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Robust Stability and Stabilization of Continuous-Time Parameter-Varying Persidskii Systems

Junfeng Zhang, Christophe Combastel, Denis Efimov, Ali Zolghadri

Abstract—This paper investigates analysis and design problems for a class of parameter-varying generalized Persidskii systems. A model within this class is simultaneously linear in time-varying parameters as in LPV framework, and contains sector nonlinearities of the state, as in Persidskii systems. The conditions for (integral) input-to-state stability and input-output-to-state stability are established. These results are developed using both, parameter-dependent and independent, Lyapunov functions. To formulate these conditions, parameterized matrix inequalities are used, which are reduced to linear ones under additional assumptions concerning the model's dependence on the time-varying parameters. Finally, a new robust nonlinear control design method is also developed. The efficacy of these results is illustrated through a numerical academic example and a more complex model describing the nonlinear dynamics of aircraft longitudinal motion.

Index Terms—Persidskii systems, LPV, Input-to-state stability, LMIs, Robust control

I. INTRODUCTION

A. Antecedents and related works

Stability and robustness are key considerations in automatic control theory [1], [2]. The input-to-state stability (ISS) framework [3], [4] offers a powerful tool for analyzing stability in the presence of disturbances, noise, and parasitic delays, and for designing robust controllers and observers. This group of methods can also be applied to address internal model uncertainty, if it can be represented by bounded disturbances entering the system dynamics.

Linear control theory offers a well-developed palette of control and estimation tools [5]. Extending these approaches to nonlinear systems, however, is often challenging due to the inherent complexities associated with stability analysis. While canonical representations like LPV systems [6], [7], homogeneous dynamics [8], or Lur'e models [9], [10] provide extensions for analysis, constructing Lyapunov functions for stability assessment in general nonlinear systems remains a complex task.

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The LPV setting provides a wide-spread and appealing approach for representing nonlinear systems in an equivalent linear form with time-varying parameters of an augmented dimension, enabling the application of a large spectrum of methods and tools of linear system theory [6], [7]. The stability of LPV systems has been intensively investigated and many important and useful results have been established. For example, parameter dependent and independent (piecewise) quadratic Lyapunov functions are frequently used [11], [12], [13], including more sophisticated quadratic or non-quadratic polynomial Lyapunov functions [14], [15]. While effective, LPV modeling requires transformation of all nonlinearities into a linear parameter form, which can introduce significant conservatism when aiming for wide-range validity. In the case of large operating domain dynamics, the global stability of a closed-loop system often cannot be guaranteed [16].

Persidskii systems are a representation of Lur'e models, and their stability analysis was initiated in [17], where a Lyapunov function was proposed as a linear combination of the integrals of the nonlinearities. This approach was further developed in [18] by introducing a Lyapunov function based on a combination of weighted absolute values of the states. This class of systems has proven to be valuable in diagonal stability analysis [19], [20], sliding mode control [21], [22], [23], robust analysis of Lur'e systems [9], opinion dynamics [24], neural networks [25], [26], [27], and numerous other domains. Nominal Lyapunov functions, expressed as a sum of quadratic state terms and integrals of nonlinearities, have been instrumental in addressing analysis and design problems for generalized Persidskii systems. In this framework, stability conditions are formulated using linear matrix inequalities (LMIs), enabling efficient computational analysis [28], [29], [30], [31], [32], [33].

B. Contributions

In this paper, we investigate Parameter-Varying Persidskii (PVP) models, an extension of the LPV framework to Persidskii systems, recently introduced in a conference communication [34]. In this class of models, the scheduling parameters can represent the additional time- or state-varying terms, parametric or signal uncertainty, etc., while the useful (passive or negative feedback) nonlinearities are kept in the model. Such a kind of dynamics may be related with the absolute stability analysis of Lur'e systems, where a set of nonlinearities from a sector is considered [35], [10], but

in the case of PVP systems, the nonlinearity is fixed and the (uncertain) scheduling parameters appear in the matrices multiplying the state and the nonlinearities.

This paper makes two key contributions:

- It provides formal proofs for the conditions of (integral) ISS and input-output-to-state stability (IOSS) for PVP systems using both parameter-dependent and independent Lyapunov functions, an aspect that was not addressed in the conference paper [34].
- It presents a novel stabilizing control design method for PVP models based on static nonlinear state feedback.
- The efficacy of these results is illustrated through a numerical academic example and a more complex model describing the nonlinear dynamics of aircraft longitudinal motion.

C. Organization

The paper is organized as follows: Section II provides the preliminaries on input-to-state stability theory and Persidskii systems. In Section III, the problem statement is introduced. Parameter-independent Lyapunov functions are investigated in Section IV, followed by the analysis of parameter-dependent Lyapunov functions in Section V. The use of these results for design of input-to-state stabilizing control law for PVP systems is considered in Section VI. Section VII demonstrates the efficiency of the obtained conditions on an academic example and on a nonlinear model of longitudinal flight for an aircraft.

Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where \mathbb{R} is the set of real numbers.
- $|\cdot|$ denotes the absolute value in \mathbb{R} , $\|\cdot\|$ is used for the Euclidean norm on \mathbb{R}^n .
- For a (Lebesgue) measurable function $d : \mathbb{R}_+ \rightarrow \mathbb{R}^m$ and $[t_0, t_1] \subset \mathbb{R}_+$, we define the norm $\|d\|_{[t_0, t_1]} = \text{ess sup}_{t \in [t_0, t_1]} \|d(t)\|$. Then, $\|d\|_\infty = \|d\|_{[0, +\infty)}$ and the set of such functions d satisfying the property $\|d\|_\infty < +\infty$ is further denoted as \mathcal{L}_∞^m (the set of essentially bounded measurable functions).
- The notation $DV(x)f(x)$ stands for the directional derivative of a continuously differentiable function $V(x)$ with respect to the vector field f evaluated at the point x .
- A finite series of integers $1, 2, \dots, n$ is denoted by $\overline{1, n}$, and $\{\overline{1, n}\} = \{1, 2, \dots, n\}$.
- Denote the identity matrix of dimension $n \times n$ by I_n , the vector of dimension n or the matrix of dimension $n \times m$ with all elements equal 1 by 1_n and $1_{n \times m}$, respectively.
- $\text{diag}\{g\} \in \mathbb{D}_+^n$ represents a diagonal matrix of dimension $n \times n$ with a vector $g \in \mathbb{R}_+^n$ on the main diagonal, where $\mathbb{D}_+^n \subset \mathbb{R}^{n \times n}$ is the set of nonnegative diagonal matrices.
- For a matrix $A \in \mathbb{R}^{n \times n}$, we denote its i^{th} row and column by $A^{(i)}$ and $A^{[i]}$, respectively, for $i = \overline{1, n}$. The relation $P \prec 0$ ($P \preceq 0$) means that a symmetric matrix $P \in \mathbb{R}^{n \times n}$ is negative (semi-)definite.

II. PRELIMINARIES

In this paper, it is conventionally assumed that if the upper limit of a summation or a sequence is smaller than the lower one, then the corresponding terms (conditions) have to be omitted.

A continuous function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to the class \mathcal{K} if $\alpha(0) = 0$ and it is strictly increasing. A function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to the class \mathcal{K}_∞ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to the class \mathcal{KL} if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_+$ and $\beta(s, \cdot)$ is decreasing to zero for each fixed $s > 0$.

Lemma (Finsler's lemma). [36] *Let $x \in \mathbb{R}^n \setminus \{0\}$ and $P, R \in \mathbb{R}^{n \times n}$ are symmetric, then $x^\top P x < 0$ whenever $x^\top R x = 0$ if and only if there exists $\rho \in \mathbb{R}$ such that $P - \rho R < 0$.*

A. Input-to-state stability

Consider a nonlinear system:

$$\dot{x}(t) = f(x(t), d(t)), \quad t \geq 0, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state, $d(t) \in \mathbb{R}^m$ is the external input, $d \in \mathcal{L}_\infty^m$, and $f : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^n$ is a locally Lipschitz continuous function, $f(0, 0) = 0$. The system (1) can be equipped with an output $y(t) \in \mathbb{R}^p$:

$$y(t) = h(x(t)), \quad (2)$$

where $h : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is a continuous function. For an initial condition $x_0 \in \mathbb{R}^n$ and input $d \in \mathcal{L}_\infty^m$, define the corresponding solutions by $x(t, x_0, d)$ for any $t \geq 0$ for which the solution exists, $y(t, x_0, d) = h(x(t, x_0, d))$.

In this work we will be interested in the following stability properties [3], [4]:

Definition 1. The system (1) is called input-to-state stable (ISS), if there are functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that

$$\|x(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \gamma(\|d\|_{[0, t]}) \quad \forall t \geq 0$$

for any $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$. The function γ is called the nonlinear asymptotic gain.

Definition 2. The system (1) is called integral ISS (iISS), if there are functions $\alpha \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that

$$\alpha(\|x(t, x_0, d)\|) \leq \beta(\|x_0\|, t) + \int_0^t \gamma(\|d(s)\|) ds \quad \forall t \geq 0$$

for any $x_0 \in \mathbb{R}^n$ and $d \in \mathcal{L}_\infty^m$.

The previous property introduced in [37] is close to the integral stability notion given in [38].

Definition 3. The system (1), (2) is called input-output-to-state stable (IOSS) if there are functions $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$ such that

$$\|x(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \gamma_1(\|d\|_{[0, t]}) + \gamma_2(\|y\|_{[0, t]})$$

for all $x_0 \in \mathbb{R}^n$, $d \in \mathcal{L}_\infty^m$ and all $t \in [0, t_{\max}(x_0, d)]$.

In this definition $t_{\max}(x_0, d) \leq +\infty$ determines the interval of existence of the solution $x(t, x_0, d)$ for the system (1), (2), and it corresponds to a nonlinear detectability notion.

