



HAL
open science

Design of Control Laws for Robust Global Stabilization of Multistable State Periodic Systems

Jesús Mendoza-Avila, Denis Efimov, José Ángel Mercado Uribe, Johannes Schiffer

► **To cite this version:**

Jesús Mendoza-Avila, Denis Efimov, José Ángel Mercado Uribe, Johannes Schiffer. Design of Control Laws for Robust Global Stabilization of Multistable State Periodic Systems. *International Journal of Robust and Nonlinear Control*, In press, <10.1016/j.ifacol.2023.10.936>. <hal-05249671>

HAL Id: hal-05249671

<https://inria.hal.science/hal-05249671v1>

Submitted on 11 Sep 2025

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY 4.0 - Attribution - International License

RESEARCH ARTICLE

Design of Control Laws for Robust Global Stabilization of Multistable State Periodic Systems

Jesus Mendoza-Avila^{1,2} | Denis Efimov^{1,3} | Jose Angel Mercado-Uribe² | Johannes Schiffer^{2,4}

¹Inria, Univ. Lille, CNRS, UMR 9189-CRISTAL, Lille, France

²Brandenburg University of Technology Cottbus-Senftenberg, Cottbus, Germany

³ITMO University, Saint Petersburg, Russia

⁴Fraunhofer Research Institution for Energy Infrastructures and Geothermal Systems (IEG), Cottbus, Germany

Correspondence

Jesus Mendoza-Avila, Brandenburg University of Technology Cottbus-Senftenberg, Cottbus, Germany.

Email: Jesus.MendozaAvila@b-tu.de

Present address

Jesus Mendoza-Avila is currently with Brandenburg University of Technology Cottbus-Senftenberg, Cottbus, Germany.

Funding Information

This research was supported by the ANR-DFG project SyNPiD - Project number 446182476.

Abstract

This paper continues the development of the input-to-state stability (ISS)- control Leonov function (CLeF) approach. The definitions of practical ISS- and integral ISS-CLeFs are refined, and the proposed methodology for control synthesis is improved to simplify the final control law. Then, it is shown that the existence of practical ISS- and integral ISS-CLeFs is a sufficient condition to guarantee the existence of a controller that endows multistable state periodic systems with the ISS and integral ISS properties, respectively. Furthermore, a methodology for the design of such a controller is provided via the well-known Sontag's universal formula. Besides, an extension of the main result is presented to connect the ISS-CLeF approach with the standard Leonov function method, such that the maximal invariant set of the closed-loop system is compact on a manifold. Finally, the proposed approach is applied to the design of an excitation controller for a synchronous generator, which guarantees global ISS properties for the closed-loop system, unlike the usual local results reported in the literature. The obtained control is also independent of the load angle. The effectiveness of the designed controller is demonstrated in simulations.

KEYWORDS

Global Stabilization, State Periodic Systems, Multistability, Input-to-State Stability, Nonlinear control, Power systems.

1 | INTRODUCTION

The concept of input-to-state stability (ISS) and its extensions form a useful framework for the analysis of stability and robustness of nonlinear systems affected by external inputs, e.g., exogenous disturbances or measurement noises.^{1,2} Particularly, ISS properties refer to the robustness of dynamical systems against bounded inputs. In contrast, the so-called integral ISS (iISS) property³ is suitable for investigating robustness against disturbances with bounded energy. Hierarchically, ISS is a stronger property than iISS, and in fact, the former implies the latter. Despite that, the results of Angeli et al.³ show that iISS is a more natural property of well-behaved systems and is, therefore, of interest for research.

Many engineering applications involve systems with multiple invariant solutions, such as power dynamics, biological systems, or chemical processes. The analysis of the stability and robustness of this kind of systems requires significant modifications and relaxations of the employed approaches; see, for example.^{4,5,6} Accordingly, Angeli and Efimov,⁷ and Forni and Angeli⁸ extend the concepts of ISS and iISS to the multistable case, respectively. Furthermore, they introduce the concepts of ISS- and iISS-Lyapunov functions for multistable systems as characterizations of the corresponding property.

A particular class of multistable systems is formed by those with periodic dynamics with respect to a part of the state, called, in short, state periodic systems. Some examples of this class of systems are the forced nonlinear pendulum⁹, power dynamics,^{10,11}

microgrids,¹² and phase-locked loop (PLL) systems.¹³ Unfortunately, the aforementioned Lyapunov-based approaches cannot be straightforwardly applied to the study of stability and robustness of state periodic systems due to their infinite number of invariant solutions. For instance, the utilization of the results of Angeli and Efimov⁷ or Forni and Angeli⁸ imply the construction of periodic ISS- or iISS-Lyapunov functions. This fact represents a strong limitation, and in most cases, the construction of such periodic functions is a challenging task in applications. As a consequence, just local stability results are usually achieved; see, for example the references.^{12,14} Therefore, a more advanced solution is required.

Recently, in the works of Efimov et al.¹⁵ and Mendoza-Avila et al.^{16,17} the concepts of *ISS-* and *iISS-Leonov functions* are introduced, respectively, as fitting characterizations of the ISS and iISS properties for state periodic systems. Typically, these functions are sign-indefinite, and their derivatives are required to be non-positive only in the region of the state space where the respective function is non-negative. Hence, they significantly relax the conditions for the study of the ISS and iISS properties in the considered class of systems, in contrast to the results mentioned above based on ISS- and iISS Lyapunov functions for multistable systems.

These works^{15,16,17} are based on the so-called *cell structure* approach¹⁸ which provides an appropriate tool to establish global boundedness of solutions for unperturbed multistable systems, whose dynamics are periodic with respect to a scalar state variable (see also^{19,20,21}). An extension of the cell structure approach for multistable systems, whose dynamics are periodic with respect to multiple state variables, was presented by Efimov and Schiffer.²² This extension is based on the concept of *Leonov functions*, which are sign-indefinite functions with respect to the periodic states, radially unbounded with respect to the non-periodic ones, and whose derivatives are sign-definite with respect to their zero level. This approach has been successfully applied to establish almost global stability properties of power systems and microgrids,^{23,24} and to ensure almost global synchronization of networks of heterogeneous Kuramoto oscillators.^{25,26}

Naturally, systems with control inputs can be endowed with the ISS and iISS properties by means of the design of feedback control laws. This task has been mainly addressed through the approach of ISS- and iISS-control Lyapunov functions.^{27,28} Especially, Librezon et al.²⁷ proposes a unified framework for ISS and iISS stabilization of nonlinear perturbed systems, where the construction of continuous feedback control laws is based on the original idea of Artstein²⁹, further formalized by Sontag³⁰ with the introduction of the so-called *control universal formula*. Recently, Barroso et al.^{31,32} have extended the concepts of ISS- and iISS-control Lyapunov functions to the case of multistable affine nonlinear systems and proposed the construction of continuous feedback controllers by using the control universal formula.³⁰

In this paper, ISS- and iISS-stabilization of state periodic systems is addressed. As mentioned above, the application of Lyapunov function-based methods entails several complications, and the Leonov function theory arises as an appropriate alternative. Therefore, we retake the ISS-control Leonov function (ISS-CLeF) approach, originally introduced in our conference paper.³³ This method proposes control synthesis for state periodic systems in terms of practical ISS- or iISS-CLeFs, such that the system under study is rendered with the ISS or iISS property, in the multistable sense, by means of a continuous and state periodic controller, which is constructed by using a mild adaptation of the control universal formula,³⁰ and a kind of small control property (SCP).

In this manuscript and in comparison to³³, we refine the definitions of practical ISS- and iISS-CLeFs to clarify important aspects in the construction of these functions. Furthermore, we improve the methodology for control design and propose a new formula for the final control law, such that the process of design is significantly simplified. Moreover, an extension of the main results is introduced to connect the ISS-CLeF approach with the standard Leonov function method, such that the maximal invariant set of the closed-loop system is compact. Also, we provide complete proofs of all results, as well as extensive discussions and illustrative examples of the critical points of our methodology. Finally, we present an application of the proposed approach for the design of an excitation controller for a synchronous generator, ensuring the ISS property for the closed-loop system. Remarkably, the new controller is independent of the load angle (the main difference with respect to the solution given in³³) and guarantees global ISS properties for the closed-loop system, unlike most results that can be found in the literature.

The outline of the paper is as follows. Section 2 presents definitions of ISS- and iISS-related notions for multistable systems, introduces the considered class of state periodic systems, and provides preliminary results about ISS- and iISS-Leonov functions. Section 3 contains our main results: definitions of practical ISS- and iISS-CLeFs, together with the methodology for control synthesis, and also provides several illustrative examples. In Section 4, our novel result to ensure compactness of the maximal invariant set of the closed-loop system is developed. Section 5 presents the design of an excitation controller for a synchronous generator based on the proposed approach. Conclusions are given in Section 6. Finally, some examples illustrating particular features of the control design based on practical ISS- and iISS-CLeFs are provided in Appendix A.

2 | PRELIMINARIES

2.1 | Notation

The sets of natural, integer and real numbers are represented by \mathbb{N} , \mathbb{Z} and \mathbb{R} , respectively. Moreover, \mathbb{R}_+ stand for the set of non-negative real numbers, i.e., $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.

A function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to the class \mathcal{K} if it is continuous, strictly increasing and $\alpha(0) = 0$. The function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to the class \mathcal{K}_∞ if $\alpha \in \mathcal{K}$ and it is unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to the class \mathcal{KL} if, for each fixed $t \in \mathbb{R}_+$, $\beta(\cdot, t) \in \mathcal{K}_\infty$ and, for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is non-increasing and it tends to zero for $t \rightarrow \infty$.

For a vector $x \in \mathbb{R}^n$, $|x|$ represents the usual Euclidean norm. Also, we introduce the vector norm $|\tilde{x}|_\infty = \max_{1 \leq i \leq n} |\tilde{x}_i|$, such that $|\tilde{x}|_\infty \leq |\tilde{x}| \leq \sqrt{n}|\tilde{x}|_\infty$. Besides, for a locally essentially bounded and measurable signal $d : \mathbb{R} \rightarrow \mathbb{R}^m$ the essential supremum norm is defined as $\|d\|_\infty = \text{ess sup}_{t \geq 0} |d(t)|$, and the set of such signals with a finite norm is denoted by \mathcal{L}_∞^m .

2.2 | Multistable systems

Let a map $f : M \times \mathbb{R}^m \rightarrow T_x M$ be continuously differentiable with respect to its arguments, where M is an n -dimensional smooth connected orientable Riemannian manifold without boundary and $T_x M$ is its tangent space. Then, consider the system

$$\dot{x} = f(x, d), \quad (1)$$

where $x \in M$ is the state and $d \in \mathcal{L}_\infty^m$ is an input signal. Moreover, for any $x_0 \in M$ and $d \in \mathcal{L}_\infty^m$ we denote by $X(t, x_0; d)$ the uniquely defined solution of (1) at time t fulfilling $X(0, x_0; d) = x_0$. Together with (1), consider its unperturbed version

$$\dot{x} = f(x, 0). \quad (2)$$

Hence, a set $S \subset M$ is invariant for the unperturbed system (2) if $X(t, x_0; 0) \in S$ for all $t \in \mathbb{R}$ and for all $x_0 \in S$. For $x_0 \in M$, the point $y \in M$ belongs to the ω -limit (α -limit) set of (2) if there is a sequence t_i , $\lim_{i \rightarrow +\infty} t_i = +\infty$, such that $\lim_{i \rightarrow +\infty} X(t_i, x_0; 0) = y$ ($\lim_{i \rightarrow +\infty} X(-t_i, x_0; 0) = y$). For any $x_0 \in M$, the α - and ω -limit sets of the unperturbed system (2) are invariant.³⁴ Furthermore, the distance from a point $x \in M$ to the set $S \subset M$ is defined as $|x|_S = \inf_{a \in S} \varpi(x, a)$, where the symbol $\varpi(x_1, x_2)$ denotes the Riemannian distance between x_1 and x_2 in M , $|x| = |x|_0$ for $x \in M$ (0 is a point selected on M).

2.3 | Decomposable sets

Let $\Lambda \subset M$ be a compact invariant set for the system (2).

Definition 1.³⁵ A decomposition of Λ is a finite and disjoint family of compact invariant sets $\Lambda_1, \dots, \Lambda_k$ such that $\Lambda = \bigcup_{i=1}^k \Lambda_i$. Each Λ_i will be called an *atom* of the decomposition. \triangle

For an invariant set Λ , its attracting and repulsing subsets are defined as follows:

$$\mathfrak{A}(\Lambda) = \{x \in M : |X(t, x; 0)|_\Lambda \rightarrow 0 \text{ as } t \rightarrow +\infty\}, \quad \text{and} \quad \mathfrak{R}(\Lambda) = \{x \in M : |X(t, x; 0)|_\Lambda \rightarrow 0 \text{ as } t \rightarrow -\infty\}.$$

Define a relation on $\mathcal{B} \subset M$ and $\mathcal{D} \subset M$ by $\mathcal{B} \prec \mathcal{D}$ if $\mathfrak{A}(\mathcal{B}) \cap \mathfrak{R}(\mathcal{D}) \neq \emptyset$.

Definition 2.³⁵ Let $\Lambda_1, \dots, \Lambda_k$ be a decomposition of Λ , then

1. An r -cycle ($r \geq 2$) is an ordered r -tuple of distinct indices i_1, \dots, i_r such that $\Lambda_{i_1} \prec \dots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$.
2. A 1-cycle is an index i such that $\mathfrak{A}(\Lambda_i) \cap \mathfrak{R}(\Lambda_i) \setminus \Lambda_i \neq \emptyset$.
3. A filtration ordering is a numbering of the Λ_i so that $\Lambda_i \prec \Lambda_j \implies i \leq j$. \triangle

Then, we have that the existence of an r -cycle with $r \geq 2$ is equivalent to the existence of a heteroclinic orbit for the system (2). Moreover, the existence of a 1-cycle implies the existence of a homoclinic orbit for the system (2).³⁴

Definition 3.⁷ The set Λ is called decomposable if it admits a finite decomposition without cycles, $\Lambda = \bigcup_{i=1}^k \Lambda_i$, for some non-empty disjoint compact sets Λ_i , which form a filtration ordering of Λ . \triangle

Then, we establish the following assumption for an invariant set $\mathcal{W} \subset M$ of the system (2).

Assumption 1. Let a set $\mathcal{W} \subset M$ contain all α - and ω -limit sets of (2), and be compact and decomposable. \triangle

2.4 | ISS and iISS properties of multistable systems

The following notions of robustness for multistable systems modeled by (1), with \mathcal{W} as in Assumption 1, have been introduced by Angeli and Efimov⁷ and Forni and Angeli⁸ (see^{1,3} for a survey on the conventional ISS and iISS frameworks).

Definition 4.⁷ The system (1) has the *practical asymptotic gain* (pAG) property if there exist $\nu \in \mathcal{K}_\infty$ and $c_1 \geq 0$, such that for all $x_0 \in M$ and all $d \in \mathcal{L}_\infty^m$ the solutions are defined for all $t \geq 0$ and

$$\limsup_{t \rightarrow \infty} |X(t, x_0; d)|_{\mathcal{W}} \leq \nu(\|d\|_\infty) + c_1.$$

If $c_1 = 0$, then we say that the *asymptotic gain* (AG) property holds. Moreover, for $d \equiv 0$, the AG property implies the so-called *zero-global attraction* (0-GATT) property for the unperturbed system (2) to the set \mathcal{W} . \triangle

Definition 5.⁷ The system (1) has the *limit* property (LIM) with respect to \mathcal{W} if there exists $\mu \in \mathcal{K}_\infty$, such that for all $x_0 \in M$ and all $d \in \mathcal{L}_\infty^m$ the solutions are defined for all $t \geq 0$ and the following holds:

$$\inf_{t \geq 0} |X(t, x_0; d)|_{\mathcal{W}} \leq \mu(\|d\|_\infty).$$

\triangle

Definition 6.⁷ The system (1) has the *practical global stability* (pGS) property with respect to \mathcal{W} if there exist $\rho \in \mathcal{K}_\infty$ and $c_2 \geq 0$, such that for all $x_0 \in M$ and all $d \in \mathcal{L}_\infty^m$ the following holds for all $t \geq 0$:

$$|X(t, x_0; d)|_{\mathcal{W}} \leq \rho(\max |x|_{\mathcal{W}}, \|d\|_\infty) + c_2.$$

\triangle

Definition 7.⁸ The system (1) is said to have the *uniform bounded-energy bounded-state* (UBEBS) property if, for some class \mathcal{K}_∞ functions χ, ς and η , and a positive constant c_3 , the following estimate holds for all $t \geq 0$, all $x_0 \in M$ and all $d \in \mathcal{L}_\infty^m$:

$$\chi(|X(t, x_0; d)|_{\mathcal{W}}) \leq \varsigma(|x|_{\mathcal{W}}) + \int_0^t \eta(|d(s)|) ds + c_3.$$

\triangle

Definition 8.⁷ The system (1) is called *ISS in the multistable sense* with respect to the set \mathcal{W} if it satisfies Assumption 1 and possesses the pAG, or the pGS and the LIM properties. \triangle

Definition 9.⁸ The system (1) is said to be *iISS in the multistable sense* with respect to the set \mathcal{W} if it satisfies Assumption 1 and possesses the UBEBS and 0-GATT properties. \triangle

Certainly, the aforementioned properties have been characterized in terms of extensions of the concepts of practical ISS- and iISS-Lyapunov functions for multistable systems.^{7,8}

2.5 | Practical ISS- and iISS-Leonov functions

In this subsection, we start by assuming that $f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ in (1) and (2), and we will denote its argument by $\tilde{x} \in \mathbb{R}^n$. Moreover, consider $k, q \in \mathbb{N}$, such that $n = k + q$. Then, let $\tilde{z} \in \mathbb{R}^k$ and $\tilde{\theta} \in \mathbb{R}^q$ be two subsets of the state vector \tilde{x} , i.e., $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$.

Assumption 2. The vector field $f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ in (2) is 2π -periodic with respect to $\tilde{\theta}$, i.e., for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$

$$f(\tilde{x}, 0) = f(\tilde{x} + \xi_j, 0),$$

where $\xi_j = [0_k^\top \ 2\pi j^\top]^\top \in \mathbb{R}^n$, with 0_k being a zero vector of dimension k and $j \in \mathbb{Z}^q$. \triangle

Accordingly, systems that satisfy Assumption 2 are called state periodic systems. Under this assumption, we can define a manifold $M = \mathbb{R}^k \times \mathbb{S}^q$, where \mathbb{S} is the unit sphere, and a coordinate projection $\mathcal{P}: \mathbb{R}^n \rightarrow M$, that is just a modulus of the $\tilde{\theta}$ coordinates over 2π . So, for any $\tilde{x} \in \mathbb{R}^n$ we have $\mathcal{P}(\tilde{x}) = x$, where $x = (z, \theta) \in M$, with $z \in \mathbb{R}^k$ and $\theta \in \mathbb{S}^q$. Hence, the system (2) can be embedded in M , i.e., its solutions $\tilde{X}(t, \tilde{x}_0; 0) \in \mathbb{R}^n$, with $\tilde{x}_0 \in \mathbb{R}^n$, can be projected to the trajectories $X(t, x_0; 0) \in M$, with $x_0 \in M$, by applying $X(t, x_0; 0) = \mathcal{P}(\tilde{X}(t, \tilde{x}_0; 0))$, such that both solutions are defined on the same time interval. Similarly, the set $\tilde{\mathcal{W}} \subset \mathbb{R}^n$, containing all α - and ω -limit sets of the system (2), can be projected to $\mathcal{W} \subset M$ by using the periodicity of the last q variables. Hence, \mathcal{W} can be compact in M but the set $\tilde{\mathcal{W}}$ becomes unbounded in \mathbb{R}^n due to the periodicity of (2). Besides, $|\tilde{x}|_{\tilde{\mathcal{W}}} = \inf_{y \in \tilde{\mathcal{W}}} |\tilde{x} - y|$ represents a distance from a point $\tilde{x} \in \mathbb{R}^n$ to the set $\tilde{\mathcal{W}}$, where $|\tilde{x}| = |\tilde{x}|_{\{0\}}$ is the usual Euclidean norm for a vector $\tilde{x} \in \mathbb{R}^n$. Due to the periodicity of (2), we have $|x|_{\mathcal{W}} = |\tilde{x}|_{\tilde{\mathcal{W}}}$.

