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ANALYSIS OF THE INTERIOR TRANSMISSION PROBLEM IN AN UNBOUNDED LOCALLY PERTURBED PERIODIC STRIP

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ABSTRACT. We analyze the interior transmission problem in a locally perturbed infinite periodic domain, considering the case where the perturbation intersects the periodic background. An equivalent formulation as coupled quasi-periodic problems is obtained by applying the Floquet-Bloch transform. We perform a discretization with respect to the Floquet-Bloch variable and prove the well-posedness of the semi-discretized problem. We then establish some a priori estimates under regularity assumptions that allow us to prove the convergence of the discrete sequence to the solution of the problem.

1. Introduction. The inverse scattering problem for an unbounded periodic medium with and without defects has received increasing attention in recent years due to its relevance in various applied fields such as mechanics, chemistry, optics, antenna arrays, etc. [3, 15, 17, 19]. In particular, many researchers have studied the application of sampling methods for imaging such media [1, 16, 11, 21, 9, 8, 2]. The theoretical justification for this type of algorithm for inhomogeneous media is generally based on the well-posedness of a so-called Interior Transmission Problem (ITP). This problem has been extensively studied in the literature for the case of bounded domains, and we refer for example to [4, 6, 5, 20, 13]. The ITP for the case of unbounded periodic domains without defects can always be reduced to a one-period cell, which leads back to the study of the problem in a bounded domain. However, the case of perturbed periodic media (not assuming that the perturbation is also periodic as in [10]) is more difficult, since one is forced to study the solvability of the problem in the whole unbounded domain. More precisely, the difficulty arises when the perturbation intersects the periodic background. In fact, for the case where the intersection is empty, one can decouple the problem into two different problems, one associated with the periodic domain and the other with the compact perturbation. Consequently, the study in this case can also be derived from results on ITP for bounded domains, this case was considered in [2].

We present in this paper a study for the ITP problem when this intersection is not empty. In this case, one can not use the Fredholm alternative in the classical

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way considered in the literature. It is also not possible to follow the methods in [15] for the solvability of the direct problem to deal with this type of problems.

The approach we propose to solve this problem in this case is as follows. First, we apply the Floquet-Bloch transformation to the problem and rewrite it as an equivalent coupled quasi-periodic problem w.r.t. the Floquet-Bloch variable. This explains why we first consider the well-posedness for the quasi-periodic ITP problems before considering our perturbed case. Some uniform estimates with respect to the Floquet-Bloch variable are proved for these problems, as they are needed for the subsequent analysis. Next, we discretize the original ITP problem with respect to the Floquet-Bloch variable and prove that the resulting semi-discretized problem is uniquely solvable. We then establish the convergence of the semi-discretized solution as the discretization step goes to zero. This step requires some a priori estimates under certain regularity assumptions on the data (with respect to the Floquet-Bloch variable). These assumptions are compatible with the use of the linear sampling method for imaging in half-space media with Dirichlet boundary conditions, as we will explain. Our analysis assumes that the refractive index has a positive imaginary part. This allows us to avoid the existence of guided modes and to easily establish the uniqueness of the solutions. The removal of this technical assumption is one of the desired perspectives of this work.

The paper is structured as follows. In the first section, we motivate our work by presenting the direct scattering problem of interest. In Section 2, we formulate the interior transmission problem and reformulate it as a coupled quasi-periodic interior transmission problem. In Section 3, we prove the well-posedness of the quasi-periodic interior transmission problem. Section 4 is dedicated to the analysis of the locally perturbed interior transmission problem as explained above.

2. The scattering problem that motivates the interior transmission problem.

In this section, we present the forward scattering problem for infinite periodic layers with defects, which has motivated us to consider the study of the interior transmission problem that will be considered later. Consider the upper half space $U^0 := \mathbb{R} \times \mathbb{R}_+ \subset \mathbb{R}^2$. We denote by k the real-valued positive wavenumber and by $n \in L^\infty(U^0)$ the refractive index satisfying $n = n_p$ outside a bounded domain \tilde{D} , with $n_p \in L^\infty(U^0)$ a refractive index 2π -periodic with respect to the first coordinate. Throughout this paper, a 2π -periodic function refers to a function that is periodic with respect to the first variable x_1 , with period 2π . It is also assumed that n and n_p have non-negative imaginary parts.

Let $R > 0$ and assume that there exists a periodic domain D^p such that $D^p \subset \Omega^R$ where $\Omega^R := \mathbb{R} \times [0, R]$ and such that $n_p = 1$ outside D^p . Additionally, we assume that $\tilde{D} \subset \Omega_0^R \cap D^p$ where $\Omega_0^R := [0, 2\pi] \times [0, R]$ as shown in Figure 1. Consider an incident field $v \in L^2(D)$. The direct scattering problem is formulated as: seek the scattered field $w \in H_{loc}^2(\Omega^R) \cap H^1(\Omega^R)$ satisfying

$$\begin{cases} \Delta w + k^2 n w = k^2 (1 - n) v & \text{in } U^0, \\ w = 0 & \text{on } \Gamma^0, \end{cases} \quad (1)$$

and the angular spectrum representation

$$w(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix \cdot \xi + i\sqrt{k^2 - |\xi|^2}(x_2 - R)} \hat{w}(\xi, R) d\xi, \quad \text{for } x_2 > R, \quad (2)$$

with $\Gamma^0 := \mathbb{R} \times \{0\}$ and $\widehat{w}(\cdot, R)$ is the Fourier transform of $w|_{\Gamma^R}$ defined as $\widehat{w}(\xi, R) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x_1} w(x_1, R) dx_1$.

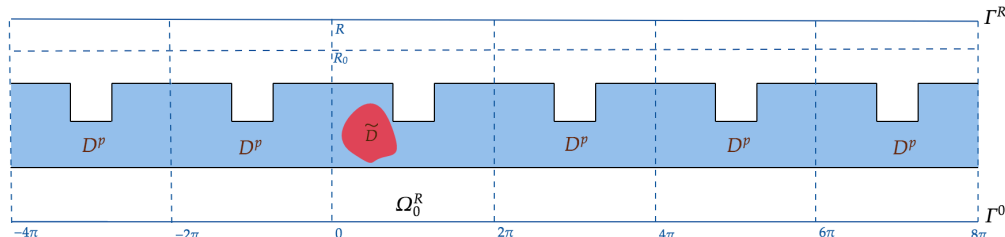


FIGURE 1. Sketch of a periodic domain D^p with embedded defect \tilde{D}

Assumption 2.1. Assume that the set $\{\Im m(n_p) > 0\}$ contains a non-empty open set \mathcal{O} .

The well-posedness of this problem has been studied in [15, 12] under the assumption 2.1 and we recall the corresponding result below. Our analysis for the interior transmission problem will also use this assumption. We should mention that the direct problem has also been studied under the weaker assumption $\Im m(n_p) \geq 0$ in [14]. However, our analysis for the interior transmission problem does not cover this case.

Theorem 2.2. Assume that assumption 2.1 holds and that $\Im m(n) \geq 0$ and let $v \in L^2(D^p)$. Then there exists a unique solution $w \in H_{loc}^2(\Omega^R) \cap H^1(\Omega^R)$ to the problem (1)-(2) such that

$$\|w\|_{H^1(\Omega^R)} \leq c \|v\|_{L^2(\Omega^R)},$$

where $c > 0$ is a constant independent of v .

This result proves that the solution is only in $H^1(\Omega^R)$. In fact, we can prove that the solution is in $H^2(\Omega^R)$, which will justify the solution space for the interior transmission problem.

Theorem 2.3. Under the same assumptions as in theorem 2.2, the solution w of the problem (1)-(2) belongs to $H^2(\Omega^R)$ for all $R > 0$ and there exists a constant $c > 0$ independent of v such that

$$\|w\|_{H^2(\Omega^R)} \leq c \|v\|_{L^2(\Omega^R)},$$

The proof of this theorem is given in the appendix. The interior transmission problem that we consider in the following section is motivated by the inverse problem of recovering D^p from measurements of the scattered field on Γ^R associated with point source incident fields. This inverse problem has been considered in [2] in the case where \tilde{D} does not intersect D^p . In this case, the interior transmission problems that arise are classical ones that have already been treated in the literature [4]. In order to extend the analysis in [2] to the case where \tilde{D} is included in D^p , one should consider a different type of interior transmission problem, namely a locally perturbed periodic interior transmission problem posed on the unbounded layer D^p . We formulate this problem in the following section.

3. Formulation of the interior transmission problems. The locally perturbed interior transmission problem is formulated as follows: search for $(u, v) \in L^2(D^p) \times L^2(D^p)$ such that $(u - v) \in H^2(D^p)$ satisfies

$$\begin{cases} \Delta u + k^2 n u = 0 & \text{in } D^p, \\ \Delta v + k^2 v = 0 & \text{in } D^p, \\ u - v = \varphi & \text{on } \partial D^p \setminus \Gamma^0, \\ \frac{\partial(u-v)}{\partial \nu} = \psi & \text{on } \partial D^p \setminus \Gamma^0, \\ u = v = 0 & \text{on } \Gamma^0, \end{cases} \quad (3)$$

for some boundary data $(\varphi, \psi) \in H^{3/2}(\partial D^p \setminus \Gamma^0) \times H^{1/2}(\partial D^p \setminus \Gamma^0)$. The variable u plays the role of a total field $w + v$ and v plays the role of the incident field in (1)-(2).

Note that the well-posedness of (3) will be shown for data (φ, ψ) which has more regularity with respect to the Floquet-Bloch variables (see theorems 5.11 and 5.12). The regularity assumptions in these theorems are compatible with the use of sampling methods, as indicated in Remark 5.13.

In order to analyse this problem, we will apply the Floquet-Bloch transform with respect to the first coordinate, which leads to coupled quasi-periodic transmission problems. We will introduce the necessary tools for this in the following.

Definition 3.1. For $\xi \in \mathbb{R}$, we say that a regular function u is ξ -quasi-periodic of period 2π if it satisfies

$$u(x_1 + 2\pi j, x_2) = e^{i2\pi\xi \cdot j} u(x_1, x_2) \quad \forall j \in \mathbb{Z} \text{ and } (x_1, x_2) \in \mathbb{R}^2. \quad (4)$$

In what follows, we do not indicate the quasi-periodicity length since it is fixed to 2π . Moreover, we introduce for a real valued $m \geq 1$ the following functional spaces

$$\begin{aligned} H_\xi^m(\Omega^R) &:= \{u \in H_{loc}^m(\overline{\Omega^R}); \text{ } u \text{ is } \xi\text{-quasi-periodic with period } 2\pi\}, \\ L_\xi^2(\Omega^R) &:= \{u \in L_{loc}^2(\Omega^R); \text{ } u \text{ is } \xi\text{-quasi-periodic with period } 2\pi\}, \\ H_\xi^1(\Omega_0^R) &:= \{u|_{\Omega_0^R}; \text{ } u \in H_\xi^1(\Omega^R)\}. \end{aligned}$$

We similarly define the spaces $H_\xi^2(D^p)$, $L_\xi^2(D^p)$ and $H_\xi^1(D_0^p)$ where $D_0^p := D^p \cap \Omega_0^R$ in a similar way.

