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A strict ISS-Lyapunov function for LTV systems with non-time-differentiable dynamics

Frédéric Mazenc Mohamed Maghenem Antonio Loria

Abstract—We study stability and robustness for a large class of linear time-varying systems under the assumption that the system possesses some kind of excitation, which is necessary for uniform attractivity of the origin, but not even boundedness of the solutions is assumed *a priori*. Our main statements provide strict Lyapunov functions, *i.e.*, having a strictly negative-definite derivative, constructed based on an initial candidate whose derivative is sign-undefined. The Lyapunov function that we construct guarantees uniform global asymptotic stability and input-to-state stability with respect to bounded additive inputs. As a byproduct of our main results, we provide a Lyapunov function for a class of systems reminiscent of model-reference adaptive control with non-differentiable regressors.

Index Terms—Adaptive control, Lyapunov functions, Time-varying systems, Persistency of excitation

I. INTRODUCTION

For continuous-time non-autonomous systems, uniform asymptotic stability is the strongest Lyapunov stability property that the solutions of the system may enjoy, since it implies local input-to-state stability (ISS) [1]. For the global counter-part of the latter, however, uniform global asymptotic stability (UGAS) is necessary but not sufficient. In turn, it is known after [2] that the existence of a positive definite radially unbounded Lyapunov function V , with negative definite derivative, is necessary and sufficient for UGAS. Such a Lyapunov function is often called *strict* and it may be useful (depending on the bound on \dot{V}) to establish the desirable property of global ISS. After [3], it is known that UGAS is equivalent to the existence of a strict Lyapunov function V and a constant $c > 0$ such that $\dot{V}(t, x) \leq -cV(t, x)$. However, in general, it is difficult to explicitly construct such a function.

In [4] a strict Lyapunov function is constructed as a function of another, non-strict, Lyapunov function V satisfying

$$\dot{V}(t, x) \leq -\varphi(t)V(t, x), \quad (1)$$

with $\varphi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ periodic. However, neither the periodicity nor the non-negativity of $\varphi(t)$ are necessary. For instance, for dimension-one systems $\dot{x} = -\varphi(t)x$, with $\varphi(t) := a(t) - b$ with $a : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ persistently exciting (PE) and $b > 0$, UGAS is established in [5]. In [6], a clever method is provided to construct Lyapunov functions for similar scalar time-varying systems, for which (1) holds,

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relying on the comparison equation $\dot{v} \leq -\varphi(t)v$ and without requiring that $\varphi(t) \geq 0$ for all $t \geq 0$; see also [7].

For linear time-varying systems, UGAS is equivalent to the existence of Lyapunov functions of the specific form $V(t, x) := x^\top P(t)x$ with $P(t) = P(t)^\top > 0$ such that $\dot{V}(t, x) \leq -x^\top Q(t)x$, with $Q(t) := cP(t)$. In some applications, such as high-gain observer design [8], the matrices P and Q may be explicitly constructed. In other cases, a candidate $P(t)$ may be inferred from the system's properties (*e.g.*, in the case of passive systems) but the corresponding $Q(t)$ is not necessarily positive definite for all $t \geq 0$, only positive semi-definite. For example, for the gradient system $\dot{x} = -B(t)B(t)^\top x$, for $P := I$, we find $Q := 2BB^\top$. Then, under PE of B , *i.e.*, the existence of T and $\mu > 0$ such that

$$\int_t^{t+T} Q(s)ds \geq \mu \quad \forall t \geq 0, \quad (2)$$

strict Lyapunov functions are constructed in [9], [10].

In this paper, we provide a strict ISS-Lyapunov function for linear time-varying systems admitting quadratic Lyapunov functions such that $\dot{V}(t, x) \leq -x^\top Q(t)x$, with $Q(t)$ not necessarily being positive semi-definite for all $t \geq 0$, so not even boundedness of solutions is assumed *a priori*. The hypothesis on Q that guarantees UGAS is expressed in the form of a bound on an integral of Q over a sliding window and it is akin to an observability Grammian. Our construction is tailored for generic linear time-varying systems, but it is particularly useful to design Lyapunov functions for systems appearing in adaptive control—*cf.* [11] and [12]. However, the constructions proposed therein require differentiability of the regressor. In [9], strict Lyapunov functions are constructed for systems whose regressors have isolated points of non-differentiability. To the best of our knowledge, the first strict Lyapunov function for such systems was proposed in [13], [14]. In [15], we present a strict Lyapunov function for systems appearing in adaptive control, but the results therein are restricted to *scalar* non-differentiable regressors.

II. PROBLEM FORMULATION AND MOTIVATION

We consider linear time-varying systems of the form

$$\dot{x} = \mathcal{A}(t)x \quad x \in \mathbb{R}^n, \quad (3)$$

and its counterpart with inputs

$$\dot{x} = \mathcal{A}(t)x + \delta(t) \quad x \in \mathbb{R}^n, \quad (4)$$

where $\delta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ and $\mathcal{A} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n \times n}$ are locally integrable functions. We address the problem of constructing

a strict Lyapunov function for (3) and a strict ISS-Lyapunov function for (4), under the following regularity hypothesis.