These properties have the following characterizations in terms of existence of Lyapunov functions:

Definition 4. A smooth $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is called ISS-Lyapunov function for the system (1) if there are $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\eta \in \mathcal{K}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$

$$DV(x)f(x, d) \leq \eta(\|d\|) - \alpha_3(\|x\|)$$

for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$. Such a V is called iISS-Lyapunov function if $\alpha_3 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is just a positive definite function.

Note that an ISS-Lyapunov function can also satisfy the following equivalent condition for some $\chi \in \mathcal{K}$:

$$\|x\| > \chi(\|d\|) \Rightarrow DV(x)f(x, d) \leq -\alpha_3(\|x\|).$$

Definition 5. A smooth $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is called IOSS-Lyapunov function for the system (1), (2) if for some $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\sigma_1, \sigma_2 \in \mathcal{K}$ the inequalities

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$

$$DV(x)f(x, d) \leq -\alpha_3(\|x\|) + \sigma_1(\|d\|) + \sigma_2(\|y\|)$$

hold for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$.

The relations between these Lyapunov characterizations and the robust stability properties are given below:

Theorem 1. *The system (1) is ISS (iISS) if and only if it admits an ISS (iISS)-Lyapunov function.*

Theorem 2. *The system (1), (2) is IOSS if and only if it admits an IOSS-Lyapunov function.*

A consequence of Theorem 1 and Definition 4 is that an ISS system (1) is also iISS.

B. Parameter-varying Persidskii systems

Consider the following class of systems [34]:

$$\dot{x}(t) = A_0(\theta(t))x(t) + \sum_{j=1}^M A_j(\theta(t))f^j(H_j x(t)) + d(t) \quad (3)$$

$$+ B(\theta(t))u(t),$$

$$y(t) = \begin{bmatrix} C_0 x(t) \\ C_1 f^1(H_1 x(t)) \\ \vdots \\ C_M f^M(H_M x(t)) \end{bmatrix}, \quad t \geq 0,$$

where $x(t) = [x_1(t) \dots x_n(t)]^\top \in \mathbb{R}^n$ is the state vector, $x(0) \in \mathbb{R}^n$; $u(t) \in \mathbb{R}^m$ is the control input, $u \in \mathcal{L}_\infty^m$; $y(t) \in \mathbb{R}^p$ is the output signal, $p = \sum_{g=0}^M p_g$ and $C_g \in \mathbb{R}^{p_g \times k_g}$ for $g = \overline{0, M}$; $d(t) \in \mathbb{R}^n$ is the external disturbance, $d \in \mathcal{L}_\infty^n$; $\theta(t) \in \mathbb{R}^q$ is the vector of time-varying parameters, $\theta \in \mathcal{L}_\infty^q$; $f^j : \mathbb{R}^{k_j} \rightarrow \mathbb{R}^{k_j}$ with diagonal structure $f^j(s) = [f_1^j(s_1) \dots f_{k_j}^j(s_{k_j})]^\top$, $j = \overline{1, M}$ are continuous functions ensuring existence of solutions of the system (3) at

least locally in the forward time; continuous matrix functions $A_g : \mathbb{R}^q \rightarrow \mathbb{R}^{n \times k_g}$ for $g = \overline{0, M}$, $B : \mathbb{R}^q \rightarrow \mathbb{R}^{n \times m}$ and matrices $H_j \in \mathbb{R}^{k_j \times n}$ for $j = \overline{1, M}$ are given. Further, for brevity and consistently with (3) we use the convention $k_0 = n$ and $H_0 = I_n$ with $f^0(x) = x$.

The model (3) belongs to the class of Persidskii system [18], [19] under the following sector condition imposed on the nonlinearities: for any $j = \overline{1, M}$ and $i = \overline{1, k_j}$,

$$s f_i^j(s) > 0 \quad \forall s \in \mathbb{R} \setminus \{0\}.$$

Consequently, all nonlinearities may take zero values at zero only. If $\theta(t) = \text{const}$, $H_1 = I_n$ and $A_r = 0$ for all $r = \overline{2, M}$, then we recover the system studied by Persidskii in the conventional framework [18]. In the case of $M = 1$ and $\theta(t) = \text{const}$, (3) belongs also to the class of Lur'e systems widely investigated in the absolute stability theory [10].

After a proper re-indexing and decomposition of f^j , there exists $\nu \in \{\overline{0, M}\}$ such that for all $z = \overline{1, \nu}$ and $i = \overline{1, k_z}$:

$$\lim_{s \rightarrow \pm\infty} f_i^z(s) = \pm\infty;$$

and there exists $\mu \in \{\overline{\nu, M}\}$ such that for all $j = \overline{1, \mu}$ and $i = \overline{1, k_j}$:

$$\lim_{s \rightarrow \pm\infty} \int_0^s f_i^j(\sigma) d\sigma = +\infty.$$

Thus, some of the nonlinearities are radially unbounded, and $\nu = 0$ corresponds to the case when all nonlinearities are bounded (at least for negative or positive argument).

III. PROBLEM STATEMENT

Consider a PVP system (3). The following assumptions are first introduced to formulate the problem addressed in this paper.

Assumption 1. *Let $\theta(t) \in \Theta$, where $\Theta \subset \mathbb{R}^q$ is a known compact set.*

Assumption 2. *Let $\dot{\theta} \in \mathcal{L}_\infty^q$ and $\|\dot{\theta}\|_\infty \leq \dot{\theta}_{\max}$ for a known constant $\dot{\theta}_{\max} > 0$.*

Assumptions 1 and 2 introduce the standard hypotheses for stability analysis of LPV systems: the former just restricts the set of admissible values for the vector of scheduling parameters, while the latter allows us to introduce in Lyapunov functions the dependence on $\theta(t)$ [6]. In general, the dependence of Lyapunov functions on $\theta(t)$ should make the stability conditions less restrictive. However, this comes at the cost of requiring a more intricate numerical procedure for verification introducing also dependence on the rate of variation of scheduling parameters $\dot{\theta}_{\max}$. In order to make the conditions more constructive, we will consider the case with linear dependence of the matrix functions in the vector of scheduling parameters:

Assumption 3. *Let $A_g(\theta) = \sum_{k=1}^q \theta_k A_{gk}$ for $g = \overline{0, M}$ and $B(\theta) = \sum_{k=1}^q \theta_k B_k$, where $A_{gk} \in \mathbb{R}^{n \times k_g}$, $B_k \in \mathbb{R}^{n \times m}$ are the known matrices and $\theta_k \in [0, 1]$ for $k = \overline{1, q}$ with $\sum_{k=1}^q \theta_k = 1$.*

The objective of this work is to propose conditions of ISS, iISS and IOSS for the system (3), using a parameter-independent Lyapunov function under Assumption 1, or parameter-dependent one introducing assumptions 1–3 (i.e., in both cases the information about the set Θ and the velocity bound $\dot{\theta}_{\max}$ should be used, but not the properties of a particular trajectory $\theta(t)$). The conventional LPV framework transforms all nonlinearities, which may be difficult and conservative, while the Persidskii or Lur'e system frameworks do not consider uncertainty presented in the matrices describing the dynamics. The PVP models combine the advantages of both these classes of systems. Another aim of this paper is to use the obtained conditions of robust stability for a stabilizing control design.

IV. PARAMETER-INDEPENDENT LYAPUNOV FUNCTIONS

Recalling [28], [29], [31], consider the following structure of a candidate Lyapunov function for (3):

$$V(x) = x^\top P x + 2 \sum_{j=1}^M \sum_{i=1}^{k_j} \Lambda_i^j \int_0^{H_j^{(i)} x} f_i^j(s) ds, \quad (4)$$

where $P \in \mathbb{R}^{n \times n}$ is a symmetric matrix and $\Lambda^j = \text{diag}[\Lambda_1^j \dots \Lambda_{k_j}^j] \in \mathbb{D}_+^{k_j}$ are parameters to be tuned in order that (4) verifies the properties stated in Definition 4.

Denote by $\mathcal{M}(H_z, H_s) \subset \mathbb{R}^{k_z \times k_s}$ a set of matrices, whose ij -th element is a) nonnegative if $H_z^{(i)}$ and $H_s^{(j)}$ are co-directional, b) nonpositive if $H_z^{(i)}$ and $H_s^{(j)}$ are anti-directional, and c) zero otherwise, for $i = \overline{1, k_z}$, $j = \overline{1, k_s}$, $z = \overline{1, M-1}$ and $s = \overline{z+1, M}$.

