponential stability can be also characterized at infinity and at the origin, as defined in [14]:

Definition 2 [14] *The system (1) is said to be*

- *hyperexponentially stable of degree $r \in \mathbb{N}$ at infinity, if it is globally asymptotically stable, and (3) is satisfied for all $x_0 \in \mathbb{R}^n : |x_0| \geq 1$;*
- *hyperexponentially stable of degree $r \in \mathbb{N}$ at origin, if it is globally asymptotically stable, and (3) is satisfied for all $x_0 \in \mathbb{R}^n : |x_0| \leq 1$.*

B. Robust hyperexponential control

Consider a control system of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + d(t, x(t)), \quad (4)$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector, $u(t) \in \mathbb{R}$ represents the control input, and:

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

The term $d : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ describes system nonlinearities, additive disturbances, and uncertainties. Such systems arise frequently in control theory, as many controllable single-input nonlinear systems can be transformed into a canonical form like (4) (see, e.g., [19]). Additionally, in some model-free approaches (see, e.g., [20]), the control design procedure reduces to analyzing a system of the form (4) with unknown dynamics in the highest derivative as matching disturbance.

Introduce the implicitly defined functions

$$Q_1(V, x) = V^{-2} x^\top D(\sigma(V)) P D(\sigma(V)) x - 1, \quad (5)$$

$$Q_2(V, x) = V^{-2} \sigma^{-2n-2} \left(\frac{1}{V} \right) x^\top D \left(\sigma \left(\frac{1}{V} \right) \right) P D \left(\sigma \left(\frac{1}{V} \right) \right) x - 1, \quad (6)$$

where $\sigma(V) = 1 - \ln V$, $D(\lambda) = \text{diag}\{\lambda^{\eta_i}\}_{i=1}^n$, $\eta_i = 1 + n - i$ for $\lambda \in \mathbb{R}_+$ and $0 < P \in \mathbb{R}^{n \times n}$. Denote $H = \text{diag}\{\eta_i\}_{i=1}^n$.

Using the ILF theorem, in the paper [14] a hyperexponential control law is proposed for the system (4) with $d \equiv 0$. In [18], the robust properties of the system (4), (9) with respect to disturbances and measurement noise were studied, demonstrating good performance:

Theorem 1 [18] *Let the LMI*

$$AX + XA^\top + BY + Y^\top B^\top + \gamma_1(2X + XH + HX) + \alpha_1 X + R_1 \leq 0, \quad (7a)$$

$$\begin{aligned} AX + XA^\top + BY + Y^\top B^\top + \gamma_2(2(n+2)X - XH - HX) \\ + \alpha_2 X + R_2 \leq 0, \end{aligned} \quad (7b)$$

$$XH + HX > 0, \quad 2(n+1)X - XH - HX > 0, \quad X > 0 \quad (7c)$$

be feasible for some fixed numbers $\gamma_1, \gamma_2, \alpha_1, \alpha_2 \in \mathbb{R}_+$ and some $R_1, R_2 \in \mathbb{R}^{n \times n}$, $R_1, R_2 > 0$, and the disturbance function d satisfies

$$\begin{aligned} d^\top D(\sigma(V)) R_1^{-1} D(\sigma(V)) d \leq \alpha_1 V^2 \sigma^2(V) \\ \text{if } x^\top P x \leq 1, \\ d^\top D(\sigma(\frac{1}{V})) R_2^{-1} D(\sigma(\frac{1}{V})) d \leq \alpha_2 V^2 \sigma^{2n+4}(\frac{1}{V}) \\ \text{if } x^\top P x \geq 1. \end{aligned} \quad (8)$$

Then the control

$$u(V, x) = \begin{cases} KD(\sigma(V))x & \text{for } x^\top P x < 1, \\ KD(\sigma(V^{-1}))x & \text{for } x^\top P x \geq 1, \end{cases} \quad (9)$$

where $V \in \mathbb{R}_+ : Q_1(V, x) = 0$ for $x^\top P x < 1$, $V \in \mathbb{R}_+ : Q_2(V, x) = 0$ for $x^\top P x \geq 1$, $K = YP$, $P = X^{-1}$, globally hyperexponentially stabilizes the system (4) with degree $r = 1$.

