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Non-overshooting quasi-continuous sliding mode control for second-order systems

Michael Ruderman^a, Denis Efimov^b

^aUniversity of Agder, Grimstad, Norway

^bInria, Univ. Lille, CNRS, CRISTAL, Lille, France

Abstract

This paper proposes a nonlinear sliding mode state feedback controller for perturbed second-order systems. In analogy to a linear proportional-derivative (PD) feedback control, the proposed nonlinear scheme uses the output of interest and its time derivative. The control has only one free design parameter, and the closed-loop system is shown to possess uniform boundedness and finite-time convergence of trajectories in the presence of matched disturbances. We derive a strict Lyapunov function for the closed-loop control system with a bounded exogenous perturbation, and use it for both, the control parameter tuning and analysis of the finite-time convergence. The essential features of the proposed control law is non-overshooting despite the unknown dynamic disturbances and the continuous control action during the convergence to zero equilibrium. Apart from the numerical results, a revealing experimental example is also shown in favor of the proposed control and in comparison with PD and sub-optimal nonlinear damping regulators.

Keywords: Sliding mode control, second-order dynamics, robust control, finite time convergence, nonlinear damping

1. Introduction

Convergence properties of a transient response and the ability to compensate for unknown (external) disturbances are often the trade-offs when designing a feedback control at large. Well known, a PD regulator, which is equivalent to the state-feedback control for second-order systems, is often sufficient for the unperturbed case provided that an exponential convergence is acceptable. When the system behavior is subject also to static errors, an integral control action can be added, thus resulting in a standard PID (or integral-plus-state feedback) scheme in case of the linear feedback regulation [1]. From another side, an integral feedback action increases inherently the system order and can also lead to the wind-up effects and even destabilization [2]. If disturbances are not static and possess certain regularity only (e.g., Lipschitz continuity or/and boundedness), then the robust feedback regulators like, for instance, high-gain (see, e.g., [3] and references therein) or finite-time (see, e.g., [4] and a recent text [5]) controllers can be required. Beyond that, the input-to-state stability (ISS) theory [6] is usually applied for studying the stability and robustness in presence of external disturbances, once the overall feedback system is nonlinear [7].

Second-order systems capture a relatively large class of the (perturbed) dynamic processes, especially in mechanical, electromagnetic, and fluid-flow domains. In that way, a perturbed double-integrator represents an often encountered system process. A comparative overview of different feedback control principles applied to a classical double-

integrator can be found in [8]. Also the sliding-mode control methods, see [9, 10], become relatively popular for compensating the unknown matched perturbations. Indeed, a feedback control action proportional to the sign of the control error has already been recognized in earlier works as particularly efficient for a time-optimal stabilization [11, 12], and disturbance compensation [13]. However, a discontinuous control signal, especially a continuously switching one, can be undesirable in some scenarios, especially with regard to the wear effects, overloading of the actuators, noise, and energy consumption. To this end, in [14] the class of quasi-continuous high order sliding mode control algorithms has been introduced, which have the discontinuity at the origin only. Therefore, a feedback control law that would allow both, a sign driven robustness to the unknown but bounded perturbations and (in addition) a continuous damping action which would minimize the overshooting and steady-state oscillations, can be desirable. The idea of an inverse to the output's norm damping was originally proposed for second-order systems in the so-called sub-optimal nonlinear damping control [15]. Recently, the sub-optimal nonlinear damping control was augmented by a linear integral term in [16], also with additional analysis so as to cope with the matched constant disturbances. Moreover, the ISS property was established for time-varying bounded perturbations in that augmented control system [16].

The present work builds up on the idea proposed in [15] and uses the sign-feedback principle of sliding mode control theory following [14], while introducing a novel nonlinear state feedback control for the perturbed second-order systems. The unknown time-varying perturbations are as-

Email address: michael.ruderman@uia.no (Michael Ruderman)

sumed to be upper bounded, and the controller has only one free design parameter, similarly to [15]. The introduced nonlinear feedback controller has the same structure as its linear counterpart (i.e., the PD one) and feeds back the output of interest and its time derivative. Moreover, it offers finite-time convergence despite unknown bounded perturbations and is continuous during convergence to the globally stable equilibrium at the origin, where the sliding mode is originated (a signature behavior for quasi-continuous sliding mode control laws). Particularly remarkable in comparison to several robust second-order sliding mode controllers is that the proposed control offers convergence without overshooting the origin (or reference value) that comes in favor of numerous applications. In addition, even if the disturbance exceeds the admissible bounds in the steady-state equilibrium at the origin, the designed controller preserves its invariance.

The paper is organized as follows. The proposed nonlinear state feedback control is introduced in Section 2, also demonstrating qualitatively the phase portrait and shape of the converging state trajectories and control value, and that in comparison with a linear state feedback control (equivalent to PD) and sub-optimal nonlinear damping control [15]. The main part of the closed-loop system analysis is delivered in Section 3. First, the solutions of the system are shown to be uniformly (in disturbances) bounded and asymptotically converging, then the finite-time convergence is formally proved, and boundedness of the control is established. Section 4 reports the experimental evaluation of the proposed control and compares it with an equivalent PD and sub-optimal nonlinear damping controllers. The results reported in this communication are briefly summarized in Section 5.