Example 1. The following systems are periodic with respect to the state variable $x_1 \in \mathbb{R}$, and have been extensively studied in the literature:

- Forced nonlinear pendulum:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -b_1 \sin(x_1) - \tau x_2 + c_1 + u + d, \end{aligned} \quad (3)$$

where $x = [x_1, x_2]^\top \in \mathbb{R}^2$ is the state, $b_1, \tau > 0$ are parameters, $c_1 \in \mathbb{R}$ is a constant bias force, $d \in \mathcal{L}_\infty$ is a perturbation, and $u \in \mathcal{L}_\infty$ is a control input.

- PLL system:

$$\begin{aligned} \dot{x}_1 &= -k_1 \sin(x_1) + x_2 + d_1, \\ \dot{x}_2 &= -k_2 \sin(x_1) + c + u + d_2, \end{aligned} \quad (4)$$

where $x = [x_1, x_2]^\top \in \mathbb{R}^2$ is the state, $k_1, k_2 > 0$ are parameters, $c \in \mathbb{R}$ is a constant bias force, $d_1, d_2 \in \mathcal{L}_\infty$ are perturbations, and $u \in \mathcal{L}_\infty$ is a control input. \triangle

Note that Assumption 2 represents a strong restriction for the global analysis of ISS properties of multistable systems via Lyapunov function-based approaches: since the set $\tilde{\mathcal{W}}$ is unbounded, Lyapunov functions must be periodic in the variables $\tilde{\theta}$, and the analyses on the manifold M cannot be straightforwardly extended to \mathbb{R}^n . The complexity of analyzing state periodic systems with Lyapunov function-based approaches can be observed in the works.^{9,12,14}

Alternatively, Efimov et al.¹⁵ and Mendoza-Avila et al.¹⁶ have recently introduced the concepts of practical ISS- and iISS-Leonov functions, respectively, as fitted characterizations of the corresponding properties for state periodic systems. Let us introduce a set $\mathcal{W}' \subseteq \mathcal{W}$ (then $\tilde{\mathcal{W}}'$ and $\tilde{\mathcal{W}}$ are their projections in \mathbb{R}^n , respectively).

Definition 10.¹⁷ A \mathcal{C}^1 function $V: \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a practical ISS-Leonov function for the system (1) if there exist functions $\vartheta_1, \vartheta_2, \sigma_1, \gamma, \lambda \in \mathcal{K}_\infty$, and scalars $g_1, \ell_1, r \geq 0$, such that for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$:

$$\vartheta_1(|\tilde{x}|_{\tilde{\mathcal{W}}'}) - \sigma_1(|\tilde{\theta}|) - g_1 \leq V(\tilde{x}) \leq \vartheta_2(|\tilde{x}|_{\tilde{\mathcal{W}}'} + \ell_1), \quad (5)$$

and the following dissipation inequality

$$\frac{\partial V(\tilde{x})}{\partial \tilde{x}} f(\tilde{x}, d) \leq -\lambda(V(\tilde{x})) + \gamma(|d|) + r, \quad (6)$$

holds for all $\tilde{x} \in \{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) \geq 0\}$ and all $d \in \mathbb{R}^m$. If the latter inequality holds for $r = 0$, then V is said to be an ISS-Leonov function. Moreover, if inequality (6) holds with $r = 0$ and $\lambda: \mathbb{R} \rightarrow \mathbb{R}_+$ being a non-negative function, then V is said to be an iISS-Leonov function. \triangle

The concept of practical ISS-Leonov functions was originally introduced by Efimov et al.,¹⁵ with $g_1 = 0$, $\mathcal{W}' = \mathcal{W}$, and the dissipation inequality (6) holding for all $\tilde{x} \in \mathbb{R}^n$, with $\lambda \in \mathcal{K}_\infty$ only for nonnegative arguments. So, Definition 10 enlarges the family of practical ISS-Leonov functions. On the other hand, the concept of iISS-Leonov functions appears for the first time in Mendoza-Avila et al.¹⁶, with $\mathcal{W}' = \mathcal{W}$.

Noticeably, ISS- and iISS-Lyapunov functions satisfy Definition 10 (for $\sigma_1(\cdot) = 0$ and $g_1 = 0$). Hence, constructing ISS- and iISS-Leonov functions would be at most as difficult as constructing ISS- and iISS-Lyapunov functions. In addition, ISS- and iISS-Leonov functions are sign-indefinite, and their derivatives are required to be non-positive only in the region of the state space where the respective function is non-negative. For these reasons, we say that ISS and iISS-Leonov functions provide relaxed characterizations of the ISS and iISS properties for systems with state periodic dynamics.

The following results establish a relationship between practical ISS- or iISS-Leonov functions described in Definition 10 and the corresponding ISS-related notions for state periodic systems.

Lemma 1. ^{15,17} *Let Assumptions 1 and 2 be satisfied. Then, the existence of a practical ISS-Leonov function as in Definition 10 for the system (1) implies the pGS and pAG properties with respect to \mathcal{W} .* \triangle

Lemma 2. ^{16,17} *Let Assumptions 1 and 2 be satisfied. Then, the existence of an iISS-Leonov function as in Definition 10 for the system (1) implies 0-GATT and UBEBS properties with respect to \mathcal{W} .* \triangle

Thus, by Lemma 1 and Definition 8, we can say that the existence of a practical ISS-Leonov function is a sufficient condition to guarantee the ISS property for the system (1). Whereas, by Lemma 2 and Definition 9 we can say that the existence of an iISS-Leonov function is a sufficient condition to guarantee the iISS property for the system (1).

Remark 1. From Proposition 5³⁵ and Assumption 1, we can say that the set \mathcal{W} is the maximal invariant set for the system (2) in M (i.e., it is invariant and contains all other invariant sets of (2)). In general, it is impossible to determine all invariant solutions of a nonlinear system. Therefore, the set \mathcal{W} may be partially unknown during design or analysis. For this reason, Definition 10 considers only a set $\mathcal{W}' \subseteq \mathcal{W}$ simplifying the study of ISS properties in the system (1). Practical ISS- and iISS-Leonov functions described in Definition 10 merely allow us to conclude the pAG or UBEBS properties for (1) with respect to \mathcal{W}' . To validate the equivalence of these properties with the ISS and iISS definitions in the multistable sense, we have to show the 0-GATT property for the system (1) with respect to \mathcal{W} . In References,^{15,17} these proofs are developed by leveraging that pAG or UBEBS with respect to \mathcal{W}' imply that the trajectories of the unperturbed system (2) are bounded in M , then by invariance arguments (see Lemma 4.1³⁶) the 0-GATT property is ensured for the system (2) with respect to its maximal invariant set \mathcal{W} , for which Assumption 1 is straightforwardly satisfied due to Proposition 5.³⁵ \triangle

For ISS-Lyapunov functions in the multistable sense, different ways to deal with Assumption 1 are discussed in Section IV⁷ through examples.

3 | MAIN RESULTS

Consider a system of the form

$$\begin{aligned}\dot{\tilde{\theta}} &= f_1(\tilde{x}, d), \\ \dot{\tilde{z}} &= f_2(\tilde{x}, d) + g(\tilde{x})u,\end{aligned}\tag{7}$$

where $\tilde{x} \in \mathbb{R}^n$ is the state vector, $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$ with $\tilde{z} \in \mathbb{R}^k$ and $\tilde{\theta} \in \mathbb{R}^q$ (thus $n = k + q$ with $k, q \in \mathbb{N}$), $u \in \mathcal{L}_\infty^p$ is an admissible control input, and $d \in \mathcal{L}_\infty^m$ is a vector of disturbance signals (perturbation, measurement noise, etc.). Furthermore, the maps $f_1 : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^q$, $f_2 : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^k$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^{k \times p}$ are supposed to be continuously differentiable with respect to their arguments.

Assumption 3. For $d \equiv 0$, the vector fields f_1, f_2 and g in (7) are 2π -periodic with respect to $\tilde{\theta}$. \triangle

Naturally, the systems presented in Example 1 belong to the family of systems described by (7) with Assumption 3.

Now, the problem is to determine conditions ensuring that the system (7) is ISS or iISS stabilizable with respect to a set $\mathcal{W}_{cl} \subset M$ (a set that contains all α - and ω -limit sets of the controlled unperturbed system (7) in M , with $d \equiv 0$ and some control law). Namely, the system (7) is said to be ISS or iISS stabilizable with respect to a set \mathcal{W}_{cl} if there exists a suitable continuous controller $u : \mathbb{R}^n \rightarrow \mathbb{R}^p$, periodic in $\tilde{\theta}$, that endows it with the respective property. To verify this kind of stabilizability for (7), we present the concepts of practical ISS- and iISS-control Leonov functions (practical ISS- or iISS-CLeFs, respectively).

3.1 | Practical ISS- and iISS-control Leonov functions

Recall the vector $\xi_j = [0_k^\top \ 2\pi j^\top]^\top \in \mathbb{R}^n$, with 0_k being a zero vector of dimension k and $j \in \mathbb{Z}^q$. Moreover, let \mathcal{W} be the maximal invariant set for the system (7) in M , with $d \equiv 0$ and $u \equiv 0$. Then, consider $\mathcal{W}' \subseteq \mathcal{W}$ as a set of known elements of the α - and ω -limit sets of (7) in M , which will be preserved by design for the controlled system, and assume that the origin $[0_k^\top \ 0_q^\top]^\top \in \mathcal{W}'$. Contrarily to Mendoza-Avila et al.,³³ we omit the assumption of decomposability (in the sense of Definition 3) of the set \mathcal{W}' , which is compact by construction but not necessarily decomposable (except if $\mathcal{W}' = \mathcal{W}$). Similarly as it was defined before, the sets $\tilde{\mathcal{W}}$ and $\tilde{\mathcal{W}}'$, in \mathbb{R}^n , include all atoms obtained by periodicity.

Definition 11. A \mathcal{C}^1 function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a practical ISS-CLeF for the system (7) if there exist functions $\vartheta, \bar{\vartheta}, \underline{\sigma}, \bar{\sigma}, \varpi, v \in \mathcal{K}_\infty$ and scalars $g, \bar{g}, \iota \geq 0$, such that for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$:

$$\vartheta(|\tilde{x}|_{\tilde{\mathcal{W}}'}) - \underline{\sigma}(|\tilde{\theta}|) - g \leq V(\tilde{x}) \leq \bar{\vartheta}(|\tilde{x}|_{\tilde{\mathcal{W}}'} + \bar{g}) - \bar{\sigma}(|\tilde{\theta}|), \quad (8)$$

and for all $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$ and all $d \in \mathbb{R}^m$:

$$\inf_{u \in \mathbb{R}^p} \{a(\tilde{x}, d) + b(\tilde{x})u\} \leq -\varpi(V(\tilde{x})) + v(|d|) + \iota, \quad (9)$$

where $a(\tilde{x}, d) = \frac{\partial V(\tilde{x})}{\partial \theta} f_1(\tilde{x}, d) + \frac{\partial V(\tilde{x})}{\partial \tilde{z}} f_2(\tilde{x}, d)$, $b(\tilde{x}) = \frac{\partial V(\tilde{x})}{\partial \tilde{z}} g(\tilde{x})$ being 2π -periodic in $\tilde{\theta}$, and $\mathcal{W}' \subseteq \{\mathcal{W}_i \subset \mathcal{W} : \tilde{\mathcal{W}}_i \cap (\Omega_1 \cup \Omega_2) \neq \emptyset\}$, where

$$\Omega_1 = \{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) < 0\} \quad \text{and} \quad \Omega_2 = \{\tilde{x} \in \mathbb{R}^n : |b(\tilde{x})| = 0\}. \quad (10)$$

If the inequality (9) holds for $\iota = 0$, then V is said to be an ISS-CLeF. Moreover, if the inequality (9) holds for $\iota = 0$ and a non-negative function $\varpi : \mathbb{R} \rightarrow \mathbb{R}_+$, then V is said to be an iISS-CLeF. \triangle

Note that the function $a(\tilde{x}, d)$ will usually be non-periodic. This property is inherited from the partial derivatives of V with respect to $\tilde{\theta}$ since practical ISS- and iISS-CLeFs use to be non-periodic. Likewise, the function $b(\tilde{x})$ is 2π -periodic in $\tilde{\theta}$ by construction. Note also that the inequality (9) is equivalent to the condition

$$b(\tilde{x}) \equiv 0 \implies a(\tilde{x}, d) \leq -\varpi(V(\tilde{x})) + v(|d|) + \iota, \quad (11)$$

for all $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$ and all $d \in \mathbb{R}^m$.

The proper definition of $\tilde{\mathcal{W}}'$ should be $\tilde{\mathcal{W}}' \subseteq \tilde{\mathcal{W}}_{cl} \cap \tilde{\mathcal{W}}$. However, $\tilde{\mathcal{W}}_{cl}$ is unknown until the design of the control law u . One solution to guarantee the property $\tilde{\mathcal{W}}_{cl} \cap \tilde{\mathcal{W}} \neq \emptyset$ is to make the controller zero at some known atoms $\tilde{\mathcal{W}}_i$ of the set $\tilde{\mathcal{W}}$, i.e., $\|u(\tilde{x})\|_\infty = 0$ for all $\tilde{x} \in \tilde{\mathcal{W}}_i$ with some $i \in \{1, 2, \dots\}$. Certainly, the main purpose of the control law u is to ensure that inequality (9) is satisfied for all $\tilde{x} \in \mathbb{R}^n \setminus (\Omega_1 \cup \Omega_2)$ (in Ω_2 inequality (9) is independent of u). Hence, the controller u can be nullified at any point of Ω_1 and Ω_2 . Thus, we can form \mathcal{W}' by taking those atoms \mathcal{W}_i of the decomposition of \mathcal{W} whose projection on \mathbb{R}^n intersects with the set $\Omega_1 \cup \Omega_2$, as introduced in Definition 11. In this way, the constraint $\|u(\tilde{x})\|_\infty = 0$ for all $\tilde{x} \in \tilde{\mathcal{W}}'$ can be established by design, and the condition $\tilde{\mathcal{W}}' \subseteq \tilde{\mathcal{W}}_{cl} \cap \tilde{\mathcal{W}} \neq \emptyset$ is guaranteed.

Remark 2. The definition of the set \mathcal{W}' given in Definition 11 suggests that we only need to know one atom $\mathcal{W}_i \subset M$ (e.g., an equilibrium point), such that $\tilde{\mathcal{W}}_i \subset (\Omega_1 \cup \Omega_2)$, to construct the set $\mathcal{W}' = \{\mathcal{W}_i\}$. \triangle

Noticeably, ISS- and iISS-control Lyapunov functions^{27,28} satisfy Definition 11. This fact together with the relaxed properties of being sign-indefinite and requiring the dissipation inequality (11) to only hold in the region of the state space where the respective function is non-negative makes ISS- and iISS-CLeFs a highly appealing solution for control synthesis of state periodic systems like (7). The next examples illustrate the construction of practical ISS-CLeFs.

Example 2. Consider the system (3), with $\tau = 0$. For $|c_1| < b_1$ and $d = u = 0$, the system (3) has equilibria at the points $[x_1^* + 2j\pi, 0]^\top$ and $[(1 + 2j)\pi - x_1^*, 0]^\top$ for all $j \in \mathbb{Z}$, where $\sin(x_1^*) = c_1 b_1^{-1}$. Moreover, define the coordinates $\zeta = [\zeta_1, \zeta_2]^\top \in \mathbb{R}^2$ as $\zeta_1 = x_1 - x_1^*$ and $\zeta_2 = x_2$. Then, the set $\tilde{\mathcal{W}} = \{[2j\pi, 0]^\top \cup [(1 + 2j)\pi - 2x_1^*, 0]^\top : j \in \mathbb{Z}\}$ contains all equilibria of the system (3) in coordinates ζ , and it is assumed to be decomposable in the sense of Definition 3. Now, consider the ISS-CLeF candidate

$$V(\zeta) = \frac{1}{2}\zeta_2^2 - \frac{1}{2}\Delta\zeta_1^2 + b_1(\cos(x_1^*) - \cos(\zeta_1 + x_1^*)) - c_1\zeta_1, \quad (12)$$

where $\Delta > 0$ is a constant parameter to be determined.

Define the function $\varrho(\zeta_1) = b_1(\cos(x_1^*) - \cos(\zeta_1 + x_1^*)) - c_1\zeta_1 - \frac{1}{2}\kappa b_1\zeta_1^2$. By analyzing its derivatives of first and second order, we can conclude that it has a global maximum at $\zeta_1 = 0$ if $\kappa > 1$, which implies that $\varrho(\zeta_1) < 0$ for all $\zeta_1 \in \mathbb{R} \setminus \{0\}$ and $\varrho(0) = 0$. Consequently, we obtain $b_1(\cos(x_1^*) - \cos(\zeta_1 + x_1^*)) - c_1\zeta_1 \leq \frac{1}{2}\kappa b_1\zeta_1^2$ with a $\kappa > 1$. Similarly, we can deduce that $b_1(\cos(x_1^*) - \cos(\zeta_1 + x_1^*)) - c_1\zeta_1 \geq -\frac{1}{2}\kappa b_1\zeta_1^2$ if $\kappa > 1$ as well. Therefore, we have

$$\frac{1}{2}\zeta_2^2 - \frac{1}{2}(\Delta + \kappa b_1)\zeta_1^2 \leq V(\zeta) \leq \frac{1}{2}\zeta_2^2 - \frac{1}{2}(\Delta - \kappa b_1)\zeta_1^2,$$

for all $\zeta \in \mathbb{R}^2$. Thus, the function (12) satisfies the condition (8) if $\Delta > \kappa b_1$. In addition, under the change of coordinates $x \mapsto \zeta$, the derivative of the function (12) along the solutions of the system (3) is given by

$$\dot{V}(\zeta) = -\Delta\zeta_1\zeta_2 + \zeta_2d + \zeta_2u,$$

and we can define $a(\zeta, d) = -\Delta\zeta_1\zeta_2 + \zeta_2d$ and $b(\zeta) = \zeta_2$. Note that the restriction (11) is straightforwardly satisfied with $\iota = 0$ and any functions $\varpi, v \in \mathcal{K}_\infty$, since $b(\zeta) = 0$ implies either $a(\zeta, d) = 0$ and $V(\zeta) \leq 0$. Therefore, we can say that (12) is an ISS-CLeF for the system (3). Besides, we can easily see that $\tilde{\mathcal{W}} \subset \{\zeta \in \mathbb{R}^2 : b(\zeta) = 0\}$. Consequently, we have $\tilde{\mathcal{W}}' = \tilde{\mathcal{W}}$ for this case. Of course, $\tilde{\mathcal{W}}' = \{[2j\pi, 0]^\top : j \in \mathbb{Z}\} \subset \tilde{\mathcal{W}}$ is also a suitable alternative. \triangle

The term ‘‘practical’’ in Definition 11 allows us to consider the existence of invariant solutions of the system (7) belonging to the region with positive values of ISS-CLeFs, i.e., the case $\{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) > 0\} \cap \tilde{\mathcal{W}} \neq \emptyset$.

