Averaged Steklov Eigenvalues, Inside Outside Duality and Application to Inverse Scattering

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Lorenzo Audibert*, Houssem Haddar†, and Fabien Pourre†

Abstract. We introduce a new family of artificial backgrounds corresponding to averaged impedance boundary conditions formulated in an abstract framework. These backgrounds are used to define a finite number of averaged Steklov eigenvalues, which are associated with inverse scattering problems from inhomogeneous media. We prove that these special eigenvalues can be determined from full-aperture, fixed-frequency far-fields using the inside-outside duality method. We then show and numerically demonstrate how this method can be used to reconstruct averaged values of the refractive index.

Key words. Averaged Steklov Eigenvalue, Inverse Scattering, Inside Outside duality, Artificial Background.

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1. Introduction.

This work is motivated by the study of fixed-frequency inverse scattering problems using multistatic data. Specifically, our goal is to develop new inversion algorithms that avoid the use of forward solvers, in the sense of linear sampling methods [21, 11], and that can be applied to the imaging of complex media. This new class of algorithms uses spectral signatures to infer either an indicator function for defects or averaged values of material properties [4, 8, 7].

An example of spectral signatures is the so-called transmission eigenvalues [11, 15]. Their connection with the material properties of the scatterer [11] and the possibility of recovering them from far-field data [10, 6] have led to an imaging method in [4] that provides a quantitative indicator of crack density. The latter inspired the development of analogous fixed-frequency imaging algorithms. In these algorithms, transmission eigenvalues are replaced by artificial spectral parameters associated with artificial backgrounds [8, 7].

The use of artificial backgrounds for the construction of spectral signatures or for the design of imaging algorithms has been explored in various ways in the literature [14, 13, 2, 18, 12, 1]. The objective of this paper is to focus on the concept of averaged Steklov eigenvalues as introduced in [8, 7]. The main advantage is that the spectrum is very simple, since it contains only one non-trivial eigenvalue. This has significant implications for the robustness and accuracy of the associated imaging algorithm.

The novel contributions of this paper are threefold. Firstly, we present a more general definition of averaged Steklov eigenvalues by formulating the underlying impedance boundary condition with an abstract boundary operator of finite rank $M \geq 1$. This results in spectral problems with M nonzero eigenvalues (the parameter M can be chosen arbitrarily). In fact, the abstract model includes the explicit models used in [7] with $M = 1$. Further examples of such operators with $M > 1$ are provided in the numerical section.

The second new contribution is the analysis of the so-called Inside-Outside Duality Method (IODM) [17, 22] which is used to retrieve these spectral parameters. This is in contrasts to

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the work in [4, 7], where a different method is used based on the generalized linear sampling method. The IODM was initially proposed for Dirichlet eigenvalues [22]. It has since been extended to the identification of transmission eigenvalues in [22, 23] under some restrictive assumptions on the material parameters. For a special design of the background parameters in [19, 3] these restrictions were removed. In the present work we will show that the simple structure of the impedance boundary conditions allows a necessary and sufficient condition to characterize the spectral parameters in terms of the phase of the eigenvalues of a modified far-field operator.

Since there are only a few spectral parameters, the IODM allows a simple numerical implementation (compared to the cases in [23, 19]) and provides a faster and more reliable alternative to the method in [6]. This is the core of our third novel contribution. The efficiency of this method is demonstrated by revisiting the imaging algorithm in [7], which uses the IODM to identify the spectral parameter and reconstruct the averaged values of the refractive index.

The paper is organized as follows. Section 2 is devoted to the definition of the artificial background with an abstract formulation of the associated impedance boundary conditions. We then define the associated averaged Steklov eigenvalues (called $\mathcal{B}-$eigenvalues). In Section 3 we introduce the inverse problem for inhomogeneous media and define a modified far field operator relative to the artificial background. A key factorization of this far field operator is then given. Section 4 contains the main theoretical result of this work, which is the characterization of the $\mathcal{B}-$eigenvalues using the IODM. In Section 5 we propose some validating numerical tests against analytical expressions obtained for circular domains. We then propose an algorithm implementing the IODM that can be used in the imaging algorithm proposed in [7]. We conclude with some validating numerical examples where we reconstruct the mean value of refractive index of the probed medium from noisy far field data. Some technical results are given in an appendix.

2. Averaged Steklov Eigenvalues.

Let D be a bounded domain in $\mathbb{R}^m, m = 2, 3$ with piecewise smooth boundary ∂D and connected complement. We denote by ν the outward normal field on ∂D. We shall consider the following scattering problem, with inhomogeneity supported in D. This inhomogeneity is characterized by a real refractive index $n \in L^\infty(\mathbb{R}^m)$ such that $n > 0$ and $n = 1$ in $\mathbb{R}^m \backslash \overline{D}$. For $k > 0$ being the wavenumber, the total field $u \in H^1_{loc}(\mathbb{R}^m)$ associated with some given incident field u^i satisfies the following equations

$$\begin{cases} \Delta u + k^2 n u = 0 \text{ in } \mathbb{R}^m, \\ u = u^s + u^i, \\ \lim\limits_{R \to +\infty} \int_{|x|=R} |\frac{\partial u^s}{\partial r} - iku^s|^2 ds = 0, \end{cases} \tag{2.1}$$

where the incident field is assumed to satisfy the Helmholtz equation in $\mathbb{R}^m$. The motivation of the following developments is to infer some macroscopic properties of the refractive index n from measurements of the far fields associated with incident plane waves. To do so, we shall further develop the approach proposed in [4] by exploiting the notion of modified transmission eigenvalues relative to some artificial background media. In the context here, these eigenvalues correspond with Steklov-like eigenvalues since we shall use a background media that correspond with a generalization of the one introduced in [8, 7]. More precisely, we

shall consider the following model for the background media: the total field for the artificial background media, denoted by $u_b \in H^1_{loc}(\mathbb{R}^m \backslash D_b)$ satisfies the following equations

$$\begin{cases} \Delta u_b + k^2 u_b = 0 \text{ in } \mathbb{R}^m \backslash D_b, \\ \mu u_b + \mathcal{B}(\partial_\nu u_b) = 0 \text{ on } \partial D_b \\ u_b = u_b^s + u^i \\ \lim\limits_{R \to +\infty} \int_{|x|=R} |\frac{\partial u_b^s}{\partial r} - iku_b^s|^2 ds = 0, \end{cases} \tag{2.2}$$

for some incident field u^i and for some parameter $\mu \in \mathbb{R}$. The domain $D_b \subset \mathbb{R}^m$ is supposed to be regular, bounded and simply connected. For simplicity, we assume that $D \subset D_b$. This assumption can be weakened as we shall later explain in Remark 4.10 (Section 4).
The operator $\mathcal{B} : H^{-\frac{1}{2}}(\partial D_b) \to H^{\frac{1}{2}}(\partial D_b)$ in the boundary condition in (2.2) is supposed to be of finite rank. More specifically, in all of the following, we assume that the following hypothesis holds true for the operator $\mathcal{B}$.