Theorem 3. *Let Assumption 1 be satisfied and there exist matrices $P^\top = P \in \mathbb{R}^{n \times n}$, $\Lambda^j \in \mathbb{D}_+^{k_j}$ for $j = \overline{1, M}$, $\Phi^\top = \Phi \in \mathbb{R}^{n \times n}$, $\Psi \in \mathbb{R}^{n \times n}$ and matrix functions $\Omega_g : \mathbb{R}^q \rightarrow \mathbb{R}^{n \times k_g}$, $\Upsilon_{g,g} : \mathbb{R}^q \rightarrow \mathbb{D}_+^{k_g}$ ($\tilde{\Upsilon}_{g,g}(\theta) = H_g^\top \Upsilon_{g,g}(\theta) H_g$) for $g = \overline{0, M}$, $\Upsilon_{0,j} : \mathbb{R}^q \rightarrow \mathbb{D}_+^{k_j}$ ($\tilde{\Upsilon}_{0,j}(\theta) = H_j^\top \Upsilon_{0,j}(\theta)$) for $j = \overline{1, M}$, $\Upsilon_{z,s} : \mathbb{R}^q \rightarrow \mathcal{M}(H_z, H_s)$ ($\tilde{\Upsilon}_{z,s}(\theta) = \Upsilon_{z,s}(\theta)$) for $z = \overline{1, M-1}$ and $s = \overline{z+1, M}$ such that the following matrix inequalities are verified for all $\theta \in \Theta$:*

$$\begin{aligned} P &\succeq 0, \quad \Phi \succ 0, \quad Q(\theta) \preceq 0, \\ P + \rho_1 \sum_{j=1}^{\mu} H_j^\top \Lambda^j H_j &\succ 0, \\ \sum_{r=0}^{\nu} \tilde{\Upsilon}_{r,r}(\theta) + \rho_2 \sum_{z=1}^{\nu-1} \sum_{s=z+1}^{\nu} H_z^\top \Upsilon_{z,s}(\theta) H_s \\ + \rho_3 \sum_{j=1}^{\nu} H_j^\top \Upsilon_{0,j}(\theta) H_j &\succ 0 \end{aligned}$$

for some $\rho_1, \rho_2, \rho_3 \in \mathbb{R}$ and $\varkappa > 0$, where $Q(\theta)$ is given in (5)¹, then the system (3) with $u \equiv 0$ is IOSS. If these matrix

inequalities are complied for $\varkappa = 0$, then it is ISS, and if additionally the last inequality is replaced with

$$\begin{aligned} \sum_{r=0}^M \tilde{\Upsilon}_{r,r}(\theta) + \rho_2 \sum_{z=1}^{M-1} \sum_{s=z+1}^M H_z^\top \Upsilon_{z,s}(\theta) H_s \\ + \rho_3 \sum_{j=1}^M H_j^\top \Upsilon_{0,j}(\theta) H_j &\succ 0 \end{aligned}$$

satisfying for all $\theta \in \Theta$, then the system is iISS.

Proof. The relations $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$ are valid for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ for all $x \in \mathbb{R}^n$ due to imposed restrictions; according to Finsler's lemma, the condition $P + \rho_1 \sum_{j=1}^{\mu} H_j^\top \Lambda^j H_j \succ 0$ satisfied for some $\rho_1 \in \mathbb{R}$ implies that the unbounded terms in (4) (μ characterizes the nonlinearities with unbounded integrals, as introduced at the end of Section II) are presented for all elements of the vector x . The time derivative of (4) for the system dynamics takes the form:

$$\begin{aligned} \dot{V} &= \dot{x}^\top P x + x^\top P \dot{x} + 2 \sum_{j=1}^M \dot{x}^\top H_j^\top \Lambda^j f^j(H_j x) \\ + 2 \left[\sum_{g=0}^M f^g(H_g x)^\top \Omega_g^\top(\theta) + \dot{x}^\top \Psi^\top \right] &\left(\sum_{g=0}^M A_g(\theta) f^g(H_g x) + d - \dot{x} \right), \end{aligned}$$

where the last line is added by applying the descriptor approach [39] (this product equals to zero for $u \equiv 0$, while helping us in optimization of the resulted matrix inequalities). Opening the brackets, adding and subtracting the terms weighted by the matrices Φ and $\Upsilon_{g,s}$ with $g = \overline{0, M}$ and $s = \overline{g, M}$, this expression can be rewritten as follows (due to Assumption 1, $\theta \in \Theta$):

$$\begin{aligned} \dot{V} &= \zeta^\top Q(\theta) \zeta - \sum_{g=0}^M f^g(H_g x)^\top \Upsilon_{g,g}(\theta) f^g(H_g x) \\ - 2 \sum_{k=0}^{M-1} \sum_{s=k+1}^M f^k(H_k x)^\top \tilde{\Upsilon}_{k,s}(\theta) f^s(H_s x) &+ \varkappa y^\top y + d^\top \Phi d \\ &\leq - \sum_{g=0}^M f^g(H_g x)^\top \Upsilon_{g,g}(\theta) f^g(H_g x) \\ - 2 \sum_{k=0}^{M-1} \sum_{s=k+1}^M f^k(H_k x)^\top \tilde{\Upsilon}_{k,s}(\theta) f^s(H_s x) &+ \varkappa y^\top y + d^\top \Phi d, \end{aligned}$$

where

$$\zeta^\top = [\dot{x}^\top \ x^\top \ f^1(H_1 x)^\top \ \dots \ f^M(H_M x)^\top \ d^\top]^\top.$$

To get the desired IOSS property we need further to substantiate the upper bound:

$$\dot{V} \leq -\alpha_3(\|x\|) + \varkappa y^\top y + d^\top \Phi d$$

for some $\alpha_3 \in \mathcal{K}_\infty$. Thus, the following property has to be demonstrated,

$$\begin{aligned} \alpha_3(\|x\|) &\leq \sum_{g=0}^{\nu} f^g(H_g x)^\top \Upsilon_{g,g}(\theta) f^g(H_g x) \\ + 2 \sum_{k=0}^{\nu-1} \sum_{s=k+1}^{\nu} f^k(H_k x)^\top \tilde{\Upsilon}_{k,s}(\theta) f^s(H_s x) \end{aligned}$$

¹For brevity, the dependence on θ may be omitted once the respective functions have been defined.

$$Q(\theta) = \begin{bmatrix} -\Psi^\top - \Psi & P + \Psi^\top A_0 - \Omega_0 & H_1^\top \Lambda^1 + \Psi^\top A_1 - \Omega_1 & \cdots & H_M^\top \Lambda^M + \Psi^\top A_M - \Omega_M & \Psi^\top \\ P + A_0^\top \Psi - \Omega_0^\top & \Xi_0 & \Omega_0^\top A_1 + A_0^\top \Omega_1 + \Upsilon_{0,1} & \cdots & \Omega_0^\top A_M + A_0^\top \Omega_M + \Upsilon_{0,M} & \Omega_0^\top \\ \Lambda^1 H_1 + A_1^\top \Psi - \Omega_1^\top & A_1^\top \Omega_0 + \Omega_1^\top A_0 + \Upsilon_{0,1}^\top & \Xi_1 & \cdots & \Omega_1^\top A_M + A_1^\top \Omega_M + \Upsilon_{1,M} & \Omega_1^\top \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Lambda^M H_M + A_M^\top \Psi - \Omega_M^\top & A_M^\top \Omega_0 + \Omega_M^\top A_0 + \Upsilon_{0,M}^\top & A_M^\top \Omega_1 + \Omega_M^\top A_1 + \Upsilon_{1,M} & \cdots & \Xi_M & \Omega_M^\top \\ \Psi & \Omega_0 & \Omega_1 & \cdots & \Omega_M & -\Phi \end{bmatrix}, \quad (5)$$

$$\Xi_g = \Omega_g^\top A_g + A_g^\top \Omega_g - \varkappa C_g^\top C_g + \Upsilon_{g,g}, \quad g = \overline{0, M}$$

for all $x \in \mathbb{R}^n$ and $\theta \in \Theta$ (only unbounded nonlinearities should be taken into account, hence, the summation until ν is kept only). The right-hand side of the above inequality is an unbounded function of x if, according to Finsler's lemma, the fulfillment of the matrix inequality

$$\sum_{r=0}^{\nu} \tilde{\Upsilon}_{r,r}(\theta) + \rho_2 \sum_{z=1}^{\nu-1} \sum_{s=z+1}^{\nu} H_z^\top \Upsilon_{z,s}(\theta) H_s \\ + \rho_3 \sum_{j=1}^{\nu} H_j^\top \Upsilon_{0,j}(\theta) H_j \succ 0$$

is guaranteed for some $\rho_2, \rho_3 \in \mathbb{R}$, for all $\theta \in \Theta$. This ensures the existence of such a function α_3 , and that the system (3) is IOSS. Clearly, in such a case for $\varkappa = 0$ the ISS property is recovered. If the latter property is verified with $\varkappa = 0$ and not for the first ν nonlinearities, but for all M , then it implies existence of a positive definite function $\alpha_3 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which due to Theorem 1 is equivalent to iISS property for (3). \square

Verifying the feasibility of the proposed parameter-dependent matrix inequalities presents a challenge, especially when dealing with a large parameter space. However, various techniques are available to address this issue, such as grid-based methods, convexification, and relaxation techniques. The choice of the appropriate approach depends on the specific application and the desired level of accuracy.

Remark 1. A necessary condition for feasibility of the matrix inequality $Q(\theta) \preceq 0$ required in the formulation of Theorem 3 is that all matrices on the main diagonal of Q are nonnegative definite, i.e., $\Omega_g^\top(\theta) A_g(\theta) + A_g^\top(\theta) \Omega_g(\theta) \preceq 0$ for $g = \overline{0, M}$, which can be easily checked separately. It may also provide a hint on the choice of the structure of the functions $\Omega_g(\theta)$ for known shape of $A_g(\theta)$.

Remark 2. The Lyapunov function (4) is independent in θ . However, utilization of the descriptor approach allowed us to introduce the analogues of parameter dependent matrix functions $\Omega_g(\theta)$, $g = \overline{0, M}$ to check stability of $A_g(\theta)$: indeed, on the main diagonal of $Q(\theta)$ we have the elements $\Omega_g^\top(\theta) A_g(\theta) + A_g^\top(\theta) \Omega_g(\theta)$ that assess stability of $A_g(\theta)$. On another side note that the terms $P, H_1^\top \Lambda^1, \dots, H_M^\top \Lambda^M$ and $\Omega_0(\theta), \Omega_1(\theta), \dots, \Omega_M(\theta)$ appear with different signs, respectively, in some of the cells of the matrix $Q(\theta)$, then a possible choice is to equate them, which leads to a natural simplification $\Psi = 0$, and results in cancellation of the terms introduced by the descriptor approach (i.e., the obtained matrix inequality will be the same as after a direct substitution in \dot{V} the expression of \dot{x} from (3)). Thus, the descriptor approach allows us to get more general matrix inequalities, in some way

combining advantages of parameter-dependent and parameter-independent Lyapunov functions.