Example 3. Consider a function $\hat{V}(\zeta) = V(\zeta) + \Delta_1$, where $V(\zeta)$ is given in (12), and Δ_1 is a positive constant satisfying $\Delta_1 > |V([\pi - 2x_1^*, 0])|$. Note that the equilibrium points $[\pm\pi - 2x_1^*, 0]^\top$ lie in the region where $\hat{V}(\zeta) > 0$. Clearly, $\dot{\hat{V}}(\zeta) = \dot{V}(\zeta)$ along the trajectories of (3). However, in this case, the condition $b(\zeta) = 0$ implies $a(\zeta, d) = 0$ and $\hat{V}(\zeta) \leq \Delta_1$. Therefore, for any functions $\varpi, v \in \mathcal{K}_\infty$, a constant $\iota \geq \varpi(\Delta_1)$ is needed to fulfill the restriction (11). Thus, $\hat{V}(\zeta)$ is a practical ISS-CLeF for the system (3). Once again, we can establish $\tilde{\mathcal{W}}' = \tilde{\mathcal{W}}$ because $\tilde{\mathcal{W}} \subset \{\zeta \in \mathbb{R}^2 : b(\zeta) = 0\}$. Of course, this illustration seems to be artificial for the system (3), but practical ISS-CLeF may be more natural for complex systems. \triangle

3.2 | Controller design

Now, we aim to show that practical ISS- and iISS-CLeFs in Definition 11 provide sufficient information to design a continuous and 2π -periodic in $\tilde{\theta}$ control law u that renders the system (7) ISS or iISS, respectively.

First, we have to deal with the non-periodicity of the function $a(\tilde{x}, d)$ and its dependence on the perturbation term d . To this end, consider an appropriate \mathcal{K}_∞ -function $\bar{v}(s) \geq v(s)$ for all $s \geq 0$, and a constant $\bar{v} \geq \iota$, with $v(s)$ and ι given in Definition 11, such that the expression $a(\tilde{x}, d) - \bar{v}(|d|) - \bar{v}$ is negative for sufficiently large values of $|d|$. Hence, we construct the function

$$\omega(\tilde{x}) = \sup_{d \in \mathbb{R}^m} \{a(\tilde{x}, d) - \bar{v}(|d|) - \bar{v}\}, \quad (13)$$

which is well-defined, i.e., at any point \tilde{x} such a supremum exists, and independent of the perturbation signals d . Hence, the function (13) allows to deal with the unknown signal d by canceling its effects in the function $a(\tilde{x}, d)$. This strategy was introduced by Liberzon et al.²⁷ for control design based on ISS- and iISS- control Lyapunov functions. Next, by adding the term $-\bar{v}(|d|) - \bar{v}$ on both sides of the inequality (9), it can be rewritten as

$$\inf_{u \in \mathbb{R}^p} \{\omega(\tilde{x}) + b(\tilde{x})u\} \leq -\varpi(V(\tilde{x})), \quad (14)$$

for all $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$.

Clearly, the function $\omega(\tilde{x})$ is expected to be non-periodic with respect to $\tilde{\theta}$ since the function $a(\tilde{x}, d)$ has usually the same property. From inequality (8), we can see that $\bar{\sigma}(|\tilde{\theta}|) \leq \bar{\vartheta}(|\tilde{x}|_{\tilde{\mathcal{W}}'} + \bar{g})$, wherever $V(\tilde{x}) \geq 0$. Hence, we define the function

$$\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z}) = \sup_{\tilde{\theta} \in \Theta} \{\omega(\tilde{x}) + \frac{1}{2}\varpi(V(\tilde{x}))\}, \quad (15)$$

where $\Theta = \{\tilde{\theta} \in \mathbb{R}^q : \bar{\sigma}(|\tilde{\theta}|) \leq \bar{\vartheta}(|\tilde{x}|_{\tilde{\mathcal{W}}'} + \bar{g})\}$. Note that the function $\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z})$ is 2π -periodic with respect to $\tilde{\theta}$ by construction since $|\tilde{x}|_{\tilde{\mathcal{W}}'} = |\tilde{x} - \xi_j|_{\tilde{\mathcal{W}}'}$, and this term dominates the non-periodic expressions of $\tilde{\theta}$ in $\omega(\tilde{x})$ and $V(\tilde{x})$ for all $\tilde{\theta} \in \Theta$. Moreover,

by adding the term $\frac{1}{2}\varpi(V(\tilde{x}))$ on both sides of the inequality(14), we can rewrite it as

$$\inf_{u \in \mathbb{R}^p} \{ \phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z}) + b(\tilde{x})u \} \leq -\frac{1}{2}\varpi(V(\tilde{x})), \quad (16)$$

for all $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$.

Let us introduce a variant of the *small control property* (SCP) in the framework of Leonov functions (SCP-LeF).

Definition 12. We say that a practical ISS- or iISS-CLeF $V(\tilde{x})$ satisfies the SCP-LeF if for some $\varepsilon > 0$ there is a $\delta > 0$, such that

$$\sup_{0 < |\tilde{x}|_{\Gamma} \leq \delta} \frac{\phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z})}{|b(\tilde{x})|^2} < \varepsilon, \quad (17)$$

where $\Gamma = \{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) = 0; |b(\tilde{x})| = 0\}$.

Note that for all $|\tilde{x}|_{\Gamma} \leq \delta$, there exists some ε' such that $\varepsilon|b(\tilde{x})| \leq \varepsilon'$. So, taking $u = -\varepsilon b^T(\tilde{x})$, the condition (17) can be understood as the existence of a $\delta > 0$ for any $\varepsilon' > 0$, such that $\phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z}) + b(\tilde{x})u < 0$, for $|u| \leq \varepsilon'$ and all $|\tilde{x}|_{\Gamma} \leq \delta$. Therefore, we can say that the presented SCP-LeF is equivalent to the conventional formulation of a SCP property.^{27,30,32}

Finally, we have that the function $\phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z})$ is 2π -periodic in $\tilde{\theta}$ and independent of the perturbation terms. Thus, we are able to proceed with the design of a controller for ISS- or iISS- stabilization of the system (7). To this end, let us recall the control universal formula³⁰ given by

$$K(\phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z}), |b(\tilde{x})|) = \begin{cases} \frac{\phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z}) + \sqrt{\phi^2(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z}) + |b(\tilde{x})|^4}}{|b(\tilde{x})|^2} & \text{if } |b(\tilde{x})| \neq 0, \\ 1 & \text{if } |b(\tilde{x})| = 0. \end{cases} \quad (18)$$

We need to verify two main properties of the function K in (18): 2π -periodicity with respect to $\tilde{\theta}$, and continuity for all $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$. We can easily see that the first one is guaranteed by construction, since $b(\tilde{x})$ and $\phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z})$ have this property. Besides, the function (18) is continuous wherever $|b(\tilde{x})| \neq 0$, because the functions $\phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z})$ and $b(\tilde{x})$ are continuous everywhere. So, we only have to check the continuity of (18) when $|b(\tilde{x})|$ approaches zero. Then, recall the property $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for any $a, b \geq 0$. Accordingly, we have

$$K(\phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z}), |b(\tilde{x})|) \leq \frac{\phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z}) + |\phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z})|}{|b(\tilde{x})|^2} + 1. \quad (19)$$

Note that for $b(\tilde{x}) \equiv 0$ inequality (16) becomes

$$\phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z}) \leq -\frac{1}{2}\varpi(V(\tilde{x})), \quad (20)$$

for all $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$. Therefore, we have that $b(\tilde{x}) \equiv 0$ implies $\phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z}) < 0$, wherever $V(\tilde{x}) > 0$. Thus, the first term in the right-hand side of (19) vanishes as $|b(\tilde{x})|$ approaches zero in all points of the set $\{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) > 0\}$, i.e., the function (18) continuously approaches one. Finally, it remains to check the continuity of the function (18) at the points where $V(\tilde{x}) = 0$, specifically for $\tilde{x} \in \Gamma$. Note that around the points in Γ , i.e., for any $|\tilde{x}|_{\Gamma} \leq \delta$, with $\delta > 0$, the inequality (19) becomes

$$K(\phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z}), |b(\tilde{x})|) \leq \frac{\max\{0, 2\phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z})\}}{|b(\tilde{x})|^2} + 1.$$

Hence, we use the SCP-LeF (17) to conclude that

$$K(\phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z}), |b(\tilde{x})|) \leq 2\varepsilon + 1.$$

for all $|\tilde{x}|_{\Gamma} \leq \delta$. If such a $\delta > 0$ exists for any $\varepsilon > 0$ arbitrarily small, then we conclude the continuity of (18) at all points of Γ . Thus, the function K in (18) is continuous in the set $\mathbb{R}^n \setminus \Omega_1$ provided that the SCP-LeF in Definition 12 is satisfied. Moreover, we can claim that there exists a function $\bar{K}(\tilde{x})$, continuous for all $\tilde{x} \in \mathbb{R}^n$ and 2π -periodic in $\tilde{\theta}$, such that

$$K(\phi(|\tilde{x}|_{\tilde{\mathcal{W}}}, \tilde{z}), |b(\tilde{x})|) \leq \bar{K}(\tilde{x}) \quad (21)$$

for all $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$. Since we aim to have a feedback control law continuous in \mathbb{R}^n , the function K in (18) is not suitable for its construction. Therefore, we shall consider the function \bar{K} .

Remark 3. If the inequality $\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z}) < 0$ is satisfied, wherever $b(\tilde{x}) \equiv 0$ for all $\tilde{x} \in \mathbb{R}^n \setminus \Gamma$, and the SCP-LeF (17) is satisfied for all $\tilde{x} \in \Gamma$, then the function (18) is continuous in \mathbb{R}^n . In this case, the control universal formula (18) can be directly applied like in the control Lyapunov function framework. \triangle

The last step in the design of our controller consists of ensuring the property $\tilde{\mathcal{W}}_{cl} \cap \tilde{\mathcal{W}} \neq \emptyset$. For this purpose, we introduce the following function

$$\psi(\tilde{x}) = \begin{cases} 1 & \text{if } \tilde{\mathcal{W}} \cap \Omega_2 \neq \emptyset, \\ \min \left\{ 1, \frac{|\tilde{x}|_{\tilde{\mathcal{W}}'}}{\epsilon} \right\} & \text{if } \tilde{\mathcal{W}} \cap \Omega_2 = \emptyset, \end{cases} \quad (22)$$

with $0 < \epsilon \leq \epsilon^*$, where $\epsilon^* > 0$ is the distance between the sets $\{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) = 0\}$, and $\tilde{\mathcal{W}}'$, i.e., $\epsilon^* = \min_{x \in \{x \in \mathbb{R}^n : V(x) = 0\}} |x|_{\tilde{\mathcal{W}}'}$. Note that we have $\tilde{\mathcal{W}}' \subseteq \tilde{\mathcal{W}} \cap \Omega_2$ if $\tilde{\mathcal{W}} \cap \Omega_2 \neq \emptyset$, or $\tilde{\mathcal{W}}' \subseteq \tilde{\mathcal{W}} \cap \Omega_1$ if $\tilde{\mathcal{W}} \cap \Omega_2 = \emptyset$.

Finally, we propose a feedback controller given by

$$u(\tilde{x}) = -\psi(\tilde{x})\bar{K}(\tilde{x})b^\top(\tilde{x}), \quad (23)$$

which is continuous everywhere and 2π -periodic in $\tilde{\theta}$. In addition, $b(\tilde{x}) \equiv 0$ for all $\tilde{x} \in \tilde{\mathcal{W}}'$ if $\tilde{\mathcal{W}} \cap \Omega_2 \neq \emptyset$, or $\psi(\tilde{x}) = 0$ for all $\tilde{x} \in \tilde{\mathcal{W}}'$ if $\tilde{\mathcal{W}} \cap \Omega_2 = \emptyset$. Consequently, $u(\tilde{x}) \equiv 0$ for all $\tilde{x} \in \tilde{\mathcal{W}}'$, ensuring that $\tilde{\mathcal{W}}' \subseteq \tilde{\mathcal{W}}_{cl} \cap \tilde{\mathcal{W}} \neq \emptyset$.

Theorem 1. *Let the system (7) satisfy Assumption 3, and the set \mathcal{W}_{cl} be decomposable in the sense of Definition 3. If there exists a practical ISS-CLeF, or an iISS-CLeF, for the system (7) satisfying the SCP-LeF (17), then the feedback control law (23) is continuous on \mathbb{R}^n and endows the system (7) with the ISS or iISS property with respect to \mathcal{W}_{cl} , respectively. \triangle*

Proof. Continuity of the control law (23) is ensured by construction, under the fulfillment of the SCP-LeF (17). Therefore, we only have to focus on showing that the closed-loop system (7),(23) given by

$$\begin{aligned} \dot{\tilde{\theta}} &= f_1(\tilde{x}, d), & \dot{\tilde{z}} &= f_2(\tilde{x}, d) - \psi(\tilde{x})\bar{K}(\tilde{x})g(\tilde{x})b^\top(\tilde{x}), \end{aligned} \quad (24)$$

has the ISS or iISS property with respect to \mathcal{W}_{cl} . To this end, consider a practical ISS-CLeF V of the system (7) as a practical ISS-Leonov function candidate for the system (24). The derivative of V along the trajectories of (24) is given by

$$\dot{V}(\tilde{x}) = a(\tilde{x}, d) - \psi(\tilde{x})\bar{K}(\tilde{x})b(\tilde{x})b^\top(\tilde{x}),$$

with $a(\tilde{x}, d)$ and $b(\tilde{x})$ as in Definition 11. Note that $\bar{K}(\tilde{x})$ and $\psi(\tilde{x})$ in (21) and (22), respectively, are scalar functions. Now, by subtracting a \mathcal{K}_∞ -function $\bar{v}(|d|)$ and a constant $\bar{t} > 0$ in both sides of the latter expression, we obtain

$$\dot{V}(\tilde{x}) - \bar{v}(|d|) - \bar{t} \leq \omega(\tilde{x}) - \psi(\tilde{x})\bar{K}(\tilde{x})|b(\tilde{x})|^2,$$

where $\omega(\tilde{x})$ is defined in (13). Similarly, adding the term $\frac{1}{2}\varpi(V(\tilde{x}))$ to both sides of the above inequality, we obtain

$$\dot{V}(\tilde{x}) + \frac{1}{2}\varpi(V(\tilde{x})) - \bar{v}(|d|) - \bar{t} \leq \phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z}) - \psi(\tilde{x})\bar{K}(\tilde{x})|b(\tilde{x})|^2, \quad (25)$$

for all $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$, where $\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z})$ is introduced in (15). Now, consider the case $b(\tilde{x}) \neq 0$. Hence, we have from (18) that $\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z}) \leq K(\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z}), |b(\tilde{x})|)|b(\tilde{x})|^2$. Then, inequality (25) becomes

$$\dot{V}(\tilde{x}) + \frac{1}{2}\varpi(V(\tilde{x})) - \bar{v}(|d|) - \bar{t} \leq K(\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z}), |b(\tilde{x})|)|b(\tilde{x})|^2 - \psi(\tilde{x})\bar{K}(\tilde{x})|b(\tilde{x})|^2.$$

Note that $\psi(\tilde{x}) = 1$ for all $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$ since the case $\tilde{\mathcal{W}} \cap \Omega_2 = \emptyset$ implies $\tilde{\mathcal{W}}' \subseteq \Omega_1$, and $|\tilde{x}|_{\tilde{\mathcal{W}}'} \geq \epsilon$ for all $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$. This fact together with inequality (21) imply that the right-hand side of the latter expression is non-positive for all $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$. Thus, we can conclude that

$$\dot{V}(\tilde{x}) \leq -\frac{1}{2}\varpi(V(\tilde{x})) + \bar{v}(|d|) + \bar{t}, \quad (26)$$

for all $\tilde{x} \in \mathbb{R}^n \setminus (\Omega_1 \cup \Omega_2)$. On the other hand, for the case $b(\tilde{x}) \equiv 0$, we have from (20) that $\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z}) \leq 0$ for all $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$. Hence, (26) is straightforwardly derived from (25). Therefore, we can say that (26) holds for all $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$ and all $d \in \mathbb{R}^m$.

Accordingly, if V is a practical ISS-CLeF for (7) then ϖ is a \mathcal{K}_∞ -function, and we can say that V is a practical ISS-Leonov function for the closed-loop system (24). Consequently, by Lemma 1, we deduce that it has the ISS property with respect to \mathcal{W}_{cl} . Finally, if V is an iISS-CLeF, then ϖ will be a non-negative function and $\bar{v} = 0$. Hence, V becomes an iISS-Leonov function for the closed-loop system (24). Consequently, by Lemma 2, we conclude that it has the iISS property with respect to \mathcal{W}_{cl} . \square

From inequality (26), we can see that the parameters $\bar{v}(\cdot) \in \mathcal{K}_\infty$ and $\bar{v} \geq 0$ quantify the robustness of the closed-loop system (24), then their tuning adjusts the degree of disturbance compensation for the closed-loop system (24). Basically, small values of these parameters make stronger the controller's gain $\bar{K}(\cdot)$, which increases the robustness for the closed-loop system (24).

Remark 4. Usually state feedback controllers have to deal with noisy state measurements $\tilde{x}_m = \tilde{x} + d_m$, where $d_m \in \mathcal{L}_\infty^n$ represents a perturbation signal. Since the controller (23) is well-defined for any point $\tilde{x} \in \mathbb{R}^n$, its properties of continuity and 2π -periodicity in $\tilde{\theta}$ are preserved also for the translated coordinates $\tilde{x}_m \in \mathbb{R}^n$. Then, the closed-loop system (24) may be able to tolerate a sufficiently small and bounded measurement noise.

The implementation of the controller (23) requires the online solution of the supremum functions (13) and (15), which could demand an additional amount of computational resources. For some particular cases, it is possible to obtain an analytical solution for these functions. Another alternative approach is to find 2π -periodic in $\tilde{\theta}$ functions $\bar{\omega}(\tilde{x})$ and $\bar{\phi}(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z})$, such that $|\bar{\omega}(\tilde{x})| \geq |\omega(\tilde{x})|$ and $|\bar{\phi}(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z})| \geq |\phi(|\tilde{x}|_{\tilde{\mathcal{W}}'}, \tilde{z})|$ for all $\tilde{x} \in \mathbb{R}^n \setminus \Omega_1$. Of course, the continuity of the formula (18) together with the fulfillment of the SCP-LeF must be verified considering these functions. To illustrate the latter, let us continue developing the examples given in the first part of this section.