Assumption 2.1. *There exists a family of M vectors $\{e_1, ..., e_M\} \subset H^{\frac{1}{2}}(\partial D_b)$ which are orthonormal with respect to the $L^2(\partial D_b)$ scalar product and a positive real constant κ such that:*

$$\mathcal{B}(\psi) = \kappa \sum_{i=1}^{M} \langle \psi, e_i \rangle_{H^{-\frac{1}{2}}(\partial D_b), H^{\frac{1}{2}}(\partial D_b)} e_i, \quad \forall \psi \in H^{-\frac{1}{2}}(\partial D_b). \tag{2.3}$$

This assumption implies in particular that the operator $\mathcal{B}$ satisfies:

$$\kappa(\mathcal{B}\psi, \mathcal{B}\phi)_{L^2(\partial D_b)} = \langle \psi, \mathcal{B}\phi \rangle_{H^{-\frac{1}{2}}(\partial D_b), H^{\frac{1}{2}}(\partial D_b)}, \tag{2.4}$$

for all $\psi, \phi \in H^{-\frac{1}{2}}(\partial D_b)$. In fact, this property (2.4) is also a sufficient condition for the operator $\mathcal{B}$ to be of the form (2.3) (see A).
Using (2.4), one can prove, by standard variational techniques, that the direct problem (2.2) is well posed. The details are also given in A.

Definition 2.2. *A non zero real μ is called a $\mathcal{B}-$averaged Steklov eigenvalue (in short $\mathcal{B}-$eigenvalue) if there exists $w_0 \in H^1(D_b)$ non trivial solution of*

$$\begin{cases} \Delta w_0 + k^2 n w_0 = 0 \text{ in } D_b, \\ \mu w_0 + \mathcal{B}(\partial_\nu w_0) = 0 \text{ on } \partial D_b. \end{cases} \tag{2.5}$$

Let us assume that k^2 is not a Dirichlet eigenvalue of the following problem:

$$\begin{cases} v \in H^1(D_b), \\ \Delta v + k^2 n v = 0 \text{ in } D_b, \qquad v = 0 \text{ on } \partial D_b. \end{cases} \tag{2.6}$$

Then one can construct M functions $w_i \in H^1(D_b)$, $i = 1, ..., M$ solution of:

$$\begin{cases} \Delta w_i + k^2 n w_i = 0 \text{ in } D_b, \\ w_i = e_i \text{ on } \partial D_b. \end{cases} \tag{2.7}$$

It is then easily verified that these functions are eigenvectors of $\mathcal{B}-$eigenvalues μ_i given by:

$$\mu_i := -\kappa \langle \partial_\nu w_i, e_i \rangle = \kappa(k^2 \int_{D_b} n|w_i|^2 dx - \int_{D_b} |\nabla w_i|^2 dx). \tag{2.8}$$

In fact, we prove that these are the only possible eigenvalues if k^2 is not a Dirichlet eigenvalue as stated in the following theorem.

Theorem 2.3. *Assume that k^2 is not an eigenvalue of (2.6), then there are only M non zeros real $\mathcal{B}-$eigenvalues (including multiplicity). These eigenvalues are given by (2.8) (and the eigenvectors by (2.7)) .*

Proof. Let μ be an eigenvalue of (2.5) and denote by u its associated eigenvector. From the boundary condition, we deduce that $u|_{\partial D_b} \in span(e_1, ..., e_M)$. Since k^2 is not a Dirichlet eigenvalue, we deduce that there exists M complex numbers $\alpha_1, ..., \alpha_M$ such that $u = \sum_{i=1}^{M} \alpha_i w_i$ where the w_i are defined in (2.7). Inserting this expression in the boundary condition, one obtains

$$\alpha_i(\mu - \mu_i) = 0, \qquad i = 1, ..., M. \tag{2.9}$$

This shows that there exists $i \leq M$ such that $\mu = \mu_i$ (otherwise u would be trivial). ∎

These $\mathcal{B}-$eigenvalues corresponds to the values of μ for which one can construct an incident field $u^i \in H^1(D_b)$ solution of $\Delta u^i + k^2 u^i = 0$ in D_b for which $u_b^s = u^s$ in $\mathbb{R}^m \backslash D_b$ where u^s, u_b^s are respectively solutions of (2.1), (2.2). More explicitly, for $\mu = \mu_j$, this incident field is given by $u^i = w_j - u_j^s$ where $u_j^s \in H^1_{loc}(\mathbb{R}^m)$ is the solution of:

$$\begin{cases} \Delta u_j^s + k^2 u_j^s = k^2(1-n) w_j \text{ in } \mathbb{R}^m, \\ \lim\limits_{r \to +\infty} \int_{|x|=r} |\frac{\partial u_j^s}{\partial r} - iku_j^s|^2 = 0. \end{cases} \tag{2.10}$$

3. The far field operators and statement of the inverse problem.

As indicated above, our goal is to recover some macroscopic properties of the refractive index n from measurements of the scattered field using $\mathcal{B}-$eigenvalues. The data for this inverse problem is formed by the far fields associated with incident planes waves of the form $u^i(x) = e^{ikx\cdot d}$, denoted $u^i(., d)$ for $d \in \mathbb{S} := \{x \in \mathbb{R}^m, |x| = 1\}$. Let us denote by $u^s(., d)$ and $u(., d)$ the scattered and total field of (2.1) with $u^i = u^i(., d)$. The scattered field has the following expansion for all $\hat{x} \in \mathbb{S}$,

$$u^s(r\hat{x}, d) = \frac{e^{ikr}}{r^{\frac{m-1}{2}}} u^\infty(\hat{x}, d) + O(\frac{1}{r^{\frac{m+1}{2}}})$$

with $u^\infty(., d)$ being the so called far field pattern of the scattered field. The data for the inverse problem is constituted by $u^\infty(\hat{x}, d)$ for all $\hat{x}, d \in \mathbb{S}$. Using this data, we define the far field operator $F : L^2(\mathbb{S}) \to L^2(\mathbb{S})$ by

$$(Fg)(\hat{x}) := \int_{\mathbb{S}} g(d) u^\infty(\hat{x}, d) ds(d), \quad \hat{x} \in \mathbb{S}. \tag{3.1}$$