2. Novel nonlinear feedback control

In this section, first, we describe the studied control problem; second, the proposed control algorithm is presented; third, we finish with an exemplary illustration of the dynamic behavior of the closed-loop system.

2.1. Problem statement

We consider a class of perturbed second-order systems:

$$\frac{d^2y(t)}{dt^2} = u(t) + d(t), \quad t \geq 0, \quad (1)$$

with the control signal $u(t) \in \mathbb{R}$. The matched unknown disturbance quantity $d(t) \in \mathbb{R}$ is assumed to be Lebesgue measurable and essentially bounded, where $\|d\|_\infty = \text{ess sup}_{t \geq 0} |d(t)| \leq D$ with a known upper bound $D > 0$. Both dynamic system states, $x(t) = (x_1(t), x_2(t))^\top \equiv (y(t), \dot{y}(t))^\top \in \mathbb{R}^2$ (including output derivative¹), are available for a feedback design.

¹We will use the dotted symbol, i.e., $\dot{x} \equiv dx/dt$, to denote the time derivative of a variable x , equally as for the first-order differential equations.

Our primary goal is a variant of robust finite-time stabilization of (1) at $x = 0$: it is required to design a state feedback $u(t) = u(x_1(t), x_2(t))$ guaranteeing that there exist $\sigma, T \in \mathcal{K}$ such that $|x(t)| \leq \sigma(|x(0)|)$ for all $t \geq 0$ and $|x(t)| = 0$ for all $t \geq T(|x(0)|)$ for any $x(0) \in \mathbb{X} = \{x \in \mathbb{R}^2 : x_1 \neq 0\} \cup \{0\}$ and all $\|d\|_\infty \leq D$. Thus, we assume that at the initial time instant, the $x_1(0)$ output state is not zero, and it is required to bring $x(t)$ to zero in a finite time independently of the presence of a properly bounded disturbance d .

Note that in case of an additional (inertial) scaling constant in the left hand-side of (1), like in the experimental study of a moving lumped mass shown in Section 4, this should be additionally incorporated as a normalization factor of the control terms, cf. [17]. A more general setting with an uncertain control coefficient, which is separated from zero with known bounds, can be treated by properly augmenting the control gain as it is usually done in the sliding mode theory. This case is skipped for brevity.

2.2. Control law

The proposed control for $x(t) \in \mathbb{X}$ has the form:

$$u(t) = \begin{cases} -|x_1(t)|^{-1}(\gamma x_1(t) + |x_2(t)|x_2(t)) & x_1(t) \neq 0 \\ -\gamma \text{sign}(x_1(t)) & x_1(t) = 0 \end{cases}, \quad (2)$$

where $\gamma > D$ is the only tuning parameter, and it yields for $x(t) \in \mathbb{X}$ the closed-loop system dynamics to

$$\begin{aligned} \frac{dx_1(t)}{dt} &= x_2(t), \\ \frac{dx_2(t)}{dt} &= \begin{cases} -|x_1(t)|^{-1}(\gamma x_1(t) + |x_2(t)|x_2(t)) + d(t) & x_1(t) \neq 0 \\ -\gamma |x_1(t)|^{-1}x_1(t) + d(t) & x_1(t) = 0 \end{cases}. \end{aligned} \quad (3)$$

As we can conclude, the proposed control is discontinuous at $x_1 = 0$, and in the set \mathbb{X} the irregularity happens at the origin only, which is the equilibrium of the system. Indeed, there are two discontinuous terms. The first one is the conventional sign function $x_1(t)|x_1(t)|^{-1} = \text{sign}(x_1(t))$ widely used in the sliding mode control theory, where discontinuity appears for $x_1(t) = 0$ only. However, in the set \mathbb{X} , having $x_1(t) = 0$ is equivalent to $x(t) = 0$. Consequently, this term introduces discontinuity just at the equilibrium, and for definition of solutions in the face of this discontinuity, the Filippov's theory can be used with convex embedding [9, 10]. The second discontinuity may come from the fraction $|x_2(t)|x_2(t)|x_1(t)|^{-1}$, but this term is active for $x_1(t) \neq 0$ only, and it stays always bounded on the trajectories of (3). The investigation of the influence of this term requires a special attention at $x_1 = 0$, so that a proper analysis approach will be utilized in the following. Below, in order to substantiate existence and uniqueness of solutions for (3), we will demonstrate similarity of the system (3) via a time change to another system

with well defined solutions, as we have already done for the sub-optimal nonlinear damping control in [16]. Next, the stability and performance of the closed-loop system can be analyzed. Before doing it, we proceed with some illustrative simulation results showing the intuition behind the proposed design and qualitative properties of the state trajectories of the controlled system.

2.3. Exemplary

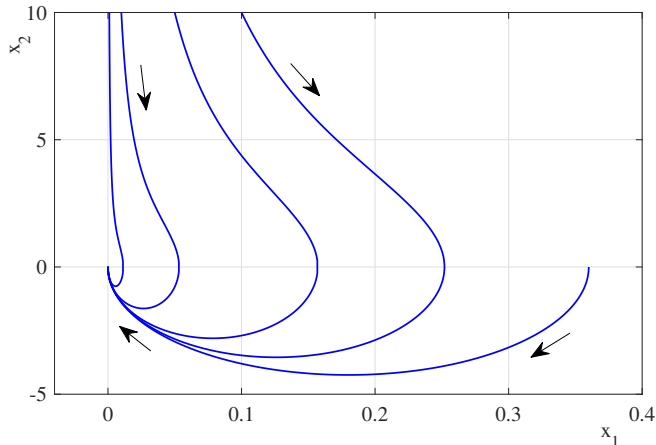


Figure 1: Exemplary phase portrait of the closed-loop system (3).