Example 4. Let us recall the forced nonlinear pendulum system (3) and the ISS-CLeF (12) obtained in Example 2. First, we construct a function $\omega(\zeta)$ as in (13). For this, consider a function $\bar{v}(|d|) = 0.5d^2$, a constant $\bar{v} = 0$ and the function $a(\zeta, d)$ from Example 2, such that

$$a(\zeta, d) - \bar{v}(|d|) = -\Delta\zeta_1\zeta_2 + \zeta_2d - 0.5d^2 = -\Delta\zeta_1\zeta_2 + 0.5\zeta_2^2 - 0.5(\zeta_2 - d)^2.$$

Then, we obtain

$$\omega(\zeta) = \sup_{d \in \mathbb{R}} \{a(\zeta, d) - \bar{v}(|d|)\} = -\Delta\zeta_1\zeta_2 + 0.5\zeta_2^2.$$

Now, similarly to (15), we define

$$\begin{aligned} \phi(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2) &= \sup_{\zeta_1 \in \Theta} \{\omega(\zeta) + 0.5\lambda V(\zeta)\} \\ &\leq \sup_{\zeta_1 \in \Theta} \{0.5(1 + 0.5\lambda)\zeta_2^2 - \Delta\zeta_1\zeta_2 - 0.25\lambda(\Delta - \kappa b_1)\zeta_1^2\} \\ &= \sup_{\zeta_1 \in \Theta} \left\{ 0.5(1 + 0.5\lambda)\zeta_2^2 + \frac{\Delta^2}{\lambda(\Delta - \kappa b_1)}\zeta_2^2 - \left(\frac{\Delta}{\sqrt{\lambda(\Delta - \kappa b_1)}}\zeta_2 + 0.5\sqrt{\lambda(\Delta - \kappa b_1)}\zeta_1 \right)^2 \right\} \\ &= 0.5(1 + 0.5\lambda)\zeta_2^2 + \frac{\Delta^2}{\lambda(\Delta - \kappa b_1)}\zeta_2^2 = \bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2), \end{aligned}$$

where $\Theta = \{\zeta_1 \in \mathbb{R} : (\Delta - \kappa b_1)\zeta_1^2 \leq \zeta_2^2\}$. Then, by substituting $b(\zeta_2) = \zeta_2$ and $\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2)$ in the formula (18) we obtain

$$K(\phi(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2), |b(\zeta_2)|) \leq \frac{\lambda(2 + \lambda)(\Delta - \kappa b_1) + 4\Delta^2}{4\lambda(\Delta - \kappa b_1)} + \sqrt{\left(\frac{\lambda(2 + \lambda)(\Delta - \kappa b_1) + 4\Delta^2}{4\lambda(\Delta - \kappa b_1)} \right)^2 + 1} = \bar{K}.$$

In this case, the resulting function \bar{K} is constant. Therefore, its continuity is immediately verified. Moreover, as established in Example 2, we have the case $\tilde{\mathcal{W}}' \cap \Omega_2 \neq \emptyset$ and consequently $\psi(\zeta) = 1$. Hence, the controller (23) takes the form

$$u(\zeta) = -\bar{K}\zeta_2. \quad (27)$$

We can say that the controller (27) adds damping to the system (3), increasing its robustness. Formally, we can see that the system (3),(27) satisfies Assumption 3, and the set $\tilde{\mathcal{W}}_{cl} = \tilde{\mathcal{W}}$ is decomposable in the sense of Definition 3. Thus, by Theorem 1, we can conclude that the closed-loop system (3),(27) has the ISS property with respect to the set \mathcal{W}_{cl} .

Note that considering the practical ISS-CLeF $\hat{V}(\zeta) = V(\zeta) + \Delta_1$ (as in Example 3), only demands to make the constant $\bar{v} = \Delta_1$, and the rest of the procedure for the design of the controller (27) remains unchanged. \triangle

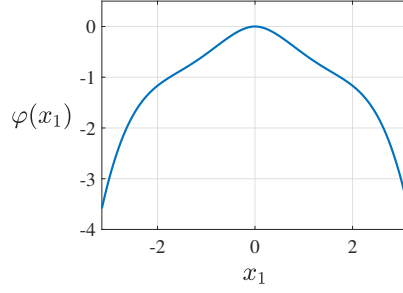


FIGURE 1 Plot of the function $\varphi(x_1) = -x_1^2 + |x_1|(1 - \cos(x_1))$ for $-\pi \leq x_1 \leq \pi$.

Further examples illustrating particular aspect of the proposed methodology for control design, like the usefulness of the function ψ in (22) and the SCP-LeF in Definition 12, are provided in Appendix A.

4 | FURTHER ANALYSIS ON THE SET \mathcal{W}_{CL}

As mentioned in Remark 1, the existence of practical ISS- and iISS-Leonov functions for the closed-loop system (1),(23) only ensures that it possesses the pAG or UBES properties with respect to \mathcal{W}_{cl} , but the 0-GATT property relies on Assumption 1. Therefore, we can say that the controller (23) is able to keep a desired behavior of the system (1) with respect to perturbations, but it cannot prescribe the set \mathcal{W}_{cl} . Indeed, \mathcal{W}_{cl} can be partially unknown, and we can only ensure that it is the maximal invariant set of the closed-loop system (1),(23). Let us illustrate this discussion through the following example.

Example 5. Consider the system

$$\dot{x}_1 = 1 - \cos(x_1) + x_2 + d, \quad \dot{x}_2 = -cx_2 + u + d, \quad (28)$$

where $[x_1, x_2]^\top \in \mathbb{R}^2$ is the vector state, $c > 0$ is a constant parameter, $d \in \mathcal{L}_\infty$ is a perturbation, and $u \in \mathcal{L}_\infty$ is a control input. Note that the system (28), with $u = d = 0$, has an equilibrium point at the origin $x = [0, 0]$ in the manifold $M = \mathbb{S} \times \mathbb{R}$, where \mathbb{S} is the unit sphere. However, as mentioned in Section IV.C,⁷ the set $\{[0, 0]\}$ does not admit a decomposition without cycles. Indeed, it is shown in Section IV.C⁷ that the finest choice of a decomposable invariant set for the system (28) is $\mathcal{W}_{u=0} = \mathbb{S} \times \{0\}$. Moreover, $V(x_1, x_2) = x_2^2$ is an ISS-Lyapunov function for the system (28), with $u = 0$, therefore it has the ISS property with respect to the set $\mathcal{W}_{u=0}$. Note that the function $V(x_1, x_2) = x_2^2 - x_1^2$ can be used as an ISS-CLeF for the systems (28) (to check the condition (11) use Figure 1), and this will lead us to the trivial conclusion that the system (28) is ISS stabilizable with respect to $\mathcal{W}_{u=0}$ with some controller like $u = Kx_2$; $K > 0$.

Besides, by setting $u = -c$, we can see that the system (28) has two equilibrium points at $\mathcal{W}_{u=-c} = \{[\frac{\pi}{2}, -1], [-\frac{\pi}{2}, -1]\}$ in the manifold M . The ISS-Lyapunov function $V(x_1, x_2) = (x_2 + 1)^2$ allows us to show that the system (28), with $u = -c$, has the ISS property with respect to the set $\mathcal{W}_{u=-c}$, which is decomposable in the sense of Definition 3 since the point $[-\frac{\pi}{2}, -1]$ attracts almost all trajectories of the system (28) in M , with $d = 0$ and $u = -c$ (this can be confirmed by applying the Lyapunov function $V_0(\hat{x}_1) = 1 + \sin(\hat{x}_1 - \frac{\pi}{2})$ to the unperturbed zero dynamics $\dot{\hat{x}}_1 = -\cos(\hat{x}_1 - \frac{\pi}{2})$, where $\hat{x}_1 = x_1 + \frac{\pi}{2}$). Note that the function $V(x_1, x_2) = (x_2 + 1)^2$ does not qualify as an ISS-CLeF for the system (28) since it was selected on the basis that $u = -c$ and the set $\mathcal{W}_{u=-c}$ is known. In contrast, practical ISS- and iISS-CLeFs should be constructed solely with information of the uncontrolled system. \triangle

From the previous example, we can see that the selection of a practical ISS- or iISS-CLeF for the system (7) determines the resultant controller (23), and consequently, the set $\mathcal{W}_{cl} \subset M$. Therefore, one could wonder if it is possible to impose certain characteristics to the set \mathcal{W}_{cl} from the selection of practical ISS- and iISS-CLeFs for the system (7). In general, we cannot specify the whole set \mathcal{W}_{cl} , but only ensure that $\mathcal{W}' \subseteq \mathcal{W}_{cl}$. Therefore, the set \mathcal{W}_{cl} cannot be forced to be decomposable in the sense of Definition 3 by design. However, it is possible to make the resultant set \mathcal{W}_{cl} compact by design. Note that in the system (1) we are considering M as a manifold without boundary. Hence, compactness is not an inherent property of the set \mathcal{W}_{cl} . Actually, the set $\mathcal{W}_{u=0} = \mathbb{S} \times \{0\}$ in Example 5 is not compact (it is not closed because $M = \mathbb{S} \times \mathbb{R}$ is a manifold without boundary).

To address this issue, first, let us recall the concept of Leonov functions introduced by Efimov and Schiffer^{22,37} to analyze boundedness of solutions in \mathbb{R}^n of state periodic systems. For the system (2) satisfying Assumption 2, we define the sets: $\mathcal{U} = \cup_{r \in \mathbb{Z}_+} \mathcal{U}_r$, where $\mathcal{U}_r = \{\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n : |\tilde{\theta}|_\infty = 2r\pi, f(\tilde{x}, 0) = 0\}$, and $\mathcal{V} \subset \mathbb{R}^n$, such that for some $\underline{c} \in (0, 2\pi)$ and $\bar{c} \in (\pi, 2\pi)$, with $\underline{c} < \bar{c}$, we have $\{\tilde{x} \in \mathbb{R}^n : \underline{c} \leq |\tilde{\theta}|_\infty \leq \bar{c}\} \subseteq \mathcal{V}$.

Definition 13.³⁷ A C^1 function $U : \mathbb{R}^n \rightarrow \mathbb{R}$ is a strong Leonov function for the system (2) if there exist a constant $g \geq 0$, functions $\alpha \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$ and a continuous function $\lambda : \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\lambda(0) = 0$, and $\lambda(s)s > 0$ for all $s \neq 0$, such that

$$\begin{aligned} \alpha(|\tilde{z}|) - \sigma(|\tilde{\theta}|) - g &\leq U(\tilde{x}) \quad \forall \tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n, \\ \inf_{\tilde{x} \in \mathcal{V}} U(\tilde{x}) &> 0, \quad \sup_{\tilde{x} \in \mathcal{U}} U(\tilde{x}) \leq 0, \end{aligned}$$

and the dissipation inequality

$$\frac{\partial U(\tilde{x})}{\partial \tilde{x}} f(\tilde{x}, 0) \leq -\lambda(U(\tilde{x})),$$

holds for all $\tilde{x} \in \{\tilde{x} \in \mathbb{R}^n : V(\tilde{x}) \geq 0\}$. △

Theorem 2.³⁷ If for the system (2) under Assumption 2, there exists a strong Leonov function as in Definition 13, then for all $\tilde{x}_0 \in \mathbb{R}^n$, the corresponding trajectories $\tilde{X}(t, \tilde{x}_0; 0)$ are bounded for all $t \geq 0$. △

By exploiting the periodicity of the system (2), the proof of Theorem 2³⁷ consists of two steps: first, it is shown that the sets $\Omega_j = \{\tilde{x} \in \mathbb{R}^n : U_j(\tilde{x}) \leq 0\}$, where $j \in \mathbb{Z}^q$ and $U_j(\tilde{x}) = U(\tilde{x} - \xi_j)$, are globally attractive and forward invariant for (2). Then, it is proven that the intersection of the sets Ω_j is composed of compact and isolated ‘‘cells’’. Thus, for any $\tilde{x}_0 \in \mathbb{R}^n$, the solution $\tilde{X}(t, \tilde{x}_0; 0)$ asymptotically enters and remains in these cells for all times. Note that $\mathcal{U} \subset \cap_{j \in \mathbb{Z}^q} \Omega_j$. Therefore, if there exists a Leonov function for the unperturbed closed-loop system (24), we can conclude that all elements of the set $\tilde{\mathcal{W}}_{cl}$ are inside of these cells. Consequently, its projection $\mathcal{W}_{cl} \subset M$ is a compact set.

Since we have established that the origin $\tilde{x} \equiv 0$ is an equilibrium point of the system (7), with $u \equiv 0$ and $d \equiv 0$, then we can define $\mathcal{U} = \{[0_k^\top, 2\pi j^\top]^\top : j \in \mathbb{Z}^q\}$ for this case. Furthermore, without loss of generality, we can make $\tilde{\mathcal{W}}' = \mathcal{U}$ in Definition 11. Then, by combining Definitions 11 and 13, and considering Theorem 2 we can establish the following result.

Corollary 1. Let the system (7) satisfy Assumptions 3, and the set \mathcal{W}_{cl} be decomposable in the sense of Definition 3. If there exists a practical ISS-CLeF (or an iISS-CLeF) satisfying the SCP-LeF (17), and additionally

$$\inf_{\tilde{x} \in \mathcal{V}} V(\tilde{x}) > 0, \quad \sup_{\tilde{x} \in \tilde{\mathcal{W}}'} V(\tilde{x}) \leq 0, \quad (29)$$

then the feedback control law (23) is continuous on \mathbb{R}^n and endows the system (7) with the ISS (iISS, respectively) property with respect to \mathcal{W}_{cl} . Moreover, the set \mathcal{W}_{cl} is compact. △

Note that the conditions in (29) do not imply any change in the proof of Theorem 1 and the construction of the controller (23). Therefore, the first part of the conclusion of Corollary 1 is directly derived from Theorem 1. On the other hand, the conditions in (29) imply that the corresponding practical ISS- or iISS-CLeF is a Leonov function for the closed-loop system (24) with $d \equiv 0$. Therefore, roughly speaking, we can say that the closed-loop system (24) has the 0-GATT property with respect to the cells, which are compact and isolated. Thus, we can conclude that the set \mathcal{W}_{cl} is compact.

Example 6. Let us revisit Example 5. Consider the system (28) and the origin $x = [0, 0]$ as its unique known equilibrium point in $M = \mathbb{S} \times \mathbb{R}$. Thus, $\tilde{\mathcal{W}}' = \{[2\pi j, 0] : j \in \mathbb{Z}\} \subset \mathbb{R}^2$. Consider also the function

$$V(x_1, x_2) = \frac{a_1}{2}x_2^2 - a_3x_2 - \frac{a_2}{2}x_1^2 - a_4 \cos(x_1), \quad (30)$$

where $a_1, a_2, a_3, a_4 > 0$ are constant parameters. By using Young’s inequality, we obtain $2|a_3x_2| \leq x_2^2 + a_3^2$. Then, we have

$$\frac{1}{2}(a_1 - 1)x_2^2 - \frac{a_2}{2}x_1^2 - \frac{1}{2}a_3^2 - a_4 \leq V(x_1, x_2) \leq \frac{1}{2}(a_1 + 1)x_2^2 - \frac{a_2}{2}x_1^2 + \frac{1}{2}a_3^2 + a_4.$$

Therefore, we can say that the function (30) satisfies inequality (8) if $a_1 > 1$. On the other hand, its derivative along the solutions of (28) can be written in the form $\dot{V}(x_1, x_2) = a(x_1, x_2, d) + b(x_2)u$, with $b(x_2) = a_1x_2 - a_3$, and

$$a(x_1, x_2, d) = -a_1cx_2^2 + (a_3c + a_4 \sin(x_1))x_2 - (a_2x_1 - a_4 \sin(x_1))(1 - \cos(x_1)) - a_2x_1x_2 + (a_1x_2 - a_3 - a_2x_1 + a_4 \sin(x_1))d.$$

By adding $cV(x_1, x_2)$ in both sides of the latter equation, we obtain

$$\begin{aligned} a(x_1, x_2, d) + cV(x_1, x_2) &= -\frac{1}{2}a_1cx_2^2 + a_4 \sin(x_1)x_2 - (a_2x_1 - a_4 \sin(x_1))(1 - \cos(x_1)) \\ &\quad - a_2x_1x_2 - \frac{a_2}{2}cx_1^2 - a_4c \cos(x_1) + (a_1x_2 - a_3 - a_2x_1 + a_4 \sin(x_1))d. \end{aligned}$$

Let us impose the restriction $a_1 \geq a_3$. The condition $b(x_2) = 0$ implies $x_2 = \frac{a_3}{a_1}$, and consequently

$$\begin{aligned} (a(x_1, x_2, d) + cV(x_1, x_2))|_{x_2=\frac{a_3}{a_1}} &= -\frac{ca_3^2}{2a_1} + a_4 \sin(x_1)\left(\frac{a_3}{a_1} + 1 - \cos(x_1)\right) - a_2x_1(1 - \cos(x_1)) \\ &\quad - a_2\frac{a_3}{a_1}x_1 - \frac{a_2}{2}cx_1^2 - a_4c \cos(x_1) + (a_4 \sin(x_1) - a_2x_1)d \\ &\leq -\frac{ca_3^2}{2a_1} + (3+c)a_4 - \frac{a_2}{2}cx_1^2 + a_4|d| + 3a_2|x_1| + a_2|x_1||d| \\ &\leq -\frac{ca_3^2}{2a_1} + (3+c)a_4 + \frac{9a_2}{2} - \frac{a_2}{2}(c-2)x_1^2 + a_4|d| + \frac{a_2}{2}d^2, \end{aligned}$$

where Young's inequality was applied in the last step to split the last two terms of the second step. Thus, if $a_1 \geq a_3 \geq \sqrt{\frac{a_1}{c}(2(3+c)a_4 + 9a_2)}$ and $c \geq 2$, the function (30) satisfies inequality (11). So, we can say that it is a practical ISS-CLeF for (28).

Now, consider in (13) that $\bar{v} = 0$, and $\bar{v}(|d|) = (a_1 + a_2)d^2 + (a_3 + a_4)|d|$, such that we obtain

$$\begin{aligned} \omega(x_1, x_2) &= \sup_{d \in \mathbb{R}^m} \{a(x_1, x_2, d) - \bar{v}(|d|)\} \\ &= \sup_{d \in \mathbb{R}^m} \left\{ -a_1cx_2^2 + (a_3c + a_4 \sin(x_1))x_2 - (a_2x_1 - a_4 \sin(x_1))(1 - \cos(x_1)) - a_2x_1x_2 \right. \\ &\quad \left. + (a_1x_2 - a_3 - a_2x_1 + a_4 \sin(x_1))d - (a_1 + a_2)d^2 - (a_3 + a_4)|d| \right\} \\ &\leq \sup_{d \in \mathbb{R}^m} \left\{ -a_1cx_2^2 + (a_3c + a_4 \sin(x_1))x_2 - (a_2x_1 - a_4 \sin(x_1))(1 - \cos(x_1)) - a_2x_1x_2 \right. \\ &\quad \left. + \frac{a_1}{4}x_2^2 - a_1\left(\frac{1}{2}x_2 - d\right)^2 + \frac{a_2}{4}x_1^2 - a_2\left(\frac{1}{2}x_1 + d\right)^2 \right\} \\ &= -\left(a_1c - \frac{a_1}{4}\right)x_2^2 + (a_3c + a_4 \sin(x_1))x_2 - (a_2x_1 - a_4 \sin(x_1))(1 - \cos(x_1)) - a_2x_1x_2 + \frac{a_2}{4}x_1^2. \end{aligned}$$

From Figure 1, we can see that $x_1^2 > |x_1|(1 - \cos(x_1))$ for all $x_1 \in \mathbb{R}$. Then, similarly to (15) we define

$$\begin{aligned} \phi(|[x_1, x_2]|_{\mathcal{W}'}, x_2) &= \sup_{x_1 \in \Theta} \left\{ \omega(x_1, x_2) + \frac{1}{2}cV(x_1, x_2) \right\} \\ &= \sup_{x_1 \in \Theta} \left\{ -\frac{a_1}{4}(3c-1)x_2^2 + \left(\frac{1}{2}a_3c + a_4 \sin(x_1)\right)x_2 - \frac{1}{2}a_4c \cos(x_1) \right. \\ &\quad \left. + a_4 \sin(x_1)(1 - \cos(x_1)) - a_2x_1x_2 - \frac{a_2}{4}(c-1)x_1^2 - a_2x_1(1 - \cos(x_1)) \right\} \\ &\leq \sup_{x_1 \in \Theta} \left\{ -\frac{a_1}{4}(3c-1)x_2^2 + \left(\frac{1}{2}a_3c + a_4 \sin(x_1)\right)x_2 - \frac{1}{2}a_4c \cos(x_1) \right. \\ &\quad \left. + a_4 \sin(x_1)(1 - \cos(x_1)) - a_2x_1x_2 - \frac{a_2}{4}(c-5)x_1^2 \right\} \\ &\leq \sup_{x_1 \in \Theta} \left\{ -\left(\frac{a_1}{4}(3c-1) - \frac{a_2}{c-5}\right)x_2^2 + \left(\frac{1}{2}a_3c + a_4 \sin(x_1)\right)x_2 - \frac{1}{2}a_4c \cos(x_1) \right. \\ &\quad \left. + a_4 \sin(x_1)(1 - \cos(x_1)) - a_2\left(\frac{1}{\sqrt{c-5}}x_2 + \frac{\sqrt{c-5}}{2}x_1\right)^2 \right\} \\ &\leq -\left(\frac{a_1}{4}(3c-1) - \frac{a_2}{c-5}\right)x_2^2 + \left(\frac{1}{2}a_3c + a_4 \sin(x_1)\right)x_2 - \frac{1}{2}a_4c \cos(x_1) + a_4 \sin(x_1)(1 - \cos(x_1)) \\ &= \bar{\phi}(|[x_1, x_2]|_{\mathcal{W}'}, x_2), \end{aligned}$$

where $\Theta = \{x_1 \in \mathbb{R} : a_2x_1^2 \leq (a_1 + 1)x_2^2 + a_3^2 + 2a_4\}$.