We remark that $u_g^\infty := Fg$ is the far field pattern of the scattered field u_g^s, solution of (2.1) with $u^i = v_g$ the Herglotz wave function defined for $g \in L^2(\mathbb{S})$ by

$$v_g(x) := \int_{\mathbb{S}} e^{ikxd} g(d) ds(d). \tag{3.2}$$

Introduce the far field constant γ such that

$$\gamma := \begin{cases} 4\pi \text{ if } m = 3, \\ e^{-i\frac{\pi}{4}}\sqrt{8\pi k} \text{ if } m = 2. \end{cases}$$

We recall that the far field operator F is normal and the scattering operator $S := I + \frac{2ik}{\overline{\gamma}} F$ is unitary [16, 21].
The algorithm we propose is based on retrieving the $\mathcal{B}-$eigenvalues from the operator F. This will be done by incorporating the far field operator associated with the artificial background defined by (2.2). We therefore similarly introduce the operator $F_b^\mu : L^2(\mathbb{S}) \to L^2(\mathbb{S})$ defined by

$$(F_b^\mu g)(\hat{x}) := \int_{\mathbb{S}} g(d) u_b^\infty(\hat{x}, d) ds(d), \quad \hat{x} \in \mathbb{S}, \tag{3.3}$$

where $u_b^\infty(., d)$ is the far field associated with $u_b^s(., d)$ the scattered field in (2.2) with $u^i = u^i(., d)$. This operator is computed numerically or analytically if D_b is a sphere. Then we define the modified far field operator

$$\mathcal{F}^\mu := F - F_b^\mu. \tag{3.4}$$

One of the main results of this paper is to show that $\mathcal{B}-$eigenvalues can be determined by applying the inside-outside algorithm to the operator

$$\mathbf{F}^\mu := \gamma S_b^* \mathcal{F}^\mu, \tag{3.5}$$

where $S_b := I + \frac{2ik}{\overline{\gamma}} F_b^\mu$ is the scattering operator associated with the background. This is done in the next section. As a preparatory material, we establish first some needed properties of these far field operators.

3.1. Properties of the far field operators. We introduce the operator $\mathcal{H}_\mathcal{B} : L^2(\mathbb{S}) \to H^{-\frac{1}{2}}(\partial D_b)$ defined as:

$$\mathcal{H}_\mathcal{B} g := \partial_\nu u_{b,g}|_{\partial D_b}, \tag{3.6}$$

where $u_{b,g}$ is solution of (2.2) with $u^i = v_g$, the Herglotz wave function defined in (3.2).
For u_g solution of (2.1) with $u^i = v_g$, set

$$\begin{cases} w^s := u_g - u_{b,g} & \text{in } \mathbb{R}^m \backslash D_b, \\ w := u_g & \text{in } D_b, \end{cases}$$

so that $\mathcal{F}^\mu g = w^\infty$ the far field associated with w^s. By linearity, we obtain that the pair $(w, w^s) \in H^1(D_b) \times H^1_{loc}(\mathbb{R}^m \backslash D_b)$ is solution of:

$$\begin{cases} \Delta w + k^2 n w = 0 \text{ in } D_b, \\ \Delta w^s + k^2 w^s = 0 \text{ in } \mathbb{R}^m \backslash D_b, \\ w^s + \frac{1}{\mu}\mathcal{B}(\partial_\nu w^s) = w + \frac{1}{\mu}\mathcal{B}(\partial_\nu w) \text{ on } \partial D_b, \\ \partial_\nu w - \partial_\nu w^s = \psi \text{ on } \partial D_b, \\ \lim\limits_{r\to+\infty} \int_{|x|=r} |\frac{\partial w^s}{\partial r} - ikw^s|^2 = 0, \end{cases} \tag{3.7}$$

where $\psi = \partial_\nu u_{b,g}$.
The solution of this problem is linear with respect to $\frac{1}{\mu}$. In fact, one can write $w = w_D + \frac{1}{\mu} w_N$ (respectively $w^s = w^s_D + \frac{1}{\mu} w^s_N$) where (w_D, w^s_D) and (w_N, w^s_N) do not depend on μ and are solution of

$$\begin{cases} \Delta w_N + k^2 n w_N = 0 \text{ in } D_b, \\ \Delta w^s_N + k^2 w^s_N = 0 \text{ in } \mathbb{R}^m \backslash D_b, \\ w_N - w^s_N = -\mathcal{B}(\psi) \text{ on } \partial D_b, \\ \partial_\nu w_N - \partial_\nu w^s_N = 0 \text{ on } \partial D_b, \\ \lim\limits_{r\to+\infty} \int_{|x|=r} |\frac{\partial w^s_N}{\partial r} - ikw^s_N|^2 = 0, \end{cases} \qquad \begin{cases} \Delta w_D + k^2 n w_D = 0 \text{ in } D_b, \\ \Delta w^s_D + k^2 w^s_D = 0 \text{ in } \mathbb{R}^m \backslash D_b, \\ w_D - w^s_D = 0 \text{ on } \partial D_b, \\ \partial_\nu w_D - \partial_\nu w^s_D = \psi \text{ on } \partial D_b, \\ \lim\limits_{r\to+\infty} \int_{|x|=r} |\frac{\partial w^s_D}{\partial r} - ikw^s_D|^2 = 0. \end{cases} \tag{3.8}$$

Lemma 3.1. *Problem (3.7) is well posed for all ψ in $H^{-\frac{1}{2}}(\partial D_b)$. Furthermore, the solution (w, w^s) of (3.7) with source term ψ satisfies the estimate for any compact set Ω that contains D_b:*

$$\|\Delta w\|^2_{L^2(D_b)} + \|\Delta w^s\|^2_{L^2(\Omega\backslash D_b)} + \|w^s\|^2_{H^1(\Omega\backslash D_b)} + \|w\|^2_{H^1(D_b)} \leq (C_1 + \frac{1}{\mu} C_2) \|\psi\|^2_{H^{-\frac{1}{2}}(\partial D_b)}, \tag{3.9}$$

where $C_1, C_2 > 0$ do not depend on ψ and μ. In addition, by elliptic regularity, for any compact set $\Omega \subset \mathbb{R}^m \backslash \overline{D_b}$, there exists a constant $C_b > 0$ independant from ψ such that

$$\|w^s\|^2_{H^2(\Omega)} \leq C_b \|\psi\|^2_{H^{-\frac{1}{2}}(\partial D_b)}. \tag{3.10}$$