Remark 1 In the case of asymptotic stability analysis, the inequalities (7a), (7b) can be replaced with a strict ones

$$AX + XA^\top + BY + Y^\top B^\top + \alpha_j X + R_j < 0, \quad j = 1, 2.$$

In the case the matching condition $d_i(t, x) = 0$, $i = \overline{1, n-1}$ holds the terms R_1, R_2 in (7) can be replaced with $r_1 E$ and $r_2 E$, correspondingly, where $r_1, r_2 \in \mathbb{R}_+$ and the matrix $E = \{e_{ij}\} \in \mathbb{R}^{n \times n}$ has only one nonzero element: $e_{n,n} = 1$. In this case, the condition (8) takes a simpler form [18]

$$\begin{aligned} |d_n| \leq \sqrt{\alpha_1 r_1} V & \quad \text{if } x^\top P x < 1, \\ |d_n| \leq \sqrt{\alpha_2 r_2} V \sigma^{1+n}(\frac{1}{V}) & \quad \text{if } x^\top P x \geq 1. \end{aligned} \quad (10)$$

III. PROBLEM FORMULATION

This paper investigates robustness with respect to input delays. Assume there is an input delay τ in the system, i.e.

$$\dot{x}(t) = Ax(t) + Bu(t - \tau) + d(t, x(t)). \quad (11)$$

The delay τ is unknown, but its upper bound $\tau_0 \in \mathbb{R}_+$ is known, i.e., $\tau \leq \tau_0$.

Based on the simulation carried out in [14], the control (9) demonstrates sufficiently less sensitivity to input delays

in comparison with finite-time counterpart [21]. There has been no previous analytical study of robustness with respect to delays for the control (9).

The main goal of the paper is to provide theoretical robustness analysis for the system (11), (9) with respect to input delays.

IV. MAIN RESULT

Define $B_\rho^c = \{\phi \in \mathbb{C}[-\tau, 0] : \|\phi\| \leq \rho\}$ as a closed ball of radius $\rho > 0$ in $\mathbb{C}[-\tau, 0]$.

Lemma 1 (Disturbance-free case) *Let $d \equiv 0$, and the system of linear matrix inequalities*

$$AX + XA^\top + BY + Y^\top B^\top + \alpha_j X + R_j < 0, \quad (12a)$$

$$XH + HX \geq \rho_1 X, \quad X > 0, \quad (12b)$$

$$2(n+1)X - XH - HX \geq \rho_2 X, \quad (12c)$$

$$\iota X \geq AX + XA^\top \geq -\iota X, \quad \begin{pmatrix} X & Y^\top \\ Y & \zeta \end{pmatrix} \geq 0, \quad (12d)$$

be feasible for $R_j = \begin{pmatrix} 0_{n-1} & 0 \\ 0 & r_j \end{pmatrix}$, $j = 1, 2$ and some r_j , α_j , ρ_j , ι , $\zeta \in \mathbb{R}_+$, $X \in \mathbb{R}^{n \times n}$, and $Y \in \mathbb{R}^{1 \times n}$. Then the system (11), (9) with $d(t, x) = 0$ is asymptotically stable in $\Omega_c \subset B_{C^{n+1}/\sqrt{\lambda_{\min}(P)}}^{2\tau}$ with respect to $\Omega_c \subset B_{c/\sqrt{\lambda_{\min}(P)}}^{2\tau}$, where $C \geq 1 \geq c > 0$, if

