



**HAL**  
open science

# Generalized Hostadter functions $G$ , $H$ and beyond: numeration systems and discrepancy

Pierre Letouzey

► **To cite this version:**

Pierre Letouzey. Generalized Hostadter functions  $G$ ,  $H$  and beyond: numeration systems and discrepancy. 2025. hal-04948022v1

**HAL Id: hal-04948022**

**<https://inria.hal.science/hal-04948022v1>**

Preprint submitted on 14 Feb 2025 (v1), last revised 18 Feb 2025 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0  
International License

# GENERALIZED HOFSTADTER FUNCTIONS G, H AND BEYOND: NUMERATION SYSTEMS AND DISCREPANCY

PIERRE LETOUZEY

ABSTRACT. Hofstadter's G function is recursively defined via  $G(0) = 0$  and then  $G(n) = n - G(G(n-1))$ . Following Hofstadter, a family  $(F_k)$  of similar functions is obtained by varying the number  $k$  of nested recursive calls in this equation. We study here some Fibonacci-like sequences that are deeply connected with these functions  $F_k$ . In particular, the Zeckendorf theorem can be adapted to provide digital expansions via sums of terms of these sequences. On these digital expansions, the functions  $F_k$  are acting as right shifts of the digits. The Fibonacci-like sequences are then expressed in terms of zeros of the polynomial  $X^k - X^{k-1} - 1$ . Thanks to that, we estimate the discrepancy of each function  $F_k$ , i.e., the maximal distance between  $F_k$  and its linear equivalent. This discrepancy is finite exactly when  $k \leq 4$ . This also solves some related questions, in particular two twenty-year-old OEIS conjectures stating how close the functions  $F_3$  and  $F_4$  are from the integer parts of their linear equivalents. Moreover  $F_k$  can coincide exactly with such an integer part only when  $k \leq 2$ , while  $F_k$  is almost additive exactly when  $k \leq 4$ . Finally, a nice fractal shape a la Rauzy has been encountered when investigating the discrepancy of  $F_3$ .

## 1. INTRODUCTION

We continue here the exploration of a family of functions started in a previous article [12]. The properties studied here are largely independent from the content of the first article, so this second article can be read without prior knowledge of the first one. In particular the reader will find again below the necessary initial definitions and basic properties. That being said, we obviously recommend reading both. For instance, the previous discussion on related works is still relevant here and will not be duplicated.

**1.1. The functions.** As in our previous article [12], we consider the following function<sup>1</sup>:

**Definition 1.1.** For each integer  $k \geq 1$ , the function  $F_k$  is defined recursively by

$$F_k : \mathbb{N} \rightarrow \mathbb{N}, \quad n \mapsto \begin{cases} 0, & \text{if } n = 0; \\ n - F_k^k(n-1), & \text{otherwise.} \end{cases}$$

This function  $F_k$  is well defined since one may prove alongside that  $0 \leq F_k(n) \leq n$  for all  $n \geq 0$ . This family of functions is due to Hofstadter [9, Chapter 5]. In particular,  $F_2$  is Hofstadter's function  $G$ , see OEIS entry [A5206](#) [1, 9, 4, 8], known to satisfy  $G(n) = \lfloor (n+1)/\varphi \rfloor$  where  $\varphi$  is the golden ratio  $(1+\sqrt{5})/2$ . Similarly,  $F_3$  is Hofstadter's function  $H$ , see OEIS [A5374](#), and the generalization to higher degrees of recursive nesting in the definition of  $F_k$  was already suggested by Hofstadter [9]. To be complete, the OEIS database already includes  $F_4$  as [A5375](#) and  $F_5$  as [A5376](#) and  $F_6$  as [A100721](#). On the other hand, we choose to start this sequence with  $F_1$  where only one recursive call is done, leading to a function that can easily be shown to verify  $F_1(n) = \lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ . Throughout this paper, we never consider the case  $k = 0$ : although the previous definition

---

<sup>1</sup>In all this work, the  $n$ -th exponent of a function denotes its  $n$ -th iterate, for instance  $F_2^2(n)$  is  $F_2(F_2(n))$ , not the square of  $F_2(n)$ .

could be extended and give a non-recursive function  $F_0$ , this  $F_0$  has too little in common with the other  $F_k$  functions to be of much interest. Small values of the functions  $F_1$  to  $F_5$  are displayed in Figure 1.1.

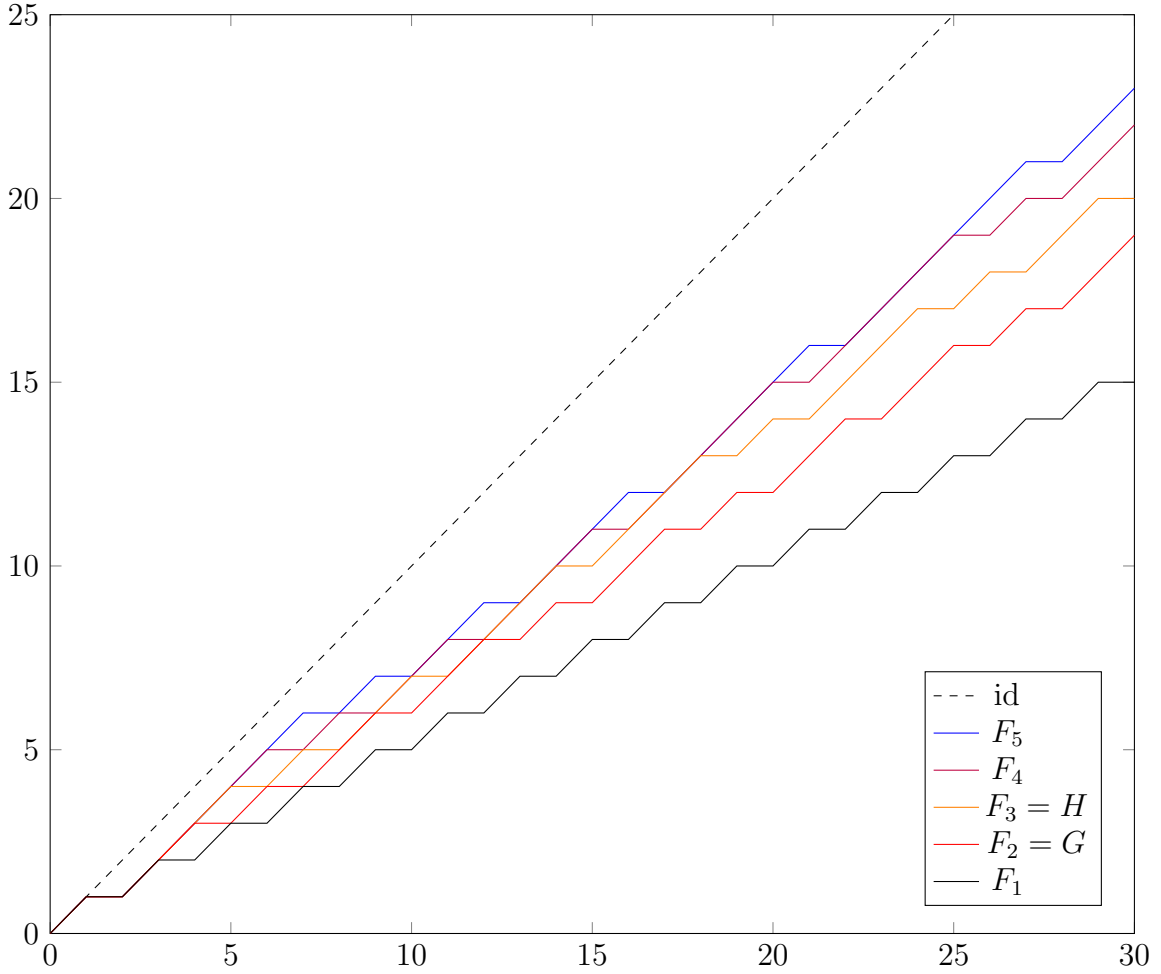


FIGURE 1.1. Plotting  $F_1, F_2, \dots, F_5$ .

**1.2. Summary of previous results.** The main result of our previous article [12] is the Theorem 7.4, stating that the  $(F_k)$  family of functions is ordered pointwise:  $F_k(n) \leq F_{k+1}(n)$  for all  $k \geq 1$  and  $n \geq 0$ . Even if we will not compare further the functions of this  $(F_k)$  family, two other aspects of the previous article will be relevant here.

First, we introduced there a family of substitutions<sup>2</sup>  $(\tau_k)$  and infinite morphic words  $(x_k)$  and length functions  $(L_k)$ :

**Definition 1.2.** For  $k \geq 1$ , let  $\tau_k$  be the substitution on the alphabet  $\{1, 2, \dots, k\}$  defined by

$$\begin{aligned} \tau_k : k &\mapsto k1, \\ i &\mapsto i+1 \quad \text{for } 1 \leq i < k. \end{aligned}$$

<sup>2</sup>A substitution (or morphism) on an alphabet  $A$  is a map  $\tau : A^* \rightarrow A^*$  satisfying  $\tau(uv) = \tau(u)\tau(v)$  for all  $u, v \in A^*$ , where  $A^*$  denotes the set of finite words with letters in  $A$ . The map  $\tau$  is therefore defined by its value on the letters of  $A$ , and it is extended in a natural way to infinite words (or sequences)  $w = w[0]w[1]\dots \in A^\infty$  by setting  $\tau(x) = \tau(x[0])\tau(x[1])\dots$ . For more on substitutions, see for instance [7].

Let  $x_k = x_k[0]x_k[1] \cdots \in \{1, 2, \dots, k\}^\infty$  be the fixed point of  $\tau_k$  and let  $L_k$  be the following function:

$$L_k : \mathbb{N} \rightarrow \mathbb{N}, \quad n \mapsto |\tau_k(x_k[0:n])|$$

where  $x_k[0:n]$  is the prefix of  $x_k$  of size  $n$ , i.e.,  $x_k[0] \cdots x_k[n-1]$ .

These notions were shown to be deeply related with  $F_k$ , in particular  $L_k$  forms a Galois connection with  $F_k$ , i.e., almost a bijection, more on that in Section 2 below. The main result of monotonicity for the  $(F_k)$  family was then converted to a similar statement on  $(L_k)$  and then proved by investigating the letter configurations in the morphic words  $(x_k)$ . In the present article, these lengths  $L_k$  will receive new descriptions, in particular as left shifts in some particular numeration systems in Section 4, while the words  $(x_k)$  will also be related with these numeration systems in Section 5.

Secondly, we studied in [12, Sect. 6] the infinitary behavior of  $F_k$  and showed in particular that  $\lim \frac{1}{n} F_k(n)$  exists and is the unique positive zero of the polynomial  $X^k + X - 1$ . Noting  $\alpha_k$  this zero, it amounts to say that  $F_k(n)$  admits  $\alpha_k n$  as linear equivalent. Once again, this result was obtained via a detour through the morphic words  $x_k$ , whose letter frequencies were expressed in terms of  $\alpha_k$ . Here, we go further and investigate in Section 8 the discrepancy of each  $F_k$ , i.e., the maximal distance between  $F_k$  and its linear equivalent. For that, the polynomial  $X^k + X - 1$  and its reciprocal  $X^k - X^{k-1} - 1$  are studied more carefully in Section 6.

**1.3. The Coq artifact.** Almost all the proofs presented in this article have been formally certified using the Coq/Rocq proof assistant [18]. The files of this Coq development are freely available [11], the recommended entry point to read alongside this article being

[https://github.com/letouzey/hofstadter\\_g/blob/main/Article2.v](https://github.com/letouzey/hofstadter_g/blob/main/Article2.v)

This formal development ensures precise definitions and statements and rules out any reasoning errors or rounding errors during the proofs. It can hence serve as a reference for the interested reader. The current article tries to be faithful to this formal work while staying readable by a large audience, at the cost of possible remaining mistakes during the transcription.

Our Coq development can be machine-checked again by any recent installation of Coq, see the joint README file (the authors used Coq version 8.16). All the first part up to Section 5 have been formalized within Coq core logic, without any extra axioms, as may be checked via the command `Print Assumptions` on our theorems. The parts corresponding to Section 6 and after involve real numbers and hence rely on some Coq standard libraries that declare four logical axioms, in particular the axiom of excluded middle.

Some differences remain between the proofs presented in this article and the corresponding Coq development:

- Some internal files in this Coq development do not use the same parameter  $k$  presented here. Instead, they rely on a parameter  $q$  actually meaning  $k-1$ . In particular, this allows to rule out the case  $k = 0$  without resorting to preconditions  $k \geq 1$  everywhere, since  $q$  belongs to the `nat` datatype encoding natural numbers. In particular, the definition of Fibonacci-like sequences `A` in file `GenFib.v` is quite lighter this way. The two files `Article2.v` and `Article1.v` serve as companion files to this article and the previous one, and hence use the parameter  $k$  just as here.
- In Proposition 6.8, we rely on the fact that the Plastic Number is the smallest Pisot number, an important result by Siegel [17]. Lacking a full Coq proof of

this result, we managed instead to formalize in Coq the first pages of the initial proof by Siegel and then adapted it to suit our need, namely that for  $k \geq 6$ , the polynomial  $X^k - X^{k-1} - 1$  does have secondary zeros of modulus strictly more than 1. More details can be found in our file `SecondRoots.v`.

- The Coq proofs corresponding to Theorems 8.2 and 8.5 only state that the discrepancies  $\Delta_k$  are infinite when  $k \geq 5$ , but do not express yet the rate of divergence, which is logarithmic for  $k = 5$  and a power function for  $k \geq 6$ . These extra details were not formalized in Coq due to lack of time, but no major difficulties is expected here should we try to do it later. For the current Coq proofs, see `Freq.delta_sup_qgen` and `LimCase4.delta_sup_q4`.
- Finally, the Appendix A has not been considered yet in Coq.

**1.4. Summary of the current article.** Section 2 recalls some earlier basic results about functions  $F_k$ . Then Section 3 investigates a family  $A_{k,p}$  of Fibonacci-like sequences and their relations with  $F_k$ . This is extended in Section 4 where numeration systems based on these  $A_{k,p}$  sequences are considered by generalizing the Zeckendorf theorem, then  $F_k$  is shown to be a right shift on these digital expansions (with a specific treatment of the lowest digit). Section 5 translates these results in term of infinite morphic words. In Section 6, we study the polynomials corresponding to all these recurrences, with a particular attention to the modulus of their zeros. Section 7 provides a detailed description of the Fibonacci-like  $A_{k,p}$  numbers as linear combinations of powers of these zeros. Finally, Section 8 assembles most of the previous pieces for a study of the discrepancy of each  $F_k$  i.e., the maximal distance between  $F_k$  and its linear equivalent. This discrepancy is shown to be finite exactly when  $k \leq 4$ , in particular we prove it to be less than 1 for  $k = 3$  and less than 2 for  $k = 4$ , before stating various consequences and encountering a nice fractal when  $k = 3$ .

## 2. BASIC PROPERTIES OF FUNCTION $F_k$

We recall now some earlier basic properties, in particular Proposition 2.1 of [12], where  $\partial F_k^j(n)$  denotes the difference  $F_k^j(n+1) - F_k^j(n)$ :

**Proposition 2.1.** *For all  $j, k \geq 1$ , the function  $F_k^j$  satisfies the following basic properties:*

- $F_k^j(1) = F_k^j(2) = 1$ ,
- $F_k^j(n) \geq 1$  whenever  $n \geq 1$ ,
- $F_k^j(n) < n$  whenever  $n \geq 2$ ,
- for all  $n > 0$ ,  $\partial F_k^j(n) = 1 - \partial F_k^k(n-1)$ ,
- hence  $\partial F_k^j(n) \in \{0, 1\}$  for all  $n \geq 0$ ,
- the function  $F_k^j$  is monotonically increasing and onto (but not one-to-one).