Proof. This result is a direct consequence on the well posedness of the problems (3.8) and the linearity of the solution of (3.7) with respect to $\frac{1}{\mu}$. ∎

Define the operator $\mathcal{G}_\mathcal{B} : H^{-\frac{1}{2}}(\partial D_b) \to L^2(\mathbb{S})$ such that: $\mathcal{G}_\mathcal{B}(\psi) := w^\infty$, with w^∞ being the far field of w^s the solution of (3.7). We then deduce the following factorization:

$$\mathcal{F}^\mu = \mathcal{G}_\mathcal{B} \mathcal{H}_\mathcal{B}. \tag{3.11}$$

In the following, we shall further expand the factorization of the operator $\mathcal{F}^\mu$ and write it in the form $S_b \mathcal{H}^*_\mathcal{B} T_\mathcal{B} \mathcal{H}_\mathcal{B}$, with a particular operator $T_\mathcal{B}$ whose injectivity is related to the $\mathcal{B}-$eigenvalues. We first state a lemma that characterizes the adjoint of $\mathcal{H}_\mathcal{B}$ denoted by $\mathcal{H}^*_\mathcal{B} : H^{\frac{1}{2}}(\partial D_b) \to L^2(\mathbb{S})$.

Lemma 3.2. *The operator $\mathcal{H}_{\mathcal{B}}$ is injective and its range is dense in $H^{-\frac{1}{2}}(\partial D_b)$. Moreover, the adjoint operator $\mathcal{H}^*_{\mathcal{B}}$ is defined by: $\mathcal{H}^*_{\mathcal{B}}\phi = \gamma S_b^* \widetilde{w}^\infty$ for all $\phi \in H^{\frac{1}{2}}(\partial D_b)$, where $\widetilde{w}^\infty$ is the far field pattern of $\widetilde{w} \in H^1_{loc}(\mathbb{R}^m \backslash D_b)$ solution of:*

$$\begin{cases} \Delta \widetilde{w} + k^2 \widetilde{w} = 0 \text{ in } \mathbb{R}^m \backslash D_b, \\ \widetilde{w} + \frac{1}{\mu}\mathcal{B}(\partial_\nu \widetilde{w}) = \phi \text{ on } \partial D_b, \\ \lim\limits_{r\to+\infty} \int_{|x|=r} |\frac{\partial \widetilde{w}}{\partial r} - ik\widetilde{w}|^2 = 0. \end{cases} \tag{3.12}$$

Proof. We first prove the last part of the lemma. Let $g \in L^2(\mathbb{S})$. Applying Green's formula twice, one obtains

$$\begin{aligned} 0 &= \int_{B_R \backslash D_b} (\Delta \overline{\widetilde{w}} + k^2 \overline{\widetilde{w}}) u^s_{b,g} dx \\ &= \int_{\partial B_R} (\partial_\nu \overline{\widetilde{w}} u^s_{b,g} - \overline{\widetilde{w}} \partial_\nu u^s_{b,g}) ds - \int_{\partial D_b} (\partial_\nu \overline{\widetilde{w}} u_{b,g} - \overline{\widetilde{w}} \partial_\nu u_{b,g}) ds \\ &\quad + \int_{\partial D_b} (\partial_\nu \overline{\widetilde{w}} v_g - \overline{\widetilde{w}} \partial_\nu v_g) ds. \end{aligned} \tag{3.13}$$

Using the Sommerfeld radiation condition, one gets

$$\lim_{R\to+\infty} \int_{\partial B_R} (\partial_\nu \overline{\widetilde{w}} u^s_{b,g} - \overline{\widetilde{w}} \partial_\nu u^s_{b,g}) ds = -2ik \int_{\mathbb{S}} \overline{\widetilde{w}^\infty}(d) u^\infty_{b,g}(d) ds(d). \tag{3.14}$$

We recall that the far field $\widetilde{w}^\infty$ admits the following expression:

$$\gamma \widetilde{w}^\infty(d) = -\int_{\partial D_b} (\partial_{\nu(x)} \widetilde{w}(x) e^{-ikx\cdot d} - \partial_{\nu(x)} e^{-ikx.d} \widetilde{w}) ds(x). \tag{3.15}$$

Thus, replacing v_g by its expression, one obtains:

$$-\overline{\gamma} \int_{\mathbb{S}} g(d) \overline{\widetilde{w}^\infty(d)} ds = \int_{\partial D_b} (\partial_\nu \overline{\widetilde{w}} v_g - \overline{\widetilde{w}} \partial_\nu v_g) ds. \tag{3.16}$$

Using the boundary condition of $\widetilde{w}$ and $u_{b,g}$, we deduce that

$$\begin{aligned} \int_{\partial D_b} (\partial_\nu \overline{\widetilde{w}} u_{b,g} - \overline{\widetilde{w}} \partial_\nu u_{b,g}) ds &= \int_{\partial D_b} (\partial_\nu \overline{\widetilde{w}} (-\tfrac{1}{\mu}\mathcal{B}(\partial_\nu u_{b,g})) - \overline{\widetilde{w}} \partial_\nu u_{b,g}) ds \\ &= -\int_{\partial D_b} \partial_\nu u_{b,g} \overline{\phi} ds, \end{aligned} \tag{3.17}$$

where, for the last equality we used the fact that $\mathcal{B}$ is self-adjoint. Equations (3.13), (3.14), (3.16) and (3.17) give

$$\int_{\partial D_b} \partial_\nu u_{b,g} \overline{\phi} ds = 2ik \int_{\mathbb{S}} \overline{\widetilde{w}^\infty}(d) u^\infty_{b,g}(d) ds(d) + \overline{\gamma} \int_{\mathbb{S}} g(d) \overline{\widetilde{w}^\infty(d)} ds. \tag{3.18}$$

Hence, using the definition of $\mathcal{H}_{\mathcal{B}}$, we get:

$$\begin{aligned} \langle \mathcal{H}_{\mathcal{B}} g, \phi \rangle_{H^{-\frac{1}{2}}(\partial D_b), H^{\frac{1}{2}}(\partial D_b)} &= \overline{\gamma}(g, \widetilde{w}^\infty)_{L^2(\mathbb{S})} + 2ik(F^b g, \widetilde{w}^\infty) \\ &= \overline{\gamma}(S_b g, \widetilde{w}^\infty)_{L^2(\mathbb{S})}. \end{aligned} \tag{3.19}$$