A drawback of the conditions formulated in Theorem 3 is that their verification for all $\theta \in \Theta$ may be computationally costly. Imposing affinity of the system matrix functions $A_g(\theta)$ in θ and restricting the matrix functions $\Omega_g(\theta)$ to be constant for all $g = \overline{0, M}$, these stability conditions can be reduced to LMIs:

Corollary 1. *Let assumptions 1 and 3 be satisfied, and there exist matrices $P^\top = P \in \mathbb{R}^{n \times n}$, $\Lambda^j \in \mathbb{D}_+^{k_j}$ for $j = \overline{1, M}$, $\Phi^\top = \Phi \in \mathbb{R}^{n \times n}$, $\Psi \in \mathbb{R}^{n \times n}$, $\Omega_g \in \mathbb{R}^{n \times k_g}$, $\Upsilon_{g,g} \in \mathbb{D}_+^{k_g}$ ($\tilde{\Upsilon}_{g,g} = H_g^\top \Upsilon_{g,g} H_g$) for $g = \overline{0, M}$, $\Upsilon_{0,j} \in \mathbb{D}_+^{k_j}$ ($\tilde{\Upsilon}_{0,j} = H_j^\top \Upsilon_{0,j}$) for $j = \overline{1, M}$, $\Upsilon_{z,s} \in \mathcal{M}(H_z, H_s)$ ($\tilde{\Upsilon}_{z,s} = \Upsilon_{z,s}$) for $z = \overline{1, M-1}$ and $s = z+1, M$ such that the following LMIs are verified:*

$$P \succeq 0, \Phi \succ 0; Q_k \preceq 0, \forall k = \overline{1, q};$$

$$P + \rho_1 \sum_{j=1}^{\mu} H_j^\top \Lambda^j H_j \succ 0,$$

$$\sum_{r=0}^{\nu} \tilde{\Upsilon}_{r,r} + \rho_2 \sum_{z=1}^{\nu-1} \sum_{s=z+1}^{\nu} H_z^\top \Upsilon_{z,s} H_s + \rho_3 \sum_{j=1}^{\nu} H_j^\top \Upsilon_{0,j} H_j \succ 0$$

for some $\rho_1, \rho_2, \rho_3 \in \mathbb{R}$ and $\varkappa > 0$, where Q_k are given in (6), then the system (3) with $u \equiv 0$ is IOSS. For $\varkappa = 0$ the ISS property can be proven. If the last LMI is replaced with

$$\sum_{r=0}^M \tilde{\Upsilon}_{r,r} + \rho_2 \sum_{z=1}^{M-1} \sum_{s=z+1}^M H_z^\top \Upsilon_{z,s} H_s \\ + \rho_3 \sum_{j=1}^M H_j^\top \Upsilon_{0,j} H_j \succ 0,$$

then the system is iISS.

Proof. The proof is a direct consequence of the result of Theorem 3 under application of Assumption 3 for decomposition of the matrix function Q . Indeed, in this case

$$\dot{V} = \sum_{k=1}^q \theta_k \zeta^\top Q_k \zeta - \sum_{g=0}^M f^g(H_g x)^\top \Upsilon_{g,g} f^g(H_g x) \\ - 2 \sum_{k=0}^{M-1} \sum_{s=k+1}^M f^k(H_k x)^\top \tilde{\Upsilon}_{k,s} f^s(H_s x) + \varkappa y^\top y + d^\top \Phi d.$$

and since $\theta_k \in [0, 1]$ the property $Q_k \preceq 0$ for all $k = \overline{1, q}$ implies $\sum_{k=1}^q \theta_k \zeta^\top Q_k \zeta \preceq 0$. All other steps and arguments are the same. \square

$$Q_k = \begin{bmatrix} -\Psi^\top - \Psi & P + \Psi^\top A_{0k} - \Omega_0 & H_1^\top \Lambda^1 + \Psi^\top A_{1k} - \Omega_1 & \cdots & H_M^\top \Lambda^M + \Psi^\top A_{Mk} - \Omega_M & \Psi^\top \\ P + A_{0k}^\top \Psi - \Omega_0^\top & \Xi_{k0} & \Omega_0^\top A_{1k} + A_{0k}^\top \Omega_1 + \tilde{\Upsilon}_{0,1} & \cdots & \Omega_0^\top A_{Mk} + A_{0k}^\top \Omega_M + \tilde{\Upsilon}_{0,M} & \Omega_0^\top \\ \Lambda^1 H_1 + A_{1k}^\top \Psi - \Omega_1^\top & A_{1k}^\top \Omega_0 + \Omega_1^\top A_{0k} + \tilde{\Upsilon}_{0,1} & \Xi_{k1} & \cdots & \Omega_1^\top A_{Mk} + A_{1k}^\top \Omega_M + \tilde{\Upsilon}_{1,M} & \Omega_1^\top \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Lambda^M H_M + A_{Mk}^\top \Psi - \Omega_M^\top & A_{Mk}^\top \Omega_0 + \Omega_M^\top A_{0k} + \tilde{\Upsilon}_{0,M} & A_{Mk}^\top \Omega_1 + \Omega_M^\top A_{1k} + \tilde{\Upsilon}_{1,M} & \cdots & \Xi_{kM} & \Omega_M^\top \\ \Psi & \Omega_0 & \Omega_1 & \cdots & \Omega_M & -\Phi \end{bmatrix}, \quad (6)$$

$$\Xi_{kg} = \Omega_g^\top A_{gk} + A_{gk}^\top \Omega_g - \varkappa C_g^\top C_g + \Upsilon_{g,g}, \quad g = \overline{0, M}$$

Note that the output y in the system (3) does not represent the variables available for measurements, but the quantity with respect to which we would like to analyze the IOSS property. For example, stabilization of the output for an IOSS system implies its state regulation to zero.

V. PARAMETER-DEPENDENT LYAPUNOV FUNCTIONS

In this case consider for (3) the following modification of (4):

$$V(x, \theta) = x^\top P(\theta)x + 2 \sum_{j=1}^M \sum_{i=1}^{k_j} \Lambda_i^j(\theta) \int_0^{H_j^{(i)}x} f_i^j(s) ds, \quad (7)$$

where $P : \mathbb{R}^q \rightarrow \mathbb{R}^{n \times n}$ is a symmetric matrix function and denote by $\Lambda^j(\theta) = \text{diag}[\Lambda_1^j(\theta) \dots \Lambda_{k_j}^j(\theta)] \in \mathbb{D}_+^{k_j}$ a nonnegative diagonal matrix function. We assume that V is differentiable with respect to θ . In such a case, since the coefficients Λ_i^j are assumed to be dependent on θ , the derivative of (7) contains the integrals of the nonlinearities, which requires the introduction of additional hypotheses on their relations with nonlinearities of the system:

Assumption 4. *Let assume that, for any $j = \overline{1, M}$ and $i = \overline{1, k_j}$, there exist $W_{g,g}^{i,j} \in \mathbb{D}_+^{k_g}$ ($\tilde{W}_{g,g}^{i,j} = H_g^\top W_{g,g}^{i,j} H_g$) for $g = \overline{0, M}$, $W_{0,z}^{i,j} \in \mathbb{D}_+^{k_j}$ ($\tilde{W}_{0,z}^{i,j} = H_z^\top W_{0,z}^{i,j}$) for $z = \overline{1, M}$, $W_{k,s}^{i,j} \in \mathcal{M}(H_k, H_s)$ ($\tilde{W}_{k,s}^{i,j} = W_{k,s}^{i,j}$) for $k = \overline{1, M-1}$ and $s = \overline{k+1, M}$ such that*

$$\int_0^{H_j^{(i)}x} f_i^j(s) ds \leq \sum_{g=0}^M f^g(H_g x)^\top W_{g,g}^{i,j} f^g(H_g x) + 2 \sum_{k=0}^{M-1} \sum_{s=k+1}^M f^k(H_k x)^\top \tilde{W}_{k,s}^{i,j} f^s(H_s x)$$

for all $x \in \mathbb{R}^n$.

Assumption 4 is naturally satisfied for polynomial functions, for example: if $f_i^j(s) = s^a$ with $a > 0$, then $\int_0^{H_j^{(i)}x} f_i^j(s) ds = \frac{1}{1+a} \left(H_j^{(i)} x \right)^{1+a} \leq \left(H_j^{(i)} f^0(H_0 x) \right) \left(W_{0,j}^{i,j} \right)_{i,i} f_i^j(H_j^{(i)} x)$ for any $\left(W_{0,j}^{i,j} \right)_{i,i} \geq \frac{1}{1+a}$, and all other elements in the matrices $W_{k,s}^{i,j}$ can be selected to be zero.