We also proved earlier [12, Prop. 4.2] that the function  $L_k$  introduced in Definition 1.2 can also be expressed in terms of  $F_k$ :

**Proposition 2.2.** *For  $k \geq 1$  and  $n \geq 0$ , we have  $L_k(n) = n + F_k^{k-1}(n)$ .*

Actually, this could have been an alternative definition for  $L_k$ , skipping all references of morphic words until Section 5. One way or the other, we have the following important relation between  $F_k$  and  $L_k$ :

**Proposition 2.3.** *For  $k \geq 1$  and  $n \geq 0$ , we have  $F_k(L_k(n)) = n$  and  $L_k(F_k(n)) \in \{n, n+1\}$ .*

*Proof.* See [12, Sect. 3]. Alternatively, here is a more direct proof, just relying on Propositions 2.1 and 2.2:

$$L_k(F_k(n)) = F_k(n) + F_k^k(n) = F_k(n) + (n+1 - F_k(n+1)) = n+1 - \partial F_k(n).$$

By Proposition 2.1,  $\partial F_k(n) \in \{0, 1\}$  hence  $L_k(F_k(n)) \in \{n, n+1\}$ .

For the first part of the statement, let us pick  $m$  such that  $F_k(m) = n$  ( $F_k$  is onto, see Proposition 2.1). Proceeding as before, but for  $m$  this time:

$$F_k(L_k(n)) = F_k(L_k(F_k(m))) = F_k(m+1 - \partial F_k(m))$$

Now, either  $\partial F_k(m) = 0$  and hence  $F_k(L_k(n)) = F_k(m+1) = F_k(m) = n$ . Or  $\partial F_k(m) = 1$  and hence  $F_k(L_k(n)) = F_k(m+1-1) = F_k(m) = n$ .  $\square$

**Corollary 2.4.** *For all  $k \geq 1$ , the functions  $F_k$  and  $L_k$  form a Galois connection between  $\mathbb{N}$  and itself (with  $F_k$  as left adjoint and  $L_k$  as right adjoint). Indeed, for all  $n, m \geq 0$  we have  $F_k(n) \leq m$  iff  $n \leq L_k(m)$ . Moreover, this Galois connection is said to be a Galois insertion since  $F_k \circ L_k = id$ .*

*Proof.* See [12, Cor. 3.2]. Or alternatively, by Proposition 2.1, the function  $F_k^{k-1}$  is increasing, and hence  $L_k(n) = n + F_k^{k-1}(n)$  is increasing as well. If  $F_k(n) \leq m$ , then  $L_k(F_k(n)) \leq L_k(m)$ . By Proposition 2.3,  $L_k(F_k(n)) \in \{n, n+1\}$  hence  $n \leq L_k(F_k(n)) \leq L_k(m)$ . Conversely,  $n \leq L_k(m)$  implies  $F_k(n) \leq F_k(L_k(m)) = m$ .  $\square$

In particular,  $L_k(n)$  is the rightmost antecedent of  $n$  by  $F_k$ . Note that there cannot exist more than two antecedents of a given point by  $F_k$ . Indeed, if  $F_k(a) = F_k(a+1)$  then  $F_k^k(a) = F_k^k(a+1)$  hence  $\partial F_k^k(a) = 0$  hence  $\partial F_k(a+1) = 1$  by Proposition 2.1, so  $F_k(a+2) > F_k(a)$ . Similarly, if  $F_k(a) = F_k(a-1)$  then  $a \neq 0$  and  $F_k(a-1) < F_k(a)$ , otherwise  $\partial F_k(a-1) = 0$  hence  $\partial F_k^k(a-1) = 0$  and  $\partial F_k(a) = 1$ .

### 3. RELATED FIBONACCI-LIKE RECURRENCE

A family of Fibonacci-like recursive sequences is of particular interest here. It is obtained by generalizing the Fibonacci recurrence in the following way: instead of adding the two previous terms of the sequence, we add the previous term and the  $k$ -th earlier term, for some  $k \geq 1$ . We will see below that these sequences could also be expressed as  $L_k^p(1)$ , and that  $F_k$  shifts down the corresponding sequence by one step.

**Definition 3.1.** For any integer  $k \geq 1$ , let  $(A_{k,p})_{p \in \mathbb{N}}$  be the following linear recurrence:

$$A_{k,p} = \begin{cases} p+1, & \text{if } 0 \leq p < k; \\ A_{k,p-1} + A_{k,p-k}, & \text{if } p \geq k. \end{cases}$$

p	0	1	2	3	4	5	6	7	8	9	10
$A_{1,p}$	1	2	4	8	16	32	64	128	256	512	1024
$A_{2,p}$	1	2	3	5	8	13	21	34	55	89	144
$A_{3,p}$	1	2	3	4	6	9	13	19	28	41	60
$A_{4,p}$	1	2	3	4	5	7	10	14	19	26	36
$A_{4,p}$	1	2	3	4	5	6	8	11	15	20	26

FIGURE 3.1. Initial values of  $A_{k,p}$ .

Hence  $A_{1,p} = 2^p$  for all  $p \geq 0$  and  $(A_{2,p})$  is the Fibonacci sequence with initial values 1 2 3 5 8 (no initial 0, only one 1). The sequence  $(A_{3,p})$  is also known as Narayana's Cows, see OEIS [A930](#) but with an index shift. Moreover, this whole family of recursive sequences

has already been considered many times [13, 10, 6]. Several of these earlier papers also use the  $A_{k,p}$  numbers as basis for alternative digital expansions, as done in the following Section 4. But we are not aware of any prior mathematical analysis of these  $A_{k,p}$  numbers as thorough as the one of Section 7, crucial for our results about discrepancy of functions  $F_k$  in Section 8.

Note that these  $A_{k,p}$  numbers should not be confused with the number of ordered  $k$ -arrangements of  $n$ , which is denoted  $A_n^k$  in French texts. Interestingly, the  $A_{k,p}$  numbers also have a nice combinatorial interpretation:

**Proposition 3.2.** *For  $k \geq 1$  and  $p \geq 0$ ,  $A_{k,p}$  is the number of subsets of  $\{1, \dots, p\}$  with a distance of at least  $k$  between elements.*

*Proof.* Let  $k \geq 1$ . We proceed by strong induction over  $p$ . For  $p < k$ , the distance constraint ensures that the considered subsets are either empty or singleton, so we have indeed  $p+1$  such subsets. And for  $p \geq k$ , we partition into the subsets that contain  $p$  and the ones that do not. By induction hypothesis, the first group has cardinal  $A_{k,p-k}$  (since the second-largest element of these subsets is at most  $p-k$  in this case), while the second group has cardinal  $A_{k,p-1}$ , hence a total cardinal of  $A_{k,p}$ .  $\square$

Obviously, all  $A_{k,p}$  are strictly positive, hence for a given  $k$ ,  $(A_{k,p})$  is strictly increasing over  $p$ . Now, as an easy consequence of the previous combinatorial interpretation, for a given  $p \geq 1$  the sequence  $(A_{k,p})_{k>0}$  is decreasing from  $A_{1,p} = 2^p$  till  $A_{p,p} = p+1$  and stationary afterwards. Indeed, the rule  $A_{k,p} = p+1$  for initial terms actually holds even when  $p = k$ , since  $A_{p,p} = A_{p,p-1} + A_{p,p-p} = p+1$ .

Interestingly, we can also extend the recurrence rule and make it handle all terms except  $A_{k,0}$ :

$$(3.1) \quad A_{k,p} = A_{k,p-1} + A_{k,p \ominus k} \quad \text{for all } p \geq 1$$

where  $a \ominus b = \max(0, a - b)$  is the total subtraction from  $\mathbb{N}^2$  to  $\mathbb{N}$  that rounds negative results to 0.

As promised, the functions  $F_k$  and  $L_k$  have a specific behavior on these  $A_{k,p}$  numbers.

**Proposition 3.3.** *For all  $k \geq 1$  and  $p, j \geq 0$ ,*

$$\begin{aligned} F_k^j(A_{k,p}) &= A_{k,p \ominus j} \\ L_k^j(A_{k,p}) &= A_{k,p+j}. \end{aligned}$$

Hence  $A_{k,p}$  can also be expressed as  $L_k^p(1)$ .

*Proof.* We prove the first equation by induction over  $p$ . The case  $p = 0$  is obvious since  $F_k^j(1) = 1$ . Now let  $p > 0$  and assume the equation is true for  $p - 1$  and for all  $j \geq 0$ . Of course  $F_k^0(A_{k,p}) = A_{k,p+0}$ . Now, for  $j \geq 0$ , note that here  $p \ominus (j+1) = p-1 \ominus j$  and  $p \ominus k = (p-1) \ominus (k-1)$ , hence

$$\begin{aligned} F_k^{j+1}(A_{k,p}) &= F_k^j(F_k(A_{k,p-1} + A_{k,p \ominus k})) && \text{(by Eq. (3.1))} \\ &= F_k^j(F_k(A_{k,p-1} + F_k^{k-1}(A_{k,p-1}))) && \text{(by I.H.)} \\ &= F_k^j(F_k(L_k(A_{k,p-1}))) \\ &= F_k^j(A_{k,p-1}) && \text{(by Prop. 2.3)} \\ &= A_{k,p-1 \ominus j} && \text{(by I.H.)} \\ &= A_{k,p \ominus (j+1)} \end{aligned}$$

Now, from the definition of  $L_k$  and Equation (3.1):

$$L_k(A_{k,p}) = A_{k,p} + F_k^{k-1}(A_{k,p}) = A_{k,p} + A_{k,p \ominus (k-1)} = A_{k,p} + A_{k,(p+1) \ominus k} = A_{k,p+1}.$$

This extends to  $L_k^j$  via a direct induction on  $j$ . Finally,  $L_k^p(1) = L_k^p(A_{k,0}) = A_{k,p}$ .  $\square$

This last proposition will be generalized in the next Section, where Theorem 4.10 and Proposition 4.12 establish the behavior of  $F_k^j$  and  $L_k^j$  on some sums of  $A_{k,p}$  numbers.

#### 4. FIBONACCI-LIKE DIGITAL EXPANSIONS

For a given  $k \geq 1$ , we can use the Fibonacci-like numbers  $(A_{k,p})$  seen in section 3 as a base for a family of digital expansions. In these numeration systems, we will see that  $L_k$  and  $F_k$  behave as bitwise operations, namely left and right shifts (with a particular treatment of the lowest digit in the case of  $F_k$ ). This extends Proposition 3.3.

Such a digital expansion can be described in a standard way through sequences of digits  $(d_i)_{i \in \mathbb{N}}$  that are ultimately null, leading to sums  $\sum_{i=0}^{\infty} d_i A_{k,i}$ . As usual for such a digital expansion, we may display a number via its digits, with the least-significant digit on the right. Nonetheless, we favor here an alternative presentation based on the positions of the non-null digits, calling this a  $k$ -decomposition. In the rare situations where some digits will be 2 or more, we repeat the corresponding positions. This approach alleviates the technical lemmas to come.

**Definition 4.1.** For  $k \geq 1$ , a  $k$ -decomposition is a finite sequence  $D = p_0 \cdots p_m$  of increasing natural numbers, with possible repetitions. Such a sequence may be empty, noted  $D = \emptyset$ . The sum of a  $k$ -decomposition  $D = p_0 \cdots p_m$  is  $\Sigma_k(D) = A_{k,p_0} + \cdots + A_{k,p_m}$ . For  $n \geq 0$ , a  $k$ -decomposition of  $n$  is any  $k$ -decomposition  $D$  such that  $\Sigma_k(D) = n$ . A  $k$ -decomposition  $D$  will be said *canonical* (resp., *lax*) when the distance between the elements of  $D$  is at least  $k$  (resp., at least  $k-1$ ).

Note that for  $k \geq 1$ , the  $k$ -decompositions that are canonical cannot contain repeated positions, and can hence be viewed as simple sets of natural numbers (with sufficient distance between them). For simplicity sake, we (ab)use set notation for these  $k$ -decompositions, when this remains clear enough, for instance  $D \cup \{p\}$  or  $D \setminus \{p\}$  for adding position  $p$  to a decomposition  $D$  (or removing it). Nonetheless, the exact description remains here through finite sequences, allowing the rare but complex situations where repeated positions are needed, see in particular the Definition 4.5 below.

As an example, let us try  $k = 2$ , i.e., decompose in sums of Fibonacci numbers: for instance the number 17 is equal to  $13 + 3 + 1 = A_{2,5} + A_{2,2} + A_{2,0}$  so 17 have a digit representation of 100101, corresponding to a (canonical) 2-decomposition of 0, 2, 5. This decomposition is indeed canonical since the positions are separated by at least 2. Other decompositions are possible, for instance  $17 = 8 + 5 + 3 + 1 = 11101$ , i.e., the lax 2-decomposition 0, 2, 3, 4. Considering higher digits allows more decompositions, for example  $17 = 2 * 8 + 1 = 20001$ , i.e., the 2-decomposition 0, 4, 4. This one is not canonical nor lax, and will hence be avoided as much as possible. Now, if we switch to  $k = 3$ , the same example 17 is now  $13 + 4 = A_{3,6} + A_{3,3} = 1001000$ , hence the canonical 3-decomposition 3, 6. Note that for  $k = 1$ , a canonical 1-decomposition is actually a sum of distinct powers of two, i.e., the usual base-2 representation, for instance  $17 = 16 + 1 = 10001$ . In this case  $k = 1$ , considering lax decompositions already implies having digits of 2 or more, i.e., repeated positions.

The least digit has always the same weight independently of  $k$ , since  $A_{k,0} = 1$ . Similarly, the second digit always weights  $A_{k,1} = 2$ . All other digits have weights that depend on  $k$ , for instance  $A_{k,2} \in \{3, 4\}$  and  $A_{k,3} \in \{4, 5, 8\}$ . For a given  $k$ , the corresponding



numeration system is clearly redundant, but with unique canonical decomposition. Indeed, Zeckendorf's theorem can be adapted for this generalized setting, as already seen for instance by Kimberling [10] or Eriksen and Anderson [6]:

**Theorem 4.2** (Zeckendorf). *For  $k \geq 1$  and  $n \geq 0$ , there exists a unique canonical  $k$ -decomposition of  $n$ . We note it  $D_k(n)$ .*

*Proof.* The proof of this theorem is quite standard. First we build by strong induction a canonical  $k$ -decomposition of  $n$ . For  $n = 0$  we take the empty decomposition, whereas for  $n > 0$  we consider first the highest position  $p$  such that  $A_{k,p} \leq n$ , then build recursively a canonical  $k$ -decomposition  $D$  of the rest  $n - A_{k,p}$ . Now  $D \cup \{p\}$  is a satisfactory decomposition of  $n$ . Indeed, the choice criterion of  $p$  implies  $A_{k,p} \leq n < A_{k,p+1}$ . If  $i < k$ , then  $A_{k,p+1} = 1 + A_{k,p}$  and hence  $n$  is exactly  $A_{k,p}$  and the decomposition stops there, with a singleton decomposition which is obviously correct and canonical. Otherwise  $i \geq k$  and so  $A_{k,p+1} = A_{k,p} + A_{k,p+1-k}$  hence  $n - A_{k,p} < A_{k,p+1-k}$ . So the largest index in the decomposition  $D$  cannot be  $p+1-k$  or more, so  $D \cup \{p\}$  is still canonical. Repeating this will hence build a decomposition that is indeed correct and canonical.