Definition 3.2. For a regular function $\phi \in C_0^\infty(U^0)$ we define the one dimensional Floquet-Bloch transform as

$$\mathcal{J}\phi(\xi, x_1, x_2) = \sum_{j \in \mathbb{Z}} \phi(x_1 + 2\pi j, x_2) e^{-i2\pi\xi \cdot j}, \quad \xi \in I := [0, 1], (x_1, x_2) \in U^0. \quad (5)$$

Let m be a non-negative integer. It is known (see for example [18, 15]) that \mathcal{J} is an isomorphism between $H^m(D^p)$ and $L^2(I, H_\xi^m(D_0^p))$

$$L^2(I, H_\xi^m(D_0^p)) := \{u \in L^2(I \times D_0^p); \int_I \|u(\xi, \cdot)\|_{H_\xi^m(D_0^p)}^2 d\xi < \infty\}. \quad (6)$$

We also have the identity

$$(u, v)_{H^m(D^p)} = \int_I (\mathcal{J}u(\xi, \cdot), \mathcal{J}v(\xi, \cdot))_{H_\xi^m(D_0^p)} d\xi. \quad (7)$$

The inverse of \mathcal{J} coincides with its L^2 adjoint and, for example, for any $u \in L^2(D^p)$ we have

$$u(x) = \int_I \mathcal{J}u(\xi, x) d\xi \quad \text{for a.e } x \in D_0^p. \quad (8)$$

For fixed $\xi \in I$, the ξ -quasi-periodic interior transmission problem can be formulated as: search for $(u_\xi, v_\xi) \in L_\xi^2(D^p) \times L_\xi^2(D^p)$ such that $(u_\xi - v_\xi) \in H_\xi^2(D^p)$ satisfies

$$\begin{cases} \Delta u_\xi + k^2 n_p u_\xi = 0 & \text{in } D_0^p, \\ \Delta v_\xi + k^2 v_\xi = 0 & \text{in } D_0^p, \\ u_\xi - v_\xi = \varphi_\xi & \text{on } \partial\check{D}_0^p, \\ \frac{\partial(u_\xi - v_\xi)}{\partial\nu} = \psi_\xi & \text{on } \partial\check{D}_0^p, \\ u_\xi = v_\xi = 0 & \text{on } \Gamma^0, \end{cases} \quad (9)$$

for given $(\varphi_\xi, \psi_\xi) \in H_\xi^{3/2}(\partial\check{D}^p) \times H_\xi^{1/2}(\partial\check{D}^p)$, where

$$\partial\check{D}_0^p := \partial D_0^p \setminus (\Gamma^0 \cup \{x_1 = 0\} \cup \{x_1 = 2\pi\}).$$

The analysis of this problem is done under the following assumption:

Assumption 3.3. We assume that

$$\Re(n_p(x)) - 1 \geq \alpha > 0 \quad \text{or} \quad 1 - \Re(n_p(x)) \geq \alpha > 0,$$

for almost every $x \in D^p$ and for some $\alpha > 0$. Moreover, we assume that $\Im m(n_p) \geq 0$ and $\{\Im m(n_p) > 0\}$ contains a non empty open set \mathcal{O} .

Under the previous assumption one can equivalently rewrite problem (9) as: Seek $w_\xi^0 := u_\xi - v_\xi \in H_\xi^2(D^p)$ satisfying

$$\begin{cases} (\Delta + k^2) \frac{1}{1-n_p} (\Delta w_\xi^0 + k^2 n_p w_\xi^0) = 0 & \text{in } D_0^p, \\ w_\xi^0 = \varphi_\xi \quad \text{and} \quad \frac{\partial w_\xi^0}{\partial\nu} = \psi_\xi & \text{on } \partial\check{D}_0^p, \\ w_\xi^0 = 0 & \text{on } \Gamma^0, \\ \frac{1}{1-n_p} (\Delta w_\xi^0 + k^2 n_p w_\xi^0) = 0 & \text{on } \Gamma^0. \end{cases} \quad (10)$$

We set

$$\tilde{H}_\xi^2(D_0^p) := \{w_\xi \in H_\xi^2(D^p) / w_\xi = \frac{\partial w_\xi}{\partial\nu} = 0 \text{ on } \partial\check{D}_0^p \text{ and } w_\xi = 0 \text{ on } \Gamma^0\}.$$

We remark that $w_\xi \rightarrow \sqrt{\|\Delta w_\xi\|_{L^2(D_0^p)}^2 + \|w_\xi\|_{L^2(D_0^p)}^2}$ defines an equivalent norm on $\tilde{H}_\xi^2(D_0^p)$. Let $\theta_\xi \in H_\xi^2(D^p)$ be a lifting function satisfying $\theta_\xi = \varphi_\xi$ and $\frac{\partial \theta_\xi}{\partial\nu} = \psi_\xi$ on $\partial\check{D}_0^p$, $\theta_\xi = 0$ on Γ^0 and

$$\|\theta_\xi\|_{H^2(D_0^p)} \leq c(\|\varphi_\xi\|_{H^{3/2}(\partial\check{D}_0^p)} + \|\psi_\xi\|_{H^{1/2}(\partial\check{D}_0^p)}),$$

for some $c > 0$.

4. Well-posedness of the quasi-periodic interior transmission problems.

The analysis of the problem (3) uses results related to the ξ -quasi-periodic interior transmission problem (9), which we will establish in this section. The existence and uniqueness of solutions follow the classical variational approach as in [4]. We also

need a bound on the solutions which are uniform with respect to the Floquet-Bloch variable ξ . This is the main theorem of this section. Define the sesquilinear form

$$a_\xi(w_\xi, w'_\xi) := \int_{D_0^p} \frac{1}{n_p - 1} (\Delta w_\xi + k^2 n_p w_\xi) \overline{(\Delta w'_\xi + k^2 w'_\xi)} dx. \quad (11)$$

Then the variational formulation equivalent to (10) can be written as Search $w_\xi := w_\xi^0 - \theta_\xi \in \tilde{H}_\xi^2(D^p)$ such that

$$a_\xi(w_\xi, w'_\xi) = - \int_{D_0^p} \frac{1}{n_p - 1} (\Delta \theta_\xi + k^2 n_p \theta_\xi) \overline{(\Delta w'_\xi + k^2 w'_\xi)} dx, \quad (12)$$

for all $w'_\xi \in \tilde{H}_\xi^2(D^p)$. We observe that

$$\mathcal{L} : w'_\xi \rightarrow \int_{D_0^p} \frac{1}{n_p - 1} (\Delta \theta_\xi + k^2 n_p \theta_\xi) \overline{(\Delta w'_\xi + k^2 w'_\xi)} dx,$$

is a bounded antilinear function on $\tilde{H}_\xi^2(D^p)$. Using the Riesz representation theorem, we consider $f_\xi \in \tilde{H}_\xi^2(D^p)$ such that $\mathcal{L}(w'_\xi) = (f_\xi, w'_\xi)_{H^2(D_0^p)}$ for all $w'_\xi \in \tilde{H}_\xi^2(D_0^p)$ and

$$\|f_\xi\|_{H^2(D_0^p)} \leq c_1 \|\theta_\xi\|_{H^2(D_0^p)} \leq c_2 \left(\|\varphi_\xi\|_{H^{3/2}(\partial \tilde{D}_0^p)} + \|\psi_\xi\|_{H^{1/2}(\partial \tilde{D}_0^p)} \right),$$

where c_1, c_2 are two positive constants independent of ξ . So the problem (12) is equivalent to

$$\mathcal{A}_\xi w_\xi = f_\xi \quad \text{in } \tilde{H}_\xi^2(D_0^p), \quad (13)$$

where $\mathcal{A}_\xi : \tilde{H}_\xi^2(D^p) \rightarrow \tilde{H}_\xi^2(D^p)$ is the operator defined by the Riesz representation Theorem as

$$(\mathcal{A}_\xi w_\xi, w'_\xi)_{H^2(D_0^p)} = a_\xi(w_\xi, w'_\xi) \quad \forall w_\xi, w'_\xi \in \tilde{H}_\xi^2(D_0^p), \quad (14)$$

Theorem 4.1. *Assume that Assumption 3.3 holds. Then the operator \mathcal{A}_ξ is invertible.*

Proof. We first prove that \mathcal{A}_ξ is a Fredholm operator of index zero. Let us define by the Riesz representation the following two operators $\mathcal{A}_{\xi,0}, \mathcal{A}_{\xi,1} : \tilde{H}_\xi^2(D^p) \rightarrow \tilde{H}_\xi^2(D^p)$ satisfying

$$(\mathcal{A}_{\xi,0} w_\xi, w'_\xi)_{H^2(D_0^p)} = \int_{D_0^p} \frac{1}{n_p - 1} \left(\Delta w_\xi \overline{\Delta w'_\xi} + w_\xi \overline{w'_\xi} \right) dx, \quad (15)$$

$$(\mathcal{A}_{\xi,1} w_\xi, w'_\xi)_{H^2(D_0^p)} = \int_{D_0^p} \frac{1}{n_p - 1} (k^2 \Delta w_\xi \overline{w'_\xi} + k^2 n_p w_\xi \overline{\Delta w'_\xi} + (k^4 n_p - 1) w_\xi \overline{w'_\xi}) dx. \quad (16)$$

We observe that

$$\Re e(\mathcal{A}_{\xi,0} w, w)_{H^2(D_0^p)} \geq \alpha_0 \left(\|\Delta w_\xi\|_{L^2(D_0^p)}^2 + \|w_\xi\|_{L^2(D_0^p)}^2 \right),$$

with $\alpha_0 := \frac{1}{n_{p,*} - 1}$ in the case $n_{p,*} > 1$ and $\alpha_0 := \frac{1}{n_{p,*} - 1}$ in the case $0 < n_{p,*} < 1$, where

$$n_{p,*} := \inf_{D^p} \Re e(n_p) \quad \text{and} \quad n^{p,*} := \sup_{D^p} \Re e(n_p). \quad (17)$$

This implies by the Lax-Milgram Theorem that $\mathcal{A}_{\xi,0}$ is invertible. On the other hand, using the compact embedding of $\tilde{H}_\xi^2(D_0^p)$ into $L_\xi^2(D_0^p)$, one observes that

$\mathcal{A}_{\xi,1}$ is compact operator. This proves that \mathcal{A}_ξ is a Fredholm operator of index zero. To conclude, we prove the injectivity of this operator. Assume that

$$(\mathcal{A}_\xi w_\xi, w'_\xi)_{H^2(D_0^p)} = 0 \quad \text{for all } w'_\xi \in \tilde{H}_\xi^2(D_0^p).$$

This implies using the definition of (14) that

$$0 = a_\xi(w_\xi, w_\xi) = \int_{D_0^p} \frac{1}{n_p - 1} |\Delta w_\xi + k^2 w_\xi|^2 - k^2 \int_{D_0^p} |\nabla w_\xi|^2 dx + k^4 \int_{D_0^p} |w_\xi|^2 dx.$$

Taking the imaginary part we get

$$\int_{D_0^p} \Im m \left(\frac{1}{n_p - 1} \right) |\Delta w_\xi + k^2 w_\xi|^2 dx = 0, \quad (18)$$

and therefore $\Delta w_\xi + k^2 w_\xi = 0$ in \mathcal{O} . Set

$$v_\xi := -\frac{1}{k^2(n_p - 1)} (\Delta w_\xi + k^2 n_p w_\xi) \quad \text{and} \quad u_\xi := w_\xi + v_\xi = \frac{1}{k^2(1 - n_p)} (\Delta w_\xi + k^2 w_\xi).$$

We then have $u_\xi = 0$ in \mathcal{O} . From the equation

$$(\Delta + k^2) \left(\frac{1}{n_p - 1} \right) (\Delta w_\xi + k^2 n_p w_\xi) = 0, \quad (19)$$

we conclude that $(u_\xi, v_\xi) \in L_\xi^2(D_0^p) \times L_\xi^2(D_0^p)$ satisfy

$$\begin{cases} \Delta u_\xi + k^2 n_p u_\xi = 0 & \text{in } D_0^p, \\ \Delta v_\xi + k^2 v_\xi = 0 & \text{in } D_0^p. \end{cases} \quad (20)$$

The unique continuation principle implies that $u_\xi = 0$ in D_0^p . So we have $\Delta w_\xi + k^2 w_\xi = 0$ in D_0^p and $w_\xi = \frac{\partial w_\xi}{\partial \nu} = 0$ on $\partial \tilde{D}^p$, which also implies by the unique continuation principle that $w_\xi \text{th}a = 0$ in D_0^p . This proves the injectivity of the operator \mathcal{A}_ξ and ends the proof. \square

We now proceed to prove a uniform bound on \mathcal{A}_ξ^{-1} with respect to the Floquet-Bloch variable ξ .

Theorem 4.2. *Assume that Assumption 3.3 holds. There exists a constant $\gamma > 0$ independent of $\xi \in I$ such that*

$$\|\mathcal{A}_\xi^{-1}\| \leq \gamma. \quad (21)$$

Proof. For given $\xi \in I$ we consider the bijective operator $\mathcal{I}_\xi : \tilde{H}_\#^2(D_0^p) \longrightarrow \tilde{H}_\xi^2(D_0^p)$ defined by

$$\mathcal{I}_\xi(v_\#)(x_1, x_2) = e^{i\xi x_1} v_\#(x_1, x_2), \quad \text{for a.e. } (x_1, x_2) \in D_0^p,$$

where $\tilde{H}_\#^2(D_0^p)$ is defined the same as $\tilde{H}_\xi^2(D_0^p)$ for $\xi = 0$. Clearly, $\|\mathcal{I}_\xi\| \leq 4$ and $\|\mathcal{I}_\xi^{-1}\| \leq 4$.