Theorem 4. *Let assumptions 1, 2 and 4 be satisfied and there exist matrices $\Phi^\top = \Phi \in \mathbb{R}^{n \times n}$, $\Psi \in \mathbb{R}^{n \times n}$, symmetric continuously differentiable matrix functions $P : \mathbb{R}^q \rightarrow \mathbb{R}^{n \times n}$, $\Lambda^j : \mathbb{R}^q \rightarrow \mathbb{D}_+^{k_j}$ for $j = \overline{1, M}$, and matrix functions $\Omega_g : \mathbb{R}^q \rightarrow \mathbb{R}^{n \times k_g}$, $\Upsilon_{g,g} : \mathbb{R}^q \rightarrow \mathbb{D}_+^{k_g}$ ($\tilde{\Upsilon}_{g,g}(\theta) = H_g^\top \Upsilon_{g,g}(\theta) H_g$) for $g = \overline{0, M}$, $\Upsilon_{0,j} : \mathbb{R}^q \rightarrow \mathbb{D}_+^{k_j}$ ($\tilde{\Upsilon}_{0,j}(\theta) = H_j^\top \Upsilon_{0,j}(\theta)$) for*

$j = \overline{1, M}$, $\Upsilon_{z,s} : \mathbb{R}^q \rightarrow \mathcal{M}(H_z, H_s)$ ($\tilde{\Upsilon}_{z,s}(\theta) = \Upsilon_{z,s}(\theta)$) for $z = \overline{1, M-1}$ and $s = \overline{z+1, M}$ such that the following matrix inequalities are verified for all $\theta \in \Theta$ and $\dot{\theta} \in [-\dot{\theta}_{\max}, \dot{\theta}_{\max}]^q$:

$$P(\theta) \succeq 0, \quad \Phi \succ 0, \quad Q(\theta, \dot{\theta}) \preceq 0, \\ P(\theta) + \rho_1 \sum_{j=1}^{\mu} H_j^\top \Lambda^j(\theta) H_j \succ 0, \\ \sum_{r=0}^{\nu} \tilde{\Upsilon}_{r,r}(\theta) + \rho_2 \sum_{z=1}^{\nu-1} \sum_{s=z+1}^{\nu} H_z^\top \Upsilon_{z,s}(\theta) H_s \\ + \rho_3 \sum_{j=1}^{\nu} H_j^\top \Upsilon_{0,j}(\theta) H_j \succ 0$$

for some $\rho_1, \rho_2, \rho_3 \in \mathbb{R}$ and $\varkappa > 0$, where the matrices are defined in (8), then the system (3) with $u \equiv 0$ is IOSS, and it is ISS for $\varkappa = 0$. If the last inequality is replaced with

$$\sum_{r=0}^M \tilde{\Upsilon}_{r,r}(\theta) + \rho_2 \sum_{z=1}^{M-1} \sum_{s=z+1}^M H_z^\top \Upsilon_{z,s}(\theta) H_s \\ + \rho_3 \sum_{j=1}^M H_j^\top \Upsilon_{0,j}(\theta) H_j \succ 0$$

satisfying for all $\theta \in \Theta$, then the system is iISS.

Proof. The relations $\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$ are verified for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ for all $x \in \mathbb{R}^n$ according to Finsler's lemma and the introduced matrix inequalities: 1) the conditions $P(\theta) \succeq 0$, $\Lambda^j(\theta) \succeq 0$ for $j = \overline{1, M}$ and $P(\theta) + \rho_1 \sum_{j=1}^{\mu} H_j^\top \Lambda^j(\theta) H_j \succ 0$ are verified for some $\rho_1 \in \mathbb{R}$ for all $\theta \in \Theta$, where Θ is a compact set (see Assumption 1), which implies that the unbounded terms in (7) (μ characterizes the nonlinearities with unbounded integrals, see end of Section II) are presented for all elements of the vector x , this justifies the existence of α_1 ; 2) since $P(\theta)$ and $\Lambda^j(\theta)$ for $j = \overline{1, M}$ are at least continuous functions, they take bounded values on a compact set Θ , hence, the function α_2 exists. The time derivative of (7) for the system dynamics takes the form:

$$\dot{V} = \dot{x}^\top P(\theta)x + x^\top P(\theta)\dot{x} + 2 \sum_{j=1}^M \dot{x}^\top H_j^\top \Lambda^j(\theta) f^j(H_j x) \\ + x^\top \dot{P}(\theta)x + 2 \sum_{j=1}^M \sum_{i=1}^{k_j} \dot{\Lambda}_i^j(\theta) \int_0^{H_j^{(i)}x} f_i^j(s) ds \\ + 2 \left[\sum_{g=0}^M f^g(H_g x)^\top \Omega_g^\top(\theta) + \dot{x}^\top \Psi^\top \right] \left(\sum_{g=0}^M A_g(\theta) f^g(H_g x) + d - \dot{x} \right),$$

$$\begin{aligned}
Q = & \begin{bmatrix} -\Psi^\top - \Psi & P + \Psi^\top A_0 - \Omega_0 & H_1^\top \Lambda^1 + \Psi^\top A_1 - \Omega_1 & \cdots & H_M^\top \Lambda^M + \Psi^\top A_M - \Omega_M & \Psi^\top \\ P + A_0^\top \Psi - \Omega_0^\top & \Omega_0^\top A_0 + A_0^\top \Omega_0 + \Gamma_{0,0} & \Omega_0^\top A_1 + A_0^\top \Omega_1 + \Gamma_{0,1} & \cdots & \Omega_0^\top A_M + A_0^\top \Omega_M + \Gamma_{0,M} & \Omega_0^\top \\ \Lambda^1 H_1 + A_1^\top \Psi - \Omega_1^\top & A_1^\top \Omega_0 + \Omega_1^\top A_0 + \Gamma_{0,1} & \Omega_1^\top A_1 + A_1^\top \Omega_1 + \Gamma_{1,1} & \cdots & \Omega_1^\top A_M + A_1^\top \Omega_M + \Gamma_{1,M} & \Omega_1^\top \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Lambda^M H_M + A_M^\top \Psi - \Omega_M^\top & A_M^\top \Omega_0 + \Omega_M^\top A_0 + \Gamma_{0,M} & A_M^\top \Omega_1 + \Omega_M^\top A_1 + \Gamma_{1,M} & \cdots & \Omega_M^\top A_M + A_M^\top \Omega_M + \Gamma_{M,M} & \Omega_M^\top \\ & \Psi & \Omega_0 & \Omega_1 & \Omega_M & -\Phi \end{bmatrix}, \\
\Gamma_{0,0}(\theta, \dot{\theta}) = & \sum_{z=1}^q \dot{\theta}_z \frac{\partial P(\theta)}{\partial \theta_z} + 2 \sum_{j=1}^M \sum_{i=1}^{k_j} W_{g,g}^{i,j} \sum_{z=1}^q \max \left\{ 0, \dot{\theta}_z \frac{\partial \Lambda_i^j(\theta)}{\partial \theta_z} \right\} - \varkappa C_0^\top C_0 + \Upsilon_{0,0}(\theta), \\
\Gamma_{j,j}(\theta, \dot{\theta}) = & 2 \sum_{j=1}^M \sum_{i=1}^{k_j} W_{j,j}^{i,j} \sum_{z=1}^q \max \left\{ 0, \dot{\theta}_z \frac{\partial \Lambda_i^j(\theta)}{\partial \theta_z} \right\} - \varkappa C_j^\top C_j + \Upsilon_{j,j}(\theta), \quad j = \overline{1, M}, \\
\Gamma_{k,s}(\theta, \dot{\theta}) = & 2 \sum_{j=1}^M \sum_{i=1}^{k_j} \tilde{W}_{k,s}^{i,j} \sum_{z=1}^q \max \left\{ 0, \dot{\theta}_z \frac{\partial \Lambda_i^j(\theta)}{\partial \theta_z} \right\} + \tilde{\Upsilon}_{k,s}
\end{aligned} \tag{8}$$

where the last line is again added by applying the descriptor approach [39] being equal zero for $u \equiv 0$. In comparison to Theorem 3, here there are two terms in the second line:

$$\begin{aligned}
x^\top \dot{P}(\theta)x &= \sum_{z=1}^q \dot{\theta}_z x^\top \frac{\partial P(\theta)}{\partial \theta_z} x, \\
\dot{\Lambda}_i^j(\theta) \int_0^{H_j^{(i)}x} f_i^j(s) ds &= \sum_{z=1}^q \dot{\theta}_z \frac{\partial \Lambda_i^j(\theta)}{\partial \theta_z} \int_0^{H_j^{(i)}x} f_i^j(s) ds \\
&\leq \sum_{z=1}^q \max \left\{ 0, \dot{\theta}_z \frac{\partial \Lambda_i^j(\theta)}{\partial \theta_z} \right\} \int_0^{H_j^{(i)}x} f_i^j(s) ds \\
&\leq \sum_{z=1}^q \max \left\{ 0, \dot{\theta}_z \frac{\partial \Lambda_i^j(\theta)}{\partial \theta_z} \right\} \left[\sum_{g=0}^M f^g(H_g x)^\top W_{g,g}^{i,j} f^g(H_g x) \right. \\
&\quad \left. + 2 \sum_{k=0}^{M-1} \sum_{s=k+1}^M f^k(H_k x)^\top \tilde{W}_{k,s}^{i,j} f^s(H_s x) \right],
\end{aligned}$$

where we utilized Assumption 4. Using these inequalities, opening the brackets, adding and subtracting the terms weighted by the matrices Φ and $\tilde{\Upsilon}_{g,s}$ with $g = \overline{0, M}$ and $s = g, \overline{M}$, the expression for \dot{V} can be rewritten as follows:

$$\begin{aligned}
\dot{V} &\leq \zeta^\top Q(\theta, \dot{\theta}) \zeta - \sum_{g=0}^M f^g(H_g x)^\top \Upsilon_{g,g}(\theta) f^g(H_g x) \\
-2 \sum_{k=0}^{M-1} \sum_{s=k+1}^M f^k(H_k x)^\top \tilde{\Upsilon}_{k,s}(\theta) f^s(H_s x) &+ \varkappa y^\top y + d^\top \Phi d \\
&\leq - \sum_{g=0}^M f^g(H_g x)^\top \Upsilon_{g,g}(\theta) f^g(H_g x) \\
-2 \sum_{k=0}^{M-1} \sum_{s=k+1}^M f^k(H_k x)^\top \tilde{\Upsilon}_{k,s}(\theta) f^s(H_s x) &+ \varkappa y^\top y + d^\top \Phi d
\end{aligned}$$

since $Q(\theta, \dot{\theta}) \preceq 0$ for all $\theta \in \Theta$ and $\dot{\theta} \in [-\dot{\theta}_{\max}, \dot{\theta}_{\max}]^q$, which are the sets of admissible values for these variables due to assumptions 1 and 2, where as before

$$\zeta^\top = [\dot{x}^\top \ x^\top \ f^1(H_1 x)^\top \ \dots \ f^M(H_M x)^\top \ d^\top]^\top.$$

The rest of the proof is the same as in Theorem 3. \square

Verification of matrix inequalities formulated in Theorem 4 is more complicated since they depend on two independent vector variables θ and $\dot{\theta}$. However, using Assumption 3 and additionally requiring a linear dependence of P and Λ^j in θ , these conditions can be reduced to verification of LMIs.