An illustration of phase-portrait of the closed-loop control system (3) is given in Figure 1. Here the control gain factor $\gamma = 100$ is arbitrary assigned. Due to a symmetric behavior with respect to the origin, the trajectories for different initial conditions are shown in the 1st and 4th quadrants only.

The trajectories of $x_2(t)$ and the control values $u(t)$ are also qualitatively compared in Figure 2 between the proposed control (3), sub-optimal nonlinear damping control [15], and critically damped linear state feedback control (equivalent to PD one). All three controllers are assigned with the same output feedback gain factor $\gamma = 100$, while the response of $x_2(t)$ is depicted on the logarithmic scale for the sake of a better comparison, see Figure 2 (a). A finite time convergence of the control (3) can be qualitatively recognized. The corresponding control values $u(t)$ are shown in Figure 2 (b). All three control laws disclose the same value at the beginning, this is due to the equal gaining factor γ and the same initial conditions $x_1(0) = -1$, $x_2(0) = 0$. Then, one can recognize that the control (3) uses the range of admissible values more efficiently during the convergence phase.

3. Analysis

We start this section with the proof of existence and uniqueness of solutions in (3). Next, the uniform boundedness of solutions and asymptotic convergence to the origin will be substantiated (uniformity is understood as independence of these properties on the properly bounded

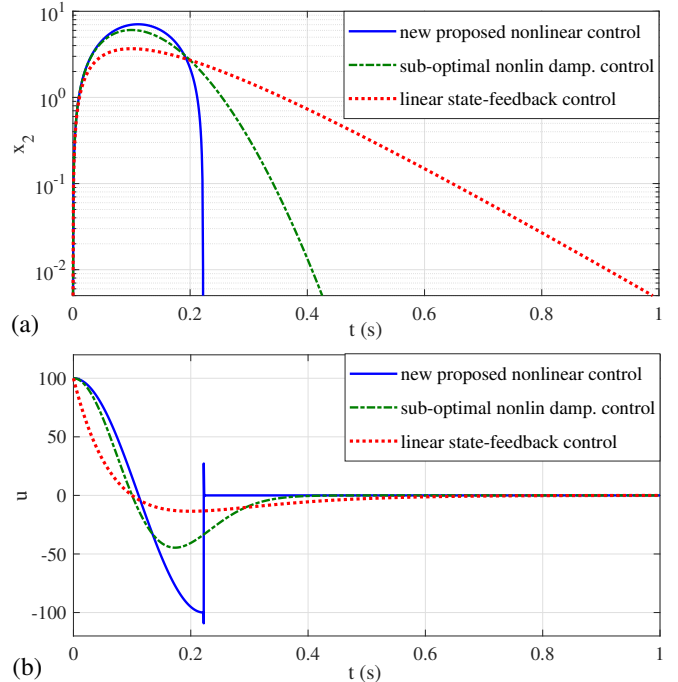


Figure 2: Response of the dynamic state $x_2(t)$ in (a) and the control signal u in (b) of the proposed nonlinear control (3), sub-optimal nonlinear damping control [15], and the critically damped linear state feedback control.

inputs). The rate of convergence is evaluated at the end of the section, together with the upper bound on the control magnitude.

3.1. Existence of solutions

In the Appendix, for the system (.1) operating in time $\tau \geq 0$ the following properties are demonstrated for any $x(0) \in \mathbb{R}^2$ and the inputs with $\|d\|_\infty \leq D$:

1. global existence and uniqueness of solutions $x(\tau) = x(\tau, x(0), d)$ for all $\tau \geq 0$,
2. invariance of the sets $\mathcal{X} = \{x \in \mathbb{R}^2 : x_1 = 0\}$ and $\mathbb{X} = (\mathbb{R}^2 \setminus \mathcal{X}) \cup \{0\}$,
3. existence of the equilibrium at the origin, which is uniformly globally asymptotically stable.

To use these facts in the analysis of (3), introduce a new (scaled) time argument for (.1):

$$t := \phi(\tau) = \int_0^\tau |x_1(s)| ds, \quad (4)$$

which is well-defined, and $t \in \mathbb{R}_+$ for $\tau \in \mathbb{R}_+$ (indeed, $t = 0$ for $\tau = 0$, and $\lim_{\tau \rightarrow +\infty} \int_0^\tau |x_1(s)| ds \leq +\infty$).

Moreover, $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a monotonously growing function for $x(0) \in \mathbb{X} \setminus \{0\}$ since the origin in such a case is reached for $\tau \rightarrow +\infty$ only, and it has an inverse $\tau = \phi^{-1}(t)$. However, the domain of values of t may

be bounded since $|x_1(\tau)| \rightarrow 0$ as $\tau \rightarrow +\infty$. That implies $dt = |x_1(\tau)|d\tau$ while transforming (.1) to the closed-loop system (3) for $x(0) \in \mathbb{X} \setminus \{0\}$. Indeed, with a slight ambiguity define $x(t) = x(\phi^{-1}(t)) = x(\tau)$ and calculate its derivative in the new time t keeping in mind that $\frac{d}{dt}\phi^{-1}(t) = \left|x_1(\phi^{-1}(t))\right|^{-1}$, which gives that $x(t)$ is a solution of (3) since

$$\frac{dx(t)}{dt} = \left|\frac{dx(\tau)}{d\tau}\right|_{\tau=\phi^{-1}(t)} \frac{d\phi^{-1}(t)}{dt}.$$

Therefore, the systems (.1) and (3) have the same solutions in $\mathbb{X} \setminus \{0\}$ (and the origin is the equilibrium for both systems), where the only difference is that each solution becomes scaled along the independent time variable based on the norm of x_1 .