We then define $\mathcal{A}_{\xi,\#} : \tilde{H}_\#^2(D_0^p) \longrightarrow \tilde{H}_\#^2(D_0^p)$ by

$$(\mathcal{A}_{\xi,\#} v_\#, v'_\#)_{H^2(D_0^p)} = a_\xi(\mathcal{I}_\xi(v_\#), \mathcal{I}_\xi(v'_\#)) = (\mathcal{A}_\xi \mathcal{I}_\xi(v_\#), \mathcal{I}_\xi(v'_\#))_{H^2(D_0^p)} \quad \forall v_\#, v'_\# \in \tilde{H}_\#^2(D_0^p). \quad (22)$$

Thus, $\mathcal{A}_{\xi,\#} = \mathcal{I}_\xi^* \mathcal{A}_\xi \mathcal{I}_\xi$, where $\mathcal{I}_\xi^* : \tilde{H}_\xi^2(D_0^p) \longrightarrow \tilde{H}_\#^2(D_0^p)$ is the adjoint of \mathcal{I}_ξ . From the definition of $\tilde{\mathcal{I}}_\xi$ we deduce that the operator $\mathcal{A}_{\xi,\#}$ is also invertible and we have

$\|\mathcal{A}_\xi^{-1}\| \leq 16\|\mathcal{A}_{\xi,\#}^{-1}\|$. Consequently, it is sufficient to prove the uniform bound (21) on the operator $\mathcal{A}_{\xi,\#}$. We have

$$\Delta \mathcal{I}_\xi(v_\#) = e^{i\xi \cdot x_1} (-|\xi|^2 v_\# + 2i\xi \nabla v_\# + \Delta v_\#).$$

Therefore,

$$(\mathcal{A}_{\xi,\#} v_\#, v'_\#)_{H^2(D_0^p)} = a_\xi^\#(v_\#, v'_\#) \quad \forall v_\#, v'_\# \in \tilde{H}_\#^2(D_0^p), \quad (23)$$

where

$$a_\xi^\#(v_\#, v'_\#) := \int_{D_0^p} \frac{1}{n_p - 1} (\Delta v_\# + 2i\xi \nabla v_\# + (k^2 n_p - |\xi|^2) v_\#) \left(\Delta \overline{v'_\#} + 2i\xi \nabla \overline{v'_\#} + (k^2 - |\xi|^2) \overline{v'_\#} \right) dx.$$

We observe that for $\xi, \xi_0 \in I$

$$\begin{aligned} \left| a_\xi^\#(v_\#, v'_\#) - a_{\xi_0}^\#(v_\#, v'_\#) \right| &\leq C (\|\xi\|^4 - \|\xi_0\|^4 + \|\xi^3 - \xi_0^3\| + \|\xi^2 - \xi_0^2\| + \|\xi - \xi_0\|) \\ &\quad \times \|v_\#\|_{H_\#^2(D_0^p)} \|v'_\#\|_{H_\#^2(D_0^p)}, \quad \forall v_\#, v'_\# \in H_\#^2(D_0^p), \end{aligned} \quad (24)$$

with C a positive constant that depends on $k^2, n_{p,*}, n^{p,*}$ (defined in (17)) and does not depend on ξ and ξ_0 . Therefore,

$$\|\mathcal{A}_{\xi,\#} - \mathcal{A}_{\xi_0,\#}\| \leq 10C |\xi - \xi_0|.$$

Now, let $\xi_0 \in I$, and set $\eta_{\xi_0} = \frac{1}{2\|\mathcal{A}_{\xi_0,\#}^{-1}\|}$. Then for $|\xi - \xi_0| \leq \eta_{\xi_0}/(10C)$

$$\|\mathcal{A}_{\xi,\#} - \mathcal{A}_{\xi_0,\#}\| \leq \eta_{\xi_0},$$

and $\|\mathcal{A}_{\xi_0,\#}^{-1}(\mathcal{A}_{\xi,\#} - \mathcal{A}_{\xi_0,\#})\| \leq \frac{1}{2}$. As a result, using a Laurent expansion,

$$\|(I + \mathcal{A}_{\xi_0,\#}^{-1}(\mathcal{A}_{\xi,\#} - \mathcal{A}_{\xi_0,\#}))^{-1}\| \leq 2.$$

Consequently, observing that $\mathcal{A}_{\xi,\#} = \mathcal{A}_{\xi_0,\#}(I + \mathcal{A}_{\xi_0,\#}^{-1}(\mathcal{A}_{\xi,\#} - \mathcal{A}_{\xi_0,\#}))$ we deduce that

$$\|\mathcal{A}_{\xi,\#}^{-1}\| \leq 2\|\mathcal{A}_{\xi_0,\#}^{-1}\|, \quad \forall |\xi - \xi_0| \leq \eta_{\xi_0},$$

Since I is a compact set, there exists a finite set $J \subset I$ for which $I = \bigcup_{\xi_0 \in J} \{\xi \in I; |\xi - \xi_0| < \eta_{\xi_0}\}$. Therefore, for all $\xi \in I$ we have

$$\|\mathcal{A}_{\xi,\#}^{-1}\| \leq 2 \sup_{\xi_0 \in J} \|\mathcal{A}_{\xi_0,\#}^{-1}\| = \gamma, \quad \forall \xi \in I,$$

which ends the proof. \square

5. Analysis of the locally perturbed interior transmission problem. We now proceed to the main purpose of this paper, which is the analysis of the interior transmission problem formulated in (3). This analysis will be carried out under the following assumption:

Assumption 5.1. We assume that there exists $\alpha > 0$ such that

$$\Re(n(x)) - 1 \geq \alpha > 0 \quad \text{or} \quad 1 - \Re(n(x)) \geq \alpha > 0 \quad \text{for a.e } x \in D^p,$$

and $n = n_p$ outside \tilde{D} . We also assume that $\Im m(n) \geq 0$.

We set

$$n_* := \inf_{D^p} \Re e(n) \quad \text{and} \quad n^* := \sup_{D^p} \Re e(n).$$

Under this assumption, one can equivalently rewrite (3) as: search for $w_0 := u - v \in H^2(D^p)$ which satisfies

$$\begin{cases} (\Delta + k^2) \frac{1}{1-n} (\Delta w_0 + k^2 n w_0) = 0 & \text{in } D^p, \\ w_0 = \varphi \text{ and } \frac{\partial w_0}{\partial \nu} = \psi & \text{on } \partial D^p \setminus \Gamma^0, \\ w_0 = 0 & \text{on } \Gamma^0, \\ \frac{1}{1-n} (\Delta w_0 + k^2 n w_0) = 0 & \text{on } \Gamma^0. \end{cases} \quad (25)$$

We introduce the functional space

$$\tilde{H}_0^2(D^p) := \{w_0 \in H^2(D^p) \mid w_0 = \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial D^p \setminus \Gamma^0 \text{ and } w_0 = 0 \text{ on } \Gamma^0\},$$

equipped with the $\tilde{H}^2(D^p)$ norm. Let $\theta \in H^2(D^p)$ be a lifting function satisfying $\theta = \varphi$ and $\frac{\partial \theta}{\partial \nu} = \psi$ on $\partial D^p \setminus \Gamma^0$ and $\theta = 0$ on Γ^0 and define the sesquilinear form

$$a(w, w') := \int_{D^p} \frac{1}{n-1} (\Delta w + k^2 n w) \overline{(\Delta w' + k^2 w')} dx. \quad (26)$$

Then the variational formulation equivalent to (25) can be written as: seek $w := w_0 - \theta \in \tilde{H}_0^2(D^p)$ such that

$$a(w, w') = \int_{D^p} f \overline{(\Delta w' + k^2 w')} dx, \quad (27)$$

for all $w' \in \tilde{H}_0^2(D^p)$ with

$$f = \frac{1}{n-1} (\Delta \theta + k^2 n \theta). \quad (28)$$

First we give the result on the uniqueness of the solutions, which follows from the assumptions on the imaginary parts of n and n_p .

Lemma 5.2. *Under the assumptions 3.3 and 5.1, the problem (27) has at most one solution.*

Proof. The uniqueness of the solutions to problem (27) can be shown by following the same steps as the uniqueness result for problem (9) given by Theorem 4.1. \square

To show the existence of solutions we will rely on the rewriting of (27) using the Floquet-Bloch transform. Setting

$$\tilde{a}(w, w') := \int_{\tilde{D}} \left(\tilde{n} (\Delta w + k^2 n_p w) + k^2 \frac{(n - n_p)}{n-1} w \right) \overline{(\Delta w' + k^2 w')} dx, \quad (29)$$

with $\tilde{n} := \left(\frac{1}{n-1} - \frac{1}{n_p-1} \right)$ we can decompose the sesquilinear form $a(w, w')$ as

$$a(w, w') = \tilde{a}(w, w') + \int_{D^p} \frac{1}{n_p-1} (\Delta w + k^2 n_p w) \overline{(\Delta w' + k^2 w')} dx. \quad (30)$$

For $\hat{w} \in L^2(I, \tilde{H}_\xi^2(D_0^p))$ and $\hat{w}' \in L^2(I, \tilde{H}_\xi^2(D_0^p))$ we define

$$\hat{a}(\hat{w}, \hat{w}') := \int_I \int_{D_0^p} \frac{1}{n_p-1} (\Delta \hat{w}_\xi + k^2 n_p \hat{w}_\xi) \overline{(\Delta \hat{w}'_\xi + k^2 \hat{w}'_\xi)} dx d\xi + \tilde{a}(w, w') \quad (31)$$

with

$$w = \int_I \hat{w}(\xi, \cdot) d\xi \quad \text{and} \quad w' = \int_I \hat{w}'(\xi, \cdot) d\xi.$$

Using properties (7) and (8), the fact that for all $\varphi \in \tilde{H}^1(D^p)$

$$\mathcal{J}(n_p\varphi) = n_p\mathcal{J}(\varphi), \quad \text{and} \quad \mathcal{J}\left(\frac{\partial\varphi}{\partial x_j}\right) = \frac{\partial(\mathcal{J}\varphi)}{\partial x_j}, \quad \text{for all } 0 \leq j \leq 2,$$

and if φ is compactly supported in D_0^p then

$$\mathcal{J}(\varphi)(x, \xi) = \varphi(x) \quad \text{for a.e. } x \in D_0^p,$$

one can easily deduce that

$$a(w, w') = \hat{a}(\mathcal{J}w, \mathcal{J}w'), \quad \forall w, w' \in \tilde{H}_0^2(D^p). \quad (32)$$

In the following, we will use the short notation $\hat{w} := \mathcal{J}w$ and $\hat{w}_\xi := \hat{w}(\xi, \cdot)$. From (32) we deduce that the variational formulation (27) is equivalent to: Search for $\hat{w} \in L^2(I, \tilde{H}_\xi^2(D_0^p))$ such that

$$\hat{a}(\hat{w}, \hat{w}') = \int_I \int_{D_0^p} \hat{f}_\xi \overline{(\Delta \hat{w}'_\xi + k^2 \hat{w}'_\xi)} dx d\xi \quad \forall \hat{w}' \in L^2(I, \tilde{H}_\xi^2(D_0^p)). \quad (33)$$

To simplify the exposition, we will study the variational problem

$$\hat{a}(\hat{w}, \hat{w}') = \int_I \int_{D_0^p} \hat{f}_\xi \overline{\hat{w}'_\xi} dx d\xi \quad \forall \hat{w}' \in L^2(I, \tilde{H}_\xi^2(D_0^p)), \quad (34)$$

for $\hat{f} \in L^2(I \times D_0^p)$, which is equivalent to the variational formulation, $w \in \tilde{H}_0^2(D^p)$.

$$a(w, w') = \int_{D^p} f \overline{w'} dx \quad \forall w' \in \tilde{H}_0^2(D^p). \quad (35)$$

The case of the right hand side as in (33) can be treated in the same way and does not change the result on the well posedness of the problem. Let us remark that by Lemma 5.2, we already have the uniqueness of the solution to problem (35) in $\tilde{H}_0^2(D^p)$.

5.1. Semi-discretized interior transmission problem. To prove the existence of a solution to problem (34) we first prove the existence of a solution for to a semi-discretized version of this problem. We then construct a convergent sequence that provides a solution to problem (34).