For proving the uniqueness, the key ingredient is the following: if a non-empty canonical decomposition  $D$  of  $n$  has  $p$  for highest position, then  $n < A_{k,p+1}$ . This is provable by induction on the cardinal of  $D$ , and then considering the sub-decomposition  $D \setminus \{p\}$ . With this key fact, one can easily prove that two canonical decompositions of the same number must have the same highest positions, and then recursively the same other elements.  $\square$

In general, a number may admit several lax  $k$ -decompositions beside the canonical one. Let us call  $\text{norm}_k(D)$  the unique canonical  $k$ -decomposition having the same sum as  $D$ . Obviously, it can be computed by sum and decompose:  $\text{norm}_k(D) = D_k(\Sigma_k(D))$ . When  $D$  is lax, a more efficient algorithm is to "repair" this almost canonical decomposition: we repeatedly find the highest pair of indices in  $D$  at distance  $k-1$ , say  $p-1$  and  $p-k$ , and replace them by  $p$ . On  $\Sigma_k(D)$ , this amounts to change terms  $A_{k,p-1} + A_{k,p-k}$  with  $A_{k,p}$ . The choice of the highest possible pair implies that the new decomposition is at least lax again, while having a strictly lower cardinal than  $D$ . Repeating the process will hence stop on a canonical decomposition of equal sum, in no more steps than the cardinal of  $D$ .

Let us express now the decomposition of  $n+1$  in terms of the decomposition of  $n$ . For that, we need to discriminate on the least element of this decomposition.

**Definition 4.3.** The rank of a decomposition  $D$  is the least element of  $D$ , or  $+\infty$  when  $D = \emptyset$ . The  $k$ -rank of a number  $n$  is the rank of  $D_k(n)$ .

**Proposition 4.4.** *For  $k \geq 1$  and  $n \geq 0$ , let us call  $r$  the  $k$ -rank of  $n$ . The canonical  $k$ -decomposition of  $n+1$  is*

$$D_k(n+1) = \begin{cases} D_k(n) \cup \{0\}, & \text{if } r \geq k; \\ \text{norm}_k(D_k(n) \setminus \{r\} \cup \{r+1\}), & \text{otherwise.} \end{cases}$$

We now generalize some bitwise operations of base 2.

**Definition 4.5.** For a  $k$ -decomposition  $D = p_0 \cdots p_m$  and  $q \geq 0$ , the left shift of  $D$  by  $q$  is adding  $q$  to all positions, i.e.,

$$D \ll q = (p_0+q) \cdots (p_m+q).$$

The upper right shift of  $D$  by  $q$  is

$$D \gg_+ q = (p_0 \ominus q) \cdots (p_m \ominus q).$$

Finally, the (usual) right shift of  $D$  by  $q$  is obtained by considering only the positions  $p_\ell \cdots p_m$  in  $D$  that are greater or equal to  $q$  (if any), and then:

$$D \gg q = (p_\ell - q) \cdots (p_m - q).$$

Recall that  $p \ominus q = \max(0, p - q)$ , so an upper right shift may produce repeated position 0, even when starting from a canonical decomposition. At least  $D \gg_+ 1$  is known to be lax when  $D$  is a canonical  $k$ -decomposition with  $k \geq 1$ , while  $D \ll q$  and  $D \gg q$  are still canonical. Still for a canonical  $k$ -decomposition  $D$ ,  $D \gg_+ 1$  is either equal to  $D \gg 1$  or differ by an extra initial 0, depending on whether  $D$  contains 0 or not. In the particular case  $k = 1$ ,  $D_k(n) \gg 1$  and  $D_k(n) \gg_+ 1$  correspond respectively to  $\lfloor n/2 \rfloor$  and  $\lceil n/2 \rceil$ .

Meek and Van Rees [13] showed that the right shift  $\gg 1$  on  $k$ -decompositions is actually a recursive function very similar to  $F_k$ , with just an extra  $-1$  in the recursive definition.

**Definition 4.6.** For  $k \geq 1$ , let  $\tilde{F}_k$  be defined recursively by

$$\tilde{F}_k : \mathbb{N} \rightarrow \mathbb{N}, \quad n \mapsto \begin{cases} 0, & \text{if } n = 0; \\ n - 1 - \tilde{F}_k^k(n-1), & \text{otherwise.} \end{cases}$$

As noticed by Meek and Van Rees,  $\tilde{F}_k$  is a translated version of  $F_k$ .

**Proposition 4.7.** For all  $k \geq 1$  and  $n \geq 0$ ,  $\tilde{F}_k(n) = F_k(n+1) - 1$ .

Here comes the result of Meek and Van Rees in our notation.

**Theorem 4.8** (Meek and Van Rees). For all  $k \geq 1$  and  $n \geq 0$ ,  $D_k(\tilde{F}_k(n)) = D_k(n) \gg 1$  hence  $\tilde{F}_k(n) = \Sigma_k(D_k(n) \gg 1)$ .

Combining these two last results, it is possible to prove that  $F_k$  is actually the upper right shift  $\gg_+ 1$ . Instead, we propose here a direct proof of this fact, without translation to  $\tilde{F}_k$ . The downside of this direct proof is that lax decompositions have to be considered, hence the following technical lemma.

**Proposition 4.9.** Let  $k \geq 1$ . For a lax  $k$ -decomposition  $D$  and a number  $q < k$ , we have

$$\Sigma_k(\text{norm}_k(D) \gg_+ q) = \Sigma_k(D \gg_+ q)$$

*Proof.* The normalization repeatedly replaces pairs of positions such as  $p-1, p-k$  into  $p$ , and exploits the equation  $A_{k,p-1} + A_{k,p-k} = A_{k,p}$  for keeping the overall sum unchanged. Here,  $p-k$  is a legal position, hence  $p \geq k > q$  and so  $p \ominus q = p - q \geq 1$ . Now, if we shift by  $\gg_+ q$  before summing, the corresponding equation  $A_{k,p-1 \ominus q} + A_{k,p-k \ominus q} = A_{k,p \ominus q}$  still hold. Indeed, Equation (3.1) for  $p \ominus q$  gives  $A_{k,p \ominus q} = A_{k,p \ominus q - 1} + A_{k,p \ominus q \ominus k}$  while here  $p \ominus q \ominus k = p \ominus k \ominus q$  and  $p \ominus q - 1 = p - 1 \ominus q$ .  $\square$

**Theorem 4.10.** For all  $k \geq 1$  and  $n \geq 0$ ,

$$F_k(n) = \Sigma_k(D_k(n) \gg_+ 1).$$

More generally, for all  $1 \leq q \leq k$ ,  $F_k^q(n) = \Sigma_k(D_k(n) \gg_+ q)$ .

*Proof.* The case  $k = 1$  is a consequence of the identity  $F_1(n) = \lfloor n/2 \rfloor$  already mentioned earlier. We now assume  $k \geq 2$ . Let us note  $G_k(n) = \Sigma_k(D_k(n) \gg_+ 1)$ . We first prove that

$$(4.1) \quad G_k^q(n) = \Sigma_k(D_k(n) \gg_+ q)$$

for all  $1 \leq q \leq k$  and  $n \geq 0$ . The case  $q = 1$  is straightforward. Let  $1 \leq q < k$  and assume (4.1) for this  $q$  and all  $n$ . By noting  $D = D_k(n) \gg_+ 1$  we have

$$G_k^{q+1}(n) = G_k^q(G_k(n)) = \Sigma_k(D_k(\Sigma_k(D))) \gg_+ q$$

Due to its definition,  $D$  is at least a lax decomposition, hence  $D_k(\Sigma_k(D)) = \text{norm}_k(D)$  and by Proposition 4.9 we obtain

$$G_k^{q+1}(n) = \Sigma_k(D \gg_+ q) = \Sigma_k(D_k(n) \gg_+ 1 \gg_+ q) = \Sigma_k(D_k(n) \gg_+ (q+1)).$$

For the last equality, note the law  $a \oplus b \oplus c = a \oplus (b + c)$ , hence two upper right shifts may be grouped into a single one. By induction on  $q$ , this concludes the proof of (4.1).

Now for proving  $F_k = G_k$  we show that  $G_k(0) = 0$  (which is obvious) and that  $G_k$  satisfies the same recursive equation as  $F_k$ . The equation about  $F_k^q$  then follows from (4.1). So let  $n \geq 0$  and consider  $G_k(n+1) + G_k^k(n)$ . If  $n = 0$  this sum is clearly  $1 = n+1$ . Assume now  $n > 0$  and note  $r = \text{rank}_k(n)$  and  $D = D_k(n) \setminus \{r\}$ . First, if  $r < k$ , then  $D_k(n+1) = \text{norm}_k(D \cup \{r+1\})$ , so

$$\begin{aligned} G_k(n+1) + G_k^k(n) &= \Sigma_k(\text{norm}_k(\{r+1\} \cup D) \gg_+ 1) + \Sigma_k(\{\{r\} \cup D\} \gg_+ k) \\ &= \Sigma_k(\{\{r+1\} \cup D\} \gg_+ 1) + \Sigma_k(\{\{r\} \cup D\} \gg_+ k) \quad (\text{by Prop. 4.9}) \\ &= A_{k,r+1 \oplus 1} + A_{k,r \oplus k} + \Sigma_k(D \gg_+ 1) + \Sigma_k(D \gg_+ k) \\ &= A_{k,r} + A_{k,0} + \Sigma_k(D) \\ &= 1 + \Sigma_k(\{r\} \cup D) \\ &= 1 + n \end{aligned}$$

We exploited here the fact that all elements in  $D$  are at least  $k+r$  hence non-zero. So  $\Sigma_k(D \gg_+ 1) + \Sigma_k(D \gg_+ k)$  is  $\Sigma_k(D)$  via repeated use of Equation 3.1.

Otherwise rank  $r \geq k$  and  $D_k(n+1) = \{0\} \cup D_k(n)$  and

$$\begin{aligned} G_k(n+1) + G_k^k(n) &= \Sigma_k(\{\{0\} \cup D_k(n)\} \gg_+ 1) + \Sigma_k(D_k(n) \gg_+ k) \\ &= A_{k,0 \oplus 1} + \Sigma_k(D_k(n) \gg_+ 1) + \Sigma_k(D_k(n) \gg_+ k) \\ &= 1 + \Sigma_k(D_k(n)) \\ &= 1 + n \end{aligned}$$

This time, it is  $D_k(n)$  whose elements are non-zero, and we handled the sum of its shifts just as before.  $\square$

As noticed earlier,  $D_k(n) \gg_+ 1$  is at least a lax  $k$ -decomposition, hence a renormalization may be necessary when computing the canonical  $k$ -decomposition of  $F_k(n)$ :

$$D_k(F_k(n)) = \text{norm}_k(D_k(n) \gg_+ 1).$$

As an alternative formulation,  $F_k(n)$  is also the (usual) right shift of the  $k$ -decomposition of  $n$  (i.e.,  $\tilde{F}_k(n)$ ), plus the least digit of this decomposition of  $n$  (i.e., its “ $k$ -parity”). When this digit is 1, this may trigger a renormalization of the decomposition.

We now relate the  $k$ -rank and the differences  $\partial F_k^q$ . From Section 2, we recall that  $\partial F_k^q(n)$  denotes  $F_k^q(n+1) - F_k^q(n)$  and that it is always either 0 or 1.

**Theorem 4.11.** *For all  $k \geq 1$  and  $n \geq 0$  and  $0 \leq q \leq k$ , we have  $\partial F_k^q(n) = 0$  iff  $\text{rank}_k(n) < q$ . In particular  $\partial F_k(n) = 0$  iff  $\text{rank}_k(n) = 0$ .*

*Proof.* First, for  $q = 0$ ,  $\partial F_k^0(n) = 1 \neq 0$  and  $\text{rank}_k(n) \not< 0$ . Moreover, for  $q = k = 1$ ,  $F_1(n) = \lceil n/2 \rceil$  and  $\partial F_1(n) = 0$  when  $n$  is odd which is indeed when  $\text{rank}_1(n) = 0$ .

Consider now  $1 \leq q < k$ , leaving away the case  $q = k$  for the moment to be able to benefit from Proposition 4.9.

If  $n = 0$ ,  $F_k^q(1) = 1 \neq 0 = F_k^q(0)$  and  $\text{rank}_k(0) = +\infty > q$ .

If  $n > 0$  and  $\text{rank}_k(n) < k$ , we reuse the notation of the last proof:  $r = \text{rank}_k(n)$  and  $D = D_k(n) \setminus \{r\}$ . The last theorem states that

$$\begin{aligned} \partial F_k^q(n) &= \Sigma_k(D_k(n+1) \gg_+ q) - \Sigma_k(D_k(n) \gg_+ q) \\ &= \Sigma_k(\text{norm}_k(\{r+1\} \cup D) \gg_+ q) - \Sigma_k(\{\{r\} \cup D\} \gg_+ q) \\ &= \Sigma_k(\{\{r+1\} \cup D\} \gg_+ q) - \Sigma_k(\{\{r\} \cup D\} \gg_+ q) \quad (\text{by Prop. 4.9}) \\ &= A_{k,r+1 \ominus q} - A_{k,r \ominus q} \end{aligned}$$

This is indeed 1 when  $q \leq r < k$ , since  $r+1 \ominus q = 1 + (r \ominus q) < k$ . Otherwise  $r < q$  and  $r+1 \ominus q = 0 = r \ominus q$  hence  $\partial F_k^q(n) = 0$ .

If  $n > 0$  and  $\text{rank}_k(n) \geq k$  and in particular  $\text{rank}_k(n) \geq q$ :

$$\begin{aligned} \partial F_k^q(n) &= \Sigma_k(D_k(n+1) \gg_+ q) - \Sigma_k(D_k(n) \gg_+ q) \\ &= \Sigma_k(\{\{0\} \cup D_k(n)\} \gg_+ q) - \Sigma_k(D_k(n) \gg_+ q) \\ &= A_{k,0} \\ &= 1 \end{aligned}$$

Finally, for the case  $q = k$ , we use the identity  $\partial F_k^k(n) = 1 - \partial F_k(n+1)$ . We have just shown that  $\partial F_k(n+1) = 0$  iff  $\text{rank}_k(n+1) < 1$ . From Proposition 4.4, we can see that  $\text{rank}_k(n+1) = 0$  iff  $\text{rank}_k(n) \geq k$ . Hence  $\partial F_k^k(n) = 0$  iff  $\text{rank}_k(n) < k$ .  $\square$

**Proposition 4.12.** *For all  $k \geq 1$  and  $n, q \geq 0$ , we have  $L_k^q(n) = \Sigma_k(D_k(n) \ll q)$  and hence  $D_k(L_k^q(n)) = D_k(n) \ll q$ .*

*Proof.* First, for  $q = 1$ , thanks to Proposition 2.2 and Theorem 4.10 and Equation (3.1):

$$\begin{aligned} L_k(n) &= n + F_k^{k-1}(n) \\ &= \Sigma_k(D_k(n)) + \Sigma_k(D_k(n) \gg_+ (k-1)) \\ &= \sum_{p \in D_k(n)} (A_{k,p} + A_{k,p \ominus (k-1)}) \\ &= \sum_{p \in D_k(n)} A_{k,p+1} \\ &= \Sigma_k(D_k(n) \ll 1) \end{aligned}$$

Since  $D_k(n) \ll 1$  is still a canonical  $k$ -decomposition, whose sum is equal to  $L_k(n)$ , we can conclude  $D_k(L_k(n)) = D_k(n) \ll 1$  by Theorem 4.2.