Then, by substituting $b(x_2)$ and $\bar{\phi}(|[x_1, x_2]|_{\mathcal{W}'}, x_2)$ in the formula (18) we obtain

$$K(\bar{\phi}(|[x_1, x_2]|_{\mathcal{W}'}, x_2), |b(x_2)|) \leq \frac{\bar{\phi}(|[x_1, x_2]|_{\mathcal{W}'}, x_2) + \sqrt{\bar{\phi}(|[x_1, x_2]|_{\mathcal{W}'}, x_2) + (a_1x_2 - a_3)^4}}{(a_1x_2 - a_3)^2} = \bar{K}(x_1, x_2). \quad (31)$$

To verify the continuity of the function (31), note that the restriction $b(x_2) = a_1x_2 - a_3 = 0$ implies

$$\bar{\phi}(|[x_1, x_2]|_{\tilde{\mathcal{W}}'}, x_2) \leq -\left(\frac{a_1}{4}(c-1) - \frac{a_2}{c-5}\right) \frac{a_3^2}{a_1^2} + a_4 \frac{a_3}{a_1} + \frac{\epsilon}{2} a_4 + 2a_4,$$

such that $\bar{\phi}(|[x_1, x_2]|_{\tilde{\mathcal{W}}'}, x_2) \leq 0$ if $a_1 \geq a_3 > a_1 \sqrt{\frac{2a_4(c+6)(c-5)}{a_1(c-1)(c-5)-4a_2}}$, $a_2 < \frac{a_1}{4}(c-1)(c-5)$, and $c > 5$. Hence, the function (31) is continuous everywhere. In this case, the SCP-LeF (17) is straightforwardly satisfied.

Moreover, we have $\tilde{\mathcal{W}}' \cap \Omega_2 = \emptyset$ for all $a_3 > 0$ (with Ω_2 as in (10)). Consequently, the function (22) becomes $\psi([x_1, x_2]) = \min \left\{ 1, \frac{|[x_1, x_2]|_{\tilde{\mathcal{W}}'}}{\epsilon} \right\}$, where $|[x_1, x_2]|_{\tilde{\mathcal{W}}'} = \sqrt{(1 - \cos(x_1))^2 + x_2^2}$, and $0 < \epsilon \leq \epsilon^*$ is a constant, with

$$\epsilon^* = \min_{[x_1, x_2] \in \{[x_1, x_2] \in \mathbb{R}^2 : V([x_1, x_2]) = 0\}} |[x_1, x_2]|_{\tilde{\mathcal{W}}'}.$$

Accordingly, the controller (23) takes the form

$$u(x_1, x_2) = -\min \left\{ 1, \frac{|[x_1, x_2]|_{\tilde{\mathcal{W}}'}}{\epsilon} \right\} \bar{K}(x_1, x_2)(a_1x_2 - a_3). \quad (32)$$

Finally, note that $V(x_1, x_2) = -a_4 < 0$ for all $[x_1, x_2] \in \mathcal{U} = \tilde{\mathcal{W}}'$. Besides, consider the function $V(\pm\pi, x_2) = \frac{a_1}{2}x_2^2 - a_3x_2 - \frac{1}{2}a_2\pi^2 + a_4$, which has a minimum at $x_2 = \frac{a_3}{a_1}$. Hence, we have $V(\pm\pi, \frac{a_3}{a_1}) = -\frac{a_3^2}{2a_1} - \frac{1}{2}a_2\pi^2 + a_4 > 0$ provided that $2a_4 > a_2\pi^2 + \frac{a_3^2}{a_1}$. Then, by continuity of (30) there exist some constants c_1 and c_2 , satisfying $0 \leq c_1 < \pi < c_2 \leq 2\pi$, such that $V(x_1, x_2) > 0$ for all $[x_1, x_2] \in \mathcal{V} = \{[x_1, x_2] \in \mathbb{R}^2 : c_1 \leq x_1 \leq c_2\}$. Thus, the function (30) satisfies the conditions (29) of Corollary 1, and we can say that it is a Leonov function for the closed-loop system (28),(32), with $d = 0$, which ensures that its invariant solutions are inside of a structure formed by compact and isolated cells. Consequently, we can say that the set $\mathcal{W}_{cl} \subset M$ is compact. Additionally, $\mathcal{W}_{cl} \supset \mathcal{W}'$ is the maximal invariant set for the closed-loop system (28),(32), and therefore, it is decomposable in the sense of Definition 3. Since the hypotheses of Corollary 1 are satisfied, we can conclude that the closed-loop system (28),(32) has the ISS property with respect to \mathcal{W}_{cl} . \triangle

The extension of the Leonov function approach to systems with controls was presented by Mendoza-Avila et al.³⁸ with the concept of control Leonov functions (CLeFs). The objective of the CLeF approach is the design of controllers for unperturbed state periodic systems that ensure boundedness of their trajectories in \mathbb{R}^n for all time $t \geq 0$. However, this property is less relevant in the presence of perturbations since they can drive the trajectories of state periodic systems to infinity. Therefore, Corollary 1 considers functions that represent the intersection between the families of practical ISS-CLeFs (iISS-CLeFs) and CLeFs, such that the properties of practical ISS-CLeFs (iISS-CLeFs) permit to deal with perturbations, and the properties of CLeFs guarantee the compactness of the set \mathcal{W}_{cl} . Since the set \mathcal{W}_{cl} is generally unknown, the advantage of its compactness lies in the fact that it could be identified, for instance, using the LaSalle's invariance principle,³⁶ which also implies that the decomposability property, in the sense of Definition 3, can be proven for it.

5 | CASE STUDY: EXCITATION CONTROL DESIGN FOR A SINGLE-MACHINE INFINITE BUS POWER SYSTEM

Clearly, the examples developed in the previous sections and in Appendix A consider simple cases of state periodic systems, or they are merely academic. In the present section, we aim to show the practical usefulness of the proposed approach for robust control of a single-machine infinite bus (SMIB) power system.

Consider the following mathematical model describing the behavior of a single synchronous generator connected to an infinite bus through a transmission line (under standard assumptions, including that the mechanical input power is constant)¹⁰:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -b_1x_3 \sin(x_1) - b_2x_2 + P + d_1, \\ \dot{x}_3 &= b_3 \cos(x_1) - b_4x_3 + E + u + d_2, \end{aligned} \quad (33)$$

where $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$ is the state vector, with x_1 as the load angle, x_2 as the shaft speed deviation from the synchronous speed, and x_3 as the quadrature axis internal voltage. Furthermore, $P, E \in \mathbb{R}$ are constants related to the mechanical power

and the bias field voltage, respectively, $b_i > 0$, $i = 1, \dots, 4$, are parameters, $d = [d_1, d_2] \in \mathcal{L}_\infty^2$ represents perturbations, and $u \in \mathcal{L}_\infty$ is the control input.

For $E > b_1^{-1}b_4P - b_3$, the unperturbed open-loop system (33) has equilibrium points at $x^* = [x_1^*, 0, x_3^*]^\top$, and $x^{**} = [x_1^{**}, 0, x_3^{**}]^\top$, where the pairs (x_1^*, x_3^*) and (x_1^{**}, x_3^{**}) are solutions of the equations

$$b_1x_3 \sin(x_1) = P, \quad E = b_4x_3 - b_3 \cos(x_1),$$

in the manifold $\{x \in \mathbb{R}^3 : -\pi \leq x_1 \leq \pi\}$. More details about the properties of the system (33), commonly known as the third-order flux-decay model of a synchronous generator, can be found in.^{10,39,40}

Now, consider the coordinates $\zeta = [\zeta_1, \zeta_2, \zeta_3]^\top \in \mathbb{R}^3$, where $\zeta_1 = x_1 - x_1^*$ and $\zeta_2 = x_2$, and $\zeta_3 = x_3 - x_3^*$. Then, the system (33) can be rewritten in the coordinates ζ as

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2, \\ \dot{\zeta}_2 &= -b_1\zeta_3 \sin(\zeta_1 + x_1^*) - b_1x_3^*\varphi_2(\zeta_1) - b_2\zeta_2 + d_1, \\ \dot{\zeta}_3 &= b_3\varphi_1(\zeta_1) - b_4\zeta_3 + u + d_2, \end{aligned} \quad (34)$$

where $\varphi_1(\zeta_1) = \cos(\zeta_1 + x_1^*) - \cos(x_1^*)$ and $\varphi_2(\zeta_1) = \sin(\zeta_1 + x_1^*) - \sin(x_1^*)$. Hence, all equilibria of the system (34) are contained in the set $\tilde{\mathcal{W}} = \{[2j\pi, 0, 0]^\top\} \cup \{[x_1^{**} - x_1^* + 2j\pi, 0, x_3^{**} - x_3^*]^\top\}$, for any $j \in \mathbb{Z}$.

5.1 | Control design with ISS-CLeFs

An ISS-CLeF candidate for the system (34) is given by

$$V(\zeta) = \frac{1}{2}\zeta_2^2 + \frac{1}{2}\zeta_3^2 - \frac{1}{2}\Delta\zeta_1^2 - b_1x_3^*\varphi_1(\zeta_1) - P\zeta_1, \quad (35)$$

where $\Delta > 0$ is a constant parameter to be determined.

Let us define $\varrho(\zeta_1) = -b_1x_3^*\varphi_1(\zeta_1) - P\zeta_1 - \frac{1}{2}\kappa b_1x_3^*\zeta_1^2$, with a constant $\kappa > 1$. Accordingly,

$$\varrho'(\zeta_1) = b_1x_3^* \sin(\zeta_1 + x_1^*) - P - \kappa b_1x_3^*\zeta_1, \quad \text{and} \quad \varrho''(\zeta_1) = b_1x_3^* \cos(\zeta_1 + x_1^*) - \kappa b_1x_3^*.$$

Hence, for any $\kappa > 1$ the equation $\varrho'(\zeta_1) = 0$ has a unique solution at $\zeta_1 = 0$, and the inequality $\varrho''(0) < 0$ holds. Moreover, the equation $\varrho''(\zeta_1) = 0$ has no solution for $\kappa > 1$, and therefore $\varrho'(\zeta_1)$ is strictly decreasing. Thus, we have that $\varrho(\zeta_1)$ is negative definite, which implies that $-b_1x_3^*\varphi_1(\zeta_1) - P\zeta_1 \leq \frac{1}{2}\kappa b_1x_3^*\zeta_1^2$, provided that $\kappa > 1$. By using similar arguments, we can deduce that $-b_1x_3^*\varphi_1(\zeta_1) - P\zeta_1 \geq -\frac{1}{2}\kappa b_1x_3^*\zeta_1^2$ for $\kappa > 1$ as well. Consequently, we have

$$\frac{1}{2}\zeta_2^2 + \frac{1}{2}\zeta_3^2 - \frac{1}{2}(\Delta + \kappa b_1x_3^*)\zeta_1^2 \leq V(\zeta) \leq \frac{1}{2}\zeta_2^2 + \frac{1}{2}\zeta_3^2 - \frac{1}{2}(\Delta - \kappa b_1x_3^*)\zeta_1^2, \quad (36)$$

for all $\zeta \in \mathbb{R}^3$. Thus, if $\Delta > \kappa b_1x_3^*$, we have that the function (35) satisfies the condition (8), with $\tilde{\mathcal{W}}' = \{[2j\pi, 0, 0]^\top : j \in \mathbb{Z}\}$.

Now, the derivative of (35) along the trajectories of (34) can be written as $\dot{V}(\zeta) = a(\zeta, d) + b(\zeta_3)u$ with $b(\zeta_3) = \zeta_3$ and

$$a(\zeta, d) = -b_2\zeta_2^2 - b_4\zeta_3^2 - b_1\zeta_2\zeta_3 \sin(\zeta_1 + x_1^*) - \Delta\zeta_1\zeta_2 + b_3\zeta_3\varphi_1(\zeta_1) + \zeta_2d_1 + \zeta_3d_2.$$

Next, consider a constant $\lambda > 0$ and inequality (36), such that

$$a(\zeta, d) + \lambda V(\zeta) \leq -\frac{\lambda}{2}(\Delta - \kappa b_1x_3^*)\zeta_1^2 - (b_2 - \frac{\lambda}{2})\zeta_2^2 - (b_4 - \frac{\lambda}{2})\zeta_3^2 - \Delta\zeta_1\zeta_2 - b_1\zeta_2\zeta_3 \sin(\zeta_1 + x_1^*) + b_3\zeta_3\varphi_1(\zeta_1) + \zeta_2d_1 + \zeta_3d_2.$$

Note that the restriction $b(\zeta_3) = 0$ implies $\zeta_3 = 0$, and consequently

$$(a(\zeta, d) + \lambda V(\zeta))|_{\zeta_3=0} \leq -\frac{\lambda}{2}(\Delta - \kappa b_1x_3^*)\zeta_1^2 - (b_2 - \frac{\lambda}{2})\zeta_2^2 - \Delta\zeta_1\zeta_2 + \zeta_2d_1 \leq -\frac{1}{2}\Phi \Xi \Phi + \nu|d|^2, \quad (37)$$

where $\Phi = [\zeta_1, \zeta_2, d_1]^\top$, and

$$\Xi = \begin{bmatrix} \lambda(\Delta - \kappa b_1x_3^*) & \Delta & 0 \\ \Delta & 2b_2 - \lambda & -1 \\ 0 & -1 & 2\nu \end{bmatrix}.$$

By applying the Schur complement, we can check that the matrix Ξ is positive semi-definite if and only if

$$\lambda(2b_2 - \lambda - \frac{1}{2v})(\Delta - \kappa b_1 x_3^*) - \Delta^2 \geq 0. \quad (38)$$

This inequality has real solutions with respect to $\Delta > \kappa b_1 x_3^*$ for any $v > 0$ and some $b_2, \lambda > 0$ satisfying

$$\lambda^2 - (2b_2 - \frac{1}{2v})\lambda + 4\kappa b_1 x_3^* \leq 0. \quad (39)$$

At the same time, this inequality has real solutions with respect to $\lambda > 0$ provided that $b_2 \geq 2\sqrt{\kappa b_1 x_3^*} + \frac{1}{4v}$.

Therefore, from (37) we have that if $b_2 \geq 2\sqrt{\kappa b_1 x_3^*} + \frac{1}{4v}$ for any $v > 0$, then there exist $\Delta > \kappa b_1 x_3^*$ and $\lambda > 0$ such that the function (35) satisfies the condition (11), with $\iota = 0$, and consequently it is an ISS-CLeF for the system (34).

Now, let us continue with the construction of a controller $u \in \mathcal{L}_\infty$, continuous in \mathbb{R}^3 and 2π -periodic in ζ_1 that ensures the ISS property for the system (34). To this end, consider a function $\bar{v}(|d|) = v(d_1^2 + d_2^2)$, such that

$$\begin{aligned} a(\zeta, d) - \bar{v}(|d|) &= -b_2 \zeta_2^2 - b_4 \zeta_3^2 - b_1 \zeta_2 \zeta_3 \sin(\zeta_1 + x_1^*) - \Delta \zeta_1 \zeta_2 + b_3 \zeta_3 \varphi_1(\zeta_1) + \zeta_2 d_1 + \zeta_3 d_2 - v d_1^2 - v d_2^2 \\ &= -(b_2 - \frac{1}{4v}) \zeta_2^2 - (b_4 - \frac{1}{4v}) \zeta_3^2 - b_1 \zeta_2 \zeta_3 \sin(\zeta_1 + x_1^*) - \Delta \zeta_1 \zeta_2 \\ &\quad + b_3 \zeta_3 \varphi_1(\zeta_1) - \left(\frac{1}{2\sqrt{v}} \zeta_2 - \sqrt{v} d_1 \right)^2 - \left(\frac{1}{2\sqrt{v}} \zeta_3 - \sqrt{v} d_2 \right)^2. \end{aligned}$$

Accordingly, from (13), with $\bar{\iota} = 0$, we obtain

$$\omega(\zeta) = \sup_{d \in \mathbb{R}^m} \{a(\zeta, d) - \bar{v}(|d|)\} = -(b_2 - \frac{1}{4v}) \zeta_2^2 - (b_4 - \frac{1}{4v}) \zeta_3^2 - b_1 \zeta_2 \zeta_3 \sin(\zeta_1 + x_1^*) - \Delta \zeta_1 \zeta_2 + b_3 \zeta_3 \varphi_1(\zeta_1).$$

Next, we define $\varrho_1(\zeta_1) = -\zeta_1^2 + 0.5\varphi_1^2(\zeta_1)$, whose derivatives of first and second order are given by $\varrho_1'(\zeta_1) = -2\zeta_1 - \sin(\zeta_1 + x_1^*)(\cos(\zeta_1 + x_1^*) - \cos(x_1^*))$, and

$$\varrho_1''(\zeta_1) = -2 + \cos(x_1^*) \cos(\zeta_1 + x_1^*) - \cos(2(\zeta_1 + x_1^*)) < -1 + |\cos(x_1^*)|.$$

Clearly, $\varrho_1''(\zeta_1) < 0$ for all $\zeta_1 \in \mathbb{R}$. Hence, $\varrho_1'(\zeta_1)$ is strictly decreasing and the equation $\varrho_1'(\zeta_1) = 0$ has a unique solution at $\zeta_1 = 0$. Thus, the function $\varrho_1(\zeta_1)$ is negative definite, which implies that $0.5\varphi_1^2(\zeta_1) \leq \zeta_1^2$ for all $\zeta_1 \in \mathbb{R}$.