Assumption 1: There exist a once-differentiable function $\mathcal{P} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n \times n}$ and a locally integrable function $\mathcal{Q} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n \times n}$ such that $\mathcal{P}(t) = \mathcal{P}(t)^\top > 0$ and $\mathcal{Q}(t) = \mathcal{Q}(t)^\top$ for all $t \geq 0$, as well as constants $c_i > 0$ for all $i \in \{1, 2, 3, 4\}$ such that, for all $t \geq 0$,

$$\dot{\mathcal{P}}(t) + \mathcal{P}(t)\mathcal{A}(t) + \mathcal{A}(t)^\top \mathcal{P}(t) \leq -\mathcal{Q}(t) \quad (5)$$

$$c_1 I \leq \mathcal{P}(t), \quad |\mathcal{P}|_\infty \leq c_2, \quad |\mathcal{Q}|_\infty \leq c_3, \quad |\mathcal{A}|_\infty \leq c_4. \quad (6)$$

In (6), $|f|_\infty := \operatorname{ess\,sup}_{t \geq 0} |f(t)|$ and $|\cdot|$ denotes the Euclidean norm of vectors or the induced norm of matrices.

Inequality (5) entails that, along trajectories of (3), the function $\nu(t) := x(t)^\top \mathcal{P}(t)x(t)$ verifies $\dot{\nu}(t) \leq -x(t)^\top \mathcal{Q}(t)x(t)$. However, we stress that $\mathcal{Q}(t)$ is not assumed to be positive semi-definite for all $t \geq 0$, where it is defined.

Example 1: Let $\{t_k\}$ be an increasing sequence of time instants and consider the system (4) with

$$\mathcal{A}(t) := \begin{cases} A_1 & t \in [t_{2k}, t_{2k+1}) \\ A_2 & t \in [t_{2k+1}, t_{2k+2}) \end{cases} \quad k \in \{0, 1, \dots\}, \quad t_0 = 0.$$

Assume that there exist matrices P_1 and Q_1 , symmetric and positive definite, such that $P_1 A_1 + A_1^\top P_1 = -Q_1$. Then, let $Q_2 := -P_1 A_2 - A_2^\top P_1$. Note that Q_2 is symmetric, but not necessarily positive semi-definite. Note that Assumption 1 is verified in this case for $\mathcal{P}(t) = P_1$ and

$$\mathcal{Q}(t) = \begin{cases} Q_1 & t \in [t_{2k}, t_{2k+1}) \\ Q_2 & t \in [t_{2k+1}, t_{2k+2}) \end{cases} \quad k \in \{0, 1, \dots\}.$$

Although $\mathcal{Q}(t)$ is not positive semi-definite for all $t \geq 0$, under some conditions on the sequence $\{t_k\}_{k=0}^\infty$, our main result enables the construction of a strict Lyapunov function. \square

Another significant example of systems covered by Assumption 1 is that of passive systems of the form

$$\begin{bmatrix} \dot{e} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} A_p(t) & -B(t)^\top \\ C(t) & 0 \end{bmatrix} \begin{bmatrix} e \\ \theta \end{bmatrix} + \begin{bmatrix} d(t) \\ 0 \end{bmatrix}, \quad (e, \theta) \in \mathbb{R}^n \times \mathbb{R}^m \quad (7)$$

satisfying the following—cf. [16], [17].

Assumption 2: The functions A_p and B are locally integrable and there exist a once-differentiable function $P_p : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n \times n}$ such that $P_p(t) = P_p(t)^\top > 0$, a locally integrable function $Q_p : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n \times n}$ such that $Q_p(t) = Q_p(t)^\top > 0$, and constants a_M, b_M, p_M, q_M , and $q_M > 0$ such that

$$C(t) = B(t)P_p(t), \quad |A_p|_\infty \leq a_M, \quad |B|_\infty \leq b_M. \quad (8)$$

$$\dot{P}_p(t) + P_p(t)A_p(t) + A_p(t)^\top P_p(t) =: -Q_p(t), \quad (9)$$

$$I \leq P_p, \quad |P_p|_\infty \leq p_M, \quad |Q_p|_\infty \leq q_M, \quad Q_p \geq q_M I. \quad (10)$$

Clearly, the system (7) is of the form (4) with

$$\mathcal{A}(t) := \begin{bmatrix} A_p(t) & -B(t)^\top \\ C(t) & 0 \end{bmatrix}, \quad (11)$$

and under Assumption 2, Assumption 1 holds with

$$\mathcal{P}(t) := \operatorname{diag}\{P_p(t), I\}, \quad \mathcal{Q}(t) := \operatorname{diag}\{Q_p(t), 0\}. \quad (12)$$

Indeed, in view of (9), the inequality in (5) holds for \mathcal{A} , \mathcal{P} , and \mathcal{Q} as in (11)–(12), respectively. On the other hand, a direct computation shows that, in view of the inequalities in (8) and (10), the inequalities in (6) hold with

$$c_1 = 1, \quad c_2 = 1 + p_M, \quad c_3 = q_M, \quad c_4 = a_M + c_2 b_M. \quad (13)$$

The system in (7) with $d \equiv 0$,

$$\begin{bmatrix} \dot{e} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} A_p(t) & -B(t)^\top \\ C(t) & 0 \end{bmatrix} \begin{bmatrix} e \\ \theta \end{bmatrix}, \quad (14)$$

has been extensively studied (most often with A_p constant and Hurwitz) in the literature of adaptive control for over fifty years. For instance, as it is well-known from [18], under the additional hypothesis that B is differentiable and has bounded derivative, it is necessary and sufficient for UGAS that B be PE. Indeed, there are many proofs of UGAS for (14); some rely on output-injection transformations [19], while others on strict Lyapunov functions [11], [20]. However, independently of the method of proof, it is often required that B in (14) be differentiable and have a uniformly bounded derivative. Yet, relaxing such assumption is possible and can be important, as we illustrate in the following example.