Since (.1) has a unique solution asymptotically approaching the origin for any initial condition $x(0) \in \mathbb{X}$ and input with $\|d\|_\infty \leq D$, the system (3) also admits this solution while demonstrating the same behavior in finite or infinite time. In the former case, since the origin is the equilibrium of (3), the zero solution can be extended for all $t \geq 0$. Therefore, the following result has been proven:

Proposition 1. *The system (3) admits unique solution for any $x(0) \in \mathbb{X}$ and $\|d\|_\infty \leq D$ well defined with $x(t) \in \mathbb{X}$ for all $t \geq 0$.*

In fact, via such a similarity of (.1) and (3) through the time scaling (4), a stronger property is substantiated: boundedness of solutions and asymptotic convergence to the origin for (3) in \mathbb{X} uniformly in properly bounded d . However, below we would like to apply the Lyapunov function method directly to (3), so as to establish this property with demonstration of the finite-time convergence rate.

3.2. Analysis of stability

We are in the position to prove the uniform boundedness of solutions and asymptotic convergence to the origin for the perturbed system (3) in \mathbb{X} .

Theorem 1. *The origin for the system (3) is uniformly attractive for any $x(0) \in \mathbb{X}$ and $\|d\|_\infty \leq D$ if*

$$\gamma > D^{1.5} + D + \frac{1}{2}, \quad (5)$$

and the solutions are bounded for all $t \geq 0$.

Proof. As in the Appendix for the system (.1), consider for (3) a strict Lyapunov function candidate

$$V(x) = \frac{1}{2} \begin{pmatrix} z_1 \\ x_2 \end{pmatrix}^\top \begin{pmatrix} 2\gamma & \varepsilon \\ \varepsilon & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ x_2 \end{pmatrix} \quad (6)$$

with $z_1 = \sqrt{|x_1|}\text{sign}(x_1)$, which is positive definite for $\sqrt{2\gamma} > \varepsilon > 0$. This Lyapunov function is continuously

differentiable for $x \in \mathbb{X} \setminus \{0\}$. Taking the time derivative of V on this set one obtains

$$\begin{aligned} \dot{V} &= x_2 d - x_2^2 \frac{|x_2|}{|x_1|} - \varepsilon(\gamma - \text{sign}(x_1)d) \sqrt{|x_1|} \\ &\quad + \varepsilon \frac{x_2^2}{\sqrt{|x_1|}} \left(0.5 - \text{sign}(x_1)\text{sign}(x_2)\right). \end{aligned}$$

Using the *Young's inequality* [18] which postulates

$$ab \leq \frac{1}{p} a^p + \frac{p-1}{p} b^{p/(p-1)}$$

for any $a, b \in \mathbb{R}_+$ and $p > 1$, one can show that

$$\dot{V} \leq -\left(\frac{2}{3} - \varepsilon\right) x_2^2 \frac{|x_2|}{|x_1|} - \left[\varepsilon\left(\gamma - \frac{1}{2} - D\right) - \frac{2}{3} D^{1.5}\right] \sqrt{|x_1|} \quad (7)$$

for any $x \in \mathbb{X}$ and $|d| \leq D$. It can be seen, that taking

$$\frac{2}{3} \frac{D^{1.5}}{\gamma - \frac{1}{2} - D} < \varepsilon < \frac{2}{3}$$

both expressions in the brackets on the right hand-side of \dot{V} are positive under (5). Hence, $\dot{V}(x, d) < 0$ for $x \in \mathbb{X} \setminus \{0\}$ and $|d| \leq D$ that completes the proof. \square

Remark 1. A consequence of the result given in Theorem 1, which establishes forward invariance of the set \mathbb{X} and global convergence of all solutions in (5) to the origin, is that it implies the absence of overshooting: the trajectories cannot cross the line $x_1 = 0$, and they reach it only at the origin.

The low bound of the control gain γ , which is satisfying the parametric condition (5), is shown as a function of the maximal disturbance magnitude, i.e., D , in Figure 3.

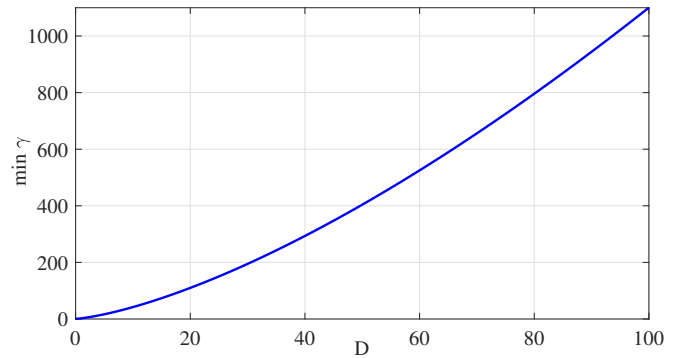


Figure 3: Low bound of the control gain depending on the perturbation bound.