$$\tau_0 < \min \left\{ \frac{\sqrt{\alpha_1 r_1}}{\frac{(1+2\sqrt{P_m \zeta})\sqrt{l_1}}{2+\rho_1 \sigma^{-1}(c)} + (\sqrt{l_2} + |k_n| \sqrt{\zeta}) \sigma(c)}, \frac{\sqrt{\alpha_2 r_2}}{\frac{(1+2\sqrt{P_m \zeta})\sqrt{l_1}}{2+\rho_2 \sigma^{-1}(c^{-1})} + (\sqrt{l_2} + |k_n| \sqrt{\zeta}) \sigma(c^{-1})} \right\}, \quad (13)$$

where $l_1 = \lambda_{\max}(X^{1/2} H K^\top K H X^{1/2})$, $l_2 = \lambda_{\max}(X^{1/2} A K^\top K A^\top X^{1/2})$, $k_n = KB$, and $P_m = B^\top P B$.

Remark 2 A less conservative estimate of the set Ω_c with respect to which the system is stable can be obtained using the numerical computations. The same applies to the set Ω_c .

Note that the form of control (9) implies increasing control gains that multiply the state:

- as $|x| \rightarrow \infty$, which provides local stability when delays are present;
- as $|x| \rightarrow 0$, which provides stability with respect to a bounded set containing the origin.

A similar situation arises when using the fixed-time homogeneous control from [21] in the presence of input delay (see [4]). The presented estimates are rather conservative, making it difficult to fairly compare the sensitivity to input delay with the control from [21]. However, based on numerical simulations [4], the sensitivity to input delay in this case is significantly lower. This can also be demonstrated numerically using the obtained estimates for sufficiently small delays. Let $n = 2$, and choose matrices $X = \begin{pmatrix} 9.9742 & -0.3992 \\ -0.3992 & 0.0661 \end{pmatrix}$, $Y = (-0.0342 \ -0.0076)$ (and $P := X^{-1} = \begin{pmatrix} 0.1322 & 0.7980 \\ 0.7980 & 19.9409 \end{pmatrix}$, $K := YP = (-0.0106 \ -0.1797)$), which satisfy (12) and the conditions from [4]. Fig. 1 and Fig. 2 show the sets Ω_c (blue color), computed on a numerical grid based on the proposed

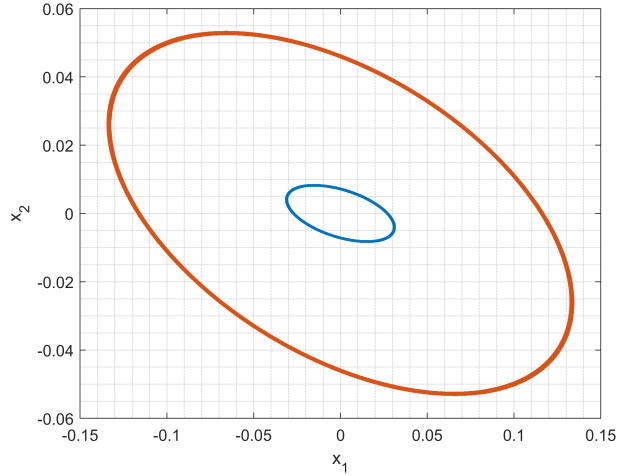


Fig. 1. The set Ω_c for hyperexponential control (blue) and finite-/fixed-time control [21] (red) with input delay $\tau_0 = 0.015$

