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► **To cite this version:**

Marianne Akian, Stéphane Gaubert, Loïc Marchesini. Escape Rate Games. 26th International Symposium on Mathematical Theory of Networks and Systems (Extended Abstract), Aug 2024, Cambridge (Angleterre), United Kingdom. hal-04912571

HAL Id: hal-04912571

<https://inria.hal.science/hal-04912571v1>

Submitted on 27 Jan 2025

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Escape Rate Games

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Abstract: We consider a new class of repeated zero-sum games in which the payoff of one player is the escape rate of a dynamical system which evolves according to a nonexpansive nonlinear operator depending on the actions of both players. Considering order preserving finite dimensional linear operators over the positive cone endowed with Hilbert's projective (hemi-)metric, we recover the matrix multiplication games, introduced by Asarin et al., which generalize the joint spectral radius of sets of nonnegative matrices and arise in some population dynamics problems (growth maximization and minimization). We establish a two-player version of Mañé's lemma characterizing the value of the game in terms of a nonlinear eigenproblem. This generalizes to the two-player case the characterization of joint spectral radii in terms of extremals norms. This also allows us to show the existence of optimal strategies of both players.

Keywords: Mean-payoff games, Stochastic Games, Joint Spectral Radius, Population dynamics, Growth optimization, Weak-KAM theory, Robust Markov Decision Processes.

1. INTRODUCTION

The joint spectral radius quantifies the maximal growth rate achievable by controlled switched linear systems. Barabanov (1988) characterized the joint spectral radius in terms of norms satisfying a functional equation. Blondel and Tsitsiklis (2000) showed that computing the joint spectral radius is generally undecidable. Approximations algorithms have been developed, see Guglielmi and Protasov (2011) or Möller and Reif (2014). See Jungers (2009) for an overview.

Asarin et al. (2016) considered the generalization to the two-player case of joint spectral radii, under the name of *matrix multiplication games*. Here, a linear switched dynamics is jointly controlled by a pair of players; one player wants to maximize the growth rate whereas the other player wants to minimize it. This was motivated by the study of topological entropies of transition systems arising in program and systems verification. Similar games arise in population dynamics, in particular in problems from mathematical biology and epidemiology, see in particular Billy et al. (2014); Calvez et al. (2014); Chakrabarti et al. (2008). They also arise in robust Markov decision processes, in which the second player represents the uncertainty, see Goyal and Grand-Clément (2023).

Entropy games constitute a somehow easier class of matrix multiplication games, in which the matrices are nonnegative and the set of transition matrices has a rectangular structure (being closed by row interchange). Akian et al. (2019) showed that entropy games reduce to stochastic mean-payoff games in which the paiement is given by a Kullback-Leibler divergence.

Beyond entropy games, matrix multiplication games appear to be a hard class of games, for which no effective results seem to be previously known.

In this paper, we study the subclass of matrix multiplication games in which the transition matrices are nonnegative, but constitute *non-rectangular* sets. Nonnegative matrices acting on the open orthant are nonexpansive in canonical weak metrics, including Hilbert's and Funk's metric. Hence, adopting a point of view from metric geometry, we study a general class of games, allowing nonexpansive possibly non-linear dynamics with respect to a weak metric. We call them the *escape rate games* since one player wants to maximize an escape rate and the other one wants to minimize it. Our main result (Theorem 6) show that an escape rate game always has a value, and even a *uniform value* in the sense of Mertens and Neyman (1981) (Corollary 8). Moreover, theorem 6 characterizes the value in terms of the largest sub-eigenvalue of an infinite dimensional non-linear fixed-point problem. This provides a generalization to the two-player case of the extremal norms arising in the study of joint spectral radii, as well as of the discrete weak-KAM solutions arising in ergodic control. In particular, Theorem 5 may be thought of as a two player version of Mañé's lemma (1992). It also extends to the "non-tubular" case a characterization of Bousch and Mairese (2001) of joint spectral radius-like functionals of families of nonexpansive mappings. It finally provides a generalization, passing from the zero-player to the two-player case, of the characterization of the escape rate of nonexpansive maps in terms of horofunctions, by Gaubert and Vigerel (2011), showing that horofunctions generally need to be replaced by 1-Lipschitz functions in the two-player case.

We illustrate these results by solving a "simple" special class of escape rate game, in which the nonexpansive dynamics consists of translations acting on the Euclidean space. We give an explicit formula for an extremal super-eigenvector, based on a result of discrete geometry (Shapley-Folkman lemma). This leads to a characteriza-

tion of the value and to explicit optimal policies (Theorem 9).