Example 2 (Observer design for rotating machines):

Consider a motor (with unitary rotor inertia) modeled by

$$\dot{\theta} = w, \quad \dot{w} = -kw + \tau - \tau_L,$$

where θ and w are, respectively, the angular position and velocity; τ is the electromagnetic torque, τ_L is a constant load, and k is a friction coefficient. To estimate (τ_L, w) , it is usual to consider $y := (y_1, y_2) := (\sin \theta, \cos \theta)$ as output [15]. Then, defining $\eta := w + \tau_L/k$, we obtain

$$\dot{y} = (y_2, -y_1)^\top (\eta - \tau_L/k), \quad \dot{\eta} = -k\eta + \tau.$$

The Luenberger-like observer we consider here is given by

$$\begin{aligned} \dot{\hat{y}} &= \begin{bmatrix} y_2 \\ -y_1 \end{bmatrix} (\hat{\eta} - \hat{\tau}_L/k) + \begin{bmatrix} k_1(y_1 - \hat{y}_1) \\ k_2(y_2 - \hat{y}_2) \end{bmatrix}, \\ \dot{\hat{\eta}} &= -k\hat{\eta} + \tau + k_{\eta 1}(y)(y_1 - \hat{y}_1) + k_{\eta 2}(y)(y_2 - \hat{y}_2), \\ \dot{\hat{\tau}}_L &= k_{\tau 1}(y)(y_1 - \hat{y}_1) + k_{\tau 2}(y)(y_2 - \hat{y}_2), \end{aligned}$$

where $k_1, k_2 > 0$, $k_{\eta 1}(y) := y_2$, $k_{\eta 2}(y) := -y_1$, $k_{\tau 1}(y) := -y_2/k$, $k_{\tau 2}(y) := y_1/k$. In the error variables $x := (\tilde{y}, \tilde{\eta}, \tilde{\tau}_L) := (\hat{y} - y, \hat{\eta} - \eta, \hat{\tau}_L - \tau_L)$, the resulting error system is of the form (14) with $B(t) = [-y_2(t)/k \quad y_1(t)/k \quad 0]$. Note that Assumption 2 is verified with $P_p = I$, and \dot{B} is function of w . Hence, the methods cited above, requiring $|w|_\infty < +\infty$, are conservative. Moreover, larger values of $|w|_\infty$ lead to more restrictive performance guarantees. \square

In [21], when dropping boundedness of \dot{B} , the following assumption is shown to be equivalent to UGAS for (14).

Assumption 3: There exist Δ , μ , and $\sigma > 0$ such that, for each $t \geq 0$ and for each unitary vector ξ , there exists $t' \in [t, t + \Delta]$ such that

$$\left| \int_{t'}^{t'+\sigma} B(s)^\top \xi ds \right| \geq \mu. \quad (15)$$

In [18], and more recently in [14], the same result is shown to hold under following assumption.

Assumption 4: There exist \bar{T} and $\bar{\mu} > 0$ such that

$$\int_t^{t+\bar{T}} \left[\int_t^s B(\tau) d\tau \right] \left[\int_t^s B(\tau) d\tau \right]^\top ds \geq \bar{\mu} I \quad \forall t \geq 0. \quad (16)$$

To the best of the authors' knowledge, the first strict Lyapunov function for (14), without assuming the differentiability of B , is provided in [14]; relying on Assumptions 2 and 4. An alternative construction is proposed in [13], relying on the solution to a bilinear Riccati equation. In this paper, we analyze ISS of systems in the general form (4), under the mild Assumption 1. Our main result consists in constructing a smooth ISS-Lyapunov function $\mathcal{V} : \mathbb{R}_{\geq T} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, for some $T \geq 0$, such that there exist positive constants α_i , with $i \in \{1, 2, 3\}$, and $\beta > 0$ such that, for all $(t, x) \in \mathbb{R}_{\geq T} \times \mathbb{R}^n$,

$$\begin{aligned} \alpha_1 |x|^2 &\leq \mathcal{V}(t, x) \leq \alpha_2 |x|^2 & (17) \\ \dot{\mathcal{V}}(t, x) &\leq -\alpha_3 |x|^2 + \beta |\delta(t)|^2. & (18) \end{aligned}$$

As a byproduct, we construct a smooth ISS-Lyapunov function for system (4) with \mathcal{A} as in (11) and satisfying Assumptions 2 and 3. That is, in particular, we provide a strict Lyapunov function for the system (14) without relying on the differentiability of the regressor B . The function that we construct cannot be obtained following [14] or [13] because even though Assumptions 3 and 4 are equivalent, the relation between the parameters T , μ and σ in Assumption 3, and the parameters \bar{T} and $\bar{\mu}$ in Assumption 4 is neither evident nor explicit in the literature. The construction in [15], in general, cannot be used for systems of arbitrary dimension.