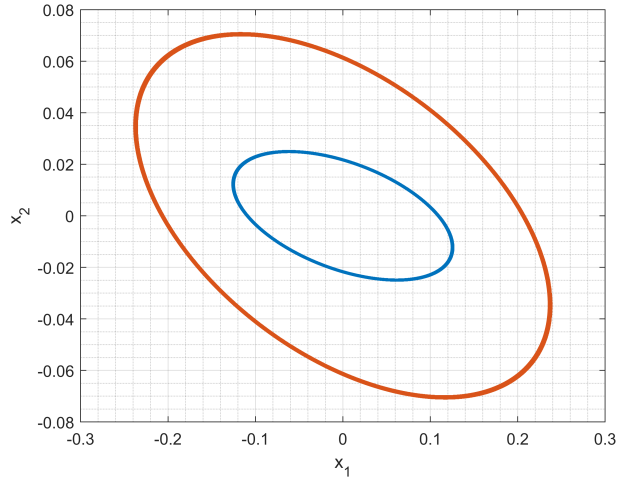


Fig. 2. The set Ω_c for hyperexponential control (blue) and finite-/fixed-time control [21] (red) with input delay $\tau_0 = 0.02$

estimates (13) for two different input delays $\tau_0 = 0.02$ and $\tau_0 = 0.015$. For comparison, the corresponding results for finite-/fixed-time control based on [4] are shown in red.

The control structure (9) enables switching from hyperexponential control at infinity to hyperexponential control at the origin on the set $\{x \in \mathbb{R}^n : x^\top P x = 1\}$. Based on [14], this structure also allows combining hyperexponential control at infinity (at the origin) with finite-time and nearly fixed-time counterparts from [21]. In the case of input delay, combining hyperexponential control at the origin with finite-time control [21] may be more relevant to ensure independent of delay stability.

Define the implicit function as

$$Q_\mu(V, x) = x^\top D_\mu(V^{-1}) P D_\mu(V^{-1}) x - 1, \quad (14)$$

where $D_\mu(\lambda)$ is a diagonal matrix given by $D_\mu(\lambda) = \text{diag}\{\lambda^{1+(n-1)\mu}\}_{i=1}^n$, $\mu \in (0, 1]$, and $P \in \mathbb{R}^{n \times n}$ is symmetric positive definite. Denote $H_\mu = \text{diag}\{1 + (n-i)\mu\}_{i=1}^n$.

Corollary 1 (Disturbance-free case) Let $d(t, x) \equiv 0$, and the system of linear matrix inequalities (12a) for $j = 1$, (12b), (12d), and

$$AX + XA^\top + BY + Y^\top B^\top + \alpha_\mu X + R_\mu \leq 0, \quad (15a)$$

$$XH_\mu + H_\mu X \geq \rho_\mu X \quad (15b)$$

be feasible for $R_\mu = \begin{pmatrix} 0_{n-1} & 0 \\ 0 & r_\mu \end{pmatrix}$ and some $r_\mu, \alpha_\mu, \rho_\mu \in \mathbb{R}_+$. Then the control

$$u(V, x) = \begin{cases} KD(\sigma(V))x & \text{for } x^\top Px < 1, \\ V^{1-\mu}KD_\mu(V^{-1})x & \text{for } x^\top Px \geq 1, \end{cases} \quad (16)$$

where $V \in \mathbb{R}_+ : \mathcal{Q}_1(V, x) = 0$ for $x^\top Px < 1$, $V \in \mathbb{R}_+ : \mathcal{Q}_\mu(V, x) = 0$ for $x^\top Px \geq 1$, $K = YP$, $P = X^{-1}$, provides:

- global asymptotic stability with respect to a bounded set

$$\{x \in \mathbb{C}_{[-2\tau, 0]} : V(x(s)) \leq c, V(x(s)) : \mathcal{Q}_1(V(x(s)), x(s)) = 0, s \in [-2\tau, 0]\}, c \in (0, 1),$$

if

$$\tau_0 < \min \left\{ \frac{\sqrt{\alpha_1 r_1}}{\frac{(1+2\sqrt{P_{nn}\zeta})\sqrt{l_1}}{2+\rho_1} + (\sqrt{l_2} + k_n|\sqrt{\zeta})\sigma(c)}, \frac{\sqrt{\alpha_\mu r_\mu}}{\rho_\mu^{-1}(1+2\sqrt{P_{nn}\zeta})\sqrt{l_\mu + \sqrt{l_2} + k_n|\sqrt{\zeta}}}} \right\}, \quad (17)$$

where $l_\mu = \lambda_{\max}(X^{1/2}H_\mu K^\top KH_\mu X^{1/2})$;