Let N be the number of discretization points with respect to the Floquet-Bloch variable. We consider a uniform partition of I into sub-domains $I_j := [\xi_j, \xi_{j+1}]$ for $0 \leq j \leq N-1$ with $\xi_j := \frac{j}{N}$. We define the discrete space

$$X^N(D_0^p) := \{\hat{W}_N = (\hat{w}_{\xi_0}, \dots, \hat{w}_{\xi_N}); \hat{w}_{\xi_j} \in \tilde{H}_{\xi_j}^2(D_0^p) \text{ for } j = 0, \dots, N-1 \text{ and } \hat{w}_{\xi_N} = \hat{w}_{\xi_0}\},$$

equipped with the norm $\|\hat{W}_N\|_{X^N(D_0^p)}^2 := \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{\tilde{H}^2(D_0^p)}^2$. For $\hat{W}_N := (\hat{w}_{\xi_0}, \dots, \hat{w}_{\xi_N}) \in$

$X^N(D_0^p)$ we denote by

$$\Sigma(\hat{W}_N) := \frac{1}{N} \sum_{j=0}^{N-1} \hat{w}_{\xi_j}.$$

Then a discretized version of (34) is given as follows: search $\hat{W}_N := (\hat{w}_{\xi_0}, \dots, \hat{w}_{\xi_N}) \in X^N(D_0^p)$ such that

$$\frac{1}{N} \sum_{j=0}^{N-1} \int_{D_0^p} \frac{1}{n_p - 1} (\Delta \hat{w}_{\xi_j} + k^2 n_p \hat{w}_{\xi_j}) \overline{(\Delta \hat{w}'_{\xi_j} + k^2 \hat{w}'_{\xi_j})} dx + \int_{\bar{D}} \frac{k^2(n - n_p)}{n - 1} w_N \overline{(\Delta w'_N + k^2 w'_N)} dx$$

$$+ \int_{\tilde{D}} \tilde{n}(\Delta w_N + k^2 n_p w_N) \overline{(\Delta w'_N + k^2 w'_N)} dx = \frac{1}{N} \sum_{j=0}^{N-1} \int_{D_0^p} \hat{f}_{\xi_j} \overline{\hat{w}'_{\xi_j}} dx, \quad (36)$$

for all $\hat{W}'_N = (\hat{w}'_{\xi_0}, \dots, \hat{w}'_{\xi_{N-1}}) \in X^N(D_0^p)$, where

$$w_N = \Sigma(\hat{W}_N) \quad \text{and} \quad w'_N = \Sigma(\hat{W}'_N).$$

With the short notation for the sesquilinear form

$$\tilde{a}(w, w') := \int_{\tilde{D}} \left(\tilde{n}(\Delta w + k^2 n_p w) + k^2 \frac{(n - n_p)}{n - 1} w \right) \overline{(\Delta w' + k^2 w')} dx, \quad (37)$$

problem (36) can be synthetically written as

$$\frac{1}{N} \sum_{j=0}^{N-1} a_{\xi_j}(\hat{w}_{\xi_j}, \hat{w}'_{\xi_j}) + \tilde{a}(w_N, w'_N) = \frac{1}{N} \sum_{j=0}^{N-1} \int_{D_0^p} \hat{f}_{\xi_j} \overline{\hat{w}'_{\xi_j}} dx. \quad (38)$$

Here, \hat{f}_{ξ_j} can be seen as pointwise values of \hat{f} at $\xi = \xi_j$, for sufficiently regular \hat{f} .

5.2. Well posedness of the discretized problem. We begin by proving the uniqueness of solutions to problem (36). Assume that $\hat{f}_{\xi_j} = 0$ for $j = 0, \dots, N-1$.

We proceed in two steps: first we prove that $w_N = 0$ and second we deduce that $\hat{w}_{\xi_j} = 0$. Let us denote by

$$D_N^p := \bigcup_{j=0}^{N-1} \{D_0^p + 2\pi j \mathbf{e}_1\},$$

where $(\mathbf{e}_1, \mathbf{e}_2)$ is the canonical basis of \mathbb{R}^2 . Let $\hat{W}_N = (\hat{w}_{\xi_0}, \dots, \hat{w}_{\xi_{N-1}}) \in X^N(D_0^p)$. Obviously, the functions \hat{w}_{ξ_j} extended by ξ_j -quasi-periodicity in \mathbf{e}_1 direction are in $\tilde{H}_{\#}^2(D_N^p)$ for $j = 0, \dots, N-1$, where $\tilde{H}_{\#}^2(D_N^p)$ is the subspace of $\tilde{H}_{loc}^2(D^p)$ of $2\pi N$ -periodic functions in the \mathbf{e}_1 direction. We will make extensive use of the following identity (which follows from the expression of ξ_ℓ)

$$\sum_{\ell=0}^{N-1} e^{i2\pi \xi_\ell (j-j')} = N \delta_j^{j'}, \quad \text{for all } 0 \leq j, j' \leq N-1 \quad (39)$$

where $\delta_j^{j'}$ is the Kronecker symbol. For example, for $\hat{w}_{\xi_j} \in L_{\xi}^2(D^p)$ and $\hat{w}_{\xi_{j'}} \in L_{\xi'}^2(D^p)$ we have

$$\int_{D_N^p} \hat{w}_{\xi_j} \overline{\hat{w}_{\xi_{j'}}} dx = \sum_{\ell=0}^{N-1} e^{i2\pi \ell (\xi_j - \xi_{j'})} \int_{D_0^p} \hat{w}_{\xi_j} \overline{\hat{w}_{\xi_{j'}}} dx = N \delta_j^{j'} \int_{D_0^p} \hat{w}_{\xi_j} \overline{\hat{w}_{\xi_{j'}}} dx. \quad (40)$$

Similarly for $\hat{w}_{\xi} \in H_{\xi}^2(D^p)$ and $\hat{w}_{\xi'} \in H_{\xi'}^2(D^p)$ we have

$$(\hat{w}_{\xi}, \hat{w}_{\xi'})_{H^2(D_N^p)} = N \delta_j^{j'} (\hat{w}_{\xi}, \hat{w}_{\xi'})_{H^2(D_0^p)}. \quad (41)$$

In particular, this proves that $\tilde{H}_{\xi_j}^2(D^p)$ and $\tilde{H}_{\xi_{j'}}^2(D^p)$ are two orthogonal subspaces of $\tilde{H}_{\#}^2(D_N^p)$. More precisely, we have the following decomposition of $\tilde{H}_{\#}^2(D_N^p)$.

Lemma 5.3. *We have the following orthogonal decomposition*

$$\tilde{H}_{\#}^2(D_N^p) = \bigoplus_{j=0}^{N-1} \tilde{H}_{\xi_j}^2(D_0^p).$$

Proof. From the previous discussion we obviously have

$$\bigoplus_{j=0}^{N-1} \tilde{H}_{\sharp}^2(D^p) \subset \tilde{H}_{\sharp}^2(D_N^p).$$

To prove the inverse inclusion we first consider the following space

$$S_{\sharp}(D_N^p) := \{\hat{w} \in \tilde{H}_{\sharp}^2(D_N^p) / \text{dist}(\text{supp}(\hat{w}), \partial D_0^p) > \delta, \text{ for some } \delta > 0\}.$$

This set is dense in $\tilde{H}_{\sharp}^2(D_N^p)$. We consider $\hat{w} \in S_{\sharp}(D_N^p)$ which we extend by zero outside D_N^p , giving a function $\hat{w} \in \tilde{H}_{\sharp}^2(\Omega_N^R)$, with $\Omega_N^R := [0, 2\pi N] \times [0, R]$. Using Fourier series we have that \hat{w} can be expanded as

$$\hat{w}(x_1, x_2) = \sum_{n=0}^{N-1} a_n(x_2) e^{inx_1}, \quad (42)$$

with $a_n(x_2)$ is defined by

$$a_n(x_2) = \frac{1}{2\pi N} \int_0^{2\pi N} \hat{w}(x_1, x_2) e^{-inx_1} dx_1.$$

Let $0 \leq j \leq N-1$, then for all $n \in \mathbb{Z}$, there exists a unique $\ell \in \mathbb{Z}$ such that $n = j + N\ell$. Hence \hat{w} can be written as

$$\hat{w} = \sum_{j=0}^{N-1} \hat{w}_{\xi_j},$$

with $\hat{w}_{\xi_j} = \sum_{\ell=0}^{N-1} a_{j+N\ell}(x_2) e^{i(j+N\ell)x_1}$ which belongs to $H_{\xi_j}^2(\Omega^R)$ since for $m \in \mathbb{Z}$ we have

$$e^{i(j+N\ell)(x_1+2\pi m)} = e^{i\xi_j m} e^{i(j+N\ell)x_1}.$$

Furthermore, $\hat{w} \in \tilde{H}_{\sharp}^2(D_N^p)$ then \hat{w} vanishes at Γ^0 . Therefore, $a_n(0) = 0$ and $\hat{w}_{\xi_j} = 0$ on Γ^0 .

Let us now define a cut-off function $\chi \in C^\infty(\mathbb{R}_+^2)$ 2π -periodic in the \mathbf{e}_1 direction such that $\chi = 1$ inside $\text{supp}(\hat{w})$ and $\chi = 0$ outside D_N^p . Therefore

$$\hat{w} = \chi \hat{w} = \sum_{j=0}^{N-1} \chi \hat{w}_{\xi_j},$$

where $\chi \hat{w}_{\xi_j} \in \tilde{H}_{\xi_j}^2(D_0^p)$. Consequently $S_{\sharp}(D_N^p) \subset \bigoplus_{j=0}^{N-1} \tilde{H}_{\xi_j}^2(D_0^p)$ which implies that

$\bigoplus_{j=0}^{N-1} \tilde{H}_{\xi_j}^2(D_0^p)$ is dense in $\tilde{H}_{\sharp}^2(D_N^p)$. Moreover, since the orthogonal sum is a closed subset of $\tilde{H}_{\sharp}^2(D_N^p)$ then we deduce the equality of the two spaces. \square

Lemma 5.4. *Let $\hat{W}_N = (\hat{w}_{\xi_0}, \dots, \hat{w}_{\xi_{N-1}}) \in X^N(D_0^p)$ be a solution of the problem (36) and let $w_N := \Sigma(\hat{W}_N)$. Then $w_N \in \tilde{H}_{\sharp}^2(D_N^p)$ and*

$$\int_{D_N^p} \frac{1}{n-1} (\Delta w_N + k^2 n w_N) \overline{(\Delta w' + k^2 w')} dx = \int_{D_N^p} f_N \overline{w'} dx, \quad \forall w' \in \tilde{H}_{\sharp}^2(D_N^p), \quad (43)$$

with $f_N := \frac{1}{N} \sum_{j=0}^{N-1} \hat{f}_{\xi_j}$.

Proof. Multiplying (36) by N and using (40) we get

$$\begin{aligned} & \frac{1}{N} \sum_{j=0}^{N-1} \int_{D_N^p} \frac{1}{n_p - 1} (\Delta \hat{w}_{\xi_j} + k^2 n_p \hat{w}_{\xi_j}) \overline{(\Delta \hat{w}'_{\xi_j} + k^2 \hat{w}'_{\xi_j})} dx + N \tilde{a}(w_N, w'_N) \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \int_{D_N^p} \hat{f}_{\xi_j} \overline{\hat{w}'_{\xi_j}} dx. \end{aligned} \quad (44)$$

On the other hand, using again (40) we deduce that for $j \neq j'$ we have

$$\int_{D_N^p} \frac{1}{n_p - 1} (\Delta \hat{w}_{\xi_j} + k^2 n_p \hat{w}_{\xi_j}) \overline{(\Delta \hat{w}_{\xi_{j'}} + k^2 \hat{w}_{\xi_{j'}})} dx = 0, \quad (45)$$

$$\int_{D_N^p} \hat{f}_{\xi_j} \overline{\hat{w}_{\xi_{j'}}} dx = 0. \quad (46)$$

Let $w' \in \tilde{H}_{\#}^2(D_N^p)$. From the Lemma 5.3 we can decompose it as $w' = \sum_{j=0}^{N-1} \hat{w}'_{\xi_j} \in \bigoplus_{j=0}^{N-1} \tilde{H}_{\xi_j}^2(D^p)$. Using (45) and (46) we deduce from (44) that

$$\frac{1}{N} \sum_{j=0}^{N-1} \int_{D_N^p} \frac{1}{n_p - 1} (\Delta \hat{w}_{\xi_j} + k^2 n_p \hat{w}_{\xi_j}) \overline{(\Delta w' + k^2 w')} dx + \tilde{a}(w_N, w') = \frac{1}{N} \sum_{j=0}^{N-1} \int_{D_N^p} \hat{f}_{\xi_j} \overline{w'} dx.$$

which gives (43) using the definition of $\Sigma(\hat{W}_N)$. \square

Lemma 5.5. *Under the assumptions 3.3 and 5.1, the problem (36) has at most one solution.*