3.3. Finite-time convergence and control boundedness

In order to prove the finite-time convergence (see, e.g., [4] for details) of the system (3), which can be supposed from the numerical results given in Figure 2 (a), we make further use of the strict Lyapunov function V given in (6).

Theorem 2. *The system (3) is finite-time convergent to the origin for any $x(0) \in \mathbb{X}$ and $\|d\|_\infty \leq D$ provided that (5) is satisfied.*

Proof. The estimate (7) implies boundedness of solutions of (3) in \mathbb{X} for the inputs with magnitude smaller than D , with their asymptotic convergence to the origin. Let us consider an auxiliary variable $\zeta(t) = x_2(t)x_1(t)^{-1}$, which is well defined on the solutions of the system (3) (in the set $\mathbb{X} \setminus \{0\}$), and it has the dynamics:

$$\begin{aligned}\dot{\zeta} &= \frac{dx_1 - (\gamma|x_1| + \text{sign}(x_1)|x_2|x_2) - x_2^2}{x_1^2} \\ &= \frac{d\text{sign}(x_1) - \gamma}{|x_1|} - |\zeta|\zeta - \zeta^2 \\ &= \frac{d\text{sign}(x_1) - \gamma}{|x_1|} - \begin{cases} 2\zeta^2 & \zeta > 0 \\ 0 & \zeta \leq 0 \end{cases}.\end{aligned}$$

Note that $(d\text{sign}(x_1) - \gamma)|x_1|^{-1} < 0$ due to (5), which implies that for any $x(0) \in \mathbb{X}$ and $\|d\|_\infty \leq D$ there is $t_1 \geq 0$ such that $\zeta(t) < 0$ for all $t > t_1$, i.e., the trajectories enter in the second or the fourth quadrants after finite transients, where the variable ζ stays separated with zero. Next, consider another auxiliary variable $z(t) = x_2^2(t)|x_1(t)|^{-1}$, which admits the dynamics:

$$\begin{aligned}\dot{z} &= \frac{|x_2|}{|x_1|} \left(2\text{sign}(x_2)(d - \gamma\text{sign}(x_1)) \right. \\ &\quad \left. - (2 + \text{sign}(x_1)\text{sign}(x_2))z \right).\end{aligned}$$

For $t > t_1$ and taking into account (5), $\text{sign}(x_2)(d - \gamma\text{sign}(x_1)) > 0$ out the origin (the lines $x_1 = 0$ and $x_2 = 0$ can be reached only there). Hence, for any $x(0) \in \mathbb{X}$ and $\|d\|_\infty \leq D$, there is $t_2 \geq 0$ such that $z(t) \geq \theta > 0$ for all $t \geq t_2$ (while the trajectories do not converge to zero), where θ depends only on the choice of γ and D . So, there are two possibilities, either the trajectories approach the origin in a finite time, or the variable $z(t)$ becomes strictly positive and separated with zero, which returning to the Lyapunov function $V(x)$ from (6) and the estimate (7) implies for $t \geq t_2$:

$$\begin{aligned}\dot{V} &\leq -\left(\frac{2}{3} - \varepsilon\right)\theta|x_2| - \left[\varepsilon\left(\gamma - \frac{1}{2} - D\right) - \frac{2}{3}D^{1.5}\right]\sqrt{|x_1|} \\ &\leq -\kappa\sqrt{V}\end{aligned}$$

for some $\kappa > 0$. The latter differential inequality ensures a finite-time convergence of V to zero according to [4]. \square

Slightly extending the arguments used in the last proof we can substantiate the boundedness of the control (2):

Proposition 2. *In the conditions of Theorem 2, the amplitude of control (2) admits an upper estimate*

$$|u(t)| \leq \gamma + \max\{x_2^2(0)|x_1(0)|^{-1}, 2(D + \gamma)\}, \quad \forall t \geq 0.$$

Proof. Following the proof of Theorem 2 and considering an auxiliary Lyapunov function $\mathcal{V}(z) = 0.5z^2$ we get the estimate on its time derivative for $t \geq 0$ while $z(t) \neq 0$ (until the origin is not reached, since it is the only invariant solution on the set $z = 0$):

$$\begin{aligned}\dot{\mathcal{V}} &= \frac{|x_2|}{|x_1|} \left(2\text{sign}(x_2)(d - \gamma\text{sign}(x_1))z \right. \\ &\quad \left. - (2 + \text{sign}(x_1)\text{sign}(x_2))z^2 \right) \\ &\leq \frac{|x_2|}{|x_1|} \left(2(D + \gamma)|z| - z^2 \right) \leq \frac{|x_2|}{|x_1|} \left(2(D + \gamma)^2 - \mathcal{V} \right),\end{aligned}$$

which implies the following boundedness estimate in the time domain: $\mathcal{V}(z(t)) \leq \max\{\mathcal{V}(z(0)), 2(D + \gamma)^2\}$ (the multiplier $|x_2||x_1|^{-1}$ may be zero at the isolated time instant t_1 when $x_2(t_1) = 0$ only), or

$$z(t) \leq \max\{z(0), 2(D + \gamma)\}$$

for all $t \geq 0$. Note that $|u(t)| \leq \gamma + z(t)$ for all $t \geq 0$ in (2) that gives the required upper bound for the control. \square

Moreover, since the set $z \leq 2(D + \gamma)$ is attractive and forward invariant (it contains the origin), it implies that for any trajectory approaching the origin (even initially $x_1(0)$ was arbitrary small) the control has an upper bound

$$|u| \leq 2D + 3\gamma,$$

and the term $\frac{|x_2|x_2}{|x_1|}$ does not impose a singularity in the system (3). In addition, for the set of initial conditions with $x_1(0) \neq 0$ and $x_2(0) = 0$ (i.e., when the control is applied to the system (1) in a steady state for $u(t) = d(t) = 0$, $t \leq 0$), the amplitude of (2) is bounded by $2D + 3\gamma$ for all $t \geq 0$.