- global asymptotic stability with respect to a bounded set

$$\{x \in \mathbb{C}_{[-2\tau, 0]} : V(x(s)) \leq c, V(x(s)) : \mathcal{Q}_\mu(V(x(s)), x(s)) = 0, s \in [-2\tau, 0]\}, c \geq 1,$$

if

$$\frac{\sqrt{\alpha_1 r_1}}{\frac{(1+2\sqrt{P_{nn}\zeta})\sqrt{l_1}}{2+\rho_1} + \sqrt{l_2} + k_n|\sqrt{\zeta}} \leq \tau_0 < \frac{\sqrt{\alpha_\mu r_\mu} c^\mu}{\rho_\mu^{-1}(1+2\sqrt{P_{nn}\zeta})\sqrt{l_\mu + \sqrt{l_2} + k_n|\sqrt{\zeta}}}, \quad (18)$$

- hyperexponential stability at the origin if $\tau = 0$.

Remark 3 Note that the control law $u(V, x) = V^{1-\mu}KD_\mu(V^{-1})x$, applied for all $x \in \mathbb{R}^n$, also ensures global stability with respect to a certain compact set in the presence of input delay (see [4]). However, as mentioned earlier, hyperexponential control at the origin is significantly less sensitive to small input delays. Therefore, control law (16) combines reduced sensitivity to small input delays by using $u(V, x) = KD(\sigma(V))x$ for $x^\top Px < 1$ and τ_0 satisfying (17), while ensuring the global stability property by $u(V, x) = V^{1-\mu}KD_\mu(V^{-1})x$ for $x^\top Px \geq 1$ and sufficiently large input delay. Thus, control law (16) may be particularly relevant in scenarios with an unreliable communication channel where unexpected increases in delay may occur.

Lemma 1, Corollary 1, and Remark 1 provide an estimate of the convergence set for the system under input delay $\tau \leq \tau_0$. This results describe the closed-loop system property analogous to ISS but with respect to input delay. It shows

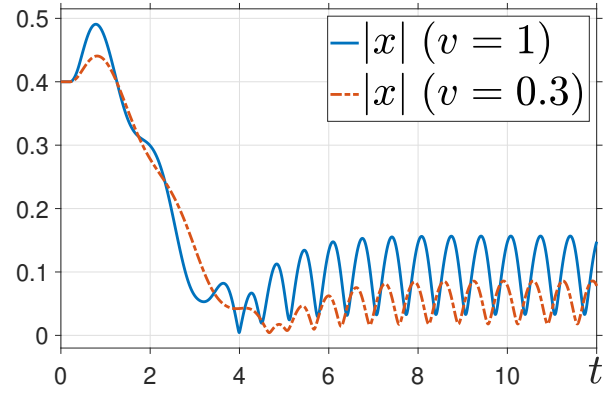


Fig. 3. Norm $|x(t)|$ for $\nu = 1$ and $\nu = 0.3$ and $\tau = 0.2$

how the size of the convergence set depends on the maximum input delay value, similar to the asymptotic gain function.

The proposed approach not only allows estimating the set to which the system converges but also enables tuning control parameters to ensure convergence to an arbitrarily small set for any bounded input delay $\tau \leq \tau_0$.