Then, similarly to (15), we define $\Theta = \{\zeta_1 \in \mathbb{R} : (\Delta - \kappa b_1 x_3^*) \zeta_1^2 \leq \zeta_2^2 + \zeta_3^2\}$ and

$$\begin{aligned} \phi(|\zeta|_{\mathbb{R}^3}, \zeta_2, \zeta_3) &= \sup_{\zeta_1 \in \Theta} \{\omega(\zeta) + 0.5\lambda V(\zeta)\} \\ &\leq \sup_{\zeta_1 \in \Theta} \left\{ -(b_2 - \frac{1}{4}\lambda - \frac{1}{4v}) \zeta_2^2 - (b_4 - \frac{1}{4}\lambda - \frac{1}{4v}) \zeta_3^2 - b_1 \zeta_2 \zeta_3 \sin(\zeta_1 + x_1^*) \right. \\ &\quad \left. - \Delta \zeta_1 \zeta_2 - \frac{1}{4}\lambda (\Delta - \kappa b_1 x_3^*) \zeta_1^2 + b_3 \zeta_3 \varphi_1(\zeta_1) \right\} \\ &\leq \sup_{\zeta_1 \in \Theta} \left\{ -(b_2 - \frac{1}{4}\lambda - \frac{1}{4v}) \zeta_2^2 - (b_4 - \frac{1}{2}b_3 - \frac{1}{4v} - \frac{1}{4}\lambda) \zeta_3^2 - b_1 \zeta_2 \zeta_3 \sin(\zeta_1 + x_1^*) \right. \\ &\quad \left. - \Delta \zeta_1 \zeta_2 - \frac{1}{4}(\lambda (\Delta - \kappa b_1 x_3^*) - 4b_3) \zeta_1^2 \right\} \\ &\leq \sup_{\zeta_1 \in \Theta} \left\{ -\left(b_2 - \frac{1}{4}\lambda - \frac{1}{4v} - \frac{\Delta^2}{\lambda(\Delta - \kappa b_1 x_3^*) - 4b_3} \right) \zeta_2^2 - \left(b_4 - \frac{1}{2}b_3 - \frac{1}{4v} - \frac{1}{4}\lambda \right) \zeta_3^2 \right. \\ &\quad \left. - b_1 \zeta_2 \zeta_3 \sin(\zeta_1 + x_1^*) - \left(\frac{1}{2} \sqrt{\lambda(\Delta - \kappa b_1 x_3^*) - 4b_3} \zeta_1 + \frac{\Delta}{\sqrt{\lambda(\Delta - \kappa b_1 x_3^*) - 4b_3}} \zeta_2 \right)^2 \right\} \\ &\leq -\left(b_2 - \frac{1}{4}\lambda - \frac{1}{4v} - \frac{\Delta^2}{\lambda(\Delta - \kappa b_1 x_3^*) - 4b_3} \right) \zeta_2^2 - \left(b_4 - \frac{1}{2}b_3 - \frac{1}{4v} - \frac{1}{4}\lambda \right) \zeta_3^2 + b_1 |\zeta_2| |\zeta_3| = \bar{\phi}(\zeta_2, \zeta_3), \end{aligned}$$

where the properties $2b_3 \zeta_3 \varphi_1(\zeta_1) \leq b_3 \zeta_3^2 + b_3 \varphi_1^2(\zeta_1)$, and $0.5b_3 \varphi_1^2(\zeta_1) \leq b_3 \zeta_1^2$ have been used in the third step to split the last term of the second step.

Hence, by substituting $b(\zeta_3)$ and $\bar{\phi}(\zeta_2, \zeta_3)$ in the formula (18), we obtain

$$K(\bar{\phi}(\zeta_2, \zeta_3), |b(\zeta_3)|) \leq \frac{\bar{\phi}(\zeta_2, \zeta_3) + \sqrt{\bar{\phi}^2(\zeta_2, \zeta_3) + |b(\zeta_3)|^4}}{|b(\zeta_3)|^2} = \bar{K}(\zeta_2, \zeta_3).$$

Note that the restriction $b(\zeta_3) = 0$ implies

$$\bar{\phi}(\zeta_2, \zeta_3) = -\left(b_2 - \frac{1}{4}\lambda - \frac{1}{4v} - \frac{\Delta^2}{\lambda(\Delta - \kappa b_1 x_3^*) - 4b_3}\right) \zeta_2^2 \leq 0,$$

for all $\zeta \in \mathbb{R}^3$. Thus, if

$$b_2 \geq \max \left\{ 2\sqrt{\kappa b_1 x_3^*} + \frac{1}{4v}, \frac{1}{4}\lambda + \frac{1}{4v} + \frac{\Delta^2}{\lambda(\Delta - \kappa b_1 x_3^*) - 4b_3} \right\}, \quad (40)$$

then we can say that the function $\bar{K}(\zeta_2, \zeta_3)$ is continuous for all $\zeta \in \mathbb{R}^3 \setminus \{\zeta \in \mathbb{R}^3 : \zeta_2 = 0, \zeta_3 = 0\}$. Next, we use the SCP-LeF (17) to analyze the behavior of $\bar{K}(\zeta_2, \zeta_3)$ at the set $\{\zeta \in \mathbb{R}^3 : \zeta_2 = 0, \zeta_3 = 0\}$. Observe that

$$\lim_{\zeta_3 \rightarrow 0} \frac{\bar{\phi}(\zeta_2, \zeta_3)}{b^2(\zeta_3)} = -\infty,$$

for any $\zeta_2 \in \mathbb{R} \setminus \{0\}$, provided that inequality (40) is satisfied, and

$$\lim_{\zeta_2 \rightarrow 0} \frac{\bar{\phi}(\zeta_2, \zeta_3)}{b^2(\zeta_3)} = \frac{1}{2}b_3 + \frac{1}{4v} + \frac{1}{4}\lambda - b_4,$$

for all $\zeta_3 \in \mathbb{R}$. In consequence, there exist a constant $\bar{\varepsilon} \geq 0$, such that for any $\delta > 0$ we have

$$\sup_{\|\zeta_2, \zeta_3\| \leq \delta} \frac{\bar{\phi}(\zeta_2, \zeta_3)}{b^2(\zeta_3)} \leq \bar{\varepsilon}.$$

Since the constant $\delta > 0$ can be chosen arbitrarily small, we can conclude that the function $\bar{K}(\zeta_2, \zeta_3)$ is continuous for all $\zeta \in \mathbb{R}^3$.

Moreover, recall that we have defined $\tilde{\mathcal{W}}' = \{[2j\pi, 0, 0]^\top : j \in \mathbb{Z}\}$ for this case. Clearly, $b(\zeta_3) = 0$ for all $\zeta \in \tilde{\mathcal{W}}'$. Therefore, from (22) we obtain $\psi(\zeta) = 1$. Finally, under the parametric conditions established above, the controller (23) takes the form

$$u(\zeta) = -\bar{K}(\zeta_2, \zeta_3)\zeta_3, \quad (41)$$

or equivalently

$$u(x) = -\frac{\bar{\phi}(x_2, x_3 - x_3^*) + \sqrt{\bar{\phi}^2(x_2, x_3 - x_3^*) + |x_3 - x_3^*|^4}}{|x_3 - x_3^*|^2} (x_3 - x_3^*), \quad (42)$$

where $\bar{\phi}(x_2, x_3 - x_3^*) = -\left(b_2 - \frac{1}{4}\lambda - \frac{1}{4v} - \frac{\Delta^2}{\lambda(\Delta - \kappa b_1 x_3^*) - 4b_3}\right) x_2^2 - \left(b_4 - \frac{1}{2}b_3 - \frac{1}{4v} - \frac{1}{4}\lambda\right) (x_3 - x_3^*)^2 + b_1 |x_2| |x_3 - x_3^*|$.

Notably, the controller (41) adds damping to the system (34), which enhances its robustness. In addition, note that $u(\zeta) = 0$ for all $\zeta \in \tilde{\mathcal{W}}'$, therefore we can ensure that $\mathcal{W}' \subset \mathcal{W}_{cl}$, where \mathcal{W}_{cl} is a set that contains all α - and ω -limit sets of the closed-loop system (34),(41). Assuming that the set \mathcal{W}_{cl} is decomposable in the sense of Definition 3, we can conclude from Theorem 1 that the closed-loop system (34),(41) has the ISS property with respect to the set \mathcal{W}_{cl} .

Let us highlight two interesting features of the proposed controller (41):

- The controller (41) is independent of the load angle x_1 , which is an attractive property since this variable is often unavailable in practice; see the discussions in the conclusions of references.^{10,41} Usually, synchronous generators are equipped with tachometers and multimeters that make available information of the states x_2 and x_3 . However, the measurement of the load angle x_1 usually requires the usage of complex electronics instruments, like phasor measurement units (PMUs), and sophisticated observation algorithms.⁴²
- The value of the damping constant b_2 is sometimes assumed to be zero in the design of excitation controllers for the third-order model of synchronous generators.^{10,39,40} However, this assumption is possible only by restricting the system's operation to the region where $0 \leq x_1 \leq 0.5\pi$. For the system (33), the state variable x_3 acts as a control input for the subsystem $[x_1, x_2]$. Then, the sine function of x_1 becomes a sign indefinite control coefficient, which implies that global stability cannot be achieved if $b_2 = 0$. Therefore, the design of the controller (41) imposes a lower bound for b_2 , rendering the controller hard to apply in a practical setting.

Remark 5. The main features of the proposed controller (41) are that it globally ensures the ISS property for the closed-loop system (34),(41) with respect to the set \mathcal{W}_{cl} , it is independent of the load angle x_1 , and it is continuous at every point of the state space. As far as we know, there is no approach in the literature that achieves similar properties. For instance, Gao et al.⁴³

present an optimal control law for global robust stabilization of a lossy n -machine power system. However, this method is based on two assumptions that can be validated only in a neighborhood of the equilibrium point of interest (these assumptions are needed to guarantee continuity of the proposed controller at every point of the state space). Kanchanaharuthai et al.⁴⁴ introduce a backstepping controller for ISS stabilization of a SMIB power system. However, this controller has a singularity in a subset of the state space. Cabrera-Vázquez et al.⁴⁵ provide a sliding mode controller for robust stabilization of a SMIB power system. In addition to the switched nature of sliding mode controllers, the proposed sliding surface is singular for certain values of the phase angle, therefore, it has a local nature. Moreover, these controllers depends on the phase angle as a direct feedback variable.

5.2 | Numerical example

Finally, we provide a numerical example to illustrate the effectiveness of the proposed controller. Fix the parameters of the system (33) as $b_1 = 34.29$, $b_2 = 15$, $b_3 = 0.149$, $b_4 = 0.3341$, $P = 28.22$, and $E = 0.2405$, which are given in per unit and are taken from¹⁰ (except that $b_2 = 0$ is considered in this reference). Hence, the unperturbed open-loop system (33) has equilibrium points at $x^* = [1.1246, 0, 0.9123]^\top$, and $x^{**} = [1.1708, 0, 0.8935]^\top$, in the manifold $\{x \in \mathbb{R}^3 : -\pi \leq x_1 \leq \pi\}$. Considering the change of coordinates $x \mapsto \zeta$, we translate the equilibrium point x^* to the origin, and we define the set $\tilde{\mathcal{W}}' = \{[2j\pi, 0, 0]^\top\}$, with $j \in \mathbb{Z}$. Moreover, the parameters of design are chosen as $\kappa = 1.1$, $\Delta = 70$, $\lambda = 18$ and $\nu = 0.125$, such that the conditions (38), (39) and (40) are satisfied. Simulations are carried out by using Matlab Simulink, with the solver Runge-Kutta and a sampling step of 1 [ms]. Figure 2 shows the simulation results of the system (34), with $\zeta_0 = [\pi, 1, 0]^\top$, $d_1 = -2 \cos(4t)$ and $d_2 = 6 \sin(10t)$, in closed-loop with the controller (41), considering either noise-free and noisy scenarios. For the noise-free case: comparing Figures 2.1 and 2.2, we can observe that the controller (41) increases the robustness of the system (34), such that, its trajectories remain in a closer vicinity of the set \mathcal{W}_{cl} . The perturbation signals $d = [d_1, d_2]^\top$ are shown in Figure 2.3. The supplied control signal is presented in Figure 2.4, where we can confirm that it is continuous. For the noisy case, a signal of white Gaussian noise, given in Figure 2.7 and whose peak amplitude represents approximately the 5% of the peak-to-peak amplitude of the state signals in steady state, is used to simulate additive measurement noise corrupting the state signals in the control loop. In Figure 2.5 we can see that the proposed controller is able to attenuate the action of the perturbations affecting in the system (34), despite the presence of measurement noise. Figure 2.6 shows that the control signal keeps continuity under the influence of measurement noise. Finally, Figure 2.8 depicts solutions of the unperturbed closed-loop system (34), (41) for different initial conditions. We can see that these trajectories converge to equilibrium points in $\tilde{\mathcal{W}}'$ (red points in the plot). Hence, we can infer that the set $\tilde{\mathcal{W}}_{cl}$ is decomposable in the sense of Definition 3. Thus, we can say that the simulation results support the conclusion that the closed-loop system (34), (41) has the ISS property with respect to \mathcal{W}_{cl} .

6 | CONCLUSIONS

In this paper, the ISS-CLeF approach is further developed. The definitions of practical ISS-, ISS- and iISS-CLeFs are refined, and the main aspects for the construction of these functions are extensively discussed and illustrated in academic examples. Also, the proposed methodology for control synthesis of state periodic systems is improved to simplify the final control law, and its main features are highlighted in the provided academic examples. Based on Sontag's universal formula, this methodology yields a continuous and state periodic controller, which endows the closed-loop system with the ISS or iISS property in the multistable sense. In summary, the main result of this paper shows that the existence of a practical ISS-, ISS- or iISS-CLeF is a sufficient condition to ensure that the system under study is ISS or iISS stabilizable with respect to the set \mathcal{W}_{cl} . Moreover, an extension of the main results to ensure compactness of the set \mathcal{W}_{cl} is presented, which imposes extra restrictions for the construction of practical ISS- and iISS-CLeFs, and connects practical ISS- and iISS-CLeFs with the standard Leonov function approach. The proposed methodology is applied to the design of an excitation controller for a SMIB power system, ensuring ISS property for the closed-loop system. The practical interest of the developed controller lies in the facts that it does not require measurements of the load angle and guarantees the ISS property globally, which are rarely achieved by other approaches reported in the literature. Simulation results supporting the effectiveness of the designed controller are provided.

The future work can be continued in three directions: developing a methodology for ISS-CLeF construction with relaxed assumptions, exploring other formulas for control synthesis based on ISS-CLeFs, which can incorporate optimization methods for improving performance and practical constraints like saturation in the control input, and implementing the ISS-CLeF approach for large-scale power systems, together with their experimental validation.

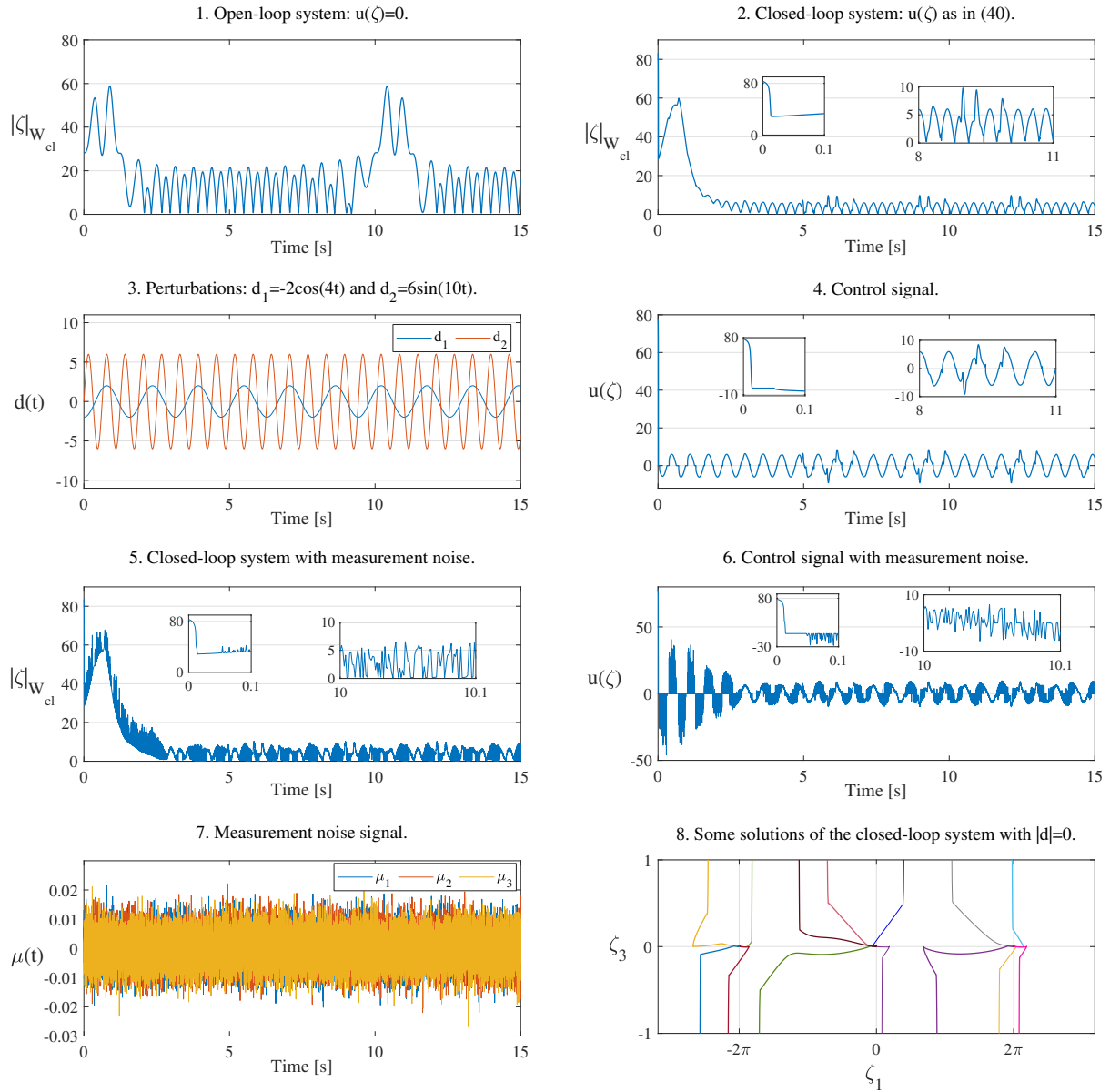


FIGURE 2 Simulation results of the closed-loop system (33),(41).

CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

No datasets were generated or analyzed during this research.

References

1. Dashkovskiy SN, Efimov D, Sontag ED. Input to state stability and allied system properties. *Automation and Remote Control*. 2011;72(8):1579–1613. doi: 10.1134/S0005117911080017
2. Sontag ED. Input to state stability: Basic concepts and results. In: Nistri P, Stefani G., eds. *Nonlinear and optimal control theory*, Berlin, Germany: Springer, 2008:163–220.