2. GAME MODEL

2.1 Hemi-metric spaces

We shall need the following notion of asymmetric metric, taken from Gaubert and Vignal (2011).

Definition 1. (Hemi-metric). A map $d : X \times X \rightarrow \mathbb{R}$ is called a *hemi-metric* on the set X if it satisfies the two following conditions:

- (1) $\forall (x, y, z) \in X^3, d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequality)
- (2) $\forall (x, y) \in X^2, d(x, y) = d(y, x) = 0$ if and only if $x = y$

Papadopoulos and Troyanov (2008) require in addition that d be nonnegative, which is undesirable in our applications. An example of hemi-metric on \mathbb{R}^n is the map $\delta_n(x, y) := \max_{i \in [n]} (x_i - y_i)$.

Given a hemi-metric d , the map

$$d^\circ(x, y) = \max(d(x, y), d(y, x)) ,$$

is always a metric on X . We will refer to it as the *symmetrized metric* of d . In the sequel, we equip X with the topology induced by d° .

Given two hemi-metric spaces (X, d) and (Y, d') , we introduce the notion of a 1-*Lipschitz* or *nonexpansive* function ν with respect to d and d' . Such a function verifies $d'(\nu(x), \nu(y)) \leq d(x, y)$ for all $x, y \in X$. We will especially be interested in the nonexpansive functions from (X, d) to (\mathbb{R}, δ_1) .

2.2 The Escape Rate Game

The *Escape Rate Game* is the following 2-player deterministic perfect information game. We fix two non-empty compact sets \mathcal{A} and \mathcal{B} , that will represent the action spaces of the two players. We also fix a hemi-metric space (X, d) , which will play the role of the state space of the game. To each pair $(a, b) \in \mathcal{A} \times \mathcal{B}$, we associate a non-expansive self-map T_{ab} of (X, d) . The initial state $x_0 \in X$ is given. We construct a sequence of states $(x_k)_{k \geq 0}$ as follows. Both players observe the current state x_k . At each turn $k \geq 1$, player *Min* chooses an action $a_k \in \mathcal{A}$ and then, after having observed the action a_k , player *Max* chooses an action $b_k \in \mathcal{B}$. The next state is given by $x_k = T_{a_k b_k}(x_{k-1})$. In the game in horizon k , denoted by Γ_k , there are k successive turns, and player *Min* pays to *Max* the following amount:

$$J_k(a_1 b_1 \dots a_k b_k) = d(T_{a_k b_k} \circ \dots \circ T_{a_1 b_1}(x_0), x_0) .$$

We will also consider the infinite horizon *mean payoff game*, denoted Γ_∞ . Then, an infinite number of turns are played, and player *Min* wishes to minimize the *escape rate*

$$J_\infty(a_1 b_1 a_2 b_2 \dots) := \limsup_{k \rightarrow \infty} \frac{J_k(a_1 b_1 \dots a_k b_k)}{k} .$$

while player *Max* wants to maximize it.

A *strategy* is a map that assigns an action to any finite history of the game. A history of the game at a given

stage is the sequence of states and actions of the game up to this stage. Since the initial state x_0 is fixed and known to both players, the history is uniquely determined by the sequence of actions of the two players up to this stage. The set of strategies of *Min* (resp *Max*) will be written \mathcal{A}^s (resp \mathcal{B}^s).

We denote by $J_k(\sigma, \tau)$ the payoff of Γ_k , assuming that the sequence of actions $a_1, b_1, a_2, b_2, \dots$ is generated according to the strategies σ and τ , and similarly, we denote by $J_\infty(\sigma, \tau)$ the escape rate defined above. We are interested in the existence of the value for each of these zero-sum games. The value is the unique quantity that the players can both guarantee. Formally a game with strategy spaces \mathcal{A}^s and \mathcal{B}^s and payoff function $\mathcal{A}^s \times \mathcal{B}^s \rightarrow \mathbb{R}, (\sigma, \tau) \mapsto J(\sigma, \tau)$, has a value $\lambda \in \mathbb{R}$ if $\forall \epsilon > 0, \exists \sigma^* \in \mathcal{A}^s, \tau^* \in \mathcal{B}^s$ such that

$$J(\sigma^*, \tau) \leq \lambda + \epsilon \text{ and } J(\sigma, \tau^*) \geq \lambda - \epsilon$$

for all strategies $\sigma \in \mathcal{A}^s$ and $\tau \in \mathcal{B}^s$.