VI. STABILIZATION OF PVP SYSTEMS

Since the instantaneous values of the vector of scheduling parameters $\theta(t)$ are assumed to be unknown (only the set of admissible values is given according to Assumption 1), the variable $\theta(t)$ cannot be used in the control design, and the following static nonlinear state feedback is proposed in this work for (3):

$$u(t) = K_0 x(t) + \sum_{j=0}^M K_j f^j(H_j x), \tag{9}$$

where $K_g \in \mathbb{R}^{m \times n}$ are the control gain matrices to be selected. Clearly, substitution of the control law (9) in the equations of (3) preserve the PVP form of the model:

$$\dot{x}(t) = \mathcal{A}_0(\theta(t))x(t) + \sum_{j=1}^M \mathcal{A}_j(\theta(t))f^j(H_j x(t)) + d(t),$$

where $\mathcal{A}_g(\theta) = A_g(\theta) + B(\theta)K_g$ for $g = \overline{0, M}$. Therefore, analysis of ISS or iISS property for the closed-loop dynamics (3), (9) can be performed using the (linear) matrix inequalities of the previously given results under substitution $A_g(\theta) \leftarrow \mathcal{A}_g(\theta)$ for any fixed set of the matrices K_g . Similarly, an output control law $u(t) = \tilde{K}y(t)$ coefficients $\tilde{K} \in \mathbb{R}^{m \times p}$ can be tested (note that it is a linear output feedback, which is however nonlinear in the state).

In order to find the gains K_g for all $g = \overline{0, M}$ as solutions of LMIs, we need to add additional restrictions to the case of Corollary 1:

Theorem 5. *Let assumptions 1 and 3 be satisfied, and there exist matrices $\bar{P} \in \mathbb{D}_+^n$, $\bar{\Lambda}^j, \bar{\Upsilon}_{0,j} \in \mathbb{D}_+^n$ for $j = \overline{1, M}$, $\bar{\Phi}^\top = \bar{\Phi} \in \mathbb{R}^{n \times n}$, $\bar{\Psi}, \bar{\Omega}_g \in \mathbb{D}_+^n$ and $U_g \in \mathbb{R}^{m \times n}$ for $g = \overline{0, M}$,*

$\bar{\Upsilon}_{z,s} \in \mathbb{D}_+^n$ for $z = \overline{0, M-1}$ and $s = \overline{z, M}$ such that the following LMIs are verified:

$$\begin{aligned} \bar{\Psi} &> 0, \bar{P} \succeq 0, \bar{\Phi} > 0; \bar{\Omega}_j > 0, \forall j = \overline{1, M}; \\ \bar{Q}_k &\preceq 0, \forall k = \overline{1, q}; \bar{P} + \sum_{j=1}^{\mu} \bar{\Lambda}^j > 0, \\ \sum_{r=0}^{\nu} \bar{\Upsilon}_{r,r} + \sum_{z=1}^{\nu-1} \sum_{s=z+1}^{\nu} \bar{\Upsilon}_{z,s} + \sum_{j=1}^{\nu} \bar{\Upsilon}_{0,j} &> 0, \end{aligned}$$

where \bar{Q}_k are given in (10), then the system (3), (9) with $H_j = I_n$ for all $j = \overline{1, M}$ and $K_g = U_g \bar{\Omega}_g^{-1}$ for $g = \overline{0, M}$ is ISS. If the last LMI is replaced with

$$\sum_{r=0}^M \bar{\Upsilon}_{r,r} + \sum_{z=1}^{M-1} \sum_{s=z+1}^M \bar{\Upsilon}_{z,s} + \sum_{j=1}^M \bar{\Upsilon}_{0,j} > 0,$$

then the system is iISS.

Proof. Based on the observed structural similarity of the closed-loop dynamics (3), (9) and the original model (3), the proof recalls the main arguments used in Corollary 1. Note that for the case of diagonal matrices the constants ρ_1, ρ_2, ρ_3 can be omitted in that corollary. Considering that $P \in \mathbb{D}_+^n$, $\Lambda^j, \Upsilon_{0,j} \in \mathbb{D}_+^n$ for $j = \overline{1, M}$ and $\Psi, \Omega_g, \Upsilon_{g,g} \in \mathbb{D}_+^n$ for $g = \overline{0, M}$ in the conditions of Corollary 1, taking $H_j = I_n$ for all $j = \overline{1, M}$, the matrix Q_k can be reduced to the form given in (11), where $A_{gk} = A_{gk} + B_k K_g$ for $k = \overline{1, q}$ and $g = \overline{0, M}$. By assuming that Ψ and Ω_j , $j = \overline{1, M}$ are nonsingular matrices; multiplying the matrix Q_k by $\text{diag}\{\Psi^{-1}, \Omega_1^{-1}, \dots, \Omega_M^{-1}, I_n\}$ from the left and the right; denoting $\bar{\Psi} = \Psi^{-1}$, $\bar{P} = \Omega_0^{-1} P \Psi^{-1}$, $\bar{\Omega}_j = \Omega_j^{-1}$ and $\bar{\Lambda}^j = \Psi^{-1} \Lambda^j \Omega_j^{-1}$ for $j = \overline{1, M}$, $\bar{\Upsilon}_{z,s} = \Omega_z^{-1} \Upsilon_{z,s} \Omega_s^{-1}$ for $z = \overline{0, M-1}$ and $s = \overline{z, M}$; $U_g = K_g \Omega_g^{-1}$ for $g = \overline{0, M}$, then (11) takes the form of (10). Since the matrices Ψ and Ω_j are nonsingular ($\bar{\Psi}$ and $\bar{\Omega}_j$, correspondingly), the proprieties of the matrices \bar{P} , $\bar{\Lambda}^j$ for $j = \overline{1, M}$ and $\bar{\Upsilon}_{z,s}$ for $z = \overline{0, M-1}$ and $s = \overline{z, M}$ imply the same characteristics for the counterparts P , Λ^j and $\Upsilon_{z,s}$, respectively. \square

The conditions of Theorem 5 seem to be restrictive (since the matrices are required to be diagonal), but taking into account the actual complexity of the studied problem (we are aiming to find a globally stabilizing control for uncertain nonlinear system via LMIs), they still provide an advancement with respect to the state of the art. Moreover, the structural restrictions for the matrices P , Λ^j and $\Upsilon_{z,s}$ to be semi-definite are preserved.

VII. EXAMPLES

In this section, an example of control design through the LMIs of Theorem 5 is first given. Then, a robust control design for the longitudinal flight of a civil aircraft is presented utilizing a highly nonlinear and relevant model. To check the LMIs, SDPT3 solver of MATLAB [40] with YALMIP [41] are used.

A. Control design (Example 1)

To illustrate the presented methodology, consider an academic example with a cubic nonlinearity:

$$\begin{aligned} \dot{x}_1(t) &= -a_1(t)x_1(t) + x_2(t) - a_2x_1^3(t) \\ &\quad - 0.5u(t) + d_1(t), \\ \dot{x}_2(t) &= -a_3(t)x_2(t) + a_4x_1^3(t) - a_5(t)x_2^3(t) \\ &\quad + u(t) + d_2(t), \end{aligned}$$

where $x_1(t), x_2(t) \in \mathbb{R}$ are the system states, $d(t) = [d_1(t) \ d_2(t)]^\top \in \mathbb{R}^2$ is a bounded disturbance, $u(t) \in \mathbb{R}$ is the control signal; $a_i(t)$ for $i = 1, 3, 5$ are positive time-varying parameters, whose instantaneous values are unknown, but the sets of admissible deviations have been identified: $a_i \min \leq a_i(t) \leq a_i \max$ for all $t \geq 0$ with some $0 < a_i \min \leq a_i \max < +\infty$ being the minimal and maximal possible values, respectively, for these parameters; $a_2, a_4 > 0$ are given constants. Our goal is to design a robustly stabilizing control for this system using the proposed conditions (with the aim of accelerating the convergence and improving the robustness). It is easy to check that this model can be rewritten in the form (3) for $n = 2$, $m = 1$, $M = 1$ and $f^1(x) = [x_1^3 \ x_2^3]^\top$, $H_1 = I_2$ satisfying the sector condition, with

$$A_g(\theta) = \sum_{k=1}^q \theta_k A_{gk}, \quad g = 0, 1, \quad q = 8,$$

$$\begin{aligned} A_{01} = A_{05} &= \begin{bmatrix} -a_{1 \min} & 1 \\ 0 & -a_{3 \min} \end{bmatrix}, \\ A_{02} = A_{06} &= \begin{bmatrix} -a_{1 \min} & 1 \\ 0 & -a_{3 \max} \end{bmatrix}, \\ A_{03} = A_{07} &= \begin{bmatrix} -a_{1 \max} & 1 \\ 0 & -a_{3 \min} \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} A_{04} = A_{08} &= \begin{bmatrix} -a_{1 \max} & 1 \\ 0 & -a_{3 \max} \end{bmatrix}, \\ A_{11} = \dots = A_{14} &= \begin{bmatrix} -a_2 & 0 \\ a_4 & -a_{5 \min} \end{bmatrix}, \\ A_{15} = \dots = A_{18} &= \begin{bmatrix} -a_2 & 0 \\ a_4 & -a_{5 \max} \end{bmatrix}, \end{aligned}$$

where $\sum_{k=1}^q \theta_k = 1$ and $\theta_k \in [0, 1]$. The compact set $\Theta \subseteq [0, 1]^q$ can be straightforwardly computed, then Assumption 1 is verified. Moreover, it is clear that Assumption 3 is also satisfied, therefore, the LMIs of Theorem 5 can be used to design the control (9) providing ISS property of this time-varying nonlinear system. Note that to apply LPV method in this example, we need to transform the cubic nonlinearities in a linear parameter-varying form, e.g., by considering:

$$x_i^3(t) = \theta_i(t)x_i(t), \quad \theta_i(t) = x_i^2(t), \quad i = 1, 2,$$

where $\theta_1(t)$ and $\theta_2(t)$ are new additional scheduling parameters that take values in \mathbb{R}_+ . Then, it may be difficult to get global stability conditions using LMIs dependent on vector $\theta(t)$ whose elements take values in unbounded sets.