Remark 2. The proven asymptotic boundedness of z and also its separation with zero, i.e., the forward attractiveness of the set $\theta \leq z \leq 2(D + \gamma)$ while the trajectory stays in $\mathbb{X} \setminus \{0\}$, implies that (3) admits an invariant cone in the second and third quadrants.

3.4. Numerical simulations

The controlled output behavior of (3) is exemplary visualized in Figure 4 (a). Here a periodic disturbance $d(t) = 50\sin(20\pi t)$ is used while the control gain is assigned as $\gamma = 100$. The most simple first-order forward Euler method (solver) is used. For numerical solvability of the closed-loop system dynamics (3), two implementation schemes are used. The first one

$$u(t) = \begin{cases} -|x_1(t)|^{-1}(\gamma x_1(t) + |x_2(t)|x_2(t)) & |x_1(t)| \geq \mu \\ -\gamma\text{sign}(x_1(t)) & |x_1(t)| < \mu \end{cases}, \quad (8)$$

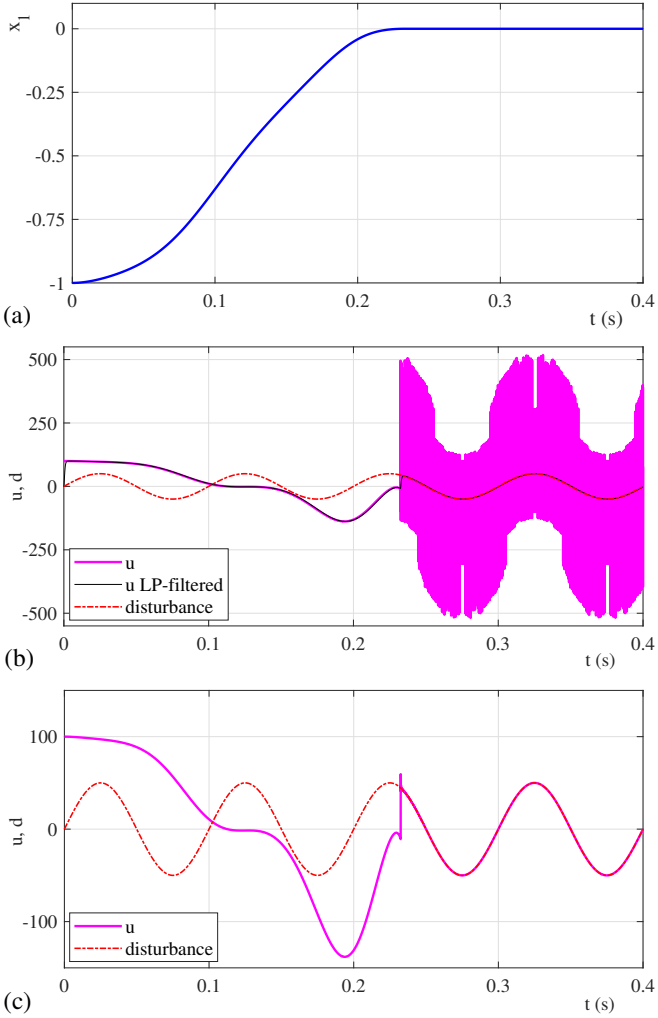


Figure 4: Output response of the perturbed control system (3) in (a), and the control signal versus disturbance value by using: thresholding scheme (8) for zero in (b), and regularization scheme (9) in (c).

incorporates a threshold value $\mu > 0$ for realizing numerical zero. The corresponding control signal is shown opposite the disturbance value in Figure 4 (b). For the sake of clarity, the same control signal but low-pass (LP) filtered with 3000 Hz cut-off frequency is also depicted. The second implementation scheme uses a regularization factor $\mu > 0$, cf. [17], so that

$$u(t) = -(|x_1(t)| + \mu)^{-1}(\gamma x_1(t) + |x_2(t)|x_2(t)). \quad (9)$$

The corresponding control signal is shown opposite the disturbance value in Figure 4 (c). Note that for both implementation schemes, a low numerical value $\mu = 10^{-9}$ is assumed.

4. Experimental control results

The proposed control (3) is experimentally evaluated on the electro-magneto-mechanical actuator system with one translational degree of freedom, see Figure 5. The system

dynamics can be reduced to the model (1). Technical de-



Figure 5: Experimental setup of the actuator system [17].

tails, including the identified system model and most relevant hardware parameters of the setup, can be found in [17]. The sampling rate is 10 kHz. We note that the overall upper-bounded perturbation d is due to the constant gravity term, but also due to the time-varying friction and cogging force by-effects of the electro-magneto-mechanical actuator.