Such strategies are called ϵ -optimal strategies. When $\epsilon = 0$, one gets *optimal strategies* σ^*, τ^* , verifying

$$\forall \sigma, \tau \quad J(\sigma^*, \tau) \leq J(\sigma^*, \tau^*) = \lambda \leq J(\sigma, \tau^*)$$

Remark 2. We can think of this game as a "traditional" *Mean Payoff Game* played on an infinite graph. Consider the set of vertices $\{x_0\} \cup \{T_{a_k b_k} \circ \dots \circ T_{a_1 b_1}(x_0) \mid k \in \mathbb{N}^*, (T_{a_1 b_1}, \dots, T_{a_k b_k}) \in (T_{ab})_{(a,b) \in \mathcal{A} \times \mathcal{B}}\}$ where there is an arc from x to y if there exists T_{ab} with $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that $y = T_{ab}(x)$. The payoff attributed to an arc $x \rightarrow y$ is $d(y, x_0) - d(x, x_0)$.

3. EXAMPLES

We now describe two examples of escape rate games.

3.1 Nonnegative Matrix Multiplication Games

We consider the state space $X := (\mathbb{R}_{>0})^n$, equipped with the *Funk* hemi-metric

$$d(y, z) = \text{Funk}(y, z) := \log \max_{i \in [n]} \frac{y_i}{z_i} .$$

This metric arises in Hilbert's geometry, see Papadopoulos and Troyanov (2008). We denote by \mathcal{M}_+^n the set of nonnegative matrices with at least one non-zero entry per column. We consider two non-empty compact subsets \mathcal{A} and \mathcal{B} of \mathcal{M}_+^n . Each pair of actions $(A, B) \in \mathcal{A} \times \mathcal{B}$ induces the self-map of $\mathbb{R}_{>0}^n$ given by

$$T_{AB}(x) := xAB ,$$

in which x is interpreted as a row vector. An elementary observation from Perron-Frobenius theory entails that T_{AB} is nonexpansive in the Funk hemi-metric. This yields an escape rate game, in which the payoff is given by

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \frac{\text{Funk}(x_0 A_1 B_1 \dots A_k B_k, x_0)}{k} \\ &= \limsup_{k \rightarrow \infty} \log \max_{i \in [n]} \left(\frac{(x_0 A_1 B_1 \dots A_k B_k)_i}{(x_0)_i} \right)^{\frac{1}{k}} \\ &= \log \left(\limsup_{k \rightarrow \infty} \|x_0 A_1 B_1 \dots A_k B_k\|^{1/k} \right) , \end{aligned} \quad (1)$$

for any choice of the norm $\|\cdot\|$ on \mathbb{R}^n . This is a special case of the *Matrix Multiplication Games* which were introduced in Asarin et al. (2016). In general matrix multiplication

games, the matrices need not be nonnegative, and the payment is defined as in (1).

Such models arise in population dynamics: x_k represents a population profile at time k ; one player wishes to minimize the growth rate of the population whereas the other player wishes to maximize it. Specific applications arise, in the one player case, in optimization of therapy, see Billy et al. (2014), or in the prion amplification, see Calvez et al. (2014). The two-player case is relevant to epidemiology (propagation of computer virus, in which hackers want to maximize spreading whereas a regulator is trying to prevent it, see Chakrabarti et al. (2008)).

3.2 Vector Addition Games

We consider the space $X = \mathbb{R}^n$ equipped with the metric d induced by a norm $\|\cdot\|$. The set of actions are now two non-empty compact subsets \mathcal{A} and \mathcal{B} of \mathbb{R}^n . Each pair of actions $(a, b) \in \mathcal{A} \times \mathcal{B}$ induces a translation T_{ab} acting on \mathbb{R}^n , given by $T_{ab}(x) = x + a + b$, which is not only nonexpansive but also an isometry. The state of the game evolves according to the dynamics $x_k = a_k + b_k + x_{k-1}$. The escape rate is given by

$$J_\infty(a_1 b_1 a_2 b_2 \dots) = \limsup_{k \rightarrow \infty} \frac{\|a_1 + b_2 + \dots + a_k + b_k\|}{k} .$$

We solve this game in §5.