This extends easily by induction on  $q$ . It is clear for  $q = 0, 1$ . Now, assume  $L^q(n) = \Sigma_k(D_k(n) \ll q)$  for a specific  $q$  and all  $n$ . Then:

$$L_k^{q+1}(n) = L_k^q(L_k(n)) = \Sigma_k(D_k(L_k(n)) \ll q) = \Sigma_k(D_k(n) \ll 1 \ll q)$$

and we can conclude by regrouping the two successive left shifts.  $\square$

## 5. DECOMPOSITIONS AND MORPHIC WORDS

We already presented in Definition 1.2 the substitution  $\tau_k$  and the infinite morphic word  $x_k$  that are intimately related with each function  $F_k$ . In this section we show that these words  $x_k$  can also be expressed in term of  $k$ -decomposition and  $k$ -rank. Before that, we recall an important link between functions  $F_k$  and words  $x_k$ , see [12, Prop. 4.4]. Note that in the word  $x_k$ , positions are indexed from 0.

**Proposition 5.1.** Consider  $1 \leq j < k$  and  $n \geq 0$ . We have  $x_k[n] = j$  if and only if both  $\partial F_k^{j-1}(n) = 1$  and  $\partial F_k^j(n) = 0$ . Moreover for  $k \geq 1$  we have  $x_k[n] = k$  iff  $\partial F_k^{k-1}(n) = 1$  (in this case  $\partial F_k^k(n)$  could be either 0 or 1).

From that, the letters of the word  $x_k$  can be expressed in terms of  $k$ -rank.

**Theorem 5.2.** For  $k \geq 1$  and  $n \geq 0$  we have  $x_k[n] = \min(k, 1 + \text{rank}_k(n))$ .

*Proof.* Combination of Proposition 5.1 and Theorem 4.11.  $\square$

We also recall that the substitution  $\tau_k$  satisfies a recursive equation extending the recursive rule of  $A_{k,p}$  numbers, see [12, Prop. 2.2].

**Proposition 5.3.** For all  $k \geq 1$  and  $j \geq 0$ , either  $j \leq k$  and  $\tau_k^j(k) = k1 \cdots j$ , or  $j \geq k$  and  $\tau_k^j(k) = \tau_k^{j-1}(k)\tau_k^{j-k}(k)$ . Hence the following equalities on words and lengths:

- $x_k[0:A_{k,j}] = \tau_k^j(k)$
- $A_{k,j} = |\tau_k^j(k)| = L_k^j(1)$ .

*Proof.* Routine induction.  $\square$

The Theorem 4.2 can be extended to express decompositions of prefixes of words  $x_k$ .

**Definition 5.4.** For a  $k$ -decomposition  $D = p_0 \cdots p_m$ , we define the corresponding word  $W_k(D)$  as

$$W_k(D) = \tau_k^{p_m}(k) \cdots \tau_k^{p_0}(k).$$

Note that  $|W_k(D)| = \Sigma_k(D)$ . We recall from our conventions that the positions  $p_0 \cdots p_m$  are increasing:  $p_0 \leq \cdots \leq p_m$ . Even more, they are strictly increasing (by steps of at least  $k$ ) for canonical  $k$ -decompositions.

**Theorem 5.5.** For  $k \geq 1$  and  $n \geq 0$ , the prefix of  $x_k$  of size  $n$  is  $x_k[0:n] = W_k(D_k(n))$ . Moreover, for  $q \geq 0$ , we have  $\tau_k^q(x_k[0:n]) = W_k(D_k(n) \ll q)$ .

*Proof.* We follow the existence part of Theorem 4.2. Once again we proceed by strong induction over  $n$ . If  $n = 0$ , then indeed  $x_k[0:0] = \varepsilon = W_k(\emptyset) = W_k(D_k(0))$ . Otherwise for  $n > 0$  let  $p$  be the highest position such that  $A_{k,p} \leq n < A_{k,p+1}$ . If  $p < k$ , then  $A_{k,p+1} = 1 + A_{k,p}$  and hence  $n = A_{k,p}$  and  $D_k(n) = \{p\}$ . So by Proposition 5.3,

$$x_k[0:n] = \tau_k^p(k) = W_k(\{p\}) = W_k(D_k(n)).$$

Otherwise  $p \geq k$ . Since  $A_{k,p} \leq n < A_{k,p+1}$ , Proposition 5.3 implies that  $\tau_k^p(k)$  is a prefix of  $x_k[0:n]$  which is a prefix of  $\tau_k^{p+1}(k) = \tau_k^p(k)\tau_k^{p-k}(k)$ . That means that  $x_k[0:n]$  can be written  $\tau_k^p(k)w$  where  $w$  is a prefix of  $\tau_k^{p-k}(k)$  and hence a prefix of  $x_k$ . Hence  $w = x_k[0:m]$  if we call  $m$  the desired length of  $|w|$ , i.e.,  $m = n - A_{k,p}$ . But here  $D_k(n) = \{p\} \cup D_k(m)$  and  $m < n$  and  $p$  is strictly above all positions in  $D_k(m)$ . By induction hypothesis,  $x_k[0:m] = W_k(D_k(m))$  and finally

$$x_k[0:n] = \tau_k^p(k)W_k(D_k(m)) = W_k(\{p\} \cup D_k(m)) = W_k(D_k(n)).$$

Now, for any  $k$ -decomposition  $D$ , we have  $\tau_k^q(W_k(D)) = W_k(D \ll q)$  by distributing the substitutions  $\tau_k^q$  into the components of  $W_k(D)$ . Hence

$$\tau_k^q(x_k[0:n]) = \tau_k^q(W_k(D_k(n))) = W_k(D_k(n) \ll q).$$

$\square$

## 6. RELATED POLYNOMIALS AND ALGEBRAIC INTEGERS

For a finer description of the Fibonacci-like  $(A_{k,p})$  numbers in Section 7, we investigate now some corresponding polynomials. This is an extension of our previous work [12, Sect. 5], where we already presented the following polynomials and their positive zeros.

**Definition 6.1.** For  $k \geq 1$ , we name  $P_k(X) = X^k + X - 1$  and  $Q_k(X) = X^k - X^{k-1} - 1$ . We name  $\alpha_k$  (resp.,  $\beta_k$ ) the unique positive zero of  $P_k$  (resp.,  $Q_k$ ). Note that  $\beta_k = 1/\alpha_k$  since  $P_k$  and  $Q_k$  are reciprocal polynomials.

**Proposition 6.2.**  $(\alpha_k)_{k \in \mathbb{N}_+}$  is a strictly increasing sequence in  $[\frac{1}{2}, 1)$  while  $(\beta_k)_{k \in \mathbb{N}_+}$  is a strictly decreasing sequence in  $(1, 2]$ . Moreover  $1 + \frac{1}{k} \leq \beta_k \leq 1 + \frac{1}{\sqrt{k}}$  hence  $\beta_k$  and  $\alpha_k$  converge to 1 when  $k \rightarrow \infty$ .

**Theorem 6.3.** For  $k \geq 1$  and  $j \geq 0$ , the following limits exist and have the given values:

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} F_k^j(n) &= \alpha_k^j \\ \lim_{n \rightarrow \infty} \frac{1}{n} L_k^j(n) &= \beta_k^j \end{aligned}$$

Said otherwise, when  $n \rightarrow \infty$  we have  $F_k^j(n) \sim \alpha_k^j n$  and  $L_k^j(n) \sim \beta_k^j n$ .

As a side remark, we are now able to provide asymptotic equivalents for  $\alpha_k$  and  $\beta_k$ , instead of just the previous bounds of Proposition 6.2. Actually, the equivalent for  $\beta_k$  was already proposed by Selmer [16, Eq. (4.4)] but without justifications.

**Proposition 6.4.** When  $k$  converges to  $+\infty$ ,

$$\begin{aligned} \alpha_k &= 1 - \frac{\ln(k)}{k} + o\left(\frac{\ln(k)}{k}\right) \\ \beta_k &= 1 + \frac{\ln(k)}{k} + o\left(\frac{\ln(k)}{k}\right) \end{aligned}$$

*Proof.* Let  $k \geq 1$ .  $\alpha_k$  can be expressed as the solution of the equation  $x = \sqrt[k]{1-x}$  in the interval  $(0, 1)$ . Even more, it is an attracting fixed point of the function  $f(x) = \sqrt[k]{1-x}$ , with the whole interval  $(0, 1)$  as basin of attraction. It is hence sufficient to iterate this function to get approximations of  $\alpha_k$ . Here, these approximations will alternate on each side of  $\alpha_k$ . We choose  $u_0 = 1 - \frac{1}{e}$  as initial point, since this cancels some constants later, then the sequence  $u_{n+1} = f(u_n)$  converges to  $\alpha_k$ . Actually,  $u_2$  and  $u_3$  already have asymptotic expansions that suit our needs. Indeed, we obtain  $u_2 < \alpha_k < u_3$  (as soon as  $k \geq 3$ ) and

$$\begin{aligned} u_2 &= f^2\left(1 - \frac{1}{e}\right) = 1 - \frac{\ln(k)}{k} + o\left(\frac{1}{k}\right) \\ u_3 &= f(u_2) = 1 - \frac{\ln(k)}{k} + \frac{\ln(\ln(k))}{k} + o\left(\frac{1}{k}\right). \end{aligned}$$

Now, the expansion of  $\beta_k$  is obtained by inverting the one of  $\alpha_k$ . □

Let us study now the complex zeros of polynomials  $P_k$  and  $Q_k$ . Since  $P_k$  has no common zero with its derivative, it admits  $k$  distinct complex zeros, including  $\alpha_k$  and a real negative zero when  $k$  is even. The  $k$  zeros of  $Q_k$  are the inverse of the zeros  $P_k$ , they are also distinct and include  $\beta_k$  and also a real negative zero when  $k$  is even. Moreover,  $\beta_k$  is the dominant zero of  $Q_k$ :

**Proposition 6.5.** Let  $z \in \mathbb{C}$  be a zero of  $Q_k$  different from  $\beta_k$ . Then  $|z| < \beta_k$ .

*Proof.* Via triangle inequality,  $|(z-1)+1| \leq |z-1|+|1|$  hence  $|z|-1 \leq |z-1|$ . Moreover this is an equality only when  $z-1 \in \mathbb{R}_+$ . But  $\beta_k$  is the unique zero of  $Q_k$  which is a positive real number. So here  $|z|-1 < |z-1|$ .

Now, if  $\beta_k \leq |z|$  we could derive the following contradiction:

$$1 = (\beta_k - 1) \cdot \beta_k^{k-1} \leq (|z| - 1) \cdot |z|^{k-1} < |z - 1| \cdot |z|^{k-1} = |z^k - z^{k-1}| = 1.$$

□

For the same kind of reasons, when  $Q_k$  admits a real negative zero (i.e., when  $k$  is even), then the absolute value of this zero is strictly smaller than the modulus of the other zeros.

**Proposition 6.6.** *For two zeros  $z$  and  $t$  of  $Q_k$ ,  $|z| = |t|$  if and only if  $\operatorname{Re}(z) = \operatorname{Re}(t)$ , which happens exactly when  $z$  and  $t$  are equal or complex conjugates. Similarly,  $|z| < |t|$  whenever  $\operatorname{Re}(z) < \operatorname{Re}(t)$ .*

*Proof.* Assume  $|z| = |t|$ .

$$|z-1| \cdot |z|^{k-1} = |(z-1)z^{k-1}| = 1 = |(t-1)t^{k-1}| = |t-1| \cdot |t|^{k-1}.$$

This simplifies into  $|z-1| = |t-1|$  here, since 0 is not a zero of  $Q_k$  and hence  $|z| = |t| \neq 0$ . Let us write  $z = a+ib$  and  $t = c+id$ . We deduce  $a^2 + b^2 = c^2 + d^2$  and  $(a-1)^2 + b^2 = (c-1)^2 + d^2$ . Developing and subtracting these two equations leads to  $a = c$ , i.e.,  $\operatorname{Re}(z) = \operatorname{Re}(t)$ , then  $b = \pm d$ , i.e.,  $t = z$  or  $t = \bar{z}$ . Similarly, when  $|z| < |t|$  we obtain  $|z|^2 < |t|^2$  and  $|t-1|^2 < |z-1|^2$  and then  $a < c$ , i.e.,  $\operatorname{Re}(z) < \operatorname{Re}(t)$ . The reciprocal statements are obtained by totality of the order on  $\mathbb{R}$ . □

**Definition 6.7.** For  $k \geq 1$ , let us name  $r_{k,0} \cdots r_{k,k-1}$  the  $k$  complex zeros of  $Q_k$ , in decreasing lexicographic order of their Cartesian coordinates.

Thanks to the previous properties, this choice of ordering implies that this sequence  $r_{k,i}$  of zeros have decreasing modulus, with equality of modulus exactly when a non-real zero is followed by its complex conjugate. More precisely:

- the positive zero comes first:  $r_{k,0} = \beta_k$ ;
- the possible negative zero comes last:  $r_{k,k-1} \in \mathbb{R}_-$  when  $k$  is even;
- the pairs of complex conjugate zeros come in between: for  $1 < 2p < k$ , we have  $r_{k,2p} = \overline{r_{k,2p-1}}$ .

For studying in Section 8 the maximal distance between  $F_k$  and its linear equivalent  $\alpha_k n$ , it will be essential to know whether the secondary zeros of  $Q_k$  may reach or even exceed 1 in modulus (by secondary, we mean different from the dominant zero  $\beta_k$ ).

**Proposition 6.8.** *Let  $k \geq 2$ . The secondary zeros of  $Q_k$  of maximal modulus are  $r_{k,1}$  and  $r_{k,2} = \overline{r_{k,1}}$  (or solely  $r_{k,1}$  when  $k = 2$ ). Moreover the modulus  $|r_{k,1}|$  is as follow:*

- $|r_{k,1}| < 1$  when  $k \in \{2, 3, 4\}$ ;
- $|r_{k,1}| = 1$  when  $k = 5$ ;
- $|r_{k,1}| > 1$  when  $k \geq 6$ .

*Proof.* Our choice of ordering for the zeros  $r_{k,i}$  implies that the secondary zeros of maximal modulus are  $r_{k,1}$  and its conjugate  $r_{k,2}$  (or solely  $r_{k,1}$  when  $k = 2$ ).

First,  $\beta_2 = \varphi \approx 1.618$  and  $\beta_3 \approx 1.465$  and  $\beta_4 \approx 1.380$  are well-known Pisot numbers [5] and  $Q_2, Q_3, Q_4$  are their minimal polynomials, hence their other zeros are all of modulus strictly below 1.

Now, the case  $k = 5$  is quite particular, since  $Q_5$  can be factorized as  $(X^2 - X + 1)(X^3 - X - 1)$ . On one side, the zeros of  $X^2 - X + 1$  are  $\frac{1+i\sqrt{3}}{2} = e^{i\pi/3}$  and its conjugate, both

of modulus 1. On the other side,  $\beta_5$  is the real zero of  $X^3 - X - 1$  and it is hence the well-known Plastic Ratio  $\approx 1.324718$ , which is the smallest Pisot number, as shown by Siegel [17]. The two remaining complex zeros of  $X^3 - X - 1$  are of modulus  $< 1$ . Finally  $r_{k,1} = e^{i\pi/3}$ .

For  $k \geq 6$ , let us name  $M_k$  the minimal polynomial of  $\beta_k$ . This polynomial  $M_k$  is monic, it has integer coefficients and it divides  $Q_k$ , so the zeros of  $M_k$  are also zeros of  $Q_k$ . The degree of  $M_k$  is at least 2, otherwise  $\beta_k$  would be an integer. So we can consider the largest modulus among the zeros of  $M_k$  distinct from  $\beta_k$  and compare this modulus with 1.