III. GENERIC LTV SYSTEMS

Consider system (4) and let $\Phi(\tau, t)$ denote the state transition matrix corresponding to $\mathcal{A}(\tau)$. That is, we let $x(\tau) = \Phi(\tau, t)x(t)$ denote the solution to system (3).

Assumption 5: For the matrix function \mathcal{Q} introduced in Assumption 1, there exist $p > 0$ and $T > 0$ such that

$$pI \leq \phi(t) := \frac{1}{T} \int_{t-T}^t \Phi(\tau, t)^\top \mathcal{Q}(\tau) \Phi(\tau, t) d\tau \quad \forall t \geq T. \quad (19)$$

Remark 1: A similar assumption is posed in [18], for $\mathcal{Q}(t) \geq 0$. To verify Assumption 5, it is not necessary to know $\Phi(\tau, t)$ —see Claim 1 below and the forthcoming Proposition 1. In particular, (19) can be expressed as

$$\nu(t-T) - \nu(t) \geq pT|x(t)|^2 \quad \forall t \geq T.$$

Such a (non-infinitesimal) decay property is verified in many situations using qualitative techniques [22], [23]. •

Then, to construct a Lyapunov function of the form

$$\mathcal{V}(t, x) := x^\top [\mathcal{P}(t) + \Upsilon(t)]x, \quad (20)$$

we consider the following assumption.

Assumption 6: Given T as in Assumption 5 and \mathcal{P} as in Assumption 1, there exists $c_5 > 0$ such that the function $\Upsilon : \mathbb{R}_{\geq T} \rightarrow \mathbb{R}^{n \times n}$, defined as

$$\Upsilon(t) := \frac{1}{T} \int_{t-T}^t \int_s^t \Phi(\tau, t)^\top \mathcal{Q}(\tau) \Phi(\tau, t) d\tau ds, \quad (21)$$

satisfies

$$c_5 I \leq \mathcal{P}(t) + \Upsilon(t) \quad \forall t \geq T. \quad (22)$$

Remark 2: In the particular case where $\mathcal{Q}(t)$ is positive semi-definite for all $t \geq 0$, so is $\Upsilon(t)$ for all $t \geq T$. Then, Assumption 6 follows from Assumption 1, with $c_5 = c_1$. •

Claim 1: For the switched system $\dot{x} = \mathcal{A}(t)x$ with \mathcal{A} as in Example 1 and appropriate dwell times T_1 and T_2 , i.e., with $t_{2k+1} - t_{2k} = T_1$ and $t_{2k+2} - t_{2k+1} = T_2$ for all $k \in \{0, 1, \dots\}$, Assumptions 5 and 6 hold.

Proof: Let $T := T_1 + T_2$. Then, for all $t \geq T$ and for all $y \in \mathbb{R}^n$, we have $Ty^\top \phi(t)y = \nu(t-T) - \nu(t)$, where $\nu(t) := x(t)^\top P_1 x(t)$ and $x(\tau) := \Phi(\tau, t)y$. Assume, without loss of generality, that $t = t_2$. Using, the positive definiteness of P_1 and Q_1 , we conclude that

$$\begin{aligned} \nu(t-T_1) - \nu(t) &\geq a_1 \int_{t-T_1}^t \nu(\tau) d\tau, \quad a_1 := \frac{\lambda_m(Q_1)}{\lambda_M(P_1)}, \\ \nu(t-T) - \nu(t-T_1) &\geq -a_2 \int_{t-T}^{t-T_1} \nu(\tau) d\tau, \quad a_2 := \frac{\lambda_M(Q_2)}{\lambda_m(P_1)}. \end{aligned}$$

Furthermore, using Gronwall-Bellman inequality, we obtain

$$\begin{aligned} \nu(t-T) - \nu(t-T_1) &\geq [\exp^{-a_2 T_2} - 1] \nu(t-T_1), \\ \nu(t) - \nu(t-T_1) &\leq [\exp^{-a_1 T_1} - 1] \nu(t-T_1), \\ \nu(t-T_1) &\geq \exp^{a_1 T_1} \nu(t) = \exp^{a_1 T_1} y^\top P_1 y. \end{aligned}$$

The latter leads to $Ty^\top \phi(t)y \geq [\exp^{(a_1 T_1 - a_2 T_2)} - 1] y^\top P_1 y$. Hence, we conclude that Assumption 5 holds if $a_1 T_1 > a_2 T_2$. To verify Assumption 6, we note that

$$\begin{aligned} P_1 + \Upsilon(t) &= P_1 + \frac{1}{T} \int_{t-T_1}^t \int_s^t \Phi(\tau, t)^\top Q_1 \Phi(\tau, t) d\tau ds \\ &\quad + \frac{1}{T} \int_{t-T}^{t-T_2} \int_s^t \Phi(\tau, t)^\top Q_2 \Phi(\tau, t) d\tau ds \\ &\geq P_1 + \frac{\lambda_m(Q_1)}{T} \int_{t-T_1}^t \int_s^t \Phi(\tau, t)^\top \Phi(\tau, t) d\tau ds \\ &\quad - \frac{\lambda_M(Q_2)}{T} \int_{t-T}^{t-T_2} \int_s^t \Phi(\tau, t)^\top \Phi(\tau, t) d\tau ds. \end{aligned}$$