We then conclude that $\mathcal{H}_{\mathcal{B}}{}^*\phi = \gamma S_b^* \widetilde{w}^\infty$.
We now prove the injectivity of $\mathcal{H}_{\mathcal{B}}$. Assume that $\mathcal{H}_{\mathcal{B}} g = 0$ for some $g \in L^2(\mathbb{S})$. This implies in particular that $\partial_\nu u_{b,g}|_{\partial D_b} = 0$ and $u_{b,g}$ vanishes on ∂D_b by the boundary condition, which gives $u^s_{b,g} = -v_g$ in $\mathbb{R}^m \backslash D_b$. Therefore the Herglotz wave function v_g which satisfies the Helmhotz equation in $\mathbb{R}^m$ also satisfies the radiation condition. This proves that $v_g = 0$ and then $g = 0$ ([16]).
We know prove that $\mathcal{H}_{\mathcal{B}}$ has dense range by showing that $\mathcal{H}^*_{\mathcal{B}}$ is injective. If $\mathcal{H}^*_{\mathcal{B}}\phi = 0$ then $\gamma S_b^* \widetilde{w}^\infty = 0$ for some $\phi \in H^{\frac{1}{2}}(\partial D_b)$. Using Rellich's lemma and the fact that S_b^* is unitary (see Proposition B.2) and therefore injective, we infer that only $\phi = 0$ satisfies the equality. ∎

The previous lemma implies in particular that $S_b \mathcal{H}_{\mathcal{B}}{}^*\phi = \gamma \widetilde{w}^\infty$. Let us introduce the operator $T_{\mathcal{B}} : H^{-\frac{1}{2}}(\partial D_b) \to H^{\frac{1}{2}}(\partial D_b)$ defined by

$$T_{\mathcal{B}}(\psi) = w + \frac{1}{\mu}\mathcal{B}(\partial_\nu w) = w^s + \frac{1}{\mu}\mathcal{B}(\partial_\nu w^s) \tag{3.20}$$

where (w, w^s) is the unique solution of (3.7). We then have $\mathcal{G}_{\mathcal{B}}\psi = S_b \mathcal{H}_{\mathcal{B}}{}^* T_{\mathcal{B}}\psi$. We deduce from (3.11) that $\mathcal{F}^\mu$ assumes the following factorization:

$$\gamma \mathcal{F}^\mu = S_b \mathcal{H}^*_{\mathcal{B}} T_{\mathcal{B}} \mathcal{H}_{\mathcal{B}}. \tag{3.21}$$

Using the fact that S_b is unitary (see Proposition B.2 in B), we infer that

$$\mathbf{F} = \mathcal{H}^*_{\mathcal{B}} T_{\mathcal{B}} \mathcal{H}_{\mathcal{B}}. \tag{3.22}$$

In addition, let us also observe that the operator $\mathbf{S} := I + \frac{2ik}{|\gamma|^2}\mathbf{F}$ is unitary due to the fact that S and S_b are unitary. This also implies in particular that $\mathbf{F}$ is a normal operator.

4. The inside-outside duality applied to F. Consider the case where $\mu \neq 0$ is not a $\mathcal{B}-$eigenvalue. Since $\mathbf{F}$ is a compact and normal operator, there exists an orthonormal complete basis $(g_j)_{j\in\mathbb{N}}$ of $L^2(\mathbb{S})$ such that $\mathbf{F}g_j = \lambda_j g_j$ where $(\lambda_j)_{j\in\mathbb{N}}$ are the eigenvalues of $\mathbf{F}$ that accumulate at 0.
Exploiting the fact that $\mathbf{S}$ is unitary, we deduce that the eigenvalues of $\mathbf{F}$ lie on the circle of radius $\frac{|\gamma|^2}{2k}$ and center $\frac{|\gamma|^2}{2ik}$. We set $\lambda_j := \frac{|\gamma|^2}{2ik}(e^{i\delta_j} - 1)$ with $e^{i\delta_j}$ being an eigenvalue of $\mathbf{S}$, $\delta_j \in [0, 2\pi)$ and define

$$\begin{cases} \delta_*(\mu) & := \max_{j\geq 1} \delta_j, \\ \lambda_* & := \frac{|\gamma|^2}{2ik}(e^{i\delta_*} - 1). \end{cases} \tag{4.1}$$

We then can state the main theorem of this section:

Theorem 4.1. *Assume that k^2 is not an eigenvalue of (2.6). μ_0 is a $\mathcal{B}-$eigenvalue if and only if $\delta_*(\mu) \to 2\pi$ as $\mu \to \mu_0$ with $\mu > \mu_0$.*

This theorem is a straightforward corollary of Proposition 4.6 (sufficient condition) and Proposition 4.8 (necessary condition). In order to prove these propositions, we first establish some properties of the operator $T_{\mathcal{B}}$ in the factorization (3.22).

4.1. Some key properties of the operator $T_{\mathcal{B}}$. One of the important ingredients of the inside-outside duality is the link between the kernel of the operator $T_{\mathcal{B}}$ and the $\mathcal{B}-$eigenvalues. The latter is proved in the following lemmas.

To shorten the notations, the duality product $\langle ., . \rangle_{H^{\frac{1}{2}}(\partial D_b), H^{-\frac{1}{2}}(\partial D_b)}$ will be denoted $\langle ., . \rangle$ or in an abuse of notation as an integral over ∂D_b.

Lemma 4.2. *The operator $T_{\mathcal{B}}$ satisfies the energy identity:*

$$\Im m \langle T_{\mathcal{B}}\psi, \psi\rangle = k \int_{\mathbb{S}} |w^\infty|^2 ds, \quad \forall \psi \in H^{-\frac{1}{2}}(\partial D_b), \tag{4.2}$$

where w^∞ is the far field pattern of w^s, with (w, w^s) being the solution of (3.7) with source term ψ.