To formulate this result, for some $\nu \in \mathbb{R}_+$ define the ILF functions:

$$\begin{aligned} \bar{\mathcal{Q}}_1(V, x) &= V^{-2}x^\top D(\sigma(V))PD(\sigma(V))x - \nu, \\ \bar{\mathcal{Q}}_\mu(V, x) &= x^\top D_\mu(V^{-1})PD_\mu(V^{-1})x - \nu. \end{aligned} \quad (19)$$

Theorem 2 Let the LMIs from Corollary 1 be feasible for some $X \in \mathbb{R}^{n \times n}$, $Y \in \mathbb{R}^{1 \times n}$, and the upper bound of the input delay τ_0 is known. Then the control

$$u(V, x) = \begin{cases} KD(\sigma(V))x & \text{for } x^\top Px < \nu, \\ V^{1-\mu}KD_\mu(V^{-1})x & \text{for } x^\top Px \geq \nu, \end{cases} \quad (20)$$

where $V \in \mathbb{R}_+ : \bar{\mathcal{Q}}_1(V, x) = 0$ for $x^\top Px < \nu$, $V \in \mathbb{R}_+ : \bar{\mathcal{Q}}_\mu(V, x) = 0$ for $x^\top Px \geq \nu$, $K = YX^{-1}$, $P = X^{-1}$ provides global asymptotic stability with respect to a bounded set $B_{c\sqrt{\nu}/\sqrt{\lambda_{\min}(P)}}^{2\tau}$ for some $c \in (0, 1)$, if (17) is satisfied, and with respect to $B_{c^{1+(n-1)\mu}\sqrt{\nu}/\sqrt{\lambda_{\min}(P)}}^{2\tau}$ for some $c \geq 1$, if (18) is satisfied.

In other words, Theorem 2 allows designing control that ensures convergence to any arbitrarily small compact set under any bounded input delay (the smaller ν , the smaller the set). In the absence of delay, it guarantees hyperexponential stability at the origin.

Example Consider an example with $n = 3$ and matrices

$$P = \begin{pmatrix} 4.2266 & 4.2427 & 1.1069 \\ 4.2427 & 5.9198 & 2.0473 \\ 1.1069 & 2.0473 & 1.2774 \end{pmatrix}, \quad K = \begin{pmatrix} -1.5642 \\ -3.0832 \\ -2.2316 \end{pmatrix}^\top,$$

obtained by solving the LMIs from Corollary 1. Fig. 3 shows the norm $|x(t)|$ for $\nu = 1$ and $\nu = 0.3$ and constant input delay $\tau = 0.2$, illustrating the reduction of the set with respect to which the system is stable as ν decreases.

Let us consider the disturbed case.

Theorem 3 (Disturbed case) *Let the system of linear matrix inequalities*

$$\begin{aligned}
& \kappa (AX + XA^\top + BY + Y^\top B^\top) + \alpha_1 X + R_1 < 0, \\
& (1 - \kappa) (AX + XA^\top + BY + Y^\top B^\top) + \tilde{\alpha}_1 X + R_{d1} < 0, \\
& \kappa_\mu (AX + XA^\top + BY + Y^\top B^\top) + \alpha_\mu X + R_\mu \leq 0, \\
& (1 - \kappa_\mu) (AX + XA^\top + BY + Y^\top B^\top) + \tilde{\alpha}_\mu X + R_{d\mu} \leq 0, \\
& XH + HX \geq \rho_1 X, \quad XH_\mu + H_\mu X \geq \rho_\mu X, \quad X > 0, \\
& \iota X \geq AX + XA^\top \geq -\iota X, \quad \begin{pmatrix} X & Y^\top \\ Y & \zeta \end{pmatrix} \geq 0, \\
& R_{d1} \leq \omega_1 X, \quad R_{d\mu} \leq \omega_\mu X
\end{aligned} \tag{21}$$

be feasible for $R_1 = \begin{pmatrix} 0_{n-1} & 0 \\ 0 & r_1 \end{pmatrix}$, $R_\mu = \begin{pmatrix} 0_{n-1} & 0 \\ 0 & r_\mu \end{pmatrix}$ and some $\kappa_1, \kappa_\mu \in (0, 1)$, $\mu \in (0, 1]$, $r_1, r_\mu, \alpha_1, \alpha_\mu, \tilde{\alpha}_1, \tilde{\alpha}_\mu, \rho_1, \rho_\mu, \omega_1, \omega_\mu, \iota, \zeta \in \mathbb{R}_+$, and the disturbance function satisfies