Having $\exp^{-2c_4(t-\tau)} \leq |\Phi(\tau, t)^\top \Phi(\tau, t)| \leq \exp^{2c_4(t-\tau)}$ for all $t \geq \tau \geq 0$, we conclude that Assumption 6 is verified if

$$\begin{aligned} 1 + \frac{a_1}{T} \int_{t-T_1}^t \int_s^t \exp^{-2c_4(t-\tau)} d\tau ds \\ - \frac{a_2}{T} \int_{t-T}^{t-T_2} \int_s^t \exp^{2c_4(t-\tau)} d\tau ds > 0. \end{aligned}$$

In turn, the latter inequality is verified if

$$1 + \frac{a_1}{2c_4T} \left[T_1 - \frac{1}{2c_4} [1 - \exp^{-2c_4T_1}] \right] + \frac{a_2}{2c_4T} \left[T_2 - \frac{\exp^{2c_4T}}{2c_4} [1 - \exp^{-2c_4T_2}] \right] > 0,$$

which holds for sufficiently large T_1 and T_2 . ■

Theorem 1 (ISS-LF): Let the system (4) satisfy Assumptions 1, 5, and 6. Then, there exist $\alpha_1, \alpha_2, \alpha_3$, and $\beta > 0$ such that \mathcal{V} in (20) satisfies (17) and (18) for all $x \in \mathbb{R}^n$ and all $t \geq T$. That is, \mathcal{V} is an ISS-Lyapunov function. □

Proof of Theorem 1: For all $t \geq T$ and all $x \in \mathbb{R}^n$, we show that

$$c_5|x|^2 \leq \mathcal{V}(t, x) \leq \left[c_2 + \frac{Tc_3e^{2c_4T}}{2} \right] |x|^2, \quad (23)$$

$$\dot{\mathcal{V}}(t, x) \leq -\frac{p}{2}|x|^2 + \frac{4}{p} \left[c_2^2 + \frac{T^2}{4} e^{4c_4T} c_3^2 \right] |\delta|^2. \quad (24)$$

The first inequality in (23) is equivalent to (22). The second follows after (6) and (21). Indeed, from the latter we have

$$|\Upsilon(t)| \leq \frac{1}{T} \int_{t-T}^t \int_s^t |\Phi(\tau, t)|^2 |Q(\tau)| d\tau ds \leq \frac{T}{2} e^{2c_4T} c_3, \quad (25)$$

for all $t \geq T$. Hence, (23) follows from (25) and (6).

On the other hand, the derivative of \mathcal{V} along the trajectories of (4) satisfies, in view of (5),

$$\dot{\mathcal{V}}(t, x) \leq x^\top [-Q(t) + \mathcal{A}(t)^\top \Upsilon(t) + \Upsilon(t)\mathcal{A}(t) + \dot{\Upsilon}(t)]x + x^\top \mathcal{P}(t)\delta + \delta^\top \mathcal{P}(t)x + \delta^\top \Upsilon(t)x + x^\top \Upsilon(t)\delta. \quad (26)$$

In the latter,

$$\begin{aligned} \dot{\Upsilon}(t) &= Q(t) - \frac{1}{T} \int_{t-T}^t \Phi(\tau, t)^\top Q(\tau) \Phi(\tau, t) d\tau \\ &\quad - \Upsilon(t)\mathcal{A}(t) - \mathcal{A}(t)^\top \Upsilon(t), \end{aligned} \quad (27)$$

so that (26) becomes

$$\begin{aligned} \dot{\mathcal{V}}(t, x) &\leq -x^\top \left[\frac{1}{T} \int_{t-T}^t \Phi(\tau, t)^\top Q(\tau) \Phi(\tau, t) d\tau \right] x \\ &\quad + x^\top \mathcal{P}(t)\delta + \delta^\top \mathcal{P}(t)x + \delta^\top \Upsilon(t)x + x^\top \Upsilon(t)\delta. \end{aligned}$$

In turn, after Assumption 5, we obtain

$$\dot{\mathcal{V}}(t, x) \leq -p|x|^2 + x^\top \mathcal{P}(t)\delta + \delta^\top \mathcal{P}(t)x + \delta^\top \Upsilon(t)x + x^\top \Upsilon(t)\delta, \quad (28)$$

so using Young's inequality to bound the cross terms $2|x||\mathcal{P}(t)\delta|$ and $2|x||\Upsilon(t)\delta|$, we obtain

$$\dot{\mathcal{V}}(t, x) \leq -\frac{p}{2}|x|^2 + \frac{4}{p} |\mathcal{P}(t)\delta|^2 + \frac{4}{p} |\Upsilon(t)\delta|^2, \quad (29)$$

which implies (24), in view of (6) and (25). ■

IV. PASSIVE LTV SYSTEMS

For the system (4) with \mathcal{A} as in (11) we provide an ISS-Lyapunov function without assuming that the regressor B is differentiable, and based on the strengthened persistency of excitation condition formulated in Assumption 3.