4. MAIN RESULTS

To prove the existence and characterize the value of escape rate games, we use an operator approach. We make the following assumption.

Assumption 3. For all $x \in X$, the maps $b \mapsto T_{ab}(x)$ and $a \mapsto T_{ab}(x)$ are continuous, and for all compact sets K , the set $\{T_{ab}(x) \mid (a, b, x) \in \mathcal{A} \times \mathcal{B} \times K\}$ is compact.

This is trivially verified if $(a, b, x) \mapsto T_{ab}(x)$ is jointly continuous.

We denote by Lip_1 the set of 1-Lipschitz functions from the hemi-metric space (X, d) to (\mathbb{R}, δ_1) (recall that $\delta_1(x, y) = x - y$). We define the following operator S acting on Lip_1 ,

$$Sv(x) := \inf_{a \in \mathcal{A}} \sup_{b \in \mathcal{B}} v(T_{ab}(x)) .$$

Assumption 3 entails that $Sv(x)$ is finite. It is also required to prove the continuity of the operator S in the topology of uniform convergence on compact sets.

We define

$$s_k := [S^k d(\cdot, x_0)](x_0) .$$

Lemma 4. We have the following subadditivity property:

$$s_{k+l} \leq s_k + s_l, \text{ for all } k, l \geq 1 .$$

Then, it follows from Fekete subadditive lemma (see Hille and Phillips (1996)) that the following limit does exist, and that we have:

$$\rho := \lim_k \frac{s_k}{k} = \inf_{k \geq 1} \frac{s_k}{k} \in \mathbb{R} .$$

Our initial assumption about the compactness of $\{T_{ab}(x_0) \mid (a, b) \in \mathcal{A} \times \mathcal{B}\}$ ensures that ρ is finite.

The following theorem, which is inspired by Mañé’s lemma in ergodic control (1992), characterizes the scalar ρ in terms of nonlinear “sub-eigenvalues” of S :

Theorem 5. We have

$$\rho = \max\{\lambda \in \mathbb{R} \mid \exists \nu \in \text{Lip}_1, \lambda + \nu \leq S\nu\} \quad (2)$$

Sketch of proof: This follows from the compactness and convexity of the “additive projective” space of Lip_1 (where two functions are equivalent if they only differ by a constant) for the topology of uniform convergence on compact subsets. Indeed, these properties allow us not only to prove the non-emptiness of $\{\lambda \in \mathbb{R} \mid \exists \nu \in \text{Lip}_1, \lambda + \nu \leq S\nu\}$ using the Leray-Schauder-Tychonoff fixed point theorem but also to extract a converging subsequence whose limit is a sub-eigenvector associated to the sub-eigenvalue ρ . This tells us that $\rho \leq \max\{\lambda \in \mathbb{R} \mid \exists \nu \in \text{Lip}_1, \lambda + \nu \leq S\nu\}$. The other inequality is an elementary consequence of the order-preserving property of S and of the fact it commutes with translations by additive constants. ■

Now we state our main theorem.

Theorem 6. The value of the escape rate game exists and is equal to ρ . Furthermore, both players have an optimal strategy.

Sketch of proof: A stationary optimal strategy for *Max* is easily constructed by playing, at turn $k + 1$, an action b_{k+1} maximizing

$$b \mapsto \nu(T_{a_{k+1}b}(x_k))$$

where $\nu \in \text{Lip}_1$ verifies $\rho + \nu \leq S\nu$.

For *Min*, it is a bit trickier as an optimal strategy is constructed by considering the optimal strategies associated with the finite horizon games $(\Gamma_{k_n})_{n \in \mathbb{N}^*}$ where, given a decreasing sequence converging to zero $(\epsilon_n)_{n \in \mathbb{N}^*} \in \mathbb{R}_{>0}^{\mathbb{N}^*}$, k_n is the smallest $k \in \mathbb{N}^*$ such that $s_k/k \leq \rho + \epsilon_n$. Then, these strategies are played successively. ■

A direct corollary of this theorem is the existence of a uniform value as defined in Bolte et al. (2014). See Mertens and Neyman (1981) for more information on the uniform value.