The step response of the control (3) (denoted further as NLG owing to the nonlinear feedback gains) is compared once with a standard proportional-derivative (PD) control and once with the sub-optimal nonlinear damping control [15] (denoted further as SOND).

The benchmarking linear PD control

$$u(t)_{\text{PD}} = -\gamma x_1 - \sigma x_2,$$

is designed first so that the unperturbed closed-loop control system has two critically damped real poles, i.e., with an optimal value of σ that corresponds to the unity damping ratio. This control is denoted by PD_1 . Then, its output feedback gain γ is enhanced while allowing also for the conjugate complex pole pair with a corresponding transient overshoot. That PD-control realization is denoted by PD_2 .

The parameters of all four evaluated controllers are listed in Table 1.

Table 1: Evaluated controllers

parameter \ control	PD_1	PD_2	SOND	NLG
x_1 -gain γ	500	750	750	25
x_2 -gain σ	2	4	—	—

Note that for a fair comparison, the SOND control is assigned with the same gaining factor $\gamma = 750$ as the faster PD-control PD_2 . The feedback gain of the NLG-control

$\gamma = 25$ is intentionally set an order of magnitude lower than the feedback gains of the PD and SOND controllers. This is for demonstrating that the NLG-control can behave less aggressive during the transient response and, at the same time, achieve a much higher accuracy at steady-state. All experimentally measured control responses, to the same reference step of $y_{\text{ref}} = 8$ mm, are shown opposite each other in Figure 6.

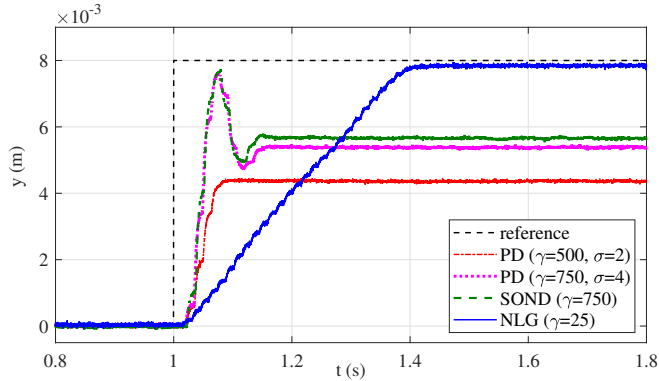


Figure 6: Measured step response of PD, SOND, and NLG controllers.

From the response of the NLG-control, one can recognize a nearly uniform behavior of $x_2(t)$ during the whole transient phase, cf. Figure 2 (a). Note that the corresponding inclination of the $x_1(t)$ slope depends on the assigned γ parameter. It is also worth noting that a residual non-zero, however minor, steady-state error of the NLG-control is due to the sensor bias and noise, while the latter is additionally impacting the used $x_2(t)$ signal, which is obtained via differentiation and filtering of the measured $x_1(t)$ value. The experimentally obtained signal $x_2(t)$ (when using the NLG controller) is also shown in Figure 7 for the sake of clarity.

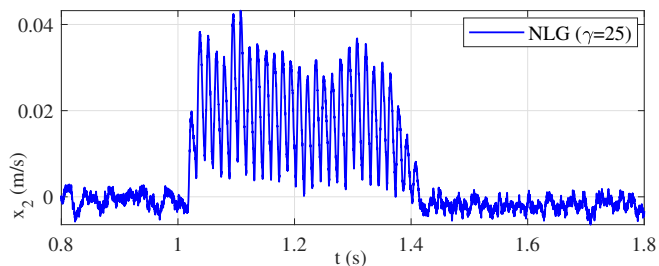


Figure 7: Velocity state $x_2(t)$ of the NLG control experiment.

5. Summary

A novel robust nonlinear state feedback control for perturbed second-order systems was introduced. The control law combines the ideas of the state-dependent nonlinear damping introduced in [15] and the sliding mode control theory, which equips the overall feedback law with

an output-sign-dependent term able to compensate for bounded perturbations. The resulted control has only one tuning parameter, the overall output feedback gain, whose selection should be based on the upper bound of the perturbation magnitude, and this closed analytic dependence (even though conservative) was shown in the paper. We provided a detailed analysis of uniform boundedness and finite-time convergence of the control system based on the proposed strict Lyapunov function. The control signal appears to be continuous during the transients, apart from the convergence instant, where the state value reaches the origin (similarly to some high order sliding mode control algorithms). Remarkable features of the proposed control are that: (i) it is equivalent (from the structural viewpoint) to a standard linear PD feedback control and, at the same time, (ii) it can guarantee (without additional integral action) for full compensation of the properly bounded matched disturbances in the second-order systems without overshoot, and arbitrary bounded in the steady state, while (iii) providing a global finite-time convergence. The proposed control was already evaluated experimentally on an electro-magneto-mechanical actuator system with considerable bounded disturbances. Analysis of its robustness abilities against process and measurement noise and development of the methods for its discretization can be considered as directions of future research.