Theorem 2: Consider the system in (4), with \mathcal{A} as in (11), and under Assumptions 2 and 3. Then, there exist $T, p > 0$ such that the function \mathcal{V} in (20), with \mathcal{P} and Υ as in (12) and (21) respectively, satisfies (23) and (24), with $c_5 = c_1$ and c_1, c_2, c_3 , and c_4 as in (13). In particular, the system (4)-(11) is input-to-state stable. □

Proof: The proof is established invoking Theorem 1. Assumption 1 in the latter is implied by Assumption 2; as a matter of fact, Assumption 1 holds with “=” in (5). Assumption 6 also holds under Assumption 2, since in this case $\Upsilon(t)$ is positive semi-definite for all $t \geq T$; see Remark 2. Therefore, the function \mathcal{V} in (20), with \mathcal{P} and Υ as in (12) and (21), respectively, satisfies (23).

That (24) holds, along (4) with \mathcal{A} as in (11), follows by showing that Assumption 5 in Theorem 1 is implied by Assumptions 2 and 3; see Proposition 1 below. ■

Proposition 1: Consider the system (14) under Assumption 2, and let the latter generate the constants a_M, b_M, p_M , and $q_m > 0$. Then, if Assumption 3 holds for the locally integrable regressor B , so does Assumption 5 with

$$p := \min\{\gamma, 1 - \kappa\}1, \quad T := T_1 + \varepsilon_2, \quad (30)$$

where, for some $\kappa, \varepsilon_1 \in (0, 1)$ and $\varepsilon_2 > 0$,

$$\rho := \sqrt{\frac{1 - \varepsilon_1}{p_M}} - \varepsilon_2 \max\{a_M, b_M\} > 0, \quad (31)$$

$$\gamma := \frac{q_m \varepsilon_2}{\kappa} \rho^2 > 0, \quad (32)$$

$$T_1 := \left[\frac{1 - \kappa \varepsilon_1 - \beta_1}{q_m \beta_1 \beta_2} + 1 \right] (\Delta + \sigma + \beta_2) > 0, \quad (33)$$

$$\beta_1 := \frac{\mu^2 \varepsilon_1 \kappa}{2 [2 + a_M \sigma + (2\Delta + \sigma) b_M^2 p_M \sigma]^2}, \quad (34)$$

$$\beta_2 := \frac{1}{q_M + b_M p_M} \log \left(\frac{b_M / \beta_3 + 2\beta_1}{b_M / \beta_3 + p_M \beta_1} \right). \quad (35)$$

□

Proof of Proposition 1: The proof follows from a chain of implications that is organized in three claims, whose proofs in turn are provided farther below.

Claim 2: Under Assumption 3, the following holds.

Property 1: Given $\varepsilon_1, \kappa \in (0, 1)$, and a solution to (14), $t \mapsto x(t)$ with $x := [e^\top \theta^\top]^\top$ and $t_o \geq 0$, there exists $t_1 \in [t_o, t_o + T_1]$, with T_1 as in (33), such that

$$|\theta(t_1)|^2 \leq \kappa \varepsilon_1 \nu(t_o), \quad (36)$$

where $\nu(t) := x(t)^\top \mathcal{P}(t)x(t)$ and \mathcal{P} is defined in (12). □

Claim 3: Property 1 implies the following:

Property 2: Given $\kappa, \varepsilon_1 \in (0, 1)$, along every solution $t \mapsto x(t)$ defined on $[t_o, +\infty)$,

$$\nu(t) \geq \kappa \nu(t_o) \quad \forall t \in [t_o, t_o + T] \quad (37)$$

implies that

$$\nu(t_o + T) \leq (1 - \gamma)\nu(t_o), \quad (38)$$

where γ and T are defined in (32) and (30), respectively. \square

Claim 4: Under Property 2, Assumption 5 holds for the same T and for p as in (30).

Proof of Claim 2: Let Assumption 3 generate Δ , μ , and $\sigma > 0$, and let $\kappa, \varepsilon_1 \in (0, 1)$ be arbitrarily fixed such that

$$|\theta(s)|^2 > \kappa\varepsilon_1\nu(t_o) \quad \forall s \in [t_o, t_o + \Delta + \sigma]. \quad (39)$$

We claim that, consequently,

$$\exists t_1 \in [t_o, t_o + \Delta + \sigma] : |e(t_1)|^2 > 2\beta_1\nu(t_o), \quad (40)$$

where β_1 is defined in (34). Therefore, after Lemma 1 in the Appendix, it follows that under Assumption 2,

$$|e(t)|^2 \geq \beta_1\nu(t_o) \quad \forall t \in [t_1, t_1 + \beta_2],$$

where β_2 is defined in (35).