3. Angeli D, Sontag ED, Wang Y. A characterization of integral input-to-state stability. *IEEE Transactions on Automatic Control*. 2000;45(6):1082–1097. doi: 10.1109/9.863594
4. Angeli D. An almost global notion of input-to-state stability. *IEEE Transactions on Automatic Control*. 2004;49(6):866–874. doi: 10.1109/TAC.2004.829594
5. Angeli D, Praly L. Stability robustness in the presence of exponentially unstable isolated equilibria. *IEEE Transactions on Automatic Control*. 2010;56(7):1582–1592. doi: 10.1109/TAC.2010.2091170
6. Efimov D. Global Lyapunov analysis of multistable nonlinear systems. *SIAM Journal on Control and Optimization*. 2012;50(5):3132–3154. doi: 10.1137/090767509
7. Angeli D, Efimov D. Characterizations of input-to-state stability for systems with multiple invariant sets. *IEEE Transactions on Automatic Control*. 2015;60(12):3242–3256. doi: 10.1109/TAC.2015.2418676
8. Forni P, Angeli D. Characterizations of integral input-to-state stability for systems with multiple invariant sets. *IEEE Transactions on Automatic Control*. 2016;62(8):3729–3743. doi: 10.1109/TAC.2016.2642782
9. Forni F, Sepulchre R. Differential analysis of nonlinear systems: Revisiting the pendulum example. In: Proceedings of the 53rd IEEE Conference on Decision and Control (CDC). IEEE. 2014:3848–3859.
10. Galaz M, Ortega R, Bazanella AS, Stankovic AM. An energy-shaping approach to the design of excitation control of synchronous generators. *Automatica*. 2003;39(1):111–119. doi: 10.1016/S0005-1098(02)00177-2
11. Schiffer J, Dörfler F. On stability of a distributed averaging PI frequency and active power controlled differential-algebraic power system model. In: Proceedings of the 2016 European Control Conference (ECC). IEEE. 2016:1487–1492.
12. Schiffer J, Ortega R, Astolfi A, Raisch J, Sezi T. Conditions for stability of droop-controlled inverter-based microgrids. *Automatica*. 2014;50(10):2457–2469. doi: 10.1016/j.automatica.2014.08.009
13. Leonov GA, Kuznetsov NV. Nonlinear analysis of phase-locked loop (PLL): Global stability analysis, hidden oscillations and simulation problems. *Mechanics and Model-Based Control of Advanced Engineering Systems*. 2014:199–207. doi: 10.1007/978-3-7091-1571-8_22
14. Dorfler F, Bullo F. Synchronization and transient stability in power networks and nonuniform Kuramoto oscillators. *SIAM Journal on Control and Optimization*. 2012;50(3):1616–1642. doi: 10.1137/110851584
15. Efimov D, Schiffer J, Barabanov N, Ortega R. A relaxed characterization of ISS for periodic systems with multiple invariant sets. *European Journal of Control*. 2017;37:1–7. doi: 10.1016/j.ejcon.2017.04.002
16. Mendoza-Avila J, Efimov D, Mercado-Uribe A, Schiffer J. On Relaxed Conditions of Integral ISS for Multistable Periodic Systems. In: Proceedings of the 61st IEEE Conference on Decision and Control (CDC). IEEE. 2022:7072–7077.
17. Mendoza-Avila J, Efimov D, Mercado-Uribe A, Schiffer J. Fitting Characterizations of Integral ISS Notions for Multistable State Periodic Systems. *Automatica*. 2023. Submitted.
18. Leonov GA. On the boundedness of the trajectories of phase systems. *Siberian Mathematical Journal*. 1974;15(3):491–495.
19. Noldus EJ. New direct Lyapunov-type method for studying synchronization problems. *Automatica*. 1977;13(2):139–151. doi: 10.1016/0005-1098(77)90038-3
20. Gelig AK, Leonov GA, Yakubovich VA. *Stability of nonlinear systems with non-unique equilibrium state*. Moscow, Russia: Nauka, 1978.
21. Yakubovich VA, Leonov GA, Gelig AK. *Stability of stationary sets in control systems with discontinuous nonlinearities*. 14 of *Series on Stability, Vibration and Control of Systems*. Singapore: World Scientific, 2004.
22. Efimov D, Schiffer J. On boundedness of solutions of state periodic systems: a multivariable cell structure approach. *IEEE Transactions on Automatic Control*. 2019;64(10):4094–4104. doi: 10.1109/TAC.2019.2892395
23. Schiffer J, Efimov D, Ortega R. Almost global synchronization in radial multi-machine power systems. In: Proceedings of the 57th IEEE Conference on Decision and Control (CDC). IEEE. 2018:800–805.
24. Schiffer J, Efimov D, Ortega R. Global synchronization analysis of droop-controlled microgrids—A multivariable cell structure approach. *Automatica*. 2019;109:108550. doi: 10.1016/j.automatica.2019.108550
25. Mercado-Uribe A, Mendoza-Avila J, Efimov D, Schiffer J. A Leonov Function for Almost Global Synchronization Conditions in Acyclic Networks of Heterogeneous Kuramoto Oscillators. *IFAC-PapersOnLine*. 2023;56(2):9505–9510. doi: 10.1016/j.ifacol.2023.10.248
26. Mercado-Uribe A, Mendoza-Avila J, Efimov D, Schiffer J. Conditions for global synchronization in networks of heterogeneous Kuramoto oscillators. *Automatica*. 2025;175:112180. doi: 10.1016/j.automatica.2025.112180
27. Liberzon D, Sontag ED, Wang Y. Universal construction of feedback laws achieving ISS and integral-ISS disturbance attenuation. *Systems & Control Letters*. 2002;46(2):111–127. doi: 10.1016/S0167-6911(02)00125-1

28. Sontag ED, Wang Y. On Characterizations of Input-to-State Stability with Respect to Compact Sets. In: Krener A, Mayne D., eds. *Nonlinear Control Systems Design 1995*, , IFAC Postprint Volume. Oxford: Pergamon, 1995:203-208.
29. Artstein Z. Stabilization with relaxed controls. *Nonlinear Analysis: Theory, Methods & Applications*. 1983;7(11):1163–1173. doi: 10.1016/0362-546X(83)90049-4
30. Sontag ED. A ‘universal’ construction of Artstein’s theorem on nonlinear stabilization. *Systems & control letters*. 1989;13(2):117–123. doi: 10.1016/0167-6911(89)90028-5
31. Barroso NF, Ushirobira R, Efimov D. Universal formula for robust stabilization of affine nonlinear multistable systems. In: Proceedings of the 58th IEEE Conference on Decision and Control (CDC). IEEE. 2019:8024–8029.
32. Barroso NF, Ushirobira R, Efimov D. Control Lyapunov function method for robust stabilization of multistable affine nonlinear systems. *International Journal of Robust and Nonlinear Control*. 2023;33(11):6354–6370. doi: 10.1002/rnc.6700
33. Mendoza-Avila J, Mercado-Uribe A, Efimov D, Schiffer J. Design of controls for ISS and Integral ISS Stabilization of Multistable State Periodic Systems. *IFAC-PapersOnLine*. 2023;56(2):4472–4477. doi: 10.1016/j.ifacol.2023.10.936
34. Guckenheimer J, Holmes P. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*. 42 of *Applied Mathematical Science*. New York, NY, USA: Springer Verlag, 1983.
35. Nitecki Z, Shub M. Filtrations, decompositions, and explosions. *American Journal of Mathematics*. 1975;97(4):1029–1047. doi: 10.2307/2373686
36. Khalil H. *Nonlinear Systems*. Upper Saddle River, NJ: Prentice Hall. 3rd ed., 2002.
37. Schiffer J, Efimov D. Strong and weak Leonov functions for global boundedness of state periodic systems. *IEEE Transactions on Automatic Control*. 2023;68(12):7958–7965. doi: 10.1109/TAC.2023.3251903
38. Mendoza-Avila J, Efimov D, Mercado-Uribe A, Schiffer J. Design of Controls for Boundedness of Trajectories of Multistable State Periodic Systems. In: Proceedings of the 62nd IEEE Conference on Decision and Control (CDC). IEEE. 2023:6647–6652
39. Kundur P. *Power Systems Stability and Control*. New York, NY: McGraw-Hill, 1994.
40. Pai MA. *Energy function analysis for power system stability*. Boston, MA: Kluwer Acad. Publishers, 1989.
41. Leon AE, Solsona JA, Valla MI. Comparison among nonlinear excitation control strategies used for damping power system oscillations. *Energy Conversion and Management*. 2012;53(1):55–67. doi: 10.1016/j.enconman.2011.08.010
42. Lorenz-Meyer N, Bobtsov A, Ortega R, Nikolaev N, Schiffer J. PMU-based decentralised mixed algebraic and dynamic state observation in multi-machine power systems. *IET Generation, Transmission & Distribution*. 2020;14(25):6267–6275. doi: 10.1049/iet-gtd.2020.1275
43. Gao J, Chaudhuri B, Astolfi A. An Optimization-Based Method for Transient Stability Assessment. In: Proceedings of the 62nd IEEE Conference on Decision and Control (CDC). IEEE. 2023:4586–4591
44. Kanchanaharuthai A, Mujjalinvimut E. Nonlinear disturbance observer-based backstepping control for a dual excitation and steam-valving system of synchronous generators with external disturbances. *International Journal of Innovative Computing, Information and Control*. 2018;14(1):111–126.
45. Cabrera-Vázquez J, Loukianov AG, Cañedo JM, Utkin VI. Robust controller for synchronous generator with local load via VSC. *International Journal of Electrical Power & Energy Systems*. 2007;29(4):348–359. doi: 10.1016/j.ijepes.2006.09.001

How to cite this article: Mendoza-Avila J., Efimov D., Mercado-Uribe A., and Schiffer J.. Design of Control Laws for Robust Global Stabilization of Multistable State Periodic Systems. *Int. J. Robust Nonlinear Control* 0000;00(00):0–00.

APPENDIX

A COMPLEMENTARY EXAMPLES

The following example illustrates the construction of iISS-CLeFs. Moreover, we will check the usefulness of the SCP-LeF (10) to guarantee the continuity of the designed controller.

Example 7. Consider the following system:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -b_1 x_3 \sin(x_1) - \tau x_2 + c_1 + \frac{x_2}{x_1^2 + 1} d_1, \quad \dot{x}_3 = -b_2 x_3 + c_2 + u + d_2, \quad (\text{A1})$$

where $x = [x_1, x_2, x_3]^\top \in \mathbb{R}^3$ is the state, $b_1, b_2, \tau > 0$ are parameters, $c_1, c_2 \in \mathbb{R}$ are constants, $d = [d_1, d_2] \in \mathcal{L}_\infty^2$ are perturbation signals, and $u \in \mathcal{L}_\infty$ is the control input. Note that this system satisfies Assumption 3 for $d \equiv 0$. Moreover, for $b_1|c_2| > b_2|c_1|$ and $u = d_1 = d_2 = 0$, the unperturbed system has two equilibrium points in the manifold $\{x \in \mathbb{R}^3 : -\pi \leq x_1 \leq \pi\}$ at $x^* = [x_1^*, 0, x_3^*]^\top$ and $x^{**} = [x_1^{**}, 0, x_3^{**}]^\top$, where $\sin(x_1^*) = c_1 b_2 (c_2 b_1)^{-1}$, $x_1^{**} = \pi - x_1^*$ and $x_3^* = c_2 b_2^{-1}$. Now, consider the change of coordinates $x \mapsto \zeta = [\zeta_1, \zeta_2, \zeta_3]^\top \in \mathbb{R}^3$, where $\zeta_1 = x_1 - x_1^*$, $\zeta_2 = x_2$ and $\zeta_3 = x_3 - x_3^*$. Then, the set $\tilde{\mathcal{W}} = \{[2j\pi, 0, 0]^\top \cup [(1+2j)\pi - 2x_1^*, 0, 0]^\top : j \in \mathbb{Z}\}$ contains all equilibria of the system (A1) in coordinates ζ , and it is assumed to be decomposable in the sense of Definition 3. Now, consider the function

$$V(\zeta) = \frac{1}{2}\zeta_2^2 + \frac{1}{2}\zeta_3^2 - \frac{1}{2}\Delta\zeta_1^2 + b_1 x_3^* (\cos(x_1^*) - \cos(\zeta_1 + x_1^*)) - c_1 \zeta_1, \quad (\text{A2})$$

where $\Delta > 0$. Recalling the function $\varrho(\zeta_1)$ introduced in Example 2, we can easily check that the function (A2) satisfies

$$\frac{1}{2}\zeta_2^2 + \frac{1}{2}\zeta_3^2 - \frac{1}{2}(\Delta + \kappa b_1 x_3^*)\zeta_1^2 \leq V(\zeta) \leq \frac{1}{2}\zeta_2^2 + \frac{1}{2}\zeta_3^2 - \frac{1}{2}(\Delta - \kappa b_1 x_3^*)\zeta_1^2, \quad (\text{A3})$$

for all $\zeta \in \mathbb{R}^3$ and $\kappa > 1$. Moreover, the derivative of the function (A2) along the trajectories of the system (A1) (in coordinates ζ) is given by $\dot{V}(\zeta) = a(\zeta, d) + b(\zeta)u$, where $b(\zeta) = \zeta_3$ and

$$a(\zeta, d) = -\Delta\zeta_1\zeta_2 - b_1\zeta_2\zeta_3 \sin(\zeta_1 + x_1^*) - \tau\zeta_2^2 - b_2\zeta_3^2 + \frac{\zeta_2^2}{(\zeta_1 + x_1^*)^2 + 1} d_1 + \zeta_3 d_2. \quad (\text{A4})$$

Then, the restriction $b(\zeta) = 0$ implies

$$a(\zeta, d) + \lambda V(\zeta) \leq -\frac{1}{2} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}^\top \begin{bmatrix} \lambda(\Delta - \kappa b_1 x_3^*) & \Delta \\ \Delta & (2\tau - \lambda) \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} + \frac{\zeta_2^2}{(\zeta_1 + x_1^*)^2 + 1} d_1 \leq \frac{\zeta_2^2}{(\zeta_1 + x_1^*)^2 + 1} d_1, \quad (\text{A5})$$

for some $\lambda > 0$ sufficiently small, and any $\tau > 0$ sufficiently large. To deal with the quadratic term multiplying the perturbation d_1 , we introduce the function

$$W(\zeta) = \begin{cases} \ln(1 + V(\zeta)) & \text{for } \zeta \in \mathbb{R}^3 \setminus \Omega_1, \\ V(\zeta) & \text{for } \zeta \in \Omega_1, \end{cases} \quad (\text{A6})$$

with Ω_1 as in (10). Note that the function $W(\zeta)$ is continuous everywhere since $W(\zeta) = \ln(1 + V(\zeta)) = V(\zeta) = 0$ for all $\zeta \in \partial\Omega_1$, where $\partial\Omega_1$ denotes the boundary of Ω_1 , and the term $\ln(1 + V(\zeta))$ is well-defined for all $\zeta \in \mathbb{R}^3 \setminus \Omega_1$. Additionally, by properties of the natural logarithm, we have $0 \leq \ln(1 + |s|) \leq |s|$. Besides, since $\ln(1 + |s|)$ is a \mathcal{K}_∞ -function, we can deduce the inequality $\ln(1 + |s_1| + |s_2|) \leq \ln(1 + 2|s_1|) + \ln(1 + 2|s_2|)$. Thus, using these properties, we obtain from inequalities (A3) that

$$\ln(1 + \frac{1}{2}\zeta_2^2 + \frac{1}{2}\zeta_3^2) - \frac{1}{2}(\Delta + \kappa b_1 x_3^*)\zeta_1^2 \leq W(\zeta) \leq \frac{1}{2}\zeta_2^2 + \frac{1}{2}\zeta_3^2 - \frac{1}{2}(\Delta - \kappa b_1 x_3^*)\zeta_1^2,$$

for all $\zeta \in \mathbb{R}^3$. Consequently, the function (A6) satisfies the inequalities (8) of Definition 11 if $\Delta > \kappa b_1 x_3^*$. Moreover, we have $\dot{W}(\zeta) = \frac{\dot{V}(\zeta)}{1+V(\zeta)}$ for all $\zeta \in \mathbb{R}^3 \setminus \Omega_1$. For $b(\zeta) = 0$, we obtain from (A5)

$$\dot{W}(\zeta) \leq \frac{1}{1+V(\zeta)} \left(-\lambda V(\zeta) + \frac{\zeta_2^2}{(\zeta_1 + x_1^*)^2 + 1} d_1 \right).$$

Clearly, $W(\zeta) = \ln(1 + V(\zeta)) \geq 0$ for all $\zeta \in \mathbb{R}^3 \setminus \Omega_1$. Additionally, the quotient $\frac{\zeta_2^2}{(1+V(\zeta))(1+(\zeta_1+x_1^*)^2)}$ is bounded for all $\zeta \in \mathbb{R}^3 \setminus \Omega_1$, too. Hence, there exists a constant $v > 0$, such that the restriction $b(\zeta) = 0$ implies

$$\dot{W}(\zeta) \leq -\lambda(1 - e^{-W(\zeta)}) + v|d|,$$

for all $\zeta \in \mathbb{R}^3 \setminus \Omega_1$. Note that $\tilde{\mathcal{W}} \subset \{\zeta \in \mathbb{R}^3 : b(\zeta) = 0\}$, hence we can select $\tilde{\mathcal{W}}' = \tilde{\mathcal{W}}$, or simply $\tilde{\mathcal{W}}' = \{[2j\pi, 0, 0]^\top : j \in \mathbb{Z}\}$. Therefore, we can say that (A6) satisfies all condition of Definition 11, and it is an iISS-CLeF for the system (A1).