$$\bar{Q}_k = \begin{bmatrix} -2\bar{\Psi} & \bar{P} + A_{0k}\bar{\Omega}_0 + B_k U_0 - \bar{\Psi} & \bar{\Lambda}^1 + A_{1k}\bar{\Omega}_1 + B_k U_1 - \bar{\Psi} & & & \\ \bar{P} + \bar{\Omega}_0 A_{0k}^\top + U_0^\top B_k^\top - \bar{\Psi} & A_{0k}\bar{\Omega}_0 + \bar{\Omega}_0 A_{0k}^\top + B_k U_0 + U_0^\top B_k^\top + \bar{\Upsilon}_{0,0} & A_{1k}\bar{\Omega}_1 + B_k U_1 + \bar{\Omega}_0 A_{0k}^\top + U_0^\top B_k^\top + \bar{\Upsilon}_{0,1} & & & \\ \bar{\Lambda}^1 + \bar{\Omega}_1 A_{1k}^\top + U_1^\top B_k^\top - \bar{\Psi} & \bar{\Omega}_1 A_{1k}^\top + U_1^\top B_k^\top + A_{0k}\bar{\Omega}_0 + B_k U_0 + \bar{\Upsilon}_{0,1} & A_{1k}\bar{\Omega}_1 + \bar{\Omega}_1 A_{1k}^\top + B_k U_1 + U_1^\top B_k^\top + \bar{\Upsilon}_{1,1} & & & \\ \vdots & \vdots & \vdots & & & \\ \bar{\Lambda}^M + \bar{\Omega}_M A_{Mk}^\top + U_M^\top B_k^\top - \bar{\Psi} & \bar{\Omega}_M A_{Mk}^\top + U_M^\top B_k^\top + A_{0k}\bar{\Omega}_0 + B_k U_0 + \bar{\Upsilon}_{0,M} & \bar{\Omega}_M A_{Mk}^\top + U_M^\top B_k^\top + A_{1k}\bar{\Omega}_1 + B_k U_1 + \bar{\Upsilon}_{1,M} & & & \\ \vdots & \vdots & \vdots & & & \\ \vdots & \vdots & \vdots & & & \\ \vdots & \vdots & \vdots & & & \\ \bar{\Lambda}^M + A_{Mk}\bar{\Omega}_M + B_k U_M - \bar{\Psi} & I_n & I_n & & & \\ \vdots & A_{Mk}\bar{\Omega}_M + B_k U_M + \bar{\Omega}_0 A_{0k}^\top + U_0^\top B_k^\top + \bar{\Upsilon}_{0,M} & I_n & & & \\ \vdots & A_{Mk}\bar{\Omega}_M + B_k U_M + \bar{\Omega}_1 A_{1k}^\top + U_1^\top B_k^\top + \bar{\Upsilon}_{1,M} & I_n & & & \\ \vdots & \vdots & \vdots & & & \\ \vdots & \vdots & \vdots & & & \\ \vdots & A_{Mk}\bar{\Omega}_M + \bar{\Omega}_M A_{Mk}^\top + B_k U_M + U_M^\top B_k^\top + \bar{\Upsilon}_{M,M} & I_n & & & \\ \vdots & \vdots & -\Phi & & & \end{bmatrix}, \quad (10)$$

$$Q_k = \begin{bmatrix} -2\Psi & P + \Psi A_{0k} - \Omega_0 & \Lambda^1 + \Psi A_{1k} - \Omega_1 & \cdots & \Lambda^M + \Psi A_{Mk} - \Omega_M & \Psi \\ P + \mathcal{A}_{0k}^\top \Psi - \Omega_0 & \Omega_0 \mathcal{A}_{0k} + \mathcal{A}_{0k}^\top \Omega_0 + \Upsilon_{0,0} & \Omega_0 \mathcal{A}_{1k} + \mathcal{A}_{0k}^\top \Omega_1 + \Upsilon_{0,1} & \cdots & \Omega_0 \mathcal{A}_{Mk} + \mathcal{A}_{0k}^\top \Omega_M + \Upsilon_{0,M} & \Omega_0 \\ \Lambda^1 + \mathcal{A}_{1k}^\top \Psi - \Omega_1 & \mathcal{A}_{1k}^\top \Omega_0 + \Omega_1 \mathcal{A}_{0k} + \Upsilon_{0,1} & \Omega_1 \mathcal{A}_{1k} + \mathcal{A}_{1k}^\top \Omega_1 + \Upsilon_{1,1} & \cdots & \Omega_1 \mathcal{A}_{Mk} + \mathcal{A}_{1k}^\top \Omega_M + \Upsilon_{1,M} & \Omega_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Lambda^M + \mathcal{A}_{Mk}^\top \Psi - \Omega_M & \mathcal{A}_{Mk}^\top \Omega_0 + \Omega_M \mathcal{A}_{0k} + \Upsilon_{0,M} & \mathcal{A}_{Mk}^\top \Omega_1 + \Omega_M \mathcal{A}_{1k} + \Upsilon_{1,M} & \cdots & \Omega_M \mathcal{A}_{Mk} + \mathcal{A}_{Mk}^\top \Omega_M + \Upsilon_{M,M} & \Omega_M \\ \Psi & \Omega_0 & \Omega_1 & \cdots & \Omega_M & -\Phi \end{bmatrix} \quad (11)$$

Let

$$\begin{aligned} a_{1 \min} &= 0.05, \quad a_{3 \min} = 0.01, \quad a_{5 \min} = 0.001, \\ a_{1 \max} &= 2, \quad a_{3 \max} = 0.5, \quad a_{5 \max} = 0.05, \\ a_2 &= 0.1, \quad a_4 = 0.01, \end{aligned}$$

then the LMIs of Theorem 5 are verified providing

$$K_0 = -[0.018 \ 37.303], \quad K_1 = -[0.079 \ 14.466]$$

as a solution. After selecting

$$\begin{aligned} d(t) &= \begin{bmatrix} 1 + \sin(6t) \\ -2 + 5 \cos(1.5t) \end{bmatrix} + 2\text{rand}(2, 1), \\ a_1(t) &= a_{1 \min} + (a_{1 \max} - a_{1 \min}) \sin^2(3t), \\ a_3(t) &= a_{3 \min} + (a_{3 \max} - a_{3 \min}) \cos^2(3t), \\ a_5(t) &= a_{5 \min} + (a_{5 \max} - a_{5 \min}) \sin^2(3t), \end{aligned}$$

the simulation results are shown in Fig. 1 for twenty different initial conditions chosen randomly in the set $[-10, 10]^2$. All trajectories converge to a vicinity of the origin proportional to the amplitude of the perturbation d , which is an illustration of ISS behavior.

B. Longitudinal flight stabilization (Example 2)

The following model is found to be relevant to describe this kind of motion in [42], [43]:

$$\begin{aligned} m\dot{v} &= -\frac{\rho S}{2} v^2 C_D(\alpha) - mg \sin(\gamma) + \cos(\alpha)F + mw_1, \\ mv\dot{\gamma} &= \frac{\rho S}{2} v^2 C_L(\alpha, r, v, \delta_e) - mg \cos(\gamma) + \sin(\alpha)F + mvw_2, \\ \dot{\eta} &= r + w_3, \\ J\dot{r} &= L\frac{\rho S}{2} v^2 C_m(\alpha, r, v, \delta_e) + z_e F + Jw_4, \end{aligned}$$

where $v \in [v_{\min}, v_{\max}]$ is the airspeed in the given limits $0 < v_{\min} < v_{\max} < +\infty$, $\gamma \in [-\frac{\pi}{20}, \frac{\pi}{20}]$ is the flight path angle, $\eta \in [-\frac{\pi}{20}, \frac{\pi}{20}]$ and $r \in \mathbb{R}$ are the pitch angle and its rate;

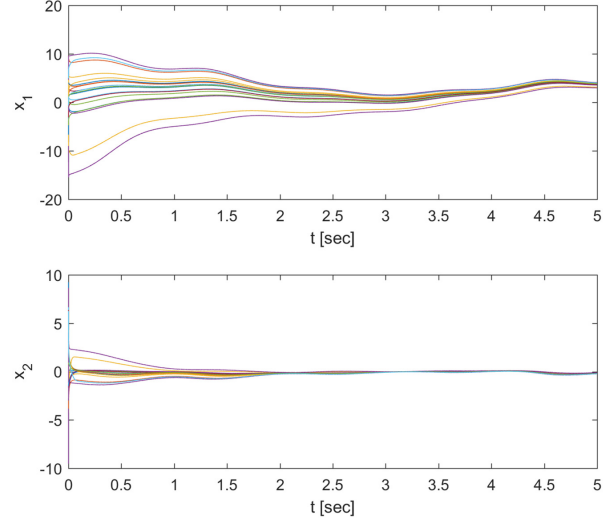


Figure 1. The results of simulation of the closed-loop academic example: $x(t)$ versus time t [sec]