Now, to verify (40), we integrate the system's equation, $\dot{e} = A_p(t)e - B(t)^\top\theta$, and introduce the unitary vector $\xi := \theta(t_o)/|\theta(t_o)|$, which is well defined under (39). We obtain

$$\begin{aligned} & -e(t + \sigma) + e(t) + \int_t^{t+\sigma} A_p(\tau)e(\tau)d\tau \\ & - \int_t^{t+\sigma} \int_{t_o}^\tau B(\tau)^\top B(s)P_p(s)e(s)dsd\tau = \int_t^{t+\sigma} B(\tau)^\top \xi d\tau |\theta(t_o)|. \end{aligned}$$

The previous expression holds for any $t \geq 0$, so let $t = t'$, where $t' \in [t_o, t_o + \Delta]$ is generated by Assumption 3. Using (39), we conclude that

$$\begin{aligned} & \left| e(t + \sigma) - e(t) - \int_{t'}^{t'+\sigma} A_p(\tau)e(\tau)d\tau \right. \\ & \left. + \int_{t'}^{t'+\sigma} \int_{t_o}^\tau B(\tau)^\top B(s)P_p(s)e(s)dsd\tau \right| > \mu\sqrt{\kappa\varepsilon_1\nu(t_o)} \end{aligned}$$

and, under Assumption 2, we obtain the lower bound

$$[2 + a_M\sigma + (2\Delta + \sigma)b_{MPM}^2] \max_{s \in [t', t'+\sigma]} |e(s)| > \mu\sqrt{\kappa\varepsilon_1\nu(t_o)}.$$

Hence, there exists $t_1 \in [t_o, t_o + \Delta + \sigma]$ such that

$$|e(t_1)| > \frac{\mu\sqrt{\kappa\varepsilon_1\nu(t_o)}}{2 + a_M\sigma + (2\Delta + \sigma)b_{MPM}^2} =: \sqrt{2\beta_1\nu(t_o)}.$$

Now, for some integer $n > 0$, we consider the interval $[t_o, t_o + n(\Delta + \sigma + \beta_2)]$ and, to proceed by contradiction, assume that

$$|\theta(s)|^2 > \kappa\varepsilon_1\nu(t_o) \quad \forall s \in [t_o, t_o + n(\Delta + \sigma + \beta_2)]. \quad (41)$$

We shall find $n^* > 0$ such that (41) cannot hold for $n = n^*$. This would complete the proof for $T_1 := n^*(\Delta + \sigma + \beta_2)$.

Under (41) and what we proved before, we conclude that, for each $i \in \{0, 1, \dots, n-1\}$, there exists

$$t_{1i} \in [t_o + i(\Delta + \sigma + \beta_2), t_o + (i+1)(\Delta + \sigma + \beta_2) - \beta_2] \quad (42)$$

such that $|\theta(s)|^2 > \kappa\varepsilon_1\nu(t_o)$ and

$$|e(s)|^2 > \beta_1\nu(t_o) \quad \forall s \in [t_{1i}, t_{1i} + \beta_2]. \quad (43)$$

Additionally, using (12), we obtain

$$\dot{\nu}(s) \leq -q_m|e(s)|^2 \quad \forall s \in [t_{1i}, t_{1i} + \beta_2].$$

Hence, using (43) for each $i \in \{0, 1, \dots, n-1\}$, we obtain

$$\nu(t_{1i} + \beta_2) - \nu(t_{1i}) \leq -q_m\beta_1\beta_2\nu(t_o). \quad (44)$$

Now, since $\dot{\nu}(t) \leq 0$ for all $t \geq t_o \geq 0$, in view of (42), we have $\nu(t_{1i} + \beta_2) \geq \nu(t_{1i+1})$ and, from (44), we get

$$\nu(t_{1i+1}) - \nu(t_{1i}) \leq -q_m\beta_1\beta_2\nu(t_o) \quad \forall i \in \{0, 1, \dots, n-1\}.$$

This implies that

$$\begin{aligned} |e(t_{1n-1})|^2 + |\theta(t_{1n-1})|^2 & \leq \nu(t_{1n-1}) \\ & \leq [1 - (n-1)q_m\beta_1\beta_2]\nu(t_o). \end{aligned}$$

Hence,

$$[\beta_1 + \kappa\varepsilon_1]\nu(t_o) < \nu(t_{1n-1}) \leq [1 - (n-1)q_m\beta_1\beta_2]\nu(t_o).$$

Setting $n = n^*$, with n^* the smallest integer such that

$$1 - (n^* - 1)q_m\beta_1\beta_2 \leq \kappa\varepsilon_1 + \beta_1,$$

contradicts (41). Hence, Property 1 follows with

$$T_1 := n^*(\Delta + \sigma + \beta_2) \text{ and } n^* := \frac{1 - \kappa\varepsilon_1 - \beta_1}{q_m\beta_1\beta_2} + 1. \quad \blacksquare$$

Proof of Claim 3: After Claim 2, for any arbitrarily fixed $\kappa, \varepsilon_1 \in (0, 1)$, there exists $t_1 \in [t_o + t_o + T_1]$ with T_1 as above, such that (36) holds. Then,

$$|\theta(t_1)|^2 \leq \varepsilon_1\kappa\nu(t_o). \quad (45)$$

Next, as defined in (30), consider $T = T_1 + \varepsilon_2$ for some $\varepsilon_2 > 0$. Since $\dot{\nu}(t) \leq 0$ we have $\nu(t) \leq \nu(t_o)$ for any $t \geq t_o$. Therefore, following Property 2, let

$$\nu(t_o) \geq \nu(t) > \kappa\nu(t_o) \quad \forall t \in [t_o, t_o + T]. \quad (46)$$

As a result, we have $\nu(t_1) \geq \kappa\nu(t_o)$, so in view of (45),

$$p_M|e(t_1)|^2 \geq \nu(t_1) - |\theta(t_1)|^2 \geq [1 - \varepsilon_1]\nu(t_1).$$