Proof. Let $\psi \in H^{-\frac{1}{2}}(\partial D_b)$, using that $\psi = \partial_\nu w - \partial_\nu w^s$, we get

$$\langle T_{\mathcal{B}}\psi, \psi\rangle = \int_{\partial D_b} \partial_\nu \overline{w}(w + \frac{1}{\mu}\mathcal{B}(\partial_\nu w))ds - \int_{\partial D_b} \partial_\nu \overline{w^s}(w^s + \frac{1}{\mu}\mathcal{B}(\partial_\nu w^s))ds. \tag{4.3}$$

Let us focus on the last term of the previous equality. Using Green's formula on a ball B_R of radius $R > 0$ large enough, one obtains:

$$\begin{aligned} 0 &= -\textstyle\int_{B_R\setminus D_b} w^s(\Delta \overline{w^s} + k^2 n \overline{w^s})dx \\ &= \textstyle\int_{B_R\setminus D_b} |\nabla w^s|^2 dx - k^2 \int_{B_R\setminus D_b} n|w^s|^2 dx \\ &+ ik \textstyle\int_{\partial B_R} |w^s|^2 ds - \int_{\partial B_R} (\partial_\nu \overline{w^s} + ik\overline{w^s}) w^s ds \\ &+ \textstyle\int_{\partial D_b} \partial_\nu \overline{w^s} w^s ds. \end{aligned} \tag{4.4}$$

Substituting the expression of $\int_{\partial D_b} \partial_\nu \overline{w^s} w^s ds$ in (4.3) gives

$$\begin{aligned} \langle T_{\mathcal{B}}\psi, \psi\rangle &= \textstyle\int_{\partial D_b} \partial_\nu \overline{w} w ds \\ &+ \textstyle\int_{B_R\setminus D_b} |\nabla w^s|^2 dx - k^2 \int_{B_R\setminus D_b} n|w^s|^2 dx \\ &+ ik \textstyle\int_{\partial B_R} |w^s|^2 ds - \int_{\partial B_R} (\partial_\nu \overline{w^s} + ik\overline{w^s}) w^s ds \\ &+ \tfrac{1}{\mu} \textstyle\int_{\partial D_b} \mathcal{B}(\partial_\nu w)\partial_\nu \overline{w} - \tfrac{1}{\mu} \int_{\partial D_b} \mathcal{B}(\partial_\nu w^s)\partial_\nu \overline{w^s}. \end{aligned} \tag{4.5}$$

Taking the imaginary part of this equality while letting $R \to +\infty$, we obtain

$$\begin{aligned} \Im m \langle T_{\mathcal{B}}\psi, \psi\rangle &= k \textstyle\int_{\mathbb{S}} |w^\infty|^2 ds + \Im m \int_{\partial D_b} \partial_\nu \overline{w} w ds, \\ &= k \textstyle\int_{\mathbb{S}} |w^\infty|^2 ds - k^2 \int_{D_b} \Im m(n)|w|^2 dx, \end{aligned} \tag{4.6}$$

where for the last equality, we used that $\Delta w + k^2 n w = 0$ in D_b. ∎

Proposition 4.3. *Let $\mu \in \mathbb{R}^*$. μ is a $\mathcal{B}-$eigenvalue if and only if there exists a non trivial $\psi \in H^{-\frac{1}{2}}(\partial D_b)$ such that $\Im m \langle T_{\mathcal{B}}\psi, \psi\rangle = 0$.*

Proof. Assume that μ is a $\mathcal{B}-$eigenvalue. We denote by $w_0 \in H^1(D_b)$ its associated eigenvector. Set $\psi = \partial_\nu w_0|_{\partial D_b}$ which is necessarily non trivial. Consider (w, w^s) the associated solution of (3.7). We then get thanks to Green's theorem:

$$\begin{aligned} \langle T_{\mathcal{B}}\psi, \psi\rangle &= \int_{\partial D_b} w\partial_\nu\overline{w_0}ds + \tfrac{1}{\mu}\int_{\partial D_b}\partial_\nu\overline{w_0}\mathcal{B}(\partial_\nu w)ds \\ &= \int_{\partial D_b}\overline{w_0}\partial_\nu w ds + \tfrac{1}{\mu}\int_{\partial D_b}\partial_\nu w\mathcal{B}(\partial_\nu\overline{w_0})ds = 0, \end{aligned} \tag{4.7}$$

where we used the fact that $\mathcal{B}$ is self-adjoint and the fact w, w_0 verify the same inhomogeneous Helmholtz equation in D_b. Conversely, assume there exists $\psi \in H^{-\frac{1}{2}}(\partial D_b)$ non trivial such that $\Im m\,\langle T_{\mathcal{B}}\psi, \psi\rangle = 0$. By Rellich lemma, this implies $w^s = 0$ in $\mathbb{R}^m\backslash D_b$ where (w, w^s) is the solution of (3.7) with source term ψ. In particular, we get $w^s|_{\partial D_b} = 0$ and $\partial_\nu w^s|_{\partial D_b} = 0$. Thus, μ is a $\mathcal{B}-$eigenvalue associated with the non trivial eigenvector w. ∎

We conclude this section by showing that $T_{\mathcal{B}}$ is a Fredholm operator of index 0.

Lemma 4.4. *$T_{\mathcal{B}}$ admits the following decomposition*

$$T_{\mathcal{B}} = T_0 + K,$$

where K is compact and T_0 is coercive. More precisely, there exists $\alpha > 0$ such that

$$\langle T_0\psi, \psi\rangle \geq \alpha\|\psi\|^2_{H^{-\frac{1}{2}}(\partial D_b)}. \tag{4.8}$$

Proof. Let $\psi, \psi' \in H^{-\frac{1}{2}}(\partial D_b)$ and (w, w^s) (respectively (w', w'^s)) the solution of (3.7) with source term ψ (respectively ψ'). Exploiting the definition of $T_{\mathcal{B}}$, one can deduce:

$$\begin{aligned} \langle T_{\mathcal{B}}\psi, \psi'\rangle &= \int_{\partial D_b}\overline{\psi'}(w + \tfrac{1}{\mu}\mathcal{B}(\partial_\nu w))ds = \int_{\partial D_b}\overline{\psi'}(w^s + \tfrac{1}{\mu}\mathcal{B}(\partial_\nu w^s))ds, \\ &= \int_{\partial D_b}\partial_\nu\overline{w'}w ds - \int_{\partial D_b}\partial_\nu\overline{w'^s}w^s ds \\ &+\tfrac{1}{\mu}\int_{\partial D_b}\partial_\nu\overline{w'}\mathcal{B}(\partial_\nu w)ds - \tfrac{1}{\mu}\int_{\partial D_b}\partial_\nu\overline{w'^s}\mathcal{B}(\partial_\nu w^s)ds. \end{aligned} \tag{4.9}$$