- If this modulus is strictly less than 1, then  $\beta_k$  is a Pisot number, but this case is impossible here, since  $\beta_k$  is strictly less than the smallest Pisot  $\beta_5$ , see Siegel [17].
- If this modulus is exactly 1, then  $\beta_k$  is a Salem number [15], but this case is also impossible here. For instance, a Salem number is known to also have its inverse as zero of its minimal polynomial (which is self-reciprocal) while here neither  $Q_k$  nor  $M_k$  have zeros in  $[0, 1]$ .
- Finally, this modulus can only be strictly more than 1. This proves the existence of a zero of  $M_k$  distinct from  $\beta_k$  with modulus  $> 1$ , and this zero is also a zero of  $Q_k$ . Considering the ordering of zeros of  $Q_k$  chosen above, we can conclude  $|r_{k,1}| > 1$ .

□

As a complement to the previous proof, it is easy to show that  $Q_k$  admits a zero  $z$  of modulus 1 only when  $|z - 1| = |(z - 1) \cdot z^{k-1}| = 1$ , and having both  $|z| = 1$  and  $|z - 1| = 1$  can only occur for  $z = e^{\pm i\pi/3}$ , the zeros of  $X^2 - X + 1$  already encountered for  $Q_5$ . In this case,  $z - 1 = z^2$  and the equation  $Q_k(z) = 0$  amounts to  $z^{k+1} = 1$  hence  $e^{\pm i(k+1)\pi/3} = 1$  hence  $k = 5 \pmod{6}$ . When  $k = 5 \pmod{6}$ , there exists a polynomial  $R_k \in \mathbb{Z}[X]$  such that  $Q_k = (X^2 - X + 1)R_k(X)$ . We have already seen  $R_5 = X^3 - X - 1$  above; the other  $R_k$  can be obtained via the recursive equation  $R_{k+6} = R_k + (X^5 - X^3 - X^2 + 1)X^{k-1}$ . Actually, this is the only possible factorization of  $Q_k$  over  $\mathbb{Z}[X]$ , or equivalently over  $\mathbb{Q}[X]$ . Indeed, the work of Selmer [16] implies that when  $k \neq 5 \pmod{6}$  the polynomial  $Q_k$  is irreducible over  $\mathbb{Z}[X]$ . This also holds for the reciprocal polynomial  $P_k$ . For a simpler proof, see Conrad [3]. In this case,  $Q_k$  is hence the minimal polynomial of  $\beta_k$ . This work of Selmer [16] also implies that for  $k = 5 \pmod{6}$ , the factor  $R_k$  of  $Q_k$  is always irreducible over  $\mathbb{Z}[X]$ , and is hence the minimal polynomial of  $\beta_k$  in this case.

## 7. ALGEBRAIC EXPRESSIONS OF FIBONACCI-LIKE SEQUENCES

For the next section, it is crucial to provide precise algebraic expressions for Fibonacci-like sequences  $(A_{k,n})$ . In particular we will distinguish here the main exponential behavior of  $(A_{k,n})$  from residual parts.

Recall from section 6 that  $Q_k(X) = X^k - X^{k-1} - 1$  and  $\beta_k \in (1, 2]$  is the unique positive zero of  $Q_k$ , as well as its dominant zero. Also recall that the  $k$  complex zeros  $r_{k,i}$  of  $Q_k$  are simple and hence distinct. In our notation,  $r_{k,0} = \beta_k$ .

**Proposition 7.1.** *For all  $k \geq 1$  and  $n \geq 0$ ,  $\beta_k^n \leq A_{k,n}$*

*Proof.* We proceed by strong induction over  $n$ . Let  $n \geq 0$  and assume  $\beta_k^m \leq A_{k,m}$  for all  $0 \leq m < n$ . Since  $Q_k(\beta_k) = 0$ , we always have

$$\beta_k^n = \beta_k^{n-k} \cdot \beta_k^k = \beta_k^{n-k} \cdot (\beta_k^{k-1} + 1) = \beta_k^{n-1} + \beta_k^{n-k}.$$



If  $n = 0$  then  $\beta_k^0 = 1 = A_{k,0}$ . If  $0 < n < k$ , then  $\beta_k^{n-k} \leq 1$  hence by induction hypothesis in  $n-1$ :

$$\beta_k^n \leq \beta_k^{n-1} + 1 \leq A_{k,n-1} + 1 = A_{k,n}.$$

Now, if  $n \geq k$ ,

$$\beta_k^n = \beta_k^{n-1} + \beta_k^{n-k} \leq A_{k,n-1} + A_{k,n-k} = A_{k,n}$$

by induction hypothesis in  $n-1$  and  $n-k$ .  $\square$

**Proposition 7.2.** *Let us name  $c_{k,i}$  the complex quantity  $r_{k,i}^k (kr_{k,i} - (k-1))^{-1}$ . Then*

$$(7.1) \quad A_{k,n} = \sum_{i=0}^{k-1} c_{k,i} r_{k,i}^n \quad \text{for all } n \geq 0.$$

Moreover all these coefficients  $c_{k,i}$  differ from 0 and  $c_{k,0}$  is a real greater or equal to 1 and

$$\lim_{n \rightarrow \infty} \frac{A_{k,n}}{\beta_k^n} = c_{k,0} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{A_{k,n+1}}{A_{k,n}} = \beta_k.$$

*Proof.* The polynomial  $Q_k$  is the characteristic polynomial of the linear recurrence  $(A_{k,n})$ , and  $Q_k$  has no multiple zeros. Hence standard theory of linear recurrence with constant coefficients states here that  $A_{k,n}$  is a linear combination of the  $n$ -th powers of the  $Q_k$  zeros  $r_{k,i}$ . We now show that the coefficients  $c_{k,i}$  of this linear combination have indeed the aforementioned values. For that we study the linear system formed by Equation (7.1) when  $n = 0, \dots, k-1$ . In this case  $A_{k,n} = n+1$ . Let us write this system via matrices<sup>3</sup>. We name  $V = (r_{k,i}^j)_{0 \leq i, j < k}$  the following  $k \times k$  Vandermonde matrix:

$$V = \begin{bmatrix} 1 & r_{k,0} & \cdots & r_{k,0}^{k-1} \\ 1 & r_{k,1} & \cdots & r_{k,1}^{k-1} \\ \vdots & \vdots & & \vdots \\ 1 & r_{k,k-1} & \cdots & r_{k,k-1}^{k-1} \end{bmatrix}.$$

The linear system to solve amounts to  $[(c_{k,0}), \dots, (c_{k,k-1})] \cdot V = [1, 2, \dots, k]$ . This Vandermonde matrix  $V$  is known to be invertible since all the numbers  $r_{k,i}$  are distinct, hence the coefficients are equal to  $[1, 2, \dots, k] \cdot V^{-1}$ . Actually, we can restrict our needs to the last line of  $V^{-1}$ , which is much simpler than the other elements of this matrix. For that, we consider a sequence  $\tilde{A}_{k,n}$  satisfying the same recursive equation as  $A_{k,n}$  but with different initial values:  $\tilde{A}_{k,0} = \dots = \tilde{A}_{k,k-2} = 0$  and  $\tilde{A}_{k,k-1} = 1$ . When computing the next terms of this sequence, we first find several 1 until index  $2k-2$ , then values  $2, 3 \dots k$  afterwards. As a consequence, one can easily prove by induction that for all  $n \geq 0$ ,  $A_{k,n} = \tilde{A}_{k,(n+2k-2)}$ . Now, for the same reasons as for  $(A_{k,n})$ , there exist coefficients  $\tilde{c}_{k,i}$  such that  $\tilde{A}_{k,n} = \sum_i \tilde{c}_{k,i} r_{k,i}^n$ . This time, these coefficients are equal to  $[0, \dots, 0, 1] \cdot V^{-1}$ , i.e., the last line of  $V^{-1}$ :

$$\tilde{c}_{k,i} = (V^{-1})_{k-1,i} = \frac{1}{\prod_{\substack{0 \leq j < n \\ j \neq i}} (r_{k,i} - r_{k,j})}.$$

This expression of  $(V^{-1})_{k-1,i}$  is quite standard, it can be obtained by dividing the cofactor of  $V$  at  $i, k-1$  by the determinant of  $V$ . This cofactor is here  $(-1)^{i+k-1}$  times the determinant of the  $(k-1) \times (k-1)$  Vandermonde matrix for the zeros of  $Q_k$  except  $r_{k,i}$ .

<sup>3</sup>Note that our indexes of matrix lines and rows starts at 0, an unorthodox choice made for keeping a low distance between this article and the corresponding Coq development [11].

Since the polynomial  $Q_k(X) = X^k - X^{k-1} - 1$  can also be written as  $\prod_i (X - r_{k,i})$ , the product above can be formulated as the derivative polynomial  $Q'_k$  evaluated at  $r_{k,i}$ , leading to a short expression for  $\tilde{c}_{k,i}$ :

$$\tilde{c}_{k,i} = \frac{1}{Q'_k(r_{k,i})} = \frac{1}{r_{k,i}^{k-2} (kr_{k,i} - (k-1))}.$$

Now, back to  $A_{k,n}$ :

$$A_{k,n} = \tilde{A}_{k,(n+2k-2)} = \sum_{i=0}^{k-1} \tilde{c}_{k,i} r_{k,i}^{(n+2k-2)} = \sum_{i=0}^{k-1} (\tilde{c}_{k,i} r_{k,i}^{2k-2}) r_{k,i}^n.$$

The coefficients  $c_{k,i}$  being uniquely determined, we have hence the desired equalities for all  $0 \leq i < k$ :

$$c_{k,i} = \tilde{c}_{k,i} r_{k,i}^{2k-2} = r_{k,i}^k (kr_{k,i} - (k-1))^{-1}.$$

Since  $r_{k,i} \neq 0$ , these coefficients  $c_{k,i}$  are all different from 0.

We now consider separately the first zero  $r_{k,0} = \beta_k$ :

$$\frac{A_{k,n}}{\beta_k^n} = c_{k,0} + \sum_{i=1}^{k-1} c_{k,i} \left( \frac{r_{k,i}}{\beta_k} \right)^n$$

where  $|r_{k,i}| < \beta_k$  for  $1 \leq i < k$ , hence the above expression converges to  $c_{k,0}$  when  $n \rightarrow \infty$ .

Now, recall from Proposition 7.1 that  $\beta_k^n \leq A_{k,n}$ . As a finite limit of real numbers greater or equal to 1,  $c_{k,0}$  is hence a real greater or equal to 1.

Finally

$$\frac{A_{k,n+1}}{A_{k,n}} = \beta_k \cdot \frac{A_{k,n+1}}{\beta_k^{n+1}} \cdot \frac{\beta_k^n}{A_{k,n}}$$

hence this ratio converges to  $\beta_k c_{k,0} c_{k,0}^{-1} = \beta_k$ .  $\square$

In particular,  $c_{1,0} = 1$  and  $c_{2,0} \approx 1.17$  and  $c_{3,0} \approx 1.31$  and  $c_{4,0} \approx 1.43$ . Despite these moderate initial values,  $c_{k,0}$  tends to  $+\infty$  when  $k$  tends to  $+\infty$ , since the definition of  $c_{k,0}$  combined with Proposition 6.4 lead to an asymptotic equivalent of  $k/(\ln(k))^2$ .

Note that for  $k = 2$ , the sequence  $(\tilde{A}_{2,n})$  used during the previous proof is actually the usual definition of the Fibonacci numbers, with initial values 0 1, and the formula  $\tilde{A}_{2,n} = \sum_i \tilde{c}_{2,i} r_{2,i}^n$  is the well-known Binet formula, since here  $r_{2,0} = \beta_2 = \varphi$  and  $r_{2,1} = 1 - \varphi = \frac{-1}{\varphi}$  and  $\tilde{c}_{2,i} = \frac{\pm 1}{\sqrt{5}}$ . The Equation (7.1) for  $A_{2,n}$  is less elegant, with  $c_{2,i} = \frac{1}{10}(5 \pm 3\sqrt{5})$ , but the choice of a sequence without 0 nor duplicated 1 helps in section 4.

Alternatively, another possible choice of initial values would be to use repeated 1, as in some non-standard definition of the Fibonacci numbers or in Narayana's Cows description on OEIS [A930](#). This leads to another shifted version of  $(\tilde{A}_{k,n})$  with no particular interest here, neither during the previous proof nor in section 4.

As a side note, the coefficients  $c_{k,i}$  are the  $k$  zeros of a polynomial in  $\mathbb{Z}[X]$  of degree  $k$ , with leading coefficient  $k^k + (k-1)^{k-1}$  when  $k > 1$  (or 1 when  $k = 1$ ) and constant coefficient  $(-1)$ . This fact is also true for the coefficients  $\tilde{c}_{k,i}$ . Here are the first such polynomials:

- $X - 1$  for  $c_{1,0} = \tilde{c}_{1,0} = 1$ .
- $5X^2 - 5X - 1$  for  $c_{2,i}$  and  $5X^2 - 1$  for  $\tilde{c}_{2,i}$ .
- $31X^3 - 31X^2 - 12X - 1$  for  $c_{3,i}$  and  $31X^3 + X - 1$  for  $\tilde{c}_{3,i}$ .
- $283X^4 - 283X^3 - 162X^2 - 24X - 1$  for  $c_{4,i}$  and  $283X^4 + 6X^2 + X - 1$  for  $\tilde{c}_{4,i}$ .

For the moment, we lack a general expression directly giving these polynomials, but their existence is proved in Appendix A. Some more coefficients can be easily determined, for instance the ones of degree  $k-1$  since  $\sum_i c_{k,i} = 1$  (see Equation (7.1) when  $n = 0$ ) and similarly  $\sum_i \tilde{c}_{k,i} = 0$ . For more on the sequence 5, 31, 283, 3381, ...  $k^k + (k-1)^{k-1}$ , see OEIS [A056788](#).

Recall now that  $\alpha_k = 1/\beta_k = 1/r_{k,0}$ .

**Proposition 7.3.** *For  $k \geq 2$  and  $0 \leq i < k$ , let us name  $d_{k,i}$  the complex quantity  $c_{k,i} (r_{k,i}^{-1} - \alpha_k)$ . Hence  $d_{k,i} = 0$  iff  $i = 0$ . Then for all  $n \geq 0$ ,*

$$A_{k,n \ominus 1} - \alpha_k A_{k,n} = \sum_{i=1}^{k-1} d_{k,i} r_{k,i}^n.$$

*Proof.* For  $n > 0$ ,  $n \ominus 1 = n-1$ , and we apply twice Proposition 7.2:

$$A_{k,n-1} - \alpha_k A_{k,n} = \sum_{i=0}^{k-1} c_{k,i} r_{k,i}^{n-1} - \alpha_k \sum_{i=0}^{k-1} c_{k,i} r_{k,i}^n = \sum_{i=0}^{k-1} c_{k,i} \left( \frac{1}{r_{k,i}} - \alpha_k \right) r_{k,i}^n = \sum_{i=0}^{k-1} d_{k,i} r_{k,i}^n$$

and the first term of this sum can be ignored since  $d_{k,0} = 0$ .