$$\begin{aligned}
& d^\top D(\sigma(V)) R_{d1}^{-1} D(\sigma(V)) d \leq \tilde{\alpha}_1 V^2 \sigma^2(V) \\
& \quad \text{if } x^\top P x \leq 1, \\
& d^\top D_\mu(V^{-1}) R_{d\mu}^{-1} D_\mu(V^{-1}) d \leq \tilde{\alpha}_\mu V^{-2\mu} \\
& \quad \text{if } x^\top P x \geq 1.
\end{aligned} \tag{22}$$

Then the control (16) with $K = YP$, $P = X^{-1}$ provides:

- global asymptotic stability with respect to a bounded set

$$\{x \in \mathbb{C}_{[-2\tau, 0]} : V(x(s)) \leq c, V(x(s)) : Q_1(V(x(s)), x(s)) = 0, s \in [-2\tau, 0]\}, \quad c \in (0, 1),$$

if

$$\tau_0 < \min \left\{ \frac{\sqrt{\alpha_1 r_1}}{\frac{(1+2\sqrt{P_{nn}\zeta+2\sqrt{\tilde{\alpha}_1\omega_1})\sqrt{r_1}}}{2+\rho_1}\sigma^{-1}(c)} + (\sqrt{r_2}+|k_n|\sqrt{\zeta} + \sqrt{\zeta\omega_1\tilde{\alpha}_1})\sigma(c)}, \frac{\sqrt{\alpha_\mu r_\mu}}{\rho_\mu^{-1}(1+2\sqrt{P_{nn}\zeta} + \sqrt{\omega_\mu\tilde{\alpha}_\mu})\sqrt{r_\mu} + \sqrt{r_2}+|k_n|\sqrt{\zeta} + \sqrt{\zeta\omega_\mu\tilde{\alpha}_\mu}} \right\};$$

- global asymptotic stability with respect to a bounded set

$$\{x \in \mathbb{C}_{[-2\tau, 0]} : V(x(s)) \leq c, V(x(s)) : Q_\mu(V(x(s)), x(s)) = 0, s \in [-2\tau, 0]\}, \quad c \geq 1,$$

if

$$\begin{aligned}
& \frac{\sqrt{\alpha_1 r_1}}{\frac{(1+2\sqrt{P_{nn}\zeta+2\sqrt{\tilde{\alpha}_1\omega_1})\sqrt{r_1}}}{2+\rho_1} + \sqrt{r_2}+|k_n|\sqrt{\zeta} + \sqrt{\zeta\omega_1\tilde{\alpha}_1}} \leq \tau_0 \\
& < \frac{\sqrt{\alpha_\mu r_\mu} c^\mu}{\rho_\mu^{-1}(1+2\sqrt{P_{nn}\zeta} + 2\sqrt{\tilde{\alpha}_\mu\omega_\mu})\sqrt{r_\mu} + \sqrt{r_2}+|k_n|\sqrt{\zeta} + \sqrt{\zeta\omega_\mu\tilde{\alpha}_\mu}};
\end{aligned}$$

- hyperexponential stability at the origin if $\tau = 0$.

V. CONCLUSIONS

The paper presents an analysis of the stability of a system with the hyperexponential control in the presence of input delay. A control is proposed that ensures global asymptotic stability with respect to a compact set containing the origin under any bounded input delay. This set can be estimated based on the solution of the given LMI and the known upper bound of the input delay. In the absence of input delay, the system is hyperexponentially stable at the origin. A parameter tuning method is also proposed to ensure stability

with respect to an arbitrarily small set containing the origin. Future research directions include extending the approach to MIMO systems and relaxing the imposed constraints.

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