Let B_R be a ball containing D_b for R large enough. We define the operator $T_0 : H^{-\frac{1}{2}}(\partial D_b) \to H^{\frac{1}{2}}(\partial D_b)$ as:

$$\begin{aligned} \langle T_0\psi, \psi'\rangle = &\int_{B_R\backslash D_b}\nabla w^s\nabla\overline{w'^s}dx + \int_{B_R\backslash D_b} w^s\overline{w'^s}dx \\ &+ \int_{D_b}\nabla w\nabla\overline{w'}dx + \int_{D_b} w\overline{w'}dx, \end{aligned} \tag{4.10}$$

and $K : H^{-\frac{1}{2}}(\partial D_b) \to H^{\frac{1}{2}}(\partial D_b)$ as:

$$\begin{aligned} \langle K\psi, \psi'\rangle = &-(1+k^2)\int_{B_R\backslash D_b} w^s\overline{w'^s}dx \\ &-(1+k^2)\int_{D_b} nw\overline{w'}dx \\ &-\int_{\partial B_R}\partial_\nu\overline{w'^s}w^s ds \\ &+\tfrac{1}{\mu}\int_{\partial D_b}\partial_\nu\overline{w'}\mathcal{B}(\partial_\nu w)ds - \tfrac{1}{\mu}\int_{\partial D_b}\partial_\nu\overline{w'^s}\mathcal{B}(\partial_\nu w^s)ds. \end{aligned} \tag{4.11}$$

Thus, one can write $T_{\mathcal{B}} = T_0 + K$.
Since $\psi = \partial_\nu w - \partial_\nu w^s$, we deduce that

$$\begin{aligned} \|\psi\|^2_{H^{-\frac{1}{2}}(\partial D_b)} &\leq 2(\|\partial_\nu w\|^2_{H^{-\frac{1}{2}}(\partial D_b)} + \|\partial_\nu w^s\|^2_{H^{-\frac{1}{2}}(\partial D_b)}), \\ &\leq 2(\|\nabla w\|^2_{L^2(D_b)} + \|\Delta w\|^2_{L^2(D_b)} \\ &\quad + \|\nabla w^s\|^2_{L^2(B_R\setminus D_b)} + \|\Delta w^s\|^2_{L^2(B_R\setminus D_b)}). \end{aligned} \tag{4.12}$$

Using the fact that w and w^s satisfy the Helmholtz equation, one can therefore deduce the existence of some $\alpha > 0$ such that

$$\begin{aligned} \alpha\|\psi\|^2_{H^{-\frac{1}{2}}(\partial D_b)} &\leq (\|w\|^2_{H^1(D_b)} + \|w^s\|^2_{H^1(B_R\setminus D_b)}), \\ &= \langle T_0\psi, \psi\rangle\,. \end{aligned} \tag{4.13}$$

The compactness of the operator K is a direct consequence of a priori estimates in Lemma 3.1, trace theorems, Rellich's compact embedding theorems and the compactness of the operator $\mathcal{B}$. ∎

One of the consequences of the previous decomposition of $T_{\mathcal{B}}$ in Lemma 4.4 is that if μ is not a $\mathcal{B}-$eigenvalue, the eigenvalues λ_j of the operator $\mathbf{F}$ accumulate at 0 from the right, that is $\Re e(\lambda_j) > 0$ for j large enough ([21, 22]). This is equivalent to say that δ_j (the argument of λ_j) converges to 0 as j goes to $+\infty$. In contrast, Theorem 4.1 states that when μ approaches a $\mathcal{B}-$eigenvalue, the eigenvalue $\lambda_*(\mu)$ approaches 0 from the left as illustrated in Figure 1. In addition, if μ is not a $\mathcal{B}-$eigenvalue, Lemma 6 and 8 imply $T_{\mathcal{B}}$ is coercive [21], that is there exists a constant $\alpha > 0$ such that for all $\psi \in H^{-\frac{1}{2}}(\partial D_b)$

$$|\langle T_{\mathcal{B}}\psi, \psi\rangle| \geq \alpha\|\psi\|^2_{H^{-\frac{1}{2}}(\partial D_b)}. \tag{4.14}$$

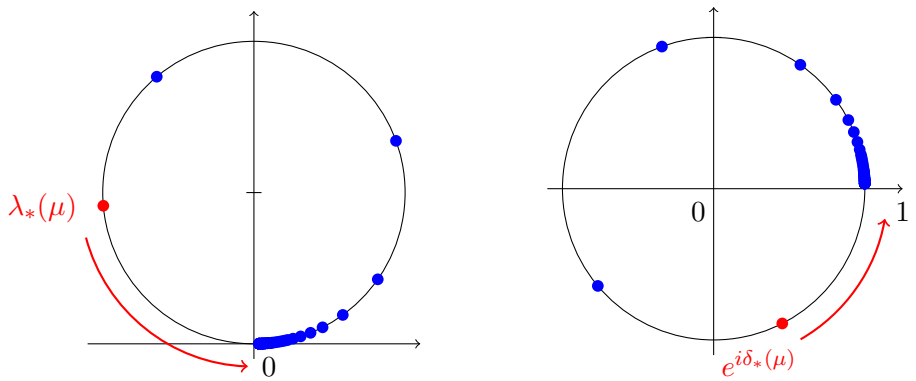

Figure 1. *Illustration of Theorem 4.1. Left: The eigenvalues of* $\mathbf{F}$ *accumulates at* 0 *from the right (in blue) while* $\lambda_*(\mu)$ *(in red) approaches* 0 *from the left as* μ *approaches the* $\mathcal{B}-$*eigenvalue. Right: The argument of the eigenvalues of* $\mathbf{S}$ *accumulates at* 0 *(in blue) while* $\delta_*(\mu)$ *(in red) approaches* 2π *as* μ *approaches the* $\mathcal{B}-$*eigenvalue.*

We dedicate the two next subsections to the proof of Theorem 4.1.

4.2. Proof of the sufficient condition in Theorem 4.1. Since we will be dealing with convergence of sequences that depend on μ, we shall explicitly indicate the dependence on μ in the notation for the operators (in particular, the operator $\mathbf{F}(\mu)$ is factorized as $\mathcal{H}^*_{\mathcal{B}}(\mu)T_{\mathcal{B}}(\mu)\mathcal{H}_{\mathcal{B}}(\mu)$).