For  $n = 0$ ,  $A_{k,0} - \alpha_k A_{k,0} = 1 - \alpha_k$  while we have (reusing the notation of the proof of Proposition 7.2):

$$\sum_{i=1}^{k-1} d_{k,i} = \sum_{i=0}^{k-1} d_{k,i} = \sum_{i=0}^{k-1} \tilde{c}_{k,i} r_{k,i}^{2k-3} - \alpha_k \sum_{i=0}^{k-1} c_{k,i} r_{k,i}^0 = \tilde{A}_{k,(2k-3)} - \alpha_k A_{k,0} = 1 - \alpha_k.$$

□

**Lemma 7.4.** *Let  $\theta, \rho \in \mathbb{R}$  and assume  $\sin \theta \neq 0$ . If the sequence  $(a_n)$  is defined by  $a_n = |\cos(n\theta + \rho)|$ , then there exist a constant  $C \in \mathbb{R}_+^*$  and a subsequence of  $(a_n)$  always above  $C$ .*

*Proof.* Since  $(a_n)$  is a sequence of real numbers in  $[0, 1]$ , it admits a subsequence  $(a_{u_n})$  that converges to some limit  $\ell \in [0, 1]$ . If  $\ell > 0$ , we may choose  $C = \ell/2$ , and find a number  $N$  such that the shifted subsequence  $(a_{u_{(n+N)}})$  is sufficiently close to  $\ell$  to be always above  $\ell/2$ . Now, if  $\ell = 0$ , we consider  $v_n = 1 + u_n$ :

$$\begin{aligned} a_{v_n} &= |\cos(v_n \theta + \rho)| \\ &= |\cos(\theta + (u_n \theta + \rho))| \\ &= |\cos \theta \cos(u_n \theta + \rho) - \sin \theta \sin(u_n \theta + \rho)| \\ &\geq |\sin \theta \sin(u_n \theta + \rho)| - |\cos \theta \cos(u_n \theta + \rho)| \\ &\geq |\sin \theta| \sqrt{1 - (a_{u_n})^2} - |\cos \theta| a_{u_n}. \end{aligned}$$

The minorant of  $a_{v_n}$  exhibited above tends to  $|\sin \theta| > 0$ , so we may choose  $C = \frac{1}{2} |\sin \theta|$  and find a shift  $N$  such that the subsequence  $(a_{v_{(n+N)}})$  works. □

**Proposition 7.5.** *For  $k \geq 6$ ,  $\sup_{n \in \mathbb{N}} |A_{k,n \ominus 1} - \alpha_k A_{k,n}| = +\infty$ . More precisely, for a given  $k \geq 6$ , there exist a constant  $C \in \mathbb{R}_+^*$  and a strictly increasing sequence of natural numbers  $(u_n)$  such that for all  $n \geq 0$ ,*

$$|A_{k,u_n \ominus 1} - \alpha_k A_{k,u_n}| > C |r_{k,1}|^{u_n}.$$

*Proof.* Let  $k \geq 6$ . Recall from Proposition 6.8 that in this case  $|r_{k,1}| > 1$  and  $r_{k,2} = \overline{r_{k,1}}$  and for  $3 \leq i < k$  we have  $|r_{k,i}| \leq |r_{k,3}| < |r_{k,1}|$ . Also note that  $d_{k,2} = \overline{d_{k,1}}$  just as

$c_{k,2} = \overline{c_{k,1}}$ , thanks to their definitions. Now, we apply Proposition 7.3 and regroup the two first terms:

$$\begin{aligned} |A_{k,n\ominus 1} - \alpha_k A_{k,n}| &= \left| \sum_{i=1}^{k-1} d_{k,i} r_{k,i}^n \right| = \left| 2 \operatorname{Re}(d_{k,1} r_{k,1}^n) + \sum_{i=3}^{k-1} d_{k,i} r_{k,i}^n \right| \\ &\geq 2 |\operatorname{Re}(d_{k,1} r_{k,1}^n)| - \left| \sum_{i=3}^{k-1} d_{k,i} r_{k,i}^n \right| \\ &\geq 2 |\operatorname{Re}(d_{k,1} r_{k,1}^n)| - C_3 |r_{k,3}|^n \end{aligned}$$

where  $C_3 \in \mathbb{R}_+^*$  is  $\sum_{3 \leq i} |d_{k,i}|$ .

By writing  $d_{k,1}$  as  $|d_{k,1}|e^{i\rho}$  and  $r_{k,1}$  as  $|r_{k,1}|e^{i\theta}$ , we get

$$|\operatorname{Re}(d_{k,1} r_{k,1}^n)| = |d_{k,1}| \cdot |r_{k,1}|^n \cdot |\cos(n\theta + \rho)|.$$

Since  $k > 2$ ,  $r_{k,1} \notin \mathbb{R}$  hence  $\sin \theta \neq 0$ , and the previous Lemma 7.4 leads to a constant  $C_1 \in \mathbb{R}_+^*$  and a strictly increasing sequence  $v_n$  such that  $|\cos(v_n \theta + \rho)| > C_1$ . So:

$$|A_{k,v_n\ominus 1} - \alpha_k A_{k,v_n}| > (2C_1 |d_{k,1}| - C_3 \cdot \chi^{v_n}) \cdot |r_{k,1}|^{v_n}$$

where  $\chi = |r_{k,3}|/|r_{k,1}| < 1$ . We can hence choose  $C = C_1 |d_{k,1}| > 0$  and  $u_n = v_{(n+N)}$  where the shift  $N$  is large enough for having  $\chi^{v_{(n+N)}} < C/C_3$  for all  $n \geq 0$ . These  $C$  and  $u_n$  ensure the desired inequality.  $\square$

## 8. DISCREPANCY: MAXIMAL DISTANCE TO THE LINEAR EQUIVALENT

From Theorem 6.3, we know that for any  $k \geq 1$ ,  $F_k(n)$  admits  $\alpha_k n$  as linear equivalent. We investigate here the distance between  $F_k$  and this linear equivalent.

**Definition 8.1.** For  $k \geq 1$ , we note  $\delta_k(n) = F_k(n) - \alpha_k n$  and call *discrepancy* the quantity  $\Delta_k = \sup_{n \in \mathbb{N}} |\delta_k(n)|$ .

We now establish that the discrepancy  $\Delta_k$  is finite (and quite small) for  $k \leq 4$  and infinite for  $k \geq 5$ .

**8.1. The general case.** In the general case  $k \geq 6$ , Proposition 7.5 implies that  $\delta_k$  diverges at least for some subsequence of the  $A_{k,n}$  numbers. More precisely:

**Theorem 8.2.** For  $k \geq 6$ ,  $\Delta_k = +\infty$ . More precisely, for a given  $k \geq 6$ , there exist a constant  $C \in \mathbb{R}_+^*$  and a subsequence  $(u_n)$  of the sequence  $(A_{k,n})_{n \in \mathbb{N}}$  such that for all  $n \geq 0$ ,

$$|\delta_k(u_n)| > C (u_n)^a$$

where  $a = \ln |r_{k,1}| / \ln \beta_k$  and hence  $0 < a < 1$ .

*Proof.* Let  $k \geq 6$ . Recall that  $r_{k,1}$  is one of the zeros of the polynomial  $Q_k$ , of maximal modulus among the zeros distinct from the positive zero  $\beta_k$ . And from Proposition 6.8, we know that here  $|r_{k,1}| > 1$ .

Let us call  $C_0$  and  $v_n$  the constant and sequence provided by Proposition 7.5 for this  $k$ , and combine it with Proposition 3.3:

$$|\delta_k(A_{k,v_n})| = |F_k(A_{k,v_n}) - \alpha_k A_{k,v_n}| = |A_{k,v_n\ominus 1} - \alpha_k A_{k,v_n}| > C_0 |r_{k,1}|^{v_n} = C_0 (\beta_k^{v_n})^a.$$

Recall from Proposition 7.2 that  $\lim_{n \rightarrow \infty} A_{k,n}/\beta_k^n = c_{k,0} > 0$ . So we can find  $N$  such that for all  $m \geq N$ ,  $A_{k,m} < 2c_{k,0}\beta_k^m$  and hence  $(\beta_k^m)^a > (2c_{k,0})^{-a}(A_{k,m})^a$ . We can now choose  $C = C_0 (2c_{k,0})^{-a}$  and  $u_n = A_{k,v_{(n+N)}}$  and put everything together (note that  $v_{(n+N)} \geq N$ ):

$$|\delta_k(u_n)| > C_0 (\beta_k^{v_{(n+N)}})^a > C_0 (2c_{k,0})^{-a} (A_{k,v_{(n+N)}})^a = C (u_n)^a.$$

□

Experimentally, the exponent is  $a \approx 0.1287$  for  $k = 6$  and  $a \approx 0.2218$  for  $k = 7$  and appears to tend to 1 when  $k$  grows.

For the remaining results of this section, we need to express  $\delta_k(n)$  thanks to the  $k$ -decomposition of  $n$ .

**Proposition 8.3.** *Let  $k \geq 2$  and  $n \geq 0$ .*

$$\delta_k(n) = \sum_{q \in D_k(n)} \sum_{i=1}^{k-1} d_{k,i} r_{k,i}^q = \sum_{i=1}^{k-1} \left( d_{k,i} \sum_{q \in D_k(n)} r_{k,i}^q \right)$$

where  $d_{k,i}$  comes from Proposition 7.3 and  $D_k(n)$  from Theorem 4.2.

*Proof.*

$$\begin{aligned} \delta_k(n) &= F_k(n) - \alpha_k n \\ &= \Sigma_k(D_k(n) \gg_+ 1) - \alpha_k n && \text{(by Thm. 4.10)} \\ &= \sum_{q \in D_k(n)} A_{k,q \oplus 1} - \alpha_k \sum_{q \in D_k(n)} A_{k,q} \\ &= \sum_{q \in D_k(n)} (A_{k,q \oplus 1} - \alpha_k A_{k,q}) \\ &= \sum_{q \in D_k(n)} \sum_{i=1}^{k-1} d_{k,i} r_{k,i}^q && \text{(by Prop. 7.3)} \\ &= \sum_{i=1}^{k-1} \left( d_{k,i} \sum_{q \in D_k(n)} r_{k,i}^q \right). \end{aligned}$$

□

For  $k \geq 6$ , we now establish that  $\delta_k$  cannot diverge at a faster pace than the behavior seen in Theorem 8.2.

**Proposition 8.4.** *For  $k \geq 6$ ,  $|\delta_k(n)| = O(n^a)$  where  $a = \ln |r_{k,1}| / \ln \beta_k$ .*

*Proof.* We exploit Proposition 8.3 and the fact that  $|r_{k,i}| \leq |r_{k,1}|$  for all  $1 \leq i < k$ :

$$|\delta_k(n)| \leq \sum_{q \in D_k(n)} \sum_{i=1}^{k-1} |d_{k,i}| |r_{k,i}|^q \leq M \sum_{q \in D_k(n)} |r_{k,1}|^q$$

where  $M = \sum_{1 \leq i < k} |d_{k,i}|$ . For  $n \neq 0$ , let us call  $m$  the largest element in  $D_k(n)$ . Since  $\Sigma_k(D_k(n)) = n$ , we have  $A_{k,m} \leq n$ , hence by Proposition 7.1  $\beta_k^m \leq n$ , so  $|r_{k,1}|^m \leq n^a$ . We over-estimate  $D_k(n)$  as  $\{0, 1, \dots, m\}$  and recall that  $|r_{k,1}| > 1$  for  $k \geq 6$ :

$$|\delta_k(n)| \leq M \sum_{q=0}^m |r_{k,1}|^q = M \frac{|r_{k,1}|^{m+1} - 1}{|r_{k,1}| - 1} \leq C n^a$$

where  $C = M |r_{k,1}| (|r_{k,1}| - 1)^{-1}$ .

□

8.2. **The case  $k = 5$ .** When  $k = 5$ , the discrepancy  $\Delta_5$  is still infinite, but the proof is different from the general case  $k \geq 6$  since here  $|r_{5,1}| = 1$  (actually  $r_{5,1} = e^{i\pi/3}$ ) and the divergence is quite slower.

**Theorem 8.5.**  $\Delta_5 = +\infty$ . More precisely, if we consider the sequence  $u_n = \sum_{p=0}^{n-1} A_{5,6p}$ , there exist some constants  $C, C', K, K' \in \mathbb{R}_+^*$  such that for all  $n \geq 0$ ,

$$|\delta_5(u_n) - Cn| < C' \quad \text{and} \quad |\delta_5(u_n) - K \ln(u_n)| < K'.$$

*Proof.* The 5-decomposition  $D = \{0, 6, \dots, 6(n-1)\}$  is canonical, since the positions  $6p$  are indeed apart by at least 5. Since the sum of this 5-decomposition is  $u_n$ , then  $D = D_5(u_n)$  by Theorem 4.2. Also note that here  $r_{5,1} = e^{i\pi/3} = \overline{r_{5,2}}$  hence  $r_{5,1}^{6p} = 1 = r_{5,2}^{6p}$ . Moreover  $d_{5,2} = \overline{d_{5,1}}$  and  $d_{5,4} = \overline{d_{5,3}}$  and  $r_{5,4} = \overline{r_{5,3}}$  and  $|r_{5,3}| < 1$ . So:

$$\begin{aligned} \delta_5(u_n) &= \sum_{i=1}^4 \left( d_{5,i} \sum_{q \in D_5(u_n)} r_{5,i}^q \right) && \text{(by Prop. 8.3)} \\ &= \sum_{i=1}^4 \left( d_{5,i} \sum_{p=0}^{n-1} r_{5,i}^{6p} \right) \\ &= 2n \operatorname{Re}(d_{5,1}) + 2 \operatorname{Re} \left( d_{5,3} \sum_{p=0}^{n-1} r_{5,3}^{6p} \right). \end{aligned}$$

From the definition of  $d_{5,1}$ , we get that  $d_{5,1} \approx 0.0189 + 0.196i$  and in particular its real part can be proved to be strictly positive. We can hence choose  $C = 2 \operatorname{Re}(d_{5,1})$ . We now give an upper bound for the last part:

$$\left| 2 \operatorname{Re} \left( d_{5,3} \sum_{p=0}^{n-1} r_{5,3}^{6p} \right) \right| \leq 2 |d_{5,3}| \sum_{p=0}^{n-1} |r_{5,3}|^{6p} < 2 |d_{5,3}| \frac{1}{1 - |r_{5,3}|^6}$$

so  $2|d_{5,3}|(1 - |r_{5,3}|^6)^{-1}$  is a suitable constant  $C'$  such that  $|\delta_5(u_n) - Cn| < C'$ .