Proof. Let $\hat{W}_N = (\hat{w}_{\xi_0}, \dots, \hat{w}_{\xi_N}) \in X^N(D_0^p)$ be a solution of the homogeneous problem (36). Then, by Lemma 5.4, w_N satisfies the $2\pi N$ periodic problem (43). Using the uniqueness result for quasi-periodic interior transmission problems given in Theorem 4.1 (replacing n_p by n and the periodicity 2π by $2\pi N$) we deduce that $w_N = 0$. Consequently, taking $\hat{w}'_{\xi_j} = 0$ for $j \neq \ell$ in (36) we get

$$\int_{D_0^p} \frac{1}{n_p - 1} (\Delta \hat{w}_{\xi_\ell} + k^2 n_p \hat{w}_{\xi_\ell}) \overline{(\Delta \hat{w}'_{\xi_\ell} + k^2 \hat{w}'_{\xi_\ell})} dx = 0,$$

for all $\hat{w}'_{\xi_\ell} \in \tilde{H}_{\xi_\ell}^2(D_0^p)$. Thus, $\hat{w}_{\xi_\ell} = 0$ by uniqueness of solutions to quasi-periodic interior transmission problems (Theorem 4.1). \square

Lemma 5.6. *Under the assumptions 3.3 and 5.1, the problem (36) has a solution $\hat{W}_N \in X^N(D_0^p)$.*

Proof. From the well-posedness of the $2\pi N$ -periodic problem (after rescaling the periodicity length given e.g. by Theorem 4.1) we consider $w_N \in \tilde{H}_{\#}^2(D_N^p)$ which satisfies (43). Let $0 \leq j \leq N-1$. Using Theorem 4.1 we define the solution $\hat{w}_{\xi_j} \in \tilde{H}_{\xi_j}^2(D_0^p)$ satisfying

$$a_{\xi_j}(\hat{w}_{\xi_j}, \hat{w}'_{\xi_j}) = \int_{D_0^p} \hat{f}_{\xi_j} \overline{\hat{w}'_{\xi_j}} dx - \tilde{a}(w_N, \hat{w}'_{\xi_j}) \quad \text{for all } \hat{w}'_{\xi_j} \in \tilde{H}_{\xi_j}^2(D_0^p). \quad (47)$$

Therefore

$$\frac{1}{N} \sum_{j=0}^{N-1} a_{\xi_j}(\hat{w}_{\xi_j}, \hat{w}'_{\xi_j}) = \frac{1}{N} \sum_{j=0}^{N-1} \int_{D_0^p} \hat{f}_{\xi_j} \overline{\hat{w}'_{\xi_j}} dx - \frac{1}{N} \sum_{j=0}^{N-1} \tilde{a}(w_N, \hat{w}'_{\xi_j}). \quad (48)$$

Let us define

$$w' = \sum_{j=0}^{N-1} \hat{w}'_{\xi_j}, \quad \tilde{w}_N := \frac{1}{N} \sum_{j=0}^{N-1} \hat{w}_{\xi_j}, \quad \text{and} \quad f_N := \frac{1}{N} \sum_{j=0}^{N-1} \hat{f}_{\xi_j}.$$

Then using (45) and (46) we deduce that (48) implies

$$\frac{1}{N} \sum_{j=0}^{N-1} \int_{D_N^p} \frac{1}{n_p - 1} (\Delta \hat{w}_{\xi_j} + k^2 n_p \hat{w}_{\xi_j}) \overline{(\Delta w' + k^2 w')} dx = -\tilde{a}(w_N, w') + \frac{1}{N} \sum_{j=0}^{N-1} \int_{D_N^p} \hat{f}_{\xi_j} \overline{w'} dx.$$

Therefore $\tilde{w}_N \in \tilde{H}_{\#}^2(D_N^p)$ and satisfies

$$\int_{D_N^p} \frac{1}{n_p - 1} (\Delta \tilde{w}_N + k^2 n_p \tilde{w}_N) \overline{(\Delta w' + k^2 w')} dx = -\tilde{a}(w_N, w') + \int_{D_N^p} f_N \overline{w'} dx, \quad (49)$$

for all $w' \in \tilde{H}_{\#}^2(D_N^p)$. Note that w_N also satisfies (49). Consequently, from the uniqueness of solutions to $2\pi N$ -periodic interior transmission problems (under the assumption 3.3) we deduce that $\tilde{w}_N = w_N$. Finally, if we set $\hat{W}_N = (\hat{w}_{\xi_0}, \dots, \hat{w}_{\xi_N}) \in X^N(D_0^p)$, we have $w_N = \Sigma(\hat{W}_N)$ and \hat{W}_N satisfies problem (36). \square

5.3. Convergence analysis. In order to construct a solution to the interior transmission problem (35) we shall prove that the function w_N constructed in the proof above locally converges to a solution of problem (35) as $N \rightarrow \infty$. Let $M > 0$ and $0 < \alpha < 1$ be given. We prove the convergence for the right hand sides $f \in \mathcal{F}_M$ where

$$\mathcal{F}_M := \left\{ f \in L^2(D^p); \mathcal{J}f \in C_{\#}^{0,\alpha}(I, L^2(D_0^p)) \text{ and} \right. \\ \left. \|\mathcal{J}f(\xi, \cdot) - \mathcal{J}f(\xi', \cdot)\|_{L^2(D_0^p)} \leq M |\xi - \xi'|^\alpha, \forall \xi, \xi' \in I \right\}.$$

For $f \in \mathcal{F}_M$ we use the short notation $\hat{f}_{\xi} := \mathcal{J}(f)(\xi, \cdot)$. Consider $\hat{W}_N = (\hat{w}_{\xi_0}, \dots, \hat{w}_{\xi_N}) \in X^N(D_0^p)$ as the solution of the problem (36) associated with \hat{f}_{ξ} and set $w_N := \Sigma(\hat{W}_N)$.

Let us introduce the continuous and piecewise affine function with respect to the ξ variable

$$\tilde{w}_{\xi}^N := N e^{i\xi x_1} (\hat{w}_{\xi_j} e^{-i\xi_j x_1} (\xi_{j+1} - \xi) + \hat{w}_{\xi_{j+1}} e^{-i\xi_{j+1} x_1} (\xi - \xi_j)), \quad (50)$$

for $\xi \in [\xi_j, \xi_{j+1}[$ and $j = 0, \dots, N-1$. We then define

$$\tilde{w}_N := \int_I \tilde{w}_{\xi}^N d\xi. \quad (51)$$

This function satisfies

$$\|\tilde{w}_N\|_{L^2(D^p)}^2 = \int_0^1 \|\tilde{w}_{\xi}\|_{L^2(D_0^p)}^2 d\xi \leq \frac{4}{3} \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{L^2(D_0^p)}^2 = \frac{4}{3} \|w_N\|_{L^2(D_N^p)}^2, \quad (52)$$

and

$$\|\check{w}_N\|_{H^2(D^p)}^2 = \int_0^1 \|\check{w}_\xi^N\|_{H^2(D_0^p)}^2 d\xi \leq \frac{16}{3} \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{H^2(D_0^p)}^2 = \frac{16}{3} \|w_N\|_{H^2(D_N^p)}^2. \quad (53)$$

We also have for $\xi \in [\xi_j, \xi_{j+1}[$

$$\begin{aligned} \|\check{w}_\xi^N\|_{L^2(D_0^p)}^2 &\geq \|\hat{w}_{\xi_j}\|_{L^2(D_0^p)}^2 (\xi - \xi_{j+1})^2 + \|\hat{w}_{\xi_{j+1}}\|_{L^2(D_0^p)}^2 (\xi - \xi_j)^2 \\ &\quad - 2 \left((\xi_{j+1} - \xi)(\xi - \xi_j) \|\hat{w}_{\xi_j}\|_{L^2(D_0^p)} \|\hat{w}_{\xi_{j+1}}\|_{L^2(D_0^p)} \right). \end{aligned}$$

Therefore,

$$\|\check{w}_N\|_{L^2(D^p)}^2 = \frac{1}{N} \sum_{j=0}^{N-1} \int_{\xi_j}^{\xi_{j+1}} \|\check{w}_\xi^N\|_{L^2(D_0^p)}^2 d\xi \geq \frac{1}{3} \|w_N\|_{L^2(D_N^p)}^2. \quad (54)$$

In fact, we shall prove that the sequence (\check{w}_N) converges to the solution of problem (35) as $N \rightarrow \infty$. The following lemma proves that the local limit of the two sequences (w_N) and (\check{w}_N) is the same.

Lemma 5.7. *Fix $N_0 \in \mathbb{N}$ and consider $N \geq N_0$. Then there exists a constant $C(N_0) > 0$ which depends only on N_0 such that*

$$\|w_N - \check{w}_N\|_{L^2(D_{N_0}^p)} \leq \frac{N_0^2 C(N_0)}{12N^2} \left(\frac{1}{N} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{L^2(D_0^p)} \right)^{1/2}.$$

Proof. Taking the difference between w_N and \check{w}_N . We have

$$\begin{aligned} w_N - \check{w}_N &= \sum_{j=0}^{N-1} \left(\frac{1}{N} \hat{w}_{\xi_j} - N \hat{w}_{\xi_{j+1}} \int_{\xi_j}^{\xi_{j+1}} e^{ix_1(\xi - \xi_{j+1})} (\xi - \xi_j) d\xi \right. \\ &\quad \left. + N \hat{w}_{\xi_j} \int_{\xi_j}^{\xi_{j+1}} e^{ix_1(\xi - \xi_j)} (\xi - \xi_{j+1}) d\xi \right) \\ &= \sum_{j=0}^{N-1} \left(\frac{1}{N} \hat{w}_{\xi_j} - N \hat{w}_{\xi_{j+1}} \left(\frac{1}{N} \frac{1}{ix_1} + \frac{1}{x_1^2} (1 - e^{-\frac{ix_1}{N}}) \right) \right. \\ &\quad \left. + N \hat{w}_{\xi_j} \left(\frac{1}{N} \frac{1}{ix_1} - \frac{1}{x_1^2} (1 - e^{\frac{ix_1}{N}}) \right) \right), \end{aligned}$$

Since $\hat{w}_{\xi_0} = \hat{w}_{\xi_N}$ then we have

$$w_N - \check{w}_N = \frac{1}{N} \sum_{j=0}^{N-1} \hat{w}_{\xi_j} \left[1 - \frac{N^2}{x_1^2} \left((1 - e^{-\frac{ix_1}{N}}) + (1 - e^{\frac{ix_1}{N}}) \right) \right] = \frac{1}{N} \sum_{j=0}^{N-1} \hat{w}_{\xi_j} \check{\theta}_N(x_1),$$

with

$$\check{\theta}_N(x_1) = \begin{cases} 1 - \frac{2N^2}{x_1^2} (1 - \cos(\frac{x_1}{N})) & \text{for } x_1 \neq 0, \\ 0 & \text{for } x_1 = 0. \end{cases} \quad (55)$$

We observe that $\check{\theta}_N(x_1)$ have the following asymptotic form as N tends to infinity

$$\check{\theta}_N(x_1) = 1 - \frac{2N^2}{x_1^2} \left[\frac{1}{2} \left(\frac{x_1}{N} \right)^2 - \frac{1}{4!} \left(\frac{x_1}{N} \right)^4 + \mathcal{O} \left(\frac{1}{N^6} \right) \right] = \frac{x_1^2}{12N^2} + \mathcal{O} \left(\frac{1}{N^4} \right),$$

for $x_1 \leq N_0$. Thus

$$\|w_N - \check{w}_N\|_{L^2(D_{N_0}^p)} \leq \frac{N_0^2 C(N_0)}{12N^3} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{L^2(D_0^p)} \leq \frac{N_0^2 C(N_0)}{12N^2} \left(\frac{1}{N} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{L^2(D_0^p)} \right)^{1/2},$$

which ends the proof. \square

The following lemma proves equicontinuity of the sequence \tilde{w}_ξ^N of $L^2(D_0^p)$ with respect to the ξ variable.