Acknowledgments

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Consider a second-order system

$$\begin{aligned}\frac{dx_1(\tau)}{d\tau} &= |x_1(\tau)|x_2(\tau), \\ \frac{dx_2(\tau)}{d\tau} &= -\gamma x_1(\tau) - |x_2(\tau)|x_2(\tau) + |x_1(\tau)|d(\tau),\end{aligned}\quad (.1)$$

in the time $\tau \geq 0$, with the state $x(\tau) = (x_1(\tau) \ x_2(\tau))^\top \in \mathbb{R}^2$ and an essentially bounded input $d(\tau) \in \mathbb{R}$ with $\|d\|_\infty \leq D$ for a given $D > 0$, and with a gain $\gamma > D$. The right hand-side of (.1) is locally Lipschitz continuous in $x(\tau)$ and $d(\tau)$, hence, the solutions are uniquely defined for any initial condition $x(0) \in \mathbb{R}^2$ and a measurable essentially bounded input d at least locally in time. Moreover, it is easy to check that the origin is the equilibrium of the system and the submanifold $\mathcal{X} = \{x \in \mathbb{R}^2 : x_1 = 0\}$ is invariant uniformly in d .

Lemma 3. *The origin for (.1) is uniformly globally asymptotically stable if*

$$\gamma > D^{1.5} + D + \frac{1}{2}.\quad (.2)$$

Proof. Let us consider the following Lyapunov function candidate

$$\begin{aligned}V(x) &= \frac{1}{2}\xi^\top(x) \begin{bmatrix} 2\gamma & \varepsilon \\ \varepsilon & 1 \end{bmatrix} \xi(x), \\ \xi(x) &= \begin{bmatrix} \sqrt{|x_1|}\text{sign}(x_1) \\ x_2 \end{bmatrix},\end{aligned}$$

which is positive definite providing that the parameter $\varepsilon \in (0, \sqrt{2\gamma})$. The function V is continuously differentiable for all $x \in \mathbb{R}^2 \setminus \mathcal{X}$. Note that $V(x) = \frac{x_2^2}{2}$ for $x \in \mathcal{X}$ (it is an invariant subspace for the system, and the trajectories of (.1) stay on it while being defined) and it is straightforward to check that (we denote $V(\tau) = V(x(\tau))$)

$$\frac{dV(\tau)}{d\tau} = -|x_2(\tau)|^3 \leq 0,$$

implying convergence of $x_2(\tau)$ to zero. For $x(0) \notin \mathcal{X}$ (note that $(\mathbb{R}^2 \setminus \mathcal{X}) \cup \{0\}$ is also invariant for (.1)), we obtain

$$\begin{aligned}\frac{dV(\tau)}{d\tau} &= -(\gamma - \text{sign}(x_1(\tau))d(\tau))\varepsilon\sqrt{|x_1(\tau)||x_1(\tau)| - |x_2(\tau)|^3} \\ &\quad + \varepsilon\sqrt{|x_1(\tau)|}x_2^2(\tau) \left(\frac{1}{2} - \text{sign}(x_1(\tau))\text{sign}(x_2(\tau)) \right) \\ &\quad \quad \quad + |x_1(\tau)|x_2(\tau)d(\tau) \\ &\leq -(\gamma - D)\varepsilon\sqrt{|x_1(\tau)||x_1(\tau)| - |x_2(\tau)|^3} \\ &\quad + \varepsilon\sqrt{|x_1(\tau)|}x_2^2(\tau) \left(\frac{1}{2} - \text{sign}(x_1(\tau))\text{sign}(x_2(\tau)) \right) \\ &\quad \quad \quad + D|x_1(\tau)||x_2(\tau)| \\ &\leq -\left[\left(\gamma - \frac{1}{2} - D\right)\varepsilon - \frac{2}{3}D^{1.5}\right]\sqrt{|x_1(\tau)||x_1(\tau)|} \\ &\quad \quad \quad - \left(\frac{2}{3} - \varepsilon\right)|x_2(\tau)|^3,\end{aligned}$$

where on the last step the following inequalities have been utilized (that are obtained from Young's inequality [18]):

$$\begin{aligned}\sqrt{|x_1(\tau)|}x_2^2(\tau) &\leq \frac{1}{3}\sqrt{|x_1(\tau)||x_1(\tau)|} + \frac{2}{3}|x_2(\tau)|^3, \\ D|x_1(\tau)||x_2(\tau)| &\leq \frac{2}{3}D^{1.5}\sqrt{|x_1(\tau)||x_1(\tau)|} + \frac{1}{3}|x_2(\tau)|^3.\end{aligned}$$

Hence, for $\gamma > D + \frac{1}{2}$ and

$$\frac{2}{3} \frac{D^{1.5}}{\gamma - \frac{1}{2} - D} < \varepsilon < \min\left\{\frac{2}{3}, \sqrt{2\gamma}\right\}$$

the desired property

$$\frac{dV(\tau)}{d\tau} < 0$$

while $V(\tau) \neq 0$ is obtained. The interval for the values of ε is not empty provided that the condition (.2) is verified. Indeed, this condition implies that $\gamma > 0.5$ for any $D > 0$, hence, $\min\{\frac{2}{3}, \sqrt{2\gamma}\} = \frac{2}{3}$ and $\frac{2}{3} \frac{D^{1.5}}{\gamma - \frac{1}{2} - D} < \frac{2}{3}$ is equivalent to (.2). Combining the derived estimates for $\frac{dV(\tau)}{d\tau}$ on the sub-manifold of \mathcal{X} and outside, the uniform global asymptotic stability property of the origin for (.1) follows the conventional results [19, 7]. \square