Now, thanks to Proposition 7.1:

$$\begin{aligned} u_n &= \sum_{p=0}^{n-1} \sum_{i=0}^4 c_{5,i} r_{5,i}^{6p} \\ &= \sum_{i=0}^4 c_{5,i} \left( \sum_{p=0}^{n-1} r_{5,i}^{6p} \right) \\ &= c_{5,0} \cdot \frac{\beta_5^{6n} - 1}{\beta_5 - 1} + 2n \operatorname{Re}(c_{5,1}) + 2 \operatorname{Re} \left( c_{5,3} \cdot \frac{1 - r_{5,3}^{6n}}{1 - r_{5,3}^6} \right) \\ &= K_0 \beta_5^{6n} + O(n) \end{aligned}$$

where  $K_0 = c_{5,0}(\beta_5 - 1)^{-1}$ . Note that  $c_{5,0} > 1$  and  $1 < \beta_5 < 2$  hence  $K_0 > 1$ . So there exists  $N$  such that for all  $n > N$  we have

$$\begin{aligned} (K_0/2) \beta_5^{6n} &< u_n < (2K_0) \beta_5^{6n} \\ 6n \ln \beta_5 + \ln(K_0/2) &< \ln(u_n) < 6n \ln \beta_5 + \ln(2K_0) \\ Cn + K \ln(K_0/2) &< K \ln(u_n) < Cn + K \ln(2K_0) \end{aligned}$$

where we noted  $K = C(6 \ln \beta_5)^{-1} > 0$ . Since  $-K \ln(2K_0) \leq K \ln(K_0/2)$ , we obtain, still for  $n > N$ :

$$|Cn - K \ln(u_n)| < K \ln(2K_0).$$

Combining this with the first part of this proof leads to:

$$|\delta_5(u_n) - K \ln(u_n)| < C' + K \ln(2K_0)$$

when  $n > N$ . A suitable constant  $K'$  is hence the maximum of  $C' + K \ln(2K_0)$  and of all the initial distances  $|\delta_5(u_n) - K \ln(u_n)|$  for  $0 \leq n \leq N$ .  $\square$

This logarithmic behavior in  $k = 5$  cannot be outpaced:

**Proposition 8.6.**  $|\delta_5(n)| = O(\ln n)$ .

*Proof.* Here,  $|r_{5,1}| = |r_{5,2}| = 1$  and  $|r_{5,3}| = |r_{5,4}| < 1$ . Thanks to Proposition 8.3:

$$|\delta_5(n)| \leq \sum_{q \in D_5(n)} \sum_{i=1}^4 |d_{5,i}| |r_{5,i}|^q \leq M \sum_{q \in D_5(n)} 1$$

where  $M = \sum_{1 \leq i} |d_{5,i}|$ . For  $n \neq 0$ , let us call  $p$  the largest element in  $D_5(n)$ . The number of elements in  $D_5(n)$  is at most  $p + 1$ . Moreover  $A_{5,p} \leq n$ , hence by Proposition 7.1  $\beta_5^p \leq n$ , so  $p+1 \leq 1 + \ln n / \ln \beta_5$ , and finally  $|\delta_5(n)| \leq C \ln n + M$  with  $C = M / \ln \beta_5$ , hence the desired asymptotic bound.  $\square$

**8.3. Finite discrepancy up to  $k = 4$ .** First, the discrepancies  $\Delta_1$  and  $\Delta_2$  could easily be determined:

- From  $F_1(n) = \lfloor \frac{n+1}{2} \rfloor$  and  $\alpha_1 = \frac{1}{2}$ , we obtain  $\delta_1(n) \in \{0, \frac{1}{2}\}$  for all  $n \geq 0$  and then  $\Delta_1 = \frac{1}{2}$ .
- From  $F_2(n) = G(n) = \lfloor (n+1)/\varphi \rfloor$  and  $\alpha_2 = \frac{1}{\varphi} = \varphi - 1$  where  $\varphi$  is the Golden Ratio, we obtain  $\Delta_2 = \varphi - 1$ .

For the two remaining cases  $k = 3$  and  $k = 4$ , we now provide upper bounds for  $\Delta_3$  and  $\Delta_4$ .

**Theorem 8.7.**  $\Delta_3 < 1$  and  $\Delta_4 < 2$ . More precisely, for all  $n \geq 0$ ,  $|\delta_3(n)| < 0.996$  and  $|\delta_4(n)| < 1.998$ .

*Proof.* First, we consider  $k = 3$ . In this case,  $r_{3,1} = \overline{r_{3,2}}$  and  $d_{3,1} = \overline{d_{3,2}}$  and  $|r_{3,1}| < 1$ . We abbreviate  $\rho$  for  $r_{3,1}$  here. From Proposition 8.3, we get

$$\delta_3(n) = 2 \operatorname{Re} \left( d_{3,1} \sum_{q \in D_3(n)} \rho^q \right).$$

From that, we could derive easily a first finite bound, by over-approximating  $D_3(n) \subseteq \mathbb{N}$ :

$$|\delta_3(n)| \leq 2 |d_{3,1}| \sum_{q \in D_3(n)} |\rho|^q \leq 2 |d_{3,1}| \sum_{q=0}^{\infty} |\rho|^q = 2 |d_{3,1}| \frac{1}{1 - |\rho|}.$$

But this first bound is far greater than 1. Fortunately, we can do much better. First note that the 3-decomposition  $D_3(n)$  is canonical: two different elements in it are at least at distance 3. Moreover the zero  $\rho$  is a complex number satisfying  $\rho^3 = \rho + 1$ . Hence its powers are not exactly periodic like the roots of unity, but we can nonetheless expect a

important amount of cancellation when considering groups of powers of  $\rho$ . In particular, for the bound 1 we look for, it is sufficient to partition  $D_3(n)$  into

$$D_3(n) = \bigcup_{p=0}^{\infty} (D_3(n) \cap \{9p, \dots, 9p+8\}).$$

Said otherwise, in this digital expansion of  $n$ , we regroup digits 9 by 9. This gives us:

$$\sum_{q \in D_3(n)} \rho^q = \sum_{p=0}^{\infty} \sum_{\substack{0 \leq q < 9 \\ 9p+q \in D_3(n)}} \rho^{9p+q} = \sum_{p=0}^{\infty} \left( \rho^{9p} \sum_{\substack{0 \leq q < 9 \\ 9p+q \in D_3(n)}} \rho^q \right).$$

Since  $D_3(n)$  is canonical, the last sums above iterate on values  $0 \leq q < 9$  that are apart by at least 3. We are hence manipulating subsets of  $\{0, \dots, 8\}$  with minimal distance 3 between elements. Let us call such a subset a *bucket*, and for a bucket  $B$ , we name here  $\Sigma(B)$  the corresponding sum  $\sum_{q \in B} \rho^q$ . We call  $\mathbb{B}$  the set of all possible buckets. This set  $\mathbb{B}$  is finite<sup>4</sup>, and we can hence pose  $M = \max_{B \in \mathbb{B}} |\Sigma(B)|$ . Back to our upper bound,

$$|\delta_3(n)| \leq 2 |d_{3,2}| \left| \sum_{q \in D_3(n)} \rho^q \right| \leq 2 |d_{3,2}| M \sum_{p=0}^{\infty} |\rho|^{9p} \leq 2 |d_{3,2}| \frac{M}{1 - |\rho|^9}.$$

Let us estimate  $M$ . First, we can skip the buckets  $B$  that do not contain 0. Indeed, in this case we can factorize more, i.e., there exists another bucket  $B'$  such that  $|\Sigma(B)| = |\rho| \cdot |\Sigma(B')| < |\Sigma(B')|$  since  $|\rho| < 1$ . In particular the only interesting bucket of cardinal 1 is  $\{0\}$ , of sum 1. There remain 6 buckets of cardinal 2 to consider and 6 ones of cardinal 3:

$$\{0, 3\} \dots \{0, 8\} \quad \{0, 3, 6\} \quad \{0, 3, 7\} \quad \{0, 3, 8\} \quad \{0, 4, 7\} \quad \{0, 4, 8\} \quad \{0, 5, 9\}.$$

Via some simplifications of powers (recall that  $\rho^3 = \rho + 1$ ) and similar symbolic work (for instance comparing squares of expressions), one can prove that the maximum  $M$  is actually  $|\Sigma\{0, 3, 7\}| = |1 + \rho^3 + \rho^7| \approx 1.6848$ . and we obtain indeed a bound  $2 |d_{3,2}| M (1 - |\rho|^9)^{-1}$  that can be shown to belong to the interval  $[0.9958; 0.9959]$ . We performed this interval arithmetic within the Coq proof assistant, hence cannot suffer from errors such as incorrect rounding. For more details, see the file `LimCase2.v` of the Coq development [11].

For the case  $k = 4$ , the proof is very similar, except that besides  $r_{4,1} = \overline{r_{4,2}}$ , we now have an extra zero  $r_{4,3} \in \mathbb{R}_-$ . We abbreviate here  $\gamma$  for  $r_{4,1}$  and  $\nu$  for  $r_{4,3}$ . So this time

$$\delta_4(n) = 2 \operatorname{Re} \left( d_{4,1} \sum_{q \in D_4(n)} \gamma^q \right) + d_{4,3} \sum_{q \in D_4(n)} \nu^q.$$

For justifying that 2 is a strict bound, we had this time to regroup 16 digits together instead of 9, i.e., regroup up to four terms together in the above sums instead of three as before. We obtain

$$|\delta_4(n)| \leq 2 |d_{4,1}| \frac{M_\gamma}{1 - |\gamma|^{16}} + |d_{4,3}| \frac{M_\nu}{1 - |\nu|^{16}}$$

where  $M_\gamma$  and  $M_\nu$  are defined similarly as  $M$  above, but for buckets of values strictly below 16, with a distance of at least 4 between them, and with sums of powers of  $\gamma$  for  $M_\gamma$  and sum of powers of  $\nu$  for  $M_\nu$ . Then we prove that  $M_\gamma = |1 + \gamma^5 + \gamma^9 + \gamma^{14}| \approx 2.58985$  and  $M_\nu = |1 + \nu^4 + \nu^8 + \nu^{12}| \approx 1.74437$  and finally  $|\delta_4(n)| \leq 1.998$ . More details can be found in the file `LimCase3.v` of the Coq development [11].  $\square$

<sup>4</sup>Actually this set  $\mathbb{B}$  of all possible buckets has a cardinal of  $A_{3,9} = 41$ , see Proposition 3.2.



Some computer experiments let us think that there is still room for improvement here, since apparently  $-0.707 < \delta_3(n) < 0.852$  and  $-1.402 < \delta_4(n) < 1.463$ . Concerning  $\delta_3$ , see in particular the Figure 8.1 below. But the bounds proved above are already enough for some nice corollaries.

**8.4. Applications of these (un)bounded distances.** In the OEIS pages [A5206](#) and [A5375](#) about  $F_3$  and  $F_4$  [1], Cloitre conjectured twenty year ago that  $F_3 - \lfloor \alpha_3 n \rfloor$  is either 0 or 1, and the same concerning  $F_4 - \lfloor \alpha_4 n \rfloor$ . We can now prove this conjecture concerning  $F_3$  and a corrected version of the conjecture about  $F_4$ .

**Corollary 8.8.** *For all  $n \geq 0$ ,  $F_3(n) - \lfloor \alpha_3 n \rfloor \in \{0, 1\}$  and  $F_4(n) - \lfloor \alpha_4 n \rfloor \in \{-1, 0, 1, 2\}$ . All these situations are reached.*

*Proof.* From the previous theorem,  $F_3(n) - 1 < \alpha_3 n < F_3(n) + 1$  hence the integer  $\lfloor \alpha_3 n \rfloor$  can only be  $F_3(n) - 1$  or  $F_3(n)$ . We handle  $F_4$  similarly. Now,  $F_3(0) = 0 = \lfloor 0 \alpha_3 \rfloor$  and  $F_3(1) = 1 = 1 + \lfloor 1 \alpha_3 \rfloor$ . For  $F_4(n)$ , taking  $n = 0$  and  $n = 1$  also exhibits a difference of 0 and 1 with  $\lfloor \alpha_4 n \rfloor$ . Differences of  $-1$  and  $2$  are less frequent, their first occurrences are  $n = 120$  for  $2$  and  $n = 243$  for  $-1$ .  $\square$

We can also express that for  $k \leq 4$ , the functions  $F_k$  are close to be additive maps.

**Definition 8.9.** A function  $f : \mathbb{N} \rightarrow \mathbb{N}$  is said to be *almost additive* when

$$\sup_{(n,m) \in \mathbb{N}^2} |f(n+m) - f(n) - f(m)| < +\infty.$$

Interestingly, A'Campo [2] called *slopes* similar functions on integers and showed that they could be used as an alternative construction for  $\mathbb{R}$ .

**Corollary 8.10.** *Both  $F_3$  and  $F_4$  are almost additive: for all  $n, m \geq 0$  we have*

$$\begin{aligned} |F_3(n+m) - F_3(n) - F_3(m)| &\leq 2 \\ |F_4(n+m) - F_4(n) - F_4(m)| &\leq 5. \end{aligned}$$

*Proof.*  $|F_3(n+m) - F_3(n) - F_3(m)|$  can be reformulated as  $|(F_3(n+m) - \alpha_3 n + \alpha_3 m) - (F_3(n) - \alpha_3 n) - (F_3(m) - \alpha_3 m)| < 3$ . Hence this integer quantity is at most 2. Similarly, for  $F_4$  a strict bound is  $3 \cdot 2$ , hence the large inequality with 5.  $\square$

For  $F_3$ , this almost-additivity bound of 2 is reached: for instance  $F_3(18+78) = 65$  while  $F_3(18) + F_3(78) = 13 + 54 = 67$  or  $F_3(39+168) = 142$  while  $F_3(39) + F_3(168) = 26 + 114 = 140$ . But the bound 5 for  $F_4$  appears to be a large over-estimate and could probably be improved to 3.

For similar reasons,  $F_1$  and  $F_2$  are also almost additive, with bounds 1. On the contrary,  $F_k$  cannot be almost additive when  $k \geq 5$ :

**Proposition 8.11.** *For  $k \geq 5$ ,  $\sup_{(n,m) \in \mathbb{N}^2} |F_k(n+m) - F_k(n) - F_k(m)| = +\infty$ .*

*Proof.* Let  $k \geq 5$ . Suppose there exists a constant  $C$  such that for all  $n, m \geq 0$  we have  $|F_k(n+m) - F_k(n) - F_k(m)| \leq C$ . Then consider  $u_n = F_k(n) + C$ . This sequence is subadditive, since  $u_{n+m} = F_k(n+m) + C \leq F_k(n) + F_k(m) + 2C = u_n + u_m$ . Hence by Fekete's subadditive lemma,  $\lim_{n \rightarrow \infty} \frac{1}{n} u_n$  exists and is equal to  $\inf_{n \rightarrow \infty} \frac{1}{n} u_n$ . Now, from Theorem 6.3 we know that  $\lim_{n \rightarrow \infty} \frac{1}{n} u_n = \alpha_k$ . Hence  $\inf_{n \rightarrow \infty} \frac{1}{n} u_n = \alpha_k$  and so  $u_n \geq \alpha_k n$  for all  $n$ , i.e.,  $F_k(n) - \alpha_k n \geq -C$ . Similarly,  $v_n = F_k(n) - C$  is superadditive and mutatis mutandis we also get  $F_k(n) - \alpha_k n \leq C$ . All in all,  $|\delta_k(n)| < C$  for all  $n$ , which contradicts either Theorem 8.2 or Theorem 8.5.  $\square$

From Theorem 8.7 and the bounds on  $\Delta_3$  and  $\Delta_4$ , one may also derive bounds on the difference between  $F_3^j$  and its linear equivalent, and similarly for  $F_4^j$ . For example:

$$|F_3^2(n) - \alpha_3^2 n| = |F_3(F_3(n)) - \alpha_3 F_3(n) + \alpha_3(F_3(n) - \alpha_3 n)| < 1 + \alpha_3 < 2.$$

Here again, the above bound is quite crude. But a strict bound of 1 would not work anyway, since  $|F_3^2(n) - \alpha_3^2 n|$  does slightly exceed 1 occasionally<sup>5</sup>.