Lemma 5.8. *Let $N_0 \in \mathbb{N}$. For all $\epsilon > 0$ there exists $\eta > 0$ such that for $|\xi - \xi'| < \eta$ we have*

$$\|\tilde{w}_\xi^N - \tilde{w}_{\xi'}^N\|_{L^2(D_0^p)} \leq \epsilon \quad \text{for all } N \geq N_0. \quad (56)$$

Proof. For $0 \leq j, j' \leq N-1$, we define

$$e_{j,j'} := N(\hat{w}_{\xi_j}^\# - \hat{w}_{\xi_{j'}}^\#).$$

We need first to prove some estimation on the L^2 -norm of $e_{j,j'}$. In order to do so, we let $\hat{w}^\# \in \tilde{H}_\#^2(D^p)$ and set $\hat{w}'_{\xi_j} = e^{ix_1 \xi_j} \hat{w}^\# \in \tilde{H}_{\xi_j}^2(D^p)$. Then, replacing \hat{w}_{ξ_j} and \hat{w}'_{ξ_j} respectively by $\hat{w}_{\xi_j}^\# e^{ix_1 \xi_j}$ and $\hat{w}'_{\xi_j} e^{ix_1 \xi_j}$ in (72) we get

$$\begin{aligned} & \int_{D_0^p} \frac{1}{n_p - 1} \left(\Delta \hat{w}_{\xi_j}^\# + 2i\xi_j \frac{\partial \hat{w}_{\xi_j}^\#}{\partial x_1} + (k^2 n_p - |\xi_j|^2) \hat{w}_{\xi_j}^\# \right) \\ & \quad \times \overline{\left(\Delta \hat{w}^\# + 2i\xi_j \frac{\partial \hat{w}^\#}{\partial x_1} + (k^2 n_p - |\xi_j|^2) \hat{w}^\# \right)} dx \\ & \quad + \int_{\tilde{D}} \left(\tilde{n}(\Delta w_N + k^2 n_p w_N) + k^2 \frac{(n - n_p)}{n - 1} w_N \right) \\ & \quad \times \overline{\left(\Delta \hat{w}^\# + 2i\xi_j \frac{\partial \hat{w}^\#}{\partial x_1} + (k^2 n_p - |\xi_j|^2) \hat{w}^\# \right)} e^{ix_1 \xi_j} dx \\ & = \int_{D_0^p} \hat{f}_{\xi_j}^\# \overline{\hat{w}^\#} dx, \end{aligned} \quad (57)$$

with $\hat{f}_{\xi_j}^\# := e^{-ix_1 \xi_j} \hat{f}_{\xi_j}$.

Taking the difference between the equations (57) satisfied for $\xi_{j'}$ and ξ_j respectively, we get

$$\begin{aligned} & \frac{1}{N} \int_{D_0^p} \frac{1}{n_p - 1} \left(\Delta e_{j,j'} + 2i\xi_{j'} \frac{\partial e_{j,j'}}{\partial x_1} + (k^2 n_p - |\xi_{j'}|^2) e_{j,j'} \right) \\ & \quad \times \overline{\left(\Delta \hat{w}^\# + 2i\xi_{j'} \frac{\partial \hat{w}^\#}{\partial x_1} + (k^2 n_p - |\xi_{j'}|^2) \hat{w}^\# \right)} dx \\ & \quad + (\xi_{j'} - \xi_j) \left(a_{p,1}(\hat{w}_{\xi_j}^\#, \hat{w}^\#) + a_{p,2}(\hat{w}_{\xi_j}^\#, \hat{w}^\#) + \tilde{a}_1(w_N, \hat{w}^\#) \right) + \tilde{a}_2(w_N, \hat{w}^\#) \\ & = \int_{D_0^p} (\hat{f}_{\xi_j}^\# - \hat{f}_{\xi_{j'}}^\#) \overline{\hat{w}^\#} dx. \end{aligned} \quad (58)$$

with $a_{p,1}(\hat{w}_{\xi_j}^\#, \hat{w}^\#)$ and $a_{p,2}(\hat{w}_{\xi_j}^\#, \hat{w}^\#)$ are the sesquilinear forms defined by

$$a_{p,1}(\hat{w}_{\xi_j}^\#, \hat{w}^\#) := \int_{D_0^p} \frac{1}{n_p - 1} \left(2i \frac{\partial \hat{w}_{\xi_j}^\#}{\partial x_1} - (|\xi_{j'}| + |\xi_j|) \hat{w}_{\xi_j}^\# \right)$$

$$\begin{aligned}
& \times \overline{\left(\Delta \hat{w}^{\sharp'} + 2i\xi_j \frac{\partial \hat{w}^{\sharp'}}{\partial x_1} + (k^2 - |\xi_j|^2) \hat{w}^{\sharp'} \right)} dx, \\
a_{p,2}(\hat{w}_{\xi_j}^{\sharp}, \hat{w}^{\sharp'}) &:= \int_{D_0^p} \frac{1}{n_p - 1} \left(\Delta \hat{w}_{\xi_j}^{\sharp} + 2i\xi_{j'} \frac{\partial \hat{w}_{\xi_j}^{\sharp}}{\partial x_1} + (k^2 n_p - |\xi_{j'}|^2) \hat{w}_{\xi_j}^{\sharp} \right) \\
& \times \overline{\left(2i \frac{\partial \hat{w}^{\sharp'}}{\partial x_1} - (|\xi_{j'}| + |\xi_j|) \hat{w}^{\sharp'} \right)} dx,
\end{aligned}$$

and $\tilde{a}_1(w_N, \hat{w}^{\sharp'})$, $\tilde{a}_2(w_N, \hat{w}^{\sharp'})$ are defined by

$$\begin{aligned}
\tilde{a}_1(w_N, \hat{w}^{\sharp'}) &:= \int_{\tilde{D}} \left(\tilde{n}(\Delta w_N + k^2 n_p w_N) + k^2 \frac{(n - n_p)}{n - 1} w_N \right) \\
& \times e^{i\xi_j x_1} \overline{\left(2i \frac{\partial \hat{w}^{\sharp'}}{\partial x_1} - (|\xi_j| + |\xi_{j'}|) \hat{w}^{\sharp'} \right)} dx \\
\tilde{a}_2(w_N, \hat{w}^{\sharp'}) &:= \int_{\tilde{D}} \left(\tilde{n}(\Delta w_N + k^2 n_p w_N) + k^2 \frac{(n - n_p)}{n - 1} w_N \right) \\
& \times \overline{\left(e^{ix_1 \xi_{j'}} - e^{ix_1 \xi_j} \right) \left(\Delta \hat{w}^{\sharp'} + 2i\xi_{j'} \frac{\partial \hat{w}^{\sharp'}}{\partial x_1} + (k^2 - |\xi_{j'}|^2) \hat{w}^{\sharp'} \right)} dx.
\end{aligned}$$

Setting $\hat{e}_{\xi_{j'}} := e^{ix_1 \xi_{j'}} e_{j'} \in \tilde{H}_{\xi_{j'}}^2(D^p)$ and choosing $w^{\sharp} = e^{-ix_1 \xi_{j'}} \hat{w}'_{\xi_{j'}}$, for some $\hat{w}'_{\xi_{j'}} \in \tilde{H}_{\xi_{j'}}^2(D^p)$ we deduce that

$$\begin{aligned}
a_{\xi_{j'}}(\hat{e}_{\xi_{j'}}, \hat{w}'_{\xi_{j'}}) &= - \left(a_{p,1}(\hat{w}_{\xi_j}^{\sharp}, \hat{w}^{\sharp'}) + a_{p,2}(\hat{w}_{\xi_j}^{\sharp}, \hat{w}^{\sharp'}) + \tilde{a}_1(w_N, \hat{w}^{\sharp'}) + \tilde{a}_2(w_N, \hat{w}^{\sharp'}) \right) \\
& + N \int_{D_0^p} \left(\hat{f}_{\xi_j}^{\sharp} - \hat{f}_{\xi_{j'}}^{\sharp} \right) \overline{\hat{w}^{\sharp'}} dx.
\end{aligned}$$

Therefore, using Theorem 4.2 we conclude that

$$\|\hat{e}_{\xi_{j'}}\|_{H^2(D_0^p)} \leq \gamma_1 \left(\|\hat{w}_{\xi_j}^{\sharp}\|_{H^2(D_0^p)} + \|w_N\|_{H^2(\tilde{D})} + N \left\| \hat{f}_{\xi_j}^{\sharp} - \hat{f}_{\xi_{j'}}^{\sharp} \right\|_{L^2(D_0^p)} \right), \quad (59)$$

with $\gamma_1 > 0$ is a constant independent from $\xi_{j'}$ and N .

Now, we first prove (56) in the case when $\xi, \xi' \in [\xi_j, \xi_{j+1}[$, i.e. $j' = j$. From (50) we have

$$\begin{aligned}
\check{w}_{\xi}^N - \check{w}_{\xi'}^N &= N e^{i\xi x_1} \hat{w}_{\xi_j}^{\sharp} (\xi_{j+1} - \xi) + N e^{i\xi x_1} \hat{w}_{\xi_{j+1}}^{\sharp} (\xi - \xi_j) \\
& - N e^{i\xi' x_1} \hat{w}_{\xi_j}^{\sharp} (\xi_{j+1} - \xi') - N e^{i\xi' x_1} \hat{w}_{\xi_j}^{\sharp} (\xi' - \xi_j),
\end{aligned} \quad (60)$$

where

$$\hat{w}_{\xi_j}^{\sharp} := e^{-ix_1 \xi_j} \hat{w}_{\xi_j} \quad \text{and} \quad \hat{w}_{\xi_{j+1}}^{\sharp} := e^{-ix_1 \xi_{j+1}} \hat{w}_{\xi_{j+1}}, \quad (61)$$

Denoting

$$\theta_{\xi_{j+1}}(\xi) := \hat{w}_{\xi_j}^{\sharp} N(\xi_{j+1} - \xi) + \hat{w}_{\xi_{j+1}}^{\sharp} N(\xi - \xi_j),$$

we observe that (60) can be given as

$$\check{w}_{\xi}^N - \check{w}_{\xi'}^N = (e^{i\xi x_1} - e^{i\xi' x_1}) \theta_{\xi_{j+1}}(\xi) + N(\xi - \xi') (\hat{w}_{\xi_{j+1}}^{\sharp} - \hat{w}_{\xi_j}^{\sharp}) \quad (62)$$

Observing that $|N(\xi_{j+1} - \xi)| \leq 1$, and $|N(\xi - \xi_j)| \leq 1$ we deduce that $\|\theta_{\xi_{j+1}}(\xi)\|_{L^2(D^p)}$ is bounded, and then from the continuity of the exponential function we deduce the existence of a constant $C > 0$ such that

$$\|(e^{i\xi x_1} - e^{i\xi' x_1}) \theta_{\xi_{j+1}}(\xi)\|_{L^2(D^p)} \leq C |\xi - \xi'|.$$

On the other hand, since $\xi \rightarrow f_\xi$ is a α -Hölderian function, then from (59) we deduce that the second term of of the right hand side of (62) satisfies

$$\begin{aligned} \|N(\xi - \xi')(\hat{w}_{\xi_{j+1}}^\sharp - \hat{w}_{\xi_j}^\sharp)\| &\leq C'N|\xi - \xi'| |\xi_{j+1} - \xi_j|^\alpha = C'N|\xi - \xi'| \frac{1}{N^\alpha} \\ &\leq C'|\xi - \xi'|N^{1-\alpha}, \end{aligned}$$

where C' is a positive constant. Since ξ and ξ' are both in $[\xi_j, \xi_{j+1}[$, then $|\xi - \xi'| \leq \frac{1}{N}$, which implies that $N^{1-\alpha} \leq \frac{1}{|\xi - \xi'|^{1-\alpha}}$ and therefore

$$\|N(\xi - \xi')(\hat{w}_{\xi_{j+1}}^\sharp - \hat{w}_{\xi_j}^\sharp)\| \leq C'|\xi - \xi'|^\alpha.$$

This implies that

$$\|\check{w}_\xi^N - \check{w}_{\xi'}^N\| \leq C'|\xi - \xi'|^\alpha.$$

We now consider the case where ξ and ξ' are not in the same interval, i.e. $j \neq j'$. We have

$$\check{w}_\xi^N - \check{w}_{\xi'}^N = (\check{w}_\xi^N - e^{i\xi x_1} \hat{w}_{\xi_{j+1}}^\sharp) - (\check{w}_{\xi'}^N - e^{i\xi' x_1} \hat{w}_{\xi_{j'}}^\sharp) + (e^{i\xi x_1} \hat{w}_{\xi_{j+1}}^\sharp - e^{i\xi' x_1} \hat{w}_{\xi_{j'}}^\sharp). \quad (63)$$

On the other hand we observe that

$$\begin{aligned} \check{w}_\xi^N - e^{i\xi x_1} \hat{w}_{\xi_{j+1}}^\sharp &= e^{i\xi x_1} \left(\hat{w}_{\xi_j}^\sharp N(\xi_{j+1} - \xi) + \hat{w}_{\xi_{j+1}}^\sharp (N(\xi - \xi_j) - 1) \right) \\ &= e^{i\xi x_1} \left(\hat{w}_{\xi_j}^\sharp N(\xi_{j+1} - \xi) + \hat{w}_{\xi_{j+1}}^\sharp (N(\xi - \xi_j) - N(\xi_{j+1} - \xi_j)) \right) \\ &= e^{i\xi x_1} N(\xi - \xi_{j+1}) (\hat{w}_{\xi_{j+1}}^\sharp - \hat{w}_{\xi_j}^\sharp). \end{aligned} \quad (64)$$