That is, it holds that

$$|e(t_1)|^2 \geq \left(\frac{1 - \varepsilon_1}{p_M}\right)\nu(t_1). \quad (47)$$

Next, we proceed to find a similar upper bound for all $t \in [t_1, t_1 + \varepsilon_2]$. To do so, we observe that for all $t \in [t_1, t_1 + \varepsilon_2]$,

$$\begin{aligned} |e(t)| & \geq |e(t_1)| - \left| \int_{t_1}^t [A(s) - B(s)^\top] x(s) ds \right| \\ & \geq |e(t_1)| - \max\{a_M, b_M\} \int_{t_1}^t \sqrt{\nu(s)} ds \\ & \geq |e(t_1)| - \varepsilon_2 \max\{a_M, b_M\} \sqrt{\nu(t_1)} \geq \rho\sqrt{\nu(t_1)}, \end{aligned}$$

where ρ is defined in (31) and is positive for any appropriate choice of $\varepsilon_1 \in (0, 1)$ and $\varepsilon_2 > 0$. Next, using the fact that $\dot{\nu}(t) \leq -q_m|e(t)|^2$ for all $t \geq t_o$, we conclude that $\nu(t_1 + \varepsilon_2) - \nu(t_1) \leq -q_m\varepsilon_2\rho^2\nu(t_1)$ and, in view of (46),

$$\nu(t_o + T) - \nu(t_o) \leq -\frac{q_m\varepsilon_2}{\kappa}\rho^2\nu(t_o).$$

Hence, in view of (32), (38) follows. \blacksquare

Proof of Claim 4: We note that Property 2 implies that

$$\Phi(t+T, t)^\top \mathcal{P}(t+T) \Phi(t+T, t) - \mathcal{P}(t) \leq -pTI \quad \forall t \geq 0, \quad (48)$$

for the same T and for $p = \min\{\gamma, 1 - \kappa\}\varepsilon_1$. Then, we show that (48) implies Assumption 5 for $\mathcal{Q}(t) = \dot{\mathcal{P}}(t) - \mathcal{P}(t)\mathcal{A}(t) - \mathcal{A}(t)^\top \mathcal{P}(t)$, which holds by Assumption 2. Indeed, the total derivative of ν along the trajectories of (3)-(11) satisfies $\dot{\nu}(t) = -x^\top \mathcal{Q}(t)x$. Given $T > 0$, by integrating the latter identity from t to $t+T$ we obtain $\nu(t+T) - \nu(t) = -\int_t^{t+T} x(\tau)^\top \mathcal{Q}(\tau)x(\tau)d\tau$, which is equivalent to

$$\nu(t+T) - \nu(t) = -\int_t^{t+T} x(\tau)^\top \Phi(\tau, t)^\top \mathcal{Q}(\tau) \Phi(\tau, t)x(\tau)d\tau.$$

It follows that

$$\begin{aligned} x(t)^\top [\Phi(t+T, t)^\top \mathcal{P}(t+T) \Phi(t+T, t) - \mathcal{P}(t)] x(t) \\ = -x(t)^\top \left[\int_t^{t+T} \Phi(\tau, t)^\top \mathcal{Q}(\tau) \Phi(\tau, t)d\tau \right] x(t). \end{aligned}$$

Now, since $x(t) = \Phi(t, 0)x(0)$ can take any value in \mathbb{R}^n due to $\Phi(t, 0)$ being non-singular, we conclude that

$$\begin{aligned} \Phi(t+T, t)^\top \mathcal{P}(t+T) \Phi(t+T, t) - \mathcal{P}(t) \\ = -\int_t^{t+T} \Phi(\tau, t)^\top \mathcal{Q}(\tau) \Phi(\tau, t)d\tau \quad \forall t, T \geq 0. \end{aligned} \quad \blacksquare$$

V. CONCLUSION

Our main result provides a strict ISS-Lyapunov function for linear time-varying systems that cover many cases from the literature that have been studied over several decades, such as systems appearing in the analysis of adaptive systems. The latter, however, are nonlinear even in the simplest cases; so stability theorems tailored for linear systems must be used with prudence [24]. On the other hand, the relations established in this paper with properties such as persistency of excitation, may give the cue to study the construction of strict Lyapunov functions for nonlinear systems with non-smooth regressors, based on appropriate PE conditions. Another interesting open problem is to establish the relations between different forms of PE, even for linear systems—see Assumptions 3 and 4.

APPENDIX

Lemma 1: Consider system (14) under Assumption 2. Let a solution $t \mapsto x(t)$, with $x := [e^\top \theta^\top]^\top$, be such that there exist $\beta_1 > 0$ and $t_1 \geq t_o \geq 0$ such that

$$|e(t_1)|^2 \geq 2\beta_1 \nu(t_o). \quad (49)$$

Then, $|e(t)|^2 \geq \beta_1 \nu(t_o)$ for all $t \in [t_1, t_1 + \beta_2]$, where

$$\beta_2 := \frac{1}{\beta_3} \log \left(\frac{b_M/\beta_3 + 2\beta_1}{b_M/\beta_3 + p_M\beta_1} \right), \quad \beta_3 := q_M + b_{MPM}. \quad (50)$$

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