Next, we proceed to the control design. To construct a function $\omega(\zeta)$ as in (13), we select $\bar{v}(|d|) = \bar{v}_1|d_1| + \bar{v}_2d_2^2$, with $\bar{v}_1, \bar{v}_2 > 0$, and a constant $\bar{v} = 0$. Then, considering the function $a(\zeta, d)$ in (A4), we have

$$a(\zeta, d) - \bar{v}(|d|) = \frac{1}{1+V(\zeta)} \left(-\Delta\zeta_1\zeta_2 - b_1\zeta_2\zeta_3 \sin(\zeta_1 + x_1^*) - \tau\zeta_2^2 - b_2\zeta_3^2 + \frac{\zeta_2^2}{(\zeta_1+x_1^*)^2+1}d_1 + \zeta_3d_2 \right) - \bar{v}_1|d_1| - \bar{v}_2d_2^2.$$

Now, by using Young's inequality we obtain $2|\zeta_3d_2| \leq \bar{v}_2^{-1}\zeta_3^2 + \bar{v}_2d_2^2$, and note that we can define $\bar{v}_1 \geq \frac{\zeta_2^2}{(1+V(\zeta))(1+(\zeta_1+x_1^*)^2)}$, wherever $V(\zeta) \geq 0$. Therefore,

$$\omega(\zeta) = \sup_{d \in \mathbb{R}^2} \{a(\zeta, d) - \bar{v}(|d|)\} \leq \frac{1}{1+V(\zeta)} \left(-\Delta\zeta_1\zeta_2 - b_1\zeta_2\zeta_3 \sin(\zeta_1 + x_1^*) - \tau\zeta_2^2 - (b_2 - \bar{v}_2^{-1})\zeta_3^2 \right).$$

for all $\zeta \in \mathbb{R}^3 \setminus \Omega_1$. Now, defining $\Theta = \{\zeta_1 \in \mathbb{R} : (\Delta - \kappa b_1 x_3^*)\zeta_1^2 \leq \zeta_2^2 + \zeta_3^2\}$ we have from (15) and for all $\zeta \in \mathbb{R}^3 \setminus \Omega_1$ (i.e., wherever $V(\zeta) \geq 0$) that

$$\begin{aligned} \phi(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3) &= \sup_{\zeta_1 \in \Theta} \{ \omega(\zeta) + 0.5\lambda(1 - e^{-W(\zeta)}) \} \\ &= \sup_{\zeta_1 \in \Theta} \left\{ \omega(\zeta) + 0.5\lambda \frac{V(\zeta)}{1+V(\zeta)} \right\} \\ &\leq \sup_{\zeta_1 \in \Theta} \left\{ \frac{1}{1+V(\zeta)} \left(-\Delta\zeta_1\zeta_2 - b_1\zeta_2\zeta_3 \sin(\zeta_1 + x_1^*) - \left(\tau - \frac{1}{4}\lambda \right) \zeta_2^2 - \left(b_2 - \frac{1}{\bar{v}_2} - \frac{1}{4}\lambda \right) \zeta_3^2 - \frac{1}{4}\lambda(\Delta - \kappa b_1 x_3^*)\zeta_1^2 \right) \right\} \\ &= \sup_{\zeta_1 \in \Theta} \left\{ \frac{1}{1+V(\zeta)} \left(-b_1\zeta_2\zeta_3 \sin(\zeta_1 + x_1^*) - \left(\tau - \frac{1}{4}\lambda - \frac{\Delta^2}{\lambda(\Delta - \kappa b_1 x_3^*)} \right) \zeta_2^2 - \left(b_2 - \frac{1}{\bar{v}_2} - \frac{1}{4}\lambda \right) \zeta_3^2 \right. \right. \\ &\quad \left. \left. - \left(\frac{\Delta}{\sqrt{\lambda(\Delta - \kappa b_1 x_3^*)}} \zeta_2 + \frac{1}{2} \sqrt{\lambda(\Delta - \kappa b_1 x_3^*)} \zeta_1 \right)^2 \right) \right\} \\ &= \bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3). \end{aligned}$$

In this case, it is not possible to obtain an analytical solution for the supremum function. Therefore, we proceed with the control design with this function $\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3)$. By substituting $b(\zeta_3)$ and $\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3)$ in the formula (18) we obtain

$$K(\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3), |b(\zeta_3)|) \leq \frac{\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3) + \sqrt{\bar{\phi}^2(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3) + \zeta_3^4}}{\zeta_3^2} = \bar{K}(\zeta). \quad (\text{A7})$$

Let us verify the continuity of the function (A7). Note that, in the surface $b(\zeta) = \zeta_3 = 0$, we have $\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3) < 0$, wherever $\zeta_2 \neq 0$, provided that $\tau > \frac{1}{4}\lambda + \frac{\Delta^2}{\lambda(\Delta - \kappa b_1 x_3^*)}$. Hence, the function (A7) is continuous for all $\zeta \in \{\zeta \in \mathbb{R}^3 : \zeta_2 \neq 0\}$. Indeed, we can see that the origin is the unique point where the function $\bar{K}(\zeta)$ could be discontinuous. Next, we use the SCP-LeF (17) to analyze the behavior of $\bar{K}(\zeta)$ at this point. Observe that the fraction $\frac{\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3)}{\zeta_3^2}$ goes to minus infinity as ζ_3 goes to zero, for any $\zeta_2 \in \mathbb{R} \setminus \{0\}$. On the other hand, if ζ_2 goes to zero, then $\frac{\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3)}{\zeta_3^2}$ goes to $-(b_2 - \frac{1}{\bar{v}_2} - \frac{1}{4}\lambda)$. Thus, we can take any $\delta > 0$ such that $\frac{\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2, \zeta_3)}{\zeta_3^2} \leq \bar{\varepsilon}$ for all $|\zeta| \leq \delta$ and some $\bar{\varepsilon} \geq 0$. Consequently, the function (A7) is continuous for all $\zeta \in \mathbb{R}^3$.

Moreover, remember that we have shown that $\tilde{\mathcal{W}}' \cap \Omega_2 \neq \emptyset$ (with Ω_2 as in (10)) and consequently we obtain $\psi(\zeta) = 1$ from (22). Therefore, the controller (23) takes the form

$$u(\zeta) = -\bar{K}(\zeta)\zeta_3. \quad (\text{A8})$$

Finally, we can see that the closed-loop system (A1),(A8) satisfies Assumption 3, and we establish above that the set $\tilde{\mathcal{W}}_{cl} = \tilde{\mathcal{W}}$ is decomposable in the sense of Definition 3. Thus, by Theorem 1, we can conclude that the closed-loop system (A1),(A8) has the iISS property with respect to the set \mathcal{W}_{cl} . \triangle

Example 8. Let us revisit Example 7. Note that we have not imposed any restriction over the parameter b_2 to conclude that the function (A6) is an iISS-CLeF for the system (A1). Indeed, we can infer that for a sufficiently large value of $b_2 > 0$, the function (A6) is an iISS-Leonov function for the uncontrolled system (A1), such that it has the iISS property with respect to \mathcal{W}_{cl} (given in Example 7), even for $u = 0$. Therefore, the cases when the parameter $b_2 > 0$ is relatively small or even $b_2 < 0$ are of more interest because the controller (A8) (whose design is independent of the value of b_2) becomes completely responsible for rendering the system (A1) iISS with respect to \mathcal{W}_{cl} .

Another interesting case is when $b_2 = c_2 = 0$. Observe that this condition implies that the system (A1) has a continuous set of equilibria, namely all solutions of the equation $b_1 x_3 \sin(x_1) = c_1$. Clearly, the set \mathcal{W} is not decomposable in this case. Despite of this issue, the equilibrium point x^* , given in Example 7, is in this continuous set of equilibria. Hence, we can still apply the change of coordinates $x \mapsto \zeta$ (as in Example 7) to translate x^* to the origin, and define $\tilde{\mathcal{W}}' = \{[2j\pi, 0, 0]^\top : j \in \mathbb{Z}\}$. Since either the conditions to ensure that the function (A6) is an iISS-CLeF for (A1) and the design of the controller (A8) are independent of the set $\tilde{\mathcal{W}}$ and the value of parameters b_2 and c_2 , the controller (A8) is still able to render the system (A1), with $b_2 = c_2 = 0$, iISS with respect to the set \mathcal{W}_{cl} , which is the same as in Example 7, and thus, decomposable in the sense of Definition 3. \triangle

In the previous examples, we have met the case $\tilde{\mathcal{W}} \cap \Omega_2 \neq \emptyset$ and the corresponding selection of the set $\tilde{\mathcal{W}}'$. This case may be the most common one in the utilization of practical ISS- and iISS-CLeF. However, the case $\tilde{\mathcal{W}} \cap \Omega_2 = \emptyset$ is also possible. For this reason, we have considered the option $\tilde{\mathcal{W}}' \subseteq \tilde{\mathcal{W}} \cap \Omega_1$ in Definition 11. Let us illustrate the latter case in the following example. Additionally, we use this example to illustrate the usefulness of the function $\psi(\tilde{x})$ in (22).

Example 9. Consider the system (4). For $|c| < k_2$ and $u = d_1 = d_2 = 0$, the system (4) has equilibria at the points $[x_1^* + 2j\pi, x_2^*]^\top$ and $[(1+2j)\pi - x_1^*, x_2^*]^\top$ for all $j \in \mathbb{Z}$, where $\sin(x_1^*) = ck_2^{-1}$ and $x_2^* = k_1 \sin(x_1^*)$. Then, define the coordinates $\zeta = [\zeta_1, \zeta_2]^\top \in \mathbb{R}^2$ as $\zeta_1 = x_1 - x_1^*$ and $\zeta_2 = x_2 - x_2^*$. Accordingly, the set $\tilde{\mathcal{W}} = \{[2j\pi, 0]^\top \cup [(1+2j)\pi - 2x_1^*, 0]^\top : j \in \mathbb{Z}\}$ contains all equilibria of the system (4) in the coordinates ζ , and it is assumed to be decomposable in the sense of Definition 3. Now, consider the function

$$V(\zeta) = \frac{1}{2}\zeta_2^2 - \frac{1}{2}k_1\zeta_1^2 + k_1 \sin(x_1^*)\zeta_2 - \frac{1}{4}k_1. \quad (\text{A9})$$

By using Young's inequality, we obtain $2|k_1 \sin(x_1^*)\zeta_2| \leq \frac{1}{2}\zeta_2^2 + 2k_1^2 \sin^2(x_1^*)$. Therefore, we have

$$\frac{1}{4}\zeta_2^2 - \frac{1}{2}k_1\zeta_1^2 - 2k_1^2 \sin^2(x_1^*) - \frac{1}{4}k_1 \leq V(\zeta) \leq \frac{3}{4}\zeta_2^2 - \frac{1}{2}k_1\zeta_1^2 + 2k_1^2 \sin^2(x_1^*) - \frac{1}{4}k_1,$$

for all $\zeta \in \mathbb{R}^2$. Thus, the function (A9) satisfies the condition (8) of Definition 11. Furthermore, the derivative of the function V along the trajectories of the system (4), under the change of coordinates $x \mapsto \zeta$, is given by $\dot{V}(\zeta) = a(\zeta, d_1, d_2) + b(\zeta)u$, where $b(\zeta) = \zeta_2 + k_1 \sin(x_1^*)$ and

$$a(\zeta, d_1, d_2) = (k_1^2\zeta_1 - k_2(\zeta_2 + k_1 \sin(x_1^*))) (\sin(\zeta_1 + x_1^*) - \sin(x_1^*)) - k_1\zeta_1\zeta_2 - k_1\zeta_1d_1 + (\zeta_2 + k_1 \sin(x_1^*))d_2. \quad (\text{A10})$$

Note that the restriction $b(\zeta) = 0$ implies $a(\zeta, d_1, d_2) = k_1^2\zeta_1 \sin(\zeta_1 + x_1^*) - k_1\zeta_1d_1$, hence

$$\begin{aligned} (a(\zeta, d_1, d_2) + 4k_1V(\zeta))|_{b=0} &= k_1^2\zeta_1 \sin(\zeta_1 + x_1^*) - k_1\zeta_1d_1 - 2k_1^2\zeta_1^2 - 2k_1^3 \sin^2(x_1^*) - k_1^2 \\ &\leq -\frac{11}{8}k_1^2\zeta_1^2 - \frac{1}{2}k_1^2(2 - \sin^2(\zeta_1 + x_1^*)) - 2k_1^3 \sin^2(x_1^*) + 2d_1^2 \\ &\leq -\frac{11}{8}k_1^2\zeta_1^2 - 2c^2k_1^3k_2^{-2} + 2d_1^2, \end{aligned}$$

where in the second step Young's inequality was used to separate the first and second terms in the right-hand side of the preceding equation, and we recall that $\sin(x_1^*) = ck_2^{-1}$ in the third step. Therefore, we can say that (A9) is a practical ISS-CLeF for the system (4). Moreover, the intersection between the sets $\tilde{\mathcal{W}}$ and Ω_2 is empty in this case. However, there exists an intersection between the sets $\tilde{\mathcal{W}}$ and Ω_1 since $V(\zeta) < 0$ for all $\zeta \in \tilde{\mathcal{W}} \setminus \{[0, 0]\}$ (with Ω_1 and Ω_2 as in (10)). Despite the origin $\zeta = [0, 0]$ is not in Ω_1 , by the periodicity of the system (4) with respect to ζ_1 the points $[2j\pi, 0]$ for all $j \in \mathbb{Z}$ are projected to the unique point $[0, 0] \in M$. Accordingly, we can establish $\tilde{\mathcal{W}}' = \{[2j\pi, 0]^\top : j \in \mathbb{Z}\} \subset \tilde{\mathcal{W}}$.

Let us continue with the controller design. Consider the functions $\bar{v}(|d|) = 2d_1^2 + 2d_2^2$ and $a(\zeta, d_1, d_2)$ given in (A10). Then,

$$\begin{aligned} a(\zeta, d_1, d_2) - \bar{v}(|d|) &= (k_1^2\zeta_1 - k_2(\zeta_2 + k_1 \sin(x_1^*))) (\sin(\zeta_1 + x_1^*) - \sin(x_1^*)) - k_1\zeta_1\zeta_2 \\ &\quad - k_1\zeta_1d_1 + (\zeta_2 + k_1 \sin(x_1^*))d_2 - 2d_1^2 - 2d_2^2 \\ &= (k_1^2\zeta_1 - k_2(\zeta_2 + k_1 \sin(x_1^*))) (\sin(\zeta_1 + x_1^*) - \sin(x_1^*)) - k_1\zeta_1\zeta_2 + \frac{1}{8}k_1^2\zeta_1^2 + \frac{1}{8}(\zeta_2 + k_1 \sin(x_1^*))^2 \\ &\quad - \left(\frac{1}{2\sqrt{2}}k_1\zeta_1 + \sqrt{2}d_1\right)^2 - \left(\frac{1}{2\sqrt{2}}(\zeta_2 + k_1 \sin(x_1^*)) + \sqrt{2}d_2\right)^2. \end{aligned}$$

Hence, substituting the latter equation and a constant $\bar{v} = 0$ in (13), we obtain

$$\begin{aligned}\omega(\zeta) &= \sup_{d \in \mathbb{R}^2} \{a(\zeta, d_1, d_2) - \bar{v}(|d|)\} \\ &= (k_1^2 \zeta_1 - k_2(\zeta_2 + k_1 \sin(x_1^*))) (\sin(\zeta_1 + x_1^*) - \sin(x_1^*)) - k_1 \zeta_1 \zeta_2 + \frac{1}{8} k_1^2 \zeta_1^2 + \frac{1}{8} (\zeta_2 + k_1 \sin(x_1^*))^2.\end{aligned}$$

Now, similarly to (15), we define

$$\begin{aligned}\phi(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2) &= \sup_{\zeta_1 \in \Theta} \{\omega(\zeta) + 2k_1 V(\zeta)\} \\ &= \sup_{\zeta_1 \in \Theta} \left\{ k_1 \zeta_2^2 + \frac{1}{8} (\zeta_2 + k_1 \sin(x_1^*))^2 - k_1 \zeta_1 (\zeta_2 + k_1 \sin(x_1^*)) - \frac{7}{8} k_1^2 \zeta_1^2 + k_1^2 \zeta_1 \sin(\zeta_1 + x_1^*) \right. \\ &\quad \left. - \frac{1}{2} k_1^2 + 2k_1^2 \sin(x_1^*) \zeta_2 - k_2 (\zeta_2 + k_1 \sin(x_1^*)) (\sin(\zeta_1 + x_1^*) - \sin(x_1^*)) \right\} \\ &\leq \sup_{\zeta_1 \in \Theta} \left\{ k_1 \zeta_2^2 + \frac{1}{8} (\zeta_2 + k_1 \sin(x_1^*))^2 - k_1 \zeta_1 (\zeta_2 + k_1 \sin(x_1^*)) - \frac{3}{8} k_1^2 \zeta_1^2 + \frac{1}{2} k_1^2 \sin^2(\zeta_1 + x_1^*) \right. \\ &\quad \left. - \frac{1}{2} k_1^2 + 2k_1^2 \sin(x_1^*) \zeta_2 - k_2 (\zeta_2 + k_1 \sin(x_1^*)) (\sin(\zeta_1 + x_1^*) - \sin(x_1^*)) \right\} \\ &= \sup_{\zeta_1 \in \Theta} \left\{ k_1 \zeta_2^2 + \frac{19}{24} (\zeta_2 + k_1 \sin(x_1^*))^2 - \frac{2}{3} ((\zeta_2 + k_1 \sin(x_1^*)) + \frac{3}{4} k_1 \zeta_1)^2 + \frac{1}{2} k_1^2 \sin^2(\zeta_1 + x_1^*) \right. \\ &\quad \left. - \frac{1}{2} k_1^2 + 2k_1^2 \sin(x_1^*) \zeta_2 - k_2 (\zeta_2 + k_1 \sin(x_1^*)) (\sin(\zeta_1 + x_1^*) - \sin(x_1^*)) \right\} \\ &\leq k_1 \zeta_2^2 + \frac{19}{24} (\zeta_2 + k_1 \sin(x_1^*))^2 - \frac{1}{2} k_1^2 (1 - \sin^2(\zeta_1 + x_1^*)) + 2k_1^2 \sin(x_1^*) \zeta_2 \\ &\quad - k_2 (\zeta_2 + k_1 \sin(x_1^*)) (\sin(\zeta_1 + x_1^*) - \sin(x_1^*)) \\ &= \bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2),\end{aligned}$$

where $\Theta = \{\zeta_1 \in \mathbb{R} : k_1 \zeta_1^2 \leq \frac{3}{2} \zeta_2^2 + 4k_1^2 \sin^2(x_1^*) - \frac{1}{4} k_1\}$. Then, by substituting $b(\zeta) = \zeta_2 + k_1 \sin(x_1^*)$ and $\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2)$ in the formula (18) we obtain

$$K(\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2), |b(\zeta)|) \leq \frac{\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2) + \sqrt{\bar{\phi}^2(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2) + (\zeta_2 + k_1 \sin(x_1^*))^4}}{(\zeta_2 + k_1 \sin(x_1^*))^2} = \bar{K}(\zeta). \quad (\text{A11})$$

Note that the restriction $b(\zeta) = \zeta_2 + k_1 \sin(x_1^*) = 0$ implies $\bar{\phi}(|\zeta|_{\tilde{\mathcal{W}}'}, \zeta_2) \leq -\frac{1}{2} k_1^2 (1 - \sin^2(\zeta_1 + x_1^*)) - k_1^3 \sin^2(x_1^*) < 0$ for all $\zeta \in \mathbb{R}^2$. Hence, the function (A11) is continuous everywhere. In this case, the SCP-LeF (17) is straightforwardly satisfied.

Moreover, we established that $\tilde{\mathcal{W}}' = \{[2j\pi, 0]^\top : j \in \mathbb{Z}\}$, and clearly, the function $b(\zeta)$ does not vanish at any element of this set. Therefore, we have $\tilde{\mathcal{W}}' \cap \Omega_2 = \emptyset$ (with Ω_2 as in (10)). Consequently the function (22) becomes $\psi(\zeta) = \min \left\{ 1, \frac{|\zeta|_{\tilde{\mathcal{W}}'}}{\epsilon} \right\}$, where $|\zeta|_{\tilde{\mathcal{W}}'} = \sqrt{(\sin(\zeta_1 + x_1^*) - \sin(x_1^*))^2 + \zeta_2^2}$, and $0 < \epsilon \leq \epsilon^*$ is a constant. We can deduce that the origin $\zeta = [0, 0]$ is the closest element of $\tilde{\mathcal{W}}'$ to the line $V(\zeta) = 0$, besides the smallest distance between the origin and the line $V(\zeta) = 0$ is on the plane $\zeta_1 = 0$, therefore ϵ^* is given by a solution of the equation $V(\zeta) = \frac{1}{2} \zeta_2^2 + k_1 \sin(x_1^*) \zeta_2 - \frac{1}{4} k_1 = 0$, such that $\epsilon^* = -k_1 \sin(x_1^*) + \frac{1}{2} \sqrt{4k_1^2 \sin^2(x_1^*) + 2k_1}$.

Finally, the controller (23) takes the form

$$u(\zeta) = -\min \left\{ 1, \frac{|\zeta|_{\tilde{\mathcal{W}}'}}{\epsilon} \right\} \bar{K}(\zeta_2 + k_1 \sin(x_1^*)). \quad (\text{A12})$$

Hence, we can say that the closed-loop system (4),(A12) satisfies Assumption 3, and the set \mathcal{W}_{cl} is assumed to be decomposable in the sense of Definition 3, and contains the set \mathcal{W}' since the function $\psi(\zeta)$ makes $u(\zeta) = 0$ for all $\zeta \in \mathcal{W}'$. Thus, by Theorem 1 we can conclude that the closed-loop system (4),(A12) has the ISS property with respect to the set \mathcal{W}_{cl} . \triangle