In the same way we have then that

$$\check{w}_{\xi'}^N - e^{i\xi' x_1} \hat{w}_{\xi_{j'}}^\sharp = e^{i\xi' x_1} N(\xi' - \xi_{j'}) (\hat{w}_{\xi_{j'+1}}^\sharp - \hat{w}_{\xi_{j'}}^\sharp).$$

This implies that (63) can be given as

$$\begin{aligned} \check{w}_\xi^N - \check{w}_{\xi'}^N &= e^{i\xi x_1} N(\xi - \xi_{j+1}) (\hat{w}_{\xi_{j+1}}^\sharp - \hat{w}_{\xi_j}^\sharp) + e^{i\xi' x_1} N(\xi' - \xi_{j'}) (\hat{w}_{\xi_{j'+1}}^\sharp - \hat{w}_{\xi_{j'}}^\sharp) \\ &\quad + (e^{i\xi x_1} - e^{i\xi' x_1}) \hat{w}_{\xi_{j+1}}^\sharp - e^{i\xi' x_1} (\hat{w}_{\xi_{j+1}}^\sharp - \hat{w}_{\xi_{j'}}^\sharp). \end{aligned} \quad (65)$$

Since ξ and ξ' are in the same interval, then $|\xi - \xi_{j+1}| \leq |\xi - \xi'|$ and $|\xi' - \xi_{j'+1}| \leq |\xi' - \xi|$, which implies that

$$\begin{aligned} \|\check{w}_\xi^N - \check{w}_{\xi'}^N\| &= N|\xi - \xi'| \|\hat{w}_{\xi_{j+1}}^\sharp - \hat{w}_{\xi_j}^\sharp\|_{L^2(D^p)} + N|\xi' - \xi| \|\hat{w}_{\xi_{j'+1}}^\sharp - \hat{w}_{\xi_{j'}}^\sharp\|_{L^2(D^p)} \\ &\quad + \|e^{i\xi x_1} - e^{i\xi' x_1}\|_{L^2(D^p)} + \|\hat{w}_{\xi_{j+1}}^\sharp - \hat{w}_{\xi_{j'}}^\sharp\|_{L^2(D^p)}. \end{aligned} \quad (66)$$

Let us first assume that $|\xi - \xi'| \leq \frac{1}{N}$, then $\xi_{j'} = \xi_{j+1}$. In this case we have

$$\begin{aligned} \|\check{w}_\xi^N - \check{w}_{\xi'}^N\| &\leq C' (N|\xi - \xi'| \times |\xi_{j+1} - \xi_j|^\alpha + |\xi' - \xi| \times |\xi_{j'+1} - \xi_{j'}|^\alpha), \\ &\leq C' (|\xi - \xi'|N^{1-\alpha} + |\xi' - \xi|N^{1-\alpha}). \\ &\leq C'|\xi - \xi'|^\alpha. \end{aligned} \quad (67)$$

with C' is a positive constant.

For the case when $|\xi - \xi'| \geq \frac{1}{N}$, from (65) observing that $|N(\xi - \xi_j)| \leq 1$ we have

$$\begin{aligned} \|\check{w}_\xi^N - \check{w}_{\xi'}^N\| &\leq C'' (|\xi_{j+1} - \xi_j|^\alpha + |\xi_{j'+1} - \xi_{j'}|^\alpha + |\xi_{j+1} - \xi_{j'}|^\alpha), \\ &\leq C'' \left(\frac{2}{N^\alpha} + |\xi - \xi'|^\alpha \right) \leq 3C''|\xi - \xi'|^\alpha. \end{aligned} \quad (68)$$

$$\leq C'' \left(\frac{2}{N^\alpha} + |\xi - \xi'|^\alpha \right) \leq 3C''|\xi - \xi'|^\alpha. \quad (69)$$

This ends the proof. \square

Lemma 5.9. *Assume that Assumptions 3.3 and 5.1 hold. Let $f \in \mathcal{F}_M$ and w_N as defined previously. Set $f_N := \frac{1}{N} \sum_{j=0}^{N-1} \hat{f}_{\xi_j}$. There exists a constant $C > 0$ independent from f and N (that may depend on M) such that*

$$\|w_N\|_{L^2(D_N^p)} \leq C \|f_N\|_{L^2(D_N^p)}. \quad (70)$$

Proof. We prove this Lemma using a contradiction argument. Suppose that for all $C \in \mathbb{N}$, there exists $f \in \mathcal{F}_M$ which depends on C and there exists N which also depends on C such that

$$\|w_N\|_{L^2(D_N^p)} \geq C \|f_N\|_{L^2(D_N^p)}.$$

The sequence N can be chosen to be monotonically increasing with respect to C . By indexing C with respect to N , replacing f_N by $\frac{f_N}{\|w_N\|_{L^2(D_N^p)}}$ and w_N by $\frac{w_N}{\|w_N\|_{L^2(D_N^p)}}$, the contradiction statement can be equivalently formulated as: there exists a monotonically increasing sequence N such that

$$\|f_N\|_{L^2(D_N^p)} \longrightarrow 0 \quad \text{as } N \rightarrow \infty, \quad (71)$$

and $\|w_N\|_{L^2(D_N^p)} = 1$ where w_N is the solution of problem (43) associated with f_N . Taking $w' = w_N$ in (43) we obtain

$$\int_{D_N^p} \frac{1}{n-1} |\Delta w_N|^2 dx = - \int_{D_N^p} (k^2 \Delta w_N \overline{w_N} + k^2 n w_N \overline{\Delta w_N} + k^2 n |w_N|^2) dx + \int_{D_N^p} f_N \overline{w_N} dx.$$

Therefore

$$\begin{aligned} \gamma_0 \|\Delta w_N\|_{L^2(D_N^p)}^2 &\leq \left(k^2 (n^* + 1) \|w_N\|_{L^2(D_N^p)} \|\Delta w_N\|_{L^2(D_N^p)} \right. \\ &\quad \left. + k^2 n^* \|w_N\|_{L^2(D_N^p)}^2 + \|f_N\|_{L^2(D_N^p)} \|w_N\|_{L^2(D_N^p)} \right), \end{aligned}$$

with $\gamma_0 := \frac{1}{n^*-1}$ in the case $n^* > 1$ and $\gamma_0 := \frac{1}{n^*-1}$ in the case $n^* < 1$. This inequality yields in particular that Δw_N is uniformly bounded with respect to N in $L^2(D_N^p)$. This implies for instance that w_N is uniformly bounded in $H^2(\tilde{D})$ (using classical elliptic regularity results). Recall that

$$a_{\xi_j}(\hat{w}_{\xi_j}, \hat{w}'_{\xi_j}) = \int_{D_0^p} \hat{f}_{\xi_j} \overline{\hat{w}'_{\xi_j}} dx - \tilde{a}(w_N, \hat{w}'_{\xi_j}) \quad \text{for all } w'_j \in \tilde{H}_{\xi_j}^2(D_0^p). \quad (72)$$

Using Lemma 4.2 we deduce that

$$\|\hat{w}_{\xi_j}\|_{H^2(D_0^p)} \leq C \left(\|w_N\|_{H^2(\tilde{D})} + \|\hat{f}_{\xi_j}\|_{L^2(D_0^p)} \right),$$

for some constant C independent from N . Therefore

$$\frac{1}{N} \sum_{j=0}^{N-1} \|\hat{w}_{\xi_j}\|_{H^2(D_0^p)}^2 \leq 2C^2 \left(\|w_N\|_{H^2(\tilde{D})}^2 + \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{f}_{\xi_j}\|_{L^2(D_0^p)}^2 \right). \quad (73)$$

Using (53) we conclude that \tilde{w}_N is bounded in $\tilde{H}^2(D^p)$ and w_N is uniformly bounded in $\tilde{H}^2(D_N^p)$. Consequently, up to subsequence, we can assume that \tilde{w}_N weakly converges in $\tilde{H}^2(D^p)$ and its limit is the same as a consequence of Lemma 5.7. Let us denote this limit by $w \in \tilde{H}^2(D^p)$. Let $N_0 \in \mathbb{N}$ be a fixed parameter. By Lemma 5.7 we can also assume (up to changing the subsequence) that w_N weakly converges to w in $\tilde{H}^2(D_{N_0}^p)$.

Now, consider $\varphi \in C_c^\infty(D_{N_0}^p \cup \Gamma^0)$, which we extend by zero in D_N^p and by $2\pi N$ periodicity in the x_1 direction. This extension gives us a test function in $\tilde{H}_\#^2(D_N^p)$ which we can use in (43) and get

$$\int_{D_{N_0}^p} \frac{1}{n-1} (\Delta w_N + k^2 n w_N) \overline{(\Delta \varphi + k^2 \varphi)} dx = \int_{D_{N_0}^p} f_N \bar{\varphi} dx. \quad (74)$$

Going to the limit as N goes to infinity in (74) shows that $w \in \tilde{H}^2(D_{N_0}^p)$ satisfies

$$\int_{D^p} \frac{1}{n-1} (\Delta w + k^2 n w) \overline{(\Delta \varphi + k^2 \varphi)} dx = 0, \quad \forall \varphi \in C_c^\infty(D_{N_0}^p \cup \Gamma^0). \quad (75)$$

Since N_0 can be chosen arbitrarily large, the last equality holds for all $\varphi \in C_c^\infty(D^p \cup \Gamma^0)$ and by denseness argument for all $\varphi \in \tilde{H}^2(D^p)$. Then (75) shows that w is a solution of the homogeneous interior transmission problem (27) and by the uniqueness result of Lemma 5.2, $w = 0$.

To obtain a contradiction, we will prove that the sequence \tilde{w}_N strongly converges to w in $L^2(D^p)$ which is equivalent to proving that $\xi \rightarrow \tilde{w}_\xi^N$ is strongly convergent in $L^2(I, L^2(D_0^p))$. From Lemma 5.8 we have that $\xi \rightarrow \tilde{w}_\xi^N$ is uniformly equicontinuous on I with value in $L^2(D_0^p)$ for all $N \geq N_0$. Then, using the Ascoli Theorem, one can extract a subsequence, which we still denote by \tilde{w}_ξ^N which strongly converges to $w_\xi = 0$ in $L^2(I, L^2(D_0^p))$. This contradicts $\|\tilde{w}_N\|_{L^2(D^p)}^2 \geq \frac{1}{3}$ by (54). \square

Theorem 5.10. *Assume that the assumptions 3.3 and 5.1 hold and let $f \in \mathcal{F}_M$. Then problem (35) has a unique solution $w \in \tilde{H}_0^2(D^p)$. Furthermore, there exists a constant $C(M)$ independent of f such that*

$$\|w\|_{H^2(D^p)} \leq C(M) \|f\|_{L^2(D^p)}.$$

Proof. Let w_N and \tilde{w}_N be the sequences introduced above. From Lemma 5.9 we deduce that w_N is bounded in $L^2(D_N^p)$. From (52) we deduce that \tilde{w}_N is bounded in $L^2(D^p)$. The same reasoning as in the proof of Lemma 5.9 allows us to conclude that (up to a subsequence) w_N and \tilde{w}_N locally converge in $L^2(D^p)$ to a same limit w in $\tilde{H}_0^2(D^p)$ which satisfies (35). Furthermore, using the same arguments that lead to (73), Lemmas 5.9, and (53) we infer the existence of a constant $C(M)$ such that

$$\|\tilde{w}_N\|_{H^2(D^p)} \leq C(M) \|f_N\|_{L^2(D_N^p)}. \quad (76)$$

This shows in particular that we can choose the subsequence \tilde{w}_N which is weakly convergent to w in $H^2(D^p)$ and going to the limit in the inequality leads to (76) since $\|w\|_{H^2(D^p)} \leq \liminf \|\tilde{w}_N\|_{L^2(D^p)}$ and

$$\|f_N\|_{L^2(D_N^p)}^2 = \frac{1}{N} \sum_{j=0}^{N-1} \|\hat{f}_{\xi_j}\|_{L^2(D_0^p)}^2 \longrightarrow \int_I \|\hat{f}_\xi\|_{L^2(D_0^p)}^2 d\xi = \|f\|_{L^2(D^p)},$$

as N goes to infinity. This ends the proof. \square

As explained earlier, the study of problem (33) follows the same lines as problem (34). The only minor difference is in the treatment of the right-hand side of the equation (57) which contains an additional term depending on ξ_j but this has no effect on the subsequent conclusions of Lemma 5.9. This leads to the following Theorem on the well-posedness of the interior transmission problem (3).