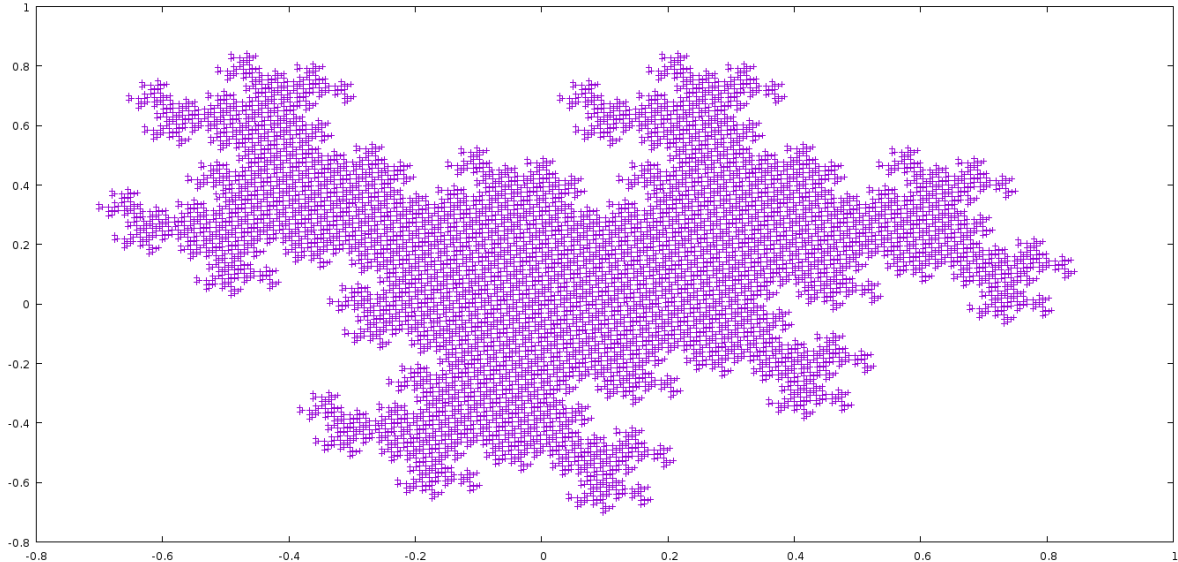


FIGURE 8.1. A Jacobi-Perron fractal: displaying  $(\delta_3(n), \delta_3(F_3(n)))$  for  $n < 10^4$ .

During this study of  $\delta_3$ , one of the authors made a serendipitous encounter with the fractal shape displayed in Figure 8.1, when displaying the points  $(\delta_3(n), \delta_3(F_3(n)))$  (for example here for  $n < 10^4$ ). Actually, this fractal happens to be the cover illustration of the reference book named after Pytheas Fogg [7]; see in particular its Chap. 8, where this fractal is associated with the modified Jacobi-Perron substitution  $\sigma(1, 0)$  (our  $\tau_3$  up to a permutation of letters). More generally, it is close to the Rauzy fractal [14] obtained from the tribonacci substitution. When giving a closer look, one may notice that our figure is slightly distorted compared with the illustrations in Pytheas Fogg [7]: the constructions done there amount to consider  $(F_3(n) - \alpha_3 n, F_3^2(n) - \alpha_3^2 n)$  or some obvious symmetry of it, and as we have just seen, this may be written as  $(\delta_3(n), \delta_3(F_3(n)) + \alpha_3 \delta_3(n))$ . So the distortion is actually the linear transformation  $(x, y) \mapsto (x, y \pm \alpha_3 x)$ . Either way, it is worth noting that such a spectacular fractal can be obtained in such a lightweight way, via solely computing some images of the recursive function  $F_3$  (i.e., Hofstadter’s  $H$ ) as well as a constant  $\alpha_3 \approx 0.6823278038280193$ , zero of  $X^3 + X - 1$ . In particular, no need for matrix reduction nor axis projection.

As a future work, the case  $k = 4$  deserves some more investigation, since it shares many aspects with the case  $k = 3$ , in particular  $\beta_4 = 1/\alpha_4$  is also a Pisot number, and  $\delta_4$  is also bounded, with similar expression in terms of number decompositions. Despite all that, our preliminary attempts lead to images that were not so visually appealing, with rather “smooth” clouds of points.

As a final remark, recall that  $F_1(n) = \lfloor \frac{n+1}{2} \rfloor = \lceil \frac{n}{2} \rceil$  and  $F_2(n) = \lfloor \frac{n+1}{\varphi} \rfloor$ . We can now prove that no such expressions exist when  $k \geq 3$ : in this case  $F_k$  cannot be anymore the exact integer part of an affine function.

<sup>5</sup>In our experiments with  $n$  up to  $10^6$ , this happens about 0.1% of the time.

**Proposition 8.12.** *For all  $k \geq 3$  and  $a, b \in \mathbb{R}$ , there exists  $n, n' \geq 0$  such that  $F_k(n) \neq \lfloor an+b \rfloor$  and  $F_k(n') \neq \lceil an'+b \rceil$ .*

*Proof.* Suppose the existence of  $a, b \in \mathbb{R}$  such that  $F_k(n) = \lfloor an+b \rfloor$  for all  $n \geq 0$  or  $F_k(n) = \lceil an+b \rceil$  for all  $n \geq 0$ . Since  $\lim_{n \rightarrow \infty} \frac{1}{n} F_k = \alpha_k$ , the coefficient  $a$  must be  $\alpha_k$ . Then any of the above expressions would lead to an finite bound on  $|F_k(n) - \alpha_k n|$ , which we proved impossible for  $k \geq 5$ . Finally, for  $k \in \{3, 4\}$ , we use some particular cases to find a contradiction:

- $F_3(5) = 4$  so  $\lfloor 5\alpha_3 + b \rfloor = 4$  would mean  $4 \leq 5\alpha_3 + b < 5$  hence  $b \geq 4 - 5\alpha_3 > 0.58$
- $F_3(8) = 5$  so  $\lfloor 8\alpha_3 + b \rfloor = 5$  would mean  $5 \leq 8\alpha_3 + b < 6$  hence  $b < 6 - 8\alpha_3 < 0.55$  which is incompatible with the previous constraint.
- The same values of  $F_3(5)$  and  $F_3(8)$  also prevent the existence of a real  $b$  such that  $F_3(n) = \lceil \alpha_3 n + b \rceil$ .
- Similarly,  $F_4(2) = 1$  so  $\lfloor 2\alpha_4 + b \rfloor = 1$  would give  $b < 0.56$  while  $F_4(6) = 5$  would give  $b \geq 0.65$ , contradiction. The same values also make it impossible to have  $F_4(n) = \lceil \alpha_4 n + b \rceil$ .

□

## 9. CONCLUSION

The Figure 9.1 provides a synthetic view of the obtained results concerning the functions  $F_k$ .

	$F_1$	$F_2$	$F_3$	$F_4$	$F_5$	$F_k$ for $k \geq 6$
Hofstadter's name		$G$	$H$			
Mean slope $\alpha_k$	0.5	$\varphi-1$	$\approx 0.682$	$\approx 0.724$	$\approx 0.754$	$\text{root}(X^k+X-1)$
Sup $ F_k(n) - \alpha_k n $	0.5	$\varphi-1$	$< 1$	$< 2$	$O(\ln(n))$	$O(n^a), 0 < a < 1$
Exact expression	$\lfloor \frac{n}{2} \rfloor = \lfloor \frac{n+1}{2} \rfloor$	$\lfloor \frac{n+1}{\varphi} \rfloor$	$\times$	$\times$	$\times$	$\times$
Almost expression			$\lfloor \alpha_3 n \rfloor + \{0, 1\}$	$\lfloor \alpha_4 n \rfloor + \{-1, 0, 1, 2\}$	$\times$	$\times$
Almost additive	✓	✓	✓	✓	$\times$	$\times$
$\beta_k = \frac{1}{\alpha_k}$ is Pisot	✓	✓	✓	✓	✓*	$\times$

FIGURE 9.1. Summary of obtained results.

Let us elaborate some more:

- Here, by “exact expression”, we mean exact integer part of an affine function, and Proposition 8.12 disproved its existence for  $F_k$  when  $k \geq 3$ .
- Similarly, an “almost expression” is here a finite amount of possible differences from such an exact expression. This cannot exist for  $F_k$  when  $k \geq 5$ , since  $|F_k(n) - \alpha_k n|$  is unbounded in this case.
- The above entry ✓\* reminds that  $F_5$  is quite specific: even though  $\beta_5$  is indeed a Pisot number (it is actually the smallest possible one, i.e., the Plastic Ratio), the polynomial  $Q_5 = X^5 - X^4 - 1$  associated with  $F_5$  is composed and admits two secondary zeros of modulus 1, leading to an unbounded  $|F_5 - \alpha_5 n|$ .

## APPENDIX A. COEFFICIENTS OF PROPOSITION 7.1 SEEN AS POLYNOMIAL ZEROS

**Proposition A.1.** *For any  $k \geq 1$ , the coefficients  $c_{k,i}$  seen in Proposition 7.1 are the  $k$  zeros of a polynomial in  $\mathbb{Z}[X]$  of degree  $k$ , with leading coefficient  $k^k + (k-1)^{k-1}$  when*

$k > 1$  (or 1 when  $k = 1$ ) and constant coefficient  $(-1)$ . This is also true for the coefficients  $\tilde{c}_{k,i}$  seen during the proof of Proposition 7.1.

*Proof.* We first prove the case of coefficients  $\tilde{c}_{k,i}$ . For that, we consider the monic polynomial  $T = \prod_i (X - (\tilde{c}_{k,i})^{-1})$ . Up to their signs, the coefficients of  $T$  are elementary symmetric polynomials on  $((\tilde{c}_{k,i})^{-1})_{0 \leq i < k} = (Q'_k(r_{k,i}))_{0 \leq i < k}$  and hence symmetric polynomials (in  $\mathbb{Z}[X]$ ) on  $(r_{k,i})_{0 \leq i < k}$ . By the fundamental theorem of symmetric polynomials, they can hence be expressed as polynomials in  $\mathbb{Z}[X]$  of the elementary symmetric polynomials on  $(r_{k,i})_{0 \leq i < k}$ . The latter are known to be integers (actually 0 or  $\pm 1$ ), since  $r_{k,i}$  are zeros of  $Q_k$ , a monic polynomial in  $\mathbb{Z}[X]$ . This allows to conclude that the polynomial  $T$  belongs to  $\mathbb{Z}[X]$ , and so does its reciprocal polynomial, whose zeros are all the  $\tilde{c}_{k,i}$ . Actually, we rather consider here the opposite of the reciprocal of  $T$ , hence its constant coefficient  $(-1)$ . Its leading coefficient is the opposite of the constant coefficient of  $T$ , i.e.,

$$\begin{aligned} (-1)^{k-1} \prod_i (\tilde{c}_{k,i})^{-1} &= (-1)^{k-1} \prod_i Q'_k(r_{k,i}) \\ &= (-1)^{k-1} \prod_{i \neq j} (r_{k,i} - r_{k,j}) \\ &= (-1)^{\frac{1}{2}(k-1)(k-2)} \prod_{i < j} (r_{k,i} - r_{k,j})^2 \\ &= (-1)^{\frac{1}{2}(k-1)(k-2)} \text{Disc}(Q_k) \end{aligned}$$

where  $\text{Disc}(Q_k)$  is the discriminant of the polynomial  $Q_k$  (or equivalently the square of the determinant of the matrix  $V$  seen in the proof of Proposition 7.1). This discriminant of  $Q_k$  is also equal to the discriminant of  $P_k$ , since  $P_k$  is the opposite of the reciprocal of  $Q_k$ . This discriminant can be proved to be

$$(-1)^{\frac{1}{2}(k-1)(k-2)} \cdot (k^k + (k-1)^{k-1})$$

for  $k > 1$  (or 1 when  $k = 1$ ). This is a consequence of the work of Selmer [16], it can also be re-obtained by computing the adequate Sylvester matrix. Finally, the desired leading coefficient is indeed  $k^k + (k-1)^{k-1}$  when  $k > 1$  and 1 otherwise.

Now, for the coefficients  $c_{k,i}$  we proceed similarly, except that this time  $(c_{k,i})^{-1}$  can be formulated as the value of an integer polynomial evaluated at  $(r_{k,i})^{-1}$ :

$$(c_{k,i})^{-1} = \frac{k r_{k,i} - (k-1)}{r_{k,i}^k} = k((r_{k,i})^{-1})^{k-1} - (k-1)((r_{k,i})^{-1})^k.$$

(Actually, with a bit more effort, we could also have exhibited another integer polynomial evaluated this time at  $r_{k,i}$ ). Since  $(r_{k,i})^{-1}$  is a zero of the polynomial  $P_k(X) = X^k + X - 1$ , we can conclude in the same way as before: the coefficients  $c_{k,i}$  are the zeros of a polynomial in  $\mathbb{Z}[X]$  with constant coefficient  $(-1)$ . Finally the leading coefficient is the same as before, since

$$\prod_i c_{k,i} = \prod_i (\tilde{c}_{k,i} r_{k,i}^{2k-2}) = \left( \prod_i \tilde{c}_{k,i} \right) \left( \prod_i r_{k,i} \right)^{2(k-1)} = \prod_i \tilde{c}_{k,i}.$$

Indeed, the product of the zeros  $r_{k,i}$  are  $\pm Q_k(0) = \pm 1$ . □

## REFERENCES

- [1] OEIS Foundation Inc. (2024). The on-line encyclopedia of integer sequences. Published electronically at <https://oeis.org>.
- [2] Norbert A'Campo. A natural construction for the real numbers, 2003.
- [3] Keith Conrad. Irreducibility of  $x^n - x - 1$ . Published electronically at <https://kconrad.math.uconn.edu/blurbs/ringtheory/irredselmerpoly.pdf>.
- [4] Peter J. Downey and Ralph E. Griswold. On a family of nested recursions. *Fibonacci Quarterly*, 22(4):310–317, 1984.
- [5] J. Dufresnoy and Ch. Pisot. étude de certaines fonctions méromorphes bornées sur le cercle unité. Application à un ensemble fermé d'entiers algébriques. *Annales scientifiques de l'École Normale Supérieure*, 3e série, 72(1):69–92, 1955.
- [6] Larry Ericksen and Peter G. Anderson. Patterns in differences between rows in k-zeckendorf arrays. *Fibonacci Quarterly*, 50(1):11–18, 2012.
- [7] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.
- [8] D. Gault and M. Clint. "Curiouser and curiouser" said Alice. Further reflections on an interesting recursive function. *Int. J. Comput. Math.*, 26(1):35–43, 1988.
- [9] Douglas R. Hofstadter. *Gödel, Escher, Bach: An Eternal Golden Braid*. Basic Books, Inc, NY, 1979.
- [10] Clark Kimberling. The Zeckendorf array equals the Wythoff array. *Fibonacci Quarterly*, 33(1):3–8, 1995.
- [11] Pierre Letouzey. Coq proofs about Hofstadter's function G. 2015–2025. [https://github.com/letouzey/hofstadter\\_g](https://github.com/letouzey/hofstadter_g).
- [12] Pierre Letouzey, Shuo Li, and Wolfgang Steiner. Pointwise order of generalized Hofstadter functions G, H and beyond. 2024. To appear. Preprint available at <https://arxiv.org/abs/2410.00529>.
- [13] D.S. Meek and G.H.J. Van Rees. The solution of an iterated recurrence. *Fibonacci Quarterly*, 22(2):101–104, 1984.
- [14] Gérard Rauzy. Nombres algébriques et substitutions. *Bulletin de la Société Mathématique de France*, 110:147–178, 1982.
- [15] Raphaël Salem and Lennart Carleson. *Algebraic numbers and Fourier analysis*. Heath Boston, 1963.
- [16] Ernst Selmer. On the irreducibility of certain trinomials. *Mathematica Scandinavica*, 4(2):287–302, 1956.
- [17] Carl Ludwig Siegel. Algebraic integers whose conjugates lie in the unit circle. *Duke Mathematical Journal*, 11:597–602, 1944.
- [18] The Coq Development Team. The Coq proof assistant. 1985–2024. <https://coq.inria.fr>.

UNIVERSITÉ PARIS CITÉ, CNRS, INRIA, IRIF, F-75013 PARIS, FRANCE  
Email address: [letouzey@irif.fr](mailto:letouzey@irif.fr)