$\alpha = \eta - \gamma \in [-\frac{\pi}{10}, \frac{\pi}{10}]$ is the angle of attack; the control inputs are represented by the engine thrust $F \in [-F_{\max}, F_{\max}]$ and the elevator deflection angle $\delta_e \in [-\delta_{\max}, \delta_{\max}]$ for known $F_{\max} > 0$ and $\delta_{\max} > 0$; $w = [w_1 \dots w_4]^\top \in \mathcal{L}_\infty^4$ is exogenous disturbance; ρ is the air density, S and L are the reference surface and length, $m \in [m_{\min}, m_{\max}]$ is the mass with given bounds $0 < m_{\min} < m_{\max} < +\infty$, g is the gravity acceleration, z_e is the shifting between the center of gravity and the thrust application point, $J > 0$ is the longitudinal inertia; the drag, lift and pitch coefficients are expressed as

follows:

$$\begin{aligned} C_D(\alpha) &= C_{D0} + C_{D\alpha}\alpha + C_{D\alpha^2}\alpha^2, \\ C_L(\alpha, r, v, \delta_e) &= C_{L0} + C_{L\alpha}\alpha + C_{Lq}\frac{r}{v} + c_L\delta_e, \\ C_m(\alpha, r, v, \delta_e) &= -C_{m0} - C_{m\alpha}\alpha - C_{mq}L\frac{r}{v} + c_m\delta_e, \end{aligned}$$

c_L , c_m and C_{xy} are positive aerodynamic coefficients that are specific to the considered aircraft, which are usually known till an interval of admissible values. It is required to stabilize the airspeed v and the flight path angle γ at their desired values v_d and γ_d chosen in the safe envelop.

Assuming for simplicity that the aerodynamic coefficients are close to their nominal values, it is straightforward to verify that for

$$\theta = \left[\frac{v}{m} \quad \frac{1}{m} \quad v^2 \quad v \quad \frac{C_D(\alpha)}{m} \quad \frac{\cos(\alpha)}{m} \quad \frac{\sin(\alpha)}{mv} \right]^\top$$

this system can be presented in the form (3) with $n = 4$, $m = 2$, $M = 1$, $x = [v \ \gamma \ \eta \ r]^\top$, $u = [F \ \delta_e]^\top$, $f^1(x) = [v|v| \ \gamma|\gamma| \ \eta|\eta| \ r|r]^\top$ (the sector condition is verified), $H_1 = I_4$ and

$$\begin{aligned} A_0(\theta) &= \begin{bmatrix} 0 & -g & 0 & 0 \\ C_{L0}\frac{\rho^S}{2}\theta_1 & -C_{L\alpha}\frac{\rho^S}{2}\theta_1 & C_{L\alpha}\frac{\rho^S}{2}\theta_1 & C_{Lq}\frac{\rho^S}{2}\theta_2 \\ 0 & 0 & 0 & 1 \\ 0 & C_{m\alpha}L\frac{\rho^S}{2J}\theta_3 & -C_{m\alpha}L\frac{\rho^S}{2J}\theta_3 & -C_{mq}L^2\frac{\rho^S}{2J}\theta_4 \end{bmatrix}, \\ A_1(\theta) &= \begin{bmatrix} -\frac{\rho^S}{2}\theta_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -C_{m0}L\frac{\rho^S}{2J} & 0 & 0 & 0 \end{bmatrix}, \quad B(\theta) = \begin{bmatrix} \theta_6 & 0 \\ \theta_7 & c_L\frac{\rho^S}{2}\theta_1 \\ 0 & 0 \\ \frac{z_e}{J} & c_mL\frac{\rho^S}{2J}\theta_3 \end{bmatrix}, \end{aligned}$$

where the perturbation signal

$$d(t) = g \begin{bmatrix} \gamma(t) - \sin(\gamma(t)) \\ -v^{-1}(t) \cos(\gamma(t)) \\ 0 \\ 0 \end{bmatrix} + w(t)$$

is composed by a nominal known part (the PVP model mismatch) and the external disturbance w . The set Θ from Assumption 1 can be easily derived from the given bounds for the angles and v_{\min} , v_{\max} , m_{\min} , m_{\max} . Assumption 3 can also be verified for a rather large value of q , thus, we prefer to avoid its use in this example. Note that the important nonlinearity v^2 presented in the original nonlinear model is preserved in the PVP representation.

For control design illustration, assume that the desired value of the pitch angle is $\eta_d = \gamma_d$ and for the pitch rate it is $r_d = 0$. After denoting the state position to stabilize as $x_d = [v_d \ \gamma_d \ \gamma_d \ 0]$, the control is then selected as

$$u(t) = K_0(x(t) - x_d) + K_1(f^1(x(t)) - f^1(x_d)) + u_0,$$

where $K_0 \in \mathbb{R}^{2 \times 4}$ is adjusted to assign the desired poles to the matrix $A_{0d} + B_d K_0$ (the system is controllable), with $A_{0d} = A_0(\theta_d)$, $B_d = B(\theta_d)$ and θ_d corresponds to the value of the scheduling parameters at x_d and mass $\frac{m_{\min} + m_{\max}}{2}$;

$$K_1 = -\kappa B_d^\top$$

for a tuning gain $\kappa > 0$; $u_0 \in \mathbb{R}^2$ is chosen to guarantee that x_d is an equilibrium in the closed-loop system. For the calculated values of the control gain, the matrix inequalities of Theorem

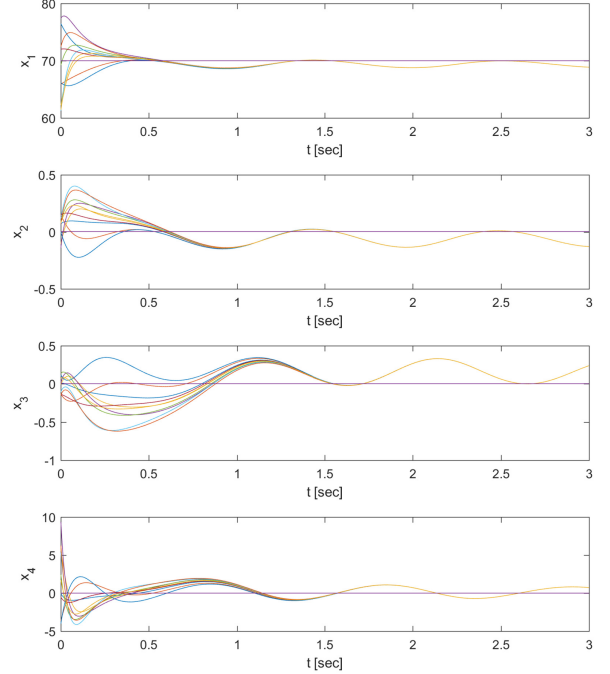


Figure 2. The results of simulation for controlled longitudinal aircraft model: $x(t)$ versus time t [sec]

3 can be used to check validity of the control in the given admissible envelop. Note that this control is nonlinear, and the term $K_1 f^1(x(t))$ introduces additional quadratic terms in the closed-loop system, which were not present in the original model.

For numerical simulation, the following parameter values inspired by [43] were taken:

$$\begin{aligned} v_{\min} &= 60, \quad v_{\max} = 80, \quad m_{\min} = 140000, \quad m_{\max} = 150000, \\ \rho &= 1.293, \quad L = 7.5, \quad S = 360, \quad g = 9.8, \quad z_e = 2, \quad J = 10^7, \\ C_{D0} &= 0.065, \quad C_{D\alpha} = 0.4, \quad C_{D\alpha^2} = 1.55, \quad C_{L0} = 0.019, \\ C_{L\alpha} &= 5.5, \quad C_{Lq} = 3.3, \quad c_L = 0.32, \\ C_{m0} &= 0.3, \quad C_{m\alpha} = 1.5, \quad C_{mq} = 12, \quad c_m = -1.2, \quad \kappa = 0.1, \end{aligned}$$

with $m(t) = \frac{m_{\min} + m_{\max}}{2} - 10t$, and

$$w(t) = \frac{1}{4} (\sin(\omega t) + \text{rand}(4, 1)), \quad \omega = [5 \dots 8]^\top$$

with $\text{rand}(4, 1)$ being a vector in \mathbb{R}_+^4 with elements uniformly distributed in the interval $[0, 1]$. For $v_d = 70$ and $\gamma_d = 0.003$ the matrix inequalities of Theorem 3 were verified on a random grid (with constant matrix functions Ω_g , $\Upsilon_{s,z}$). The simulation results for the closed-loop system are shown in Fig. 2, where each state component is plotted for ten trajectories with random initial conditions selected within the admissible bounds (and also one trajectory at the equilibrium in the disturbance-free case to confirm existence of the steady-state solution). As expected, the controlled system demonstrates an ISS behavior.

VIII. CONCLUSION

This work considered a class of Parameter-Varying Persidskii (PVP) systems, combining features of LPV and Persidskii systems to model complex dynamics with multiple nonlinearities and uncertainties. Conditions for ISS, iISS, and IOSS were derived using both parameter-independent and dependent Lyapunov functions and formulated as matrix inequalities. Imposing additional restrictions on the structure of the matrices appearing in the inequalities and the Lyapunov function, these conditions were reduced to an explicit set of LMI constraints. A robust control design methodology was also developed. Future directions for research include investigation of numeric aspects and relaxations for the presented matrix inequalities, extension of the passivity/dissipativity theory and the design of observers for PVP systems.

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