Theorem 5.11. *Assume that the assumptions 3.3 and 5.1 hold and that $M > 0$ is a given constant. Suppose also that the boundary data φ and ψ are respectively the trace and the normal trace on $\partial D^p \setminus \Gamma^0$ of some function $\theta \in \tilde{H}_0^2(D^p)$ such that $f := \frac{1}{n-1}(\Delta\theta + k^2 n\theta) \in \mathcal{F}_M$. Then the problem (25) has a unique solution $w_0 \in H^2(D^p)$ satisfying*

$$\|w_0\|_{H^2(D^p)} \leq C(M) (\|\varphi\|_{H^{3/2}(\partial D^p \setminus \Gamma^0)} + \|\psi\|_{H^{1/2}(\partial D^p \setminus \Gamma^0)}),$$

with a constant $C(M)$ independent of φ and ψ .

In the proof of the previous theorems, the Holder regularity assumptions on the right hand side f with respect to the Floquet-Bloch variable can be replaced by H^1 regularity. In this case, the arguments are simpler, since one can directly apply the Rellich compact embedding theorem. More precisely, for given $M > 0$ we define

$$\tilde{\mathcal{F}}_M := \{f \in L^2(D^p); \mathcal{J}f \in H^1(I, L^2(D_0^p)) \text{ et } \|\mathcal{J}f\|_{H^1(I, L^2(D_0^p))} \leq M\}.$$

Then, using similar arguments as in the proof of Lemma 5.8, one can prove that the sequence \tilde{w}_N is bounded in $H^1(I \times D_0^p)$. We also refer to [12, Chapter 5, Lemma 54] for a written proof of this case. This allows us to simplify the compactness arguments used in Lemma 5.8 by replacing Ascoli's theorem with Rellich's compact embedding theorem. Thus, the results of theorems 5.10 and 5.11 still hold if we replace \mathcal{F}_M with $\tilde{\mathcal{F}}_M$. In the following, we give the result similar to Theorem 5.11.

Theorem 5.12. *Assume that the assumptions 3.3 and 5.1 hold and that $M > 0$ is a given constant. Suppose further that the boundary data φ and ψ are the trace and the normal trace, respectively, on $\partial D^p \setminus \Gamma^0$ of some function $\theta \in \tilde{H}_0^2(D^p)$ such that $f := \frac{1}{n-1}(\Delta\theta + k^2 n\theta) \in \tilde{\mathcal{F}}_M$. Then the problem (25) has a unique solution $w_0 \in H^2(D^p)$ which satisfies*

$$\|w_0\|_{H^2(D^p)} \leq C(M) (\|\varphi\|_{H^{3/2}(\partial D^p \setminus \Gamma^0)} + \|\psi\|_{H^{1/2}(\partial D^p \setminus \Gamma^0)}),$$

with a constant $C(M)$ independent of φ and ψ .

The following remark indicates how theorems 5.11 and 5.12 can be applied in the justification of the linear sampling method [2].

Remark 5.13. For the theoretical justification of the linear sampling method, we use the existence of solutions to (25) with the right hand sides φ and ψ of the form

$$\varphi = \Phi(\cdot, z) \quad \text{and} \quad \psi = \frac{\partial \Phi(\cdot, z)}{\partial \nu} \quad \text{on } \partial D. \quad (77)$$

for $z \in D$ where $\Phi(\cdot, z)$ is the fundamental solution of the half-space Dirichlet problem given by

$$\Phi(\cdot, z) := \frac{i}{4} \left[H_0^{(1)}(k|\cdot - z|) - H_0^{(1)}(k|\cdot - z'|) \right],$$

where $H_0^{(1)}$ is the Hankel function and $z := (z_1, z_2)$, $z' := (z_1, -z_2)$ and $z_2 > 0$. One can use as a lifting function $\theta = \Phi\chi$, where χ is a cutoff function equal to zero in one neighborhood of z and one outside another neighborhood, both contained in D^p . The Floquet-Bloch transform of Φ is given by $\Phi_z(x, \xi) := (\mathcal{J}\Phi(\cdot, z))(\xi, x)$

$$\hat{\Phi}_z(x, \xi) := \frac{i}{4\pi} \sum_{j \in \mathbb{Z}} e^{i(\xi+j)(x_1-z_1)} \theta_j(x_2, z_2, \xi), \quad z_2 < x_2, \quad (78)$$

with

$$\theta_j(x_2, z_2, \xi) := e^{i\sqrt{k^2 - |\xi + j|^2}x_2} \left[\frac{e^{-i\sqrt{k^2 - |\xi + j|^2}z_2} - e^{i\sqrt{k^2 - |\xi + j|^2}z_2}}{\sqrt{k^2 - |\xi + j|^2}} \right]. \quad (79)$$

$\hat{\Phi}_z$ can be decomposed as

$$\hat{\Phi}_{z,1}(x, \xi) = \sum_{|j| \leq k+1} e^{i(\xi+j)(x_1-z_1)} \theta_\xi(j, x_2, z_2)$$

and

$$\hat{\Phi}_{z,2}(x, \xi) = \sum_{|j| > k+1} e^{i(\xi+j)(x_1-z_1)} \theta_\xi(j, x_2, z_2).$$

Then one can prove that $\hat{\Phi}_{z,1} \in C^{0,\alpha}(I, L^2(\Omega_{z_2}^R))$ for $\alpha \leq \frac{1}{2}$, while $\hat{\Phi}_{z,2} \in H^1(I, L^2(\Omega_{z_2}^R))$ and then obtain by symmetry with respect to x and z that $\hat{\Phi}_{z,1} \in C^{0,\alpha}(I, L^2(\Omega_0^R))$ for $\alpha \leq \frac{1}{2}$ and $\hat{\Phi}_{z,2} \in H^1(I, L^2(\Omega_0^R))$.

Since $\Delta\theta = -k^2\chi\Phi + \chi\Delta\Phi + 2\nabla\chi \cdot \nabla\Phi + \Phi\Delta\chi$ we deduce with the same arguments as in the theorem 2.3 that $\theta \in \tilde{H}_0^2(D^p)$ and from $\Delta\theta + k^2n\theta = -k^2\chi\Phi + 2\nabla\chi \cdot \nabla\Phi + \Phi\Delta\chi + k^2n\chi\Phi$ we deduce that $f = \frac{1}{n-1}(\Delta\theta + k^2n\theta)$ can be decomposed as $f = f_1 + f_2$ with $f_1 \in \mathcal{F}_M$ and $f_2 \in \tilde{\mathcal{F}}_M$ for some given $M > 0$.

Then one can apply Theorems 5.11 and 5.12 and deduce the existence of solutions to the interior transmission problem (25) with φ and ψ given by (77).

Appendix A.

Proof of Theorem 2.3. We consider

$$\tilde{w}(x_1, x_2) := \begin{cases} w(x_1, x_2) & \text{for } x_2 \geq 0, \\ -w(x_1, -x_2) & \text{in } x_2 \leq 0. \end{cases}$$

Then \tilde{w} is continuous on Γ^0 and satisfies the first equation in (1) in $\widehat{\Omega}^R := \mathbb{R} \times [-R, R]$. Let us define the cut-off function in the x_1 -direction $\chi \in C^\infty(U^0)$ such that $\chi(x_1) = 1$ for $x_1 \leq 0$ and $\chi(x_1) = 0$ for $x_1 \geq 1$, and we consider $\tilde{\chi} \in C^\infty(U^0)$ the cut-off truncation function on the x_2 -direction such that $\tilde{\chi}(x_2) = 1$ as $|x_2| \leq R/2$ and $\tilde{\chi}(x_2) = 0$ as $|x_2| \geq R$. Moreover, for $N \in \mathbb{N}$ we define

$$\phi_N(x_1, x_2) := \chi_N(x_1)\tilde{\chi}(x_2),$$

with

$$\chi_N(x_1) := \begin{cases} \chi(x_1 - N) & \text{for } x_1 \geq 0, \\ \chi(-x_1 - N) & \text{for } x_1 \leq 0. \end{cases}$$

Let us denote by

$$\Omega_N^{R/2} := [-R/2, R/2] \times [-N, N], \quad \text{and} \quad \tilde{\Omega}_N^R := [-R, R] \times [-N-1, N+1].$$

Since $\phi_N \tilde{w} = \tilde{w}$ in $\Omega_N^{R/2}$, then for $1 \leq i, j \leq 2$ we have

$$\left\| \frac{\partial^2 \tilde{w}}{\partial x_i \partial x_j} \right\|_{L^2(\Omega_N^{R/2})} = \left\| \frac{\partial^2 (\phi_N \tilde{w})}{\partial x_i \partial x_j} \right\|_{L^2(\Omega_N^{R/2})} \leq \left\| \frac{\partial^2 (\phi_N \tilde{w})}{\partial x_i \partial x_j} \right\|_{L^2(\tilde{\Omega}_N^R)}. \quad (80)$$

Observing that $\phi_N \tilde{w} \in H_0^2(\tilde{\Omega}_N^R)$, we conclude from (80) that

$$\left\| \frac{\partial^2 \tilde{w}}{\partial x_i \partial x_j} \right\|_{L^2(\Omega_N^{R/2})} \leq \|\Delta(\phi_N \tilde{w})\|_{L^2(\tilde{\Omega}_N^R)}.$$

Furthermore, we have $\Delta(\phi_N \tilde{w}) = \phi_N \Delta \tilde{w} + 2\nabla \phi_N \cdot \nabla \tilde{w} + u \Delta \phi_N$, therefore

$$\begin{aligned} \left\| \frac{\partial^2 \tilde{w}}{\partial x_i \partial x_j} \right\|_{L^2(\Omega_N^{R/2})} &\leq \|\phi_N\|_{L^\infty(\tilde{\Omega}_N^R)} \|\Delta \tilde{w}\|_{L^2(\tilde{\Omega}_N^R)} + 2 \|\nabla \phi_N\|_{L^\infty(\tilde{\Omega}_N^R)} \|\nabla \tilde{w}\|_{L^2(\tilde{\Omega}_N^R)} \\ &\quad + \|\Delta \phi_N\|_{L^\infty(\tilde{\Omega}_N^R)} \|\tilde{w}\|_{L^2(\tilde{\Omega}_N^R)}. \end{aligned} \quad (81)$$

On the other hand, since $\tilde{\chi}$ is independent from N and

$$\|\chi_N\|_{L^\infty(\tilde{\Omega}_N^R)} \leq \|\chi\|_{L^\infty(\tilde{\Omega}_N^R)}, \quad \|\nabla \chi_N\|_{L^\infty(\tilde{\Omega}_N^R)} \leq \|\nabla \chi\|_{L^\infty(\tilde{\Omega}_N^R)}, \quad \text{and} \quad \|\Delta \chi_N\|_{L^\infty(\tilde{\Omega}_N^R)} \leq \|\Delta \chi\|_{L^\infty(\tilde{\Omega}_N^R)},$$

we deduce that $\|\phi_N\|_{L^\infty(\tilde{\Omega}_N^R)}$, $\|\nabla \phi_N\|_{L^\infty(\tilde{\Omega}_N^R)}$ and $\|\Delta \phi_N\|_{L^\infty(\tilde{\Omega}_N^R)}$ are bounded with a constant independent from N . Therefore from (81) we obtain

$$\left\| \frac{\partial^2 \tilde{w}}{\partial x_i \partial x_j} \right\|_{L^2(\Omega_N^{R/2})} \leq C \left(\|\Delta \tilde{w}\|_{L^2(\tilde{\Omega}_N^R)} + \|\nabla \tilde{w}\|_{L^2(\tilde{\Omega}_N^R)} + \|\tilde{w}\|_{L^2(\tilde{\Omega}_N^R)} \right), \quad (82)$$

with C is a positive constant independent from N , and since we have that the solution $\tilde{w} \in H^1(\hat{\Omega}^R)$ and $\Delta \tilde{w} \in L^2(\hat{\Omega}^R)$, then we deduce from (82) that $\left\| \frac{\partial^2 \tilde{w}}{\partial x_i \partial x_j} \right\|_{L^2(\Omega_N^{R/2})}$ is bounded independently from N . Therefore

$$\lim_{N \rightarrow \infty} \left\| \frac{\partial^2 \tilde{w}}{\partial x_i \partial x_j} \right\|_{L^2(\Omega_N^{R/2})} = \left\| \frac{\partial^2 \tilde{w}}{\partial x_i \partial x_j} \right\|_{L^2(\hat{\Omega}^{R/2})}$$

by Lebesgue's dominated convergence theorem. This implies that $\left\| \frac{\partial^2 \tilde{w}}{\partial x_i \partial x_j} \right\|_{L^2(\hat{\Omega}^{R/2})}$ is also bounded with a constant independent from N for all $1 \leq i, j \leq 2$ which gives that $\tilde{w} \in H^2(\Omega^{R/2})$.

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