Definition 7. (Uniform Value). A zero-sum repeated game is said to have a *uniform value* v_∞ if both players can guarantee v_∞ provided that they play for a sufficiently long time. Formally v_∞ is the uniform value of the game if $\forall \epsilon > 0$, there exist a couple of strategies $(\sigma_\epsilon, \tau_\epsilon)$ and a time N such that $\forall k \geq N$ and $\forall(\sigma, \tau)$

$$\frac{J_k(\sigma_\epsilon, \tau)}{k} \leq v_\infty + \epsilon, \quad \frac{J_k(\sigma, \tau_\epsilon)}{k} \geq v_\infty - \epsilon .$$

Corollary 8. The escape rate game Γ_∞ has a uniform value, which coincides with ρ .

5. APPLICATION TO VECTOR ADDITION GAMES

As an illustration, we give a complete solution of the vector addition game. We denote by $((\mathbb{R}^n)^*, \|\cdot\|_*)$ the dual normed space of $(\mathbb{R}^n, \|\cdot\|)$.

Theorem 9. The vector addition game has a uniform value equal to

$$\lambda := \max_{b \in \mathcal{B}} \min_{a \in \text{co}(\mathcal{A})} \|a + b\| . \quad (3)$$

Any linear form ℓ achieving the maximum in

$$\max_{\ell \in (\mathbb{R}^n)^*, \|\ell\|_* \leq 1} \max_{b \in \mathcal{B}} \min_{a \in \text{co}(\mathcal{A})} \ell(a + b)$$

is a solution of the eigenproblem $S\ell = \lambda + \ell$. Furthermore Max has a constant optimal strategy consisting in playing any action

$$b^* \in \arg \max_{b \in \mathcal{B}} \min_{a \in \text{co}(\mathcal{A})} \|a + b\| , \quad (4)$$

at every stage. Finally, the function

$$\phi(x) = \text{dist}(x, -(n+1) \text{co}(\mathcal{A})) .$$

where $-(n+1) \text{co}(\mathcal{A})$ has to be understood as a Minkowski sum, satisfies $S\phi \leq \lambda + \phi$. By playing at state $x \in \mathbb{R}^n$ an action a which achieves the minimum in the expression of $S\phi(x)$, we obtain a stationary optimal strategy of Min .

Sketch of proof: Using the *Shapley-Folkman lemma* (see Starr (1981)), we can show that as soon as

$$\phi(x_k) = \text{dist}(x_k, -(n+1) \text{co}(\mathcal{A})) \leq k\lambda \quad (5)$$

there exists $a_{k+1} \in \mathcal{A}$ such that

$$\text{dist}(x_k + a_{k+1}, -n \text{co}(\mathcal{A})) \leq k\lambda .$$

We deduce that, for every $b \in \mathcal{B}$,

$$\text{dist}(x_k + a_{k+1} + b, -(n+1) \text{co}(\mathcal{A})) \leq (k+1)\lambda .$$

Since the initial state doesn't matter in this game, we can always consider the game which starts in $x_0 \in -(n+1) \text{co}(\mathcal{A})$, thus verifying the inequality (5) when $k = 0$. Therefore, by selecting the action a_{k+1} at turn $k+1$, Min guarantees λ . The proof that Max guarantees λ by playing the explicit constant control b^* is elementary. Alternatively, we can take at turn $k+1$ a control b_{k+1} (independent of k or not) maximizing $\ell(a_{k+1} + b + x_k)$, as in the initial part of the proof of Theorem 6. ■

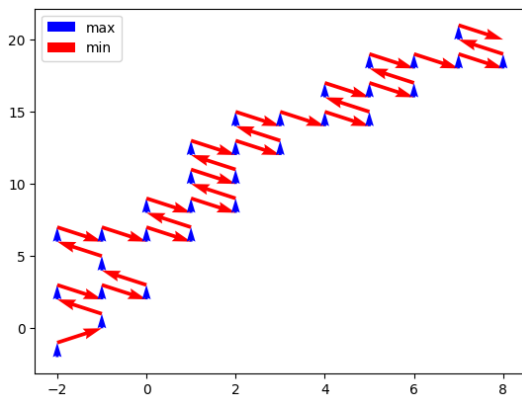


Fig. 1. First 30 optimal moves in the game where $\mathcal{A} = \{(1, 1), (1, -1), (-1, 1)\}$, $\mathcal{B} = \{(0, 1), (0, -1), (-1, 0)\}$

6. CONCLUSION

We studied escape rate games in the two-player setting. We provided a strong duality result characterizing the value. This leads to effective approximation algorithms, to be explored elsewhere. We also wish to extend these results in the continuous time case, in which the value of the escape game rate should be related to the subsolutions of Hamilton-Jacobi-Isaacs-type PDE.

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