Proposition 4.5. *The mapping $\mu \to T_{\mathcal{B}}(\mu)$ is continuous from $\mathbb{R}^*$ to the space of linear bounded operators from $H^{-\frac{1}{2}}(\partial D_b)$ to $H^{\frac{1}{2}}(\partial D_b)$ endowed with the usual operator norm.*

Proof. The proof is a direct consequence of the linear dependence with respect to $\frac{1}{\mu}$ of the solution (w, w^s) of problem (3.7) (see 18) and the expression of $T_{\mathcal{B}}(\mu)$. ∎

Proposition 4.6. *Let $\mu_0 \in \mathbb{R}^*$ and $I = (\mu_0 - \varepsilon, \mu_0 + \varepsilon)\backslash\{\mu_0\}$ for some $\varepsilon > 0$ sufficiently small such that no $\mu \in I$ is a $\mathcal{B}-$eigenvalue and such that $0 \notin I$. Assume there is a sequence $(\mu_j)_j$ of elements of I such that*

$$\mu_j \longrightarrow \mu_0 \qquad \text{and} \qquad \delta_*(\mu_j) \longrightarrow 2\pi.$$

Then μ_0 is a $\mathcal{B}-$eigenvalue.

Proof. Consider the sequence $\varphi_j = \frac{1}{\sqrt{|\lambda_*|}}\mathcal{H}_{\mathcal{B}}(\mu_j)g_j$, where g_j is the normalized eigenvector of $\mathbf{F}(\mu_j)$ associated with $\lambda_*(\mu_j)$. We then have by assumptions

$$\langle T_{\mathcal{B}}(\mu_j)\varphi_i, \varphi_i\rangle = \frac{\lambda_*(\mu_j)}{|\lambda_*(\mu_j)|} \longrightarrow_{j\to+\infty} -1. \tag{4.15}$$

Assume by contradiction that μ_0 is not a $\mathcal{B}-$eigenvalue.
In that case, the operator $T_{\mathcal{B}}(\mu_j)$ is coercive for every $\mu_j \in I$. Using that $\mu \to T_{\mathcal{B}}(\mu)$ is continuous in the operator norm, we infer that the coercivity constant can be chosen independent from $\mu \in \overline{I}$. The coercivity of $T_{\mathcal{B}}(\mu_j)$ and identity (4.15) show that $(\varphi_j)_j$ is bounded and therefore weakly converges (up to a subsequence) to some φ_0 in $H^{-\frac{1}{2}}(\partial D_b)$. Denote by (w_j, w_j^s) (respectively (w_0, w_0^s)) the solution of (3.7) with $\psi = \varphi_j$ (respectively $\psi = \varphi_0$). From Lemma 4.2, one gets:

$$\Im m\, \langle T_{\mathcal{B}}(\mu_j)\varphi_j, \varphi_j\rangle = k \int_{\mathbb{S}} |w_j^\infty|^2 ds. \tag{4.16}$$

Since the application $\psi \in H^{-\frac{1}{2}}(\partial D_b) \to w^\infty(\psi) \in L^2(\mathbb{S})$ (where $w^\infty(\psi)$ is the far field pattern of w^s solution of (3.7) with source term ψ) is compact, then $\Im m\, \langle T_{\mathcal{B}}(\mu_j)\varphi_j, \varphi_j\rangle$ converges to $\Im m\, \langle T_{\mathcal{B}}(\mu_0)\varphi_0, \varphi_0\rangle$. Identity (4.15) implies that $\lim_{j\to+\infty} \Im m\, \langle T_{\mathcal{B}}(\mu_j)\varphi_j, \varphi_j\rangle = 0$ and therefore $\varphi_0 = 0$ by Proposition 4.3.
Recall that $T_{\mathcal{B}}(\mu_j)$ assumes the decomposition $T_{\mathcal{B}}(\mu_j) = T_0(\mu_j) + K(\mu_j)$, where $T_0(\mu_j)$ is real coercive and $K(\mu_j)$ is compact. Let us show that $\lim_{j\to+\infty} \langle K(\mu_j)\varphi_j, \varphi_j\rangle = 0$. From the

decomposition shown in Lemma 4.4, we get that

$$\begin{aligned}\langle K(\mu_j)\varphi_j,\varphi_j\rangle = \quad & -(1+k^2)\|w_j^s\|^2_{H^1(B_R\setminus D_b)}\\ & -(1+k^2)(nw_j,w_j)_{H^1(D_b)}\\ & -\textstyle\int_{\partial B_R}\partial_\nu\overline{w_j^s}w_j^s ds\\ & +\frac{1}{\mu_j}\textstyle\int_{\partial D_b}\partial_\nu\overline{w_j}\mathcal{B}(\partial_\nu w_j)ds-\frac{1}{\mu_j}\int_{\partial D_b}\partial_\nu\overline{w_j^s}\mathcal{B}(\partial_\nu w_j^s)ds.\end{aligned}\tag{4.17}$$

The first three terms converge to 0 by the same arguments as in the proof of the compactness of $K(\mu_j)$ for μ_j fixed in Lemma 4.4. We conclude in the same way for the last two terms using that the sequence $\frac{1}{\mu_j}$ converges to $\frac{1}{\mu_0}$. This shows that $\lim_{j\to+\infty}\langle K(\mu_j)\varphi_j,\varphi_j\rangle = 0$. We deduce that (up to a subsequence) $\langle T_0(\mu_j)\varphi_j,\varphi_j\rangle$ converges to -1 which is a contradiction since $\langle T_0(\mu_j)\varphi_j,\varphi_j\rangle \geq 0$ for all $j\in\mathbb{N}$ (Lemma 4.4). ∎

4.3. Proof of the necessary condition in Theorem 4.1. For the proof of the necessary condition, we use the Cayley transform associated with $\mathbf{S}$ ([22]).
Assume that μ is not a $\mathcal{B}-$eigenvalue. Then 1 is not an eigenvalue of $\mathbf{S}$ because $\mathbf{F}$ is injective and we can define the Cayley transform:

$$\mathcal{T} := i(Id+\mathbf{S})(Id-\mathbf{S})^{-1}.$$

$\mathcal{T}$ is self-adjoint and has a discrete spectrum. We have the equivalence $e^{i\delta_*}$ is an eigenvalue of $\mathbf{S}$ if and only if $\cot(\delta_*/2)\in\mathbb{R}$ is an eigenvalue of $\mathcal{T}$. Applying Courant Fischer min max principle to $\mathcal{T}$, we get:

$$\cot(\delta_*/2) = \inf_{\psi\in H^{-\frac{1}{2}}(\partial D_b)}\frac{\Re e\,\langle\mathcal{T}\psi,\psi\rangle}{\Im m\,\langle\mathcal{T}\psi,\psi\rangle}.\tag{4.18}$$

Let μ_0 be a $\mathcal{B}-$eigenvalue associated with the eigenvector w_0. For $\mu\in\mathbb{R}^*$, define $(w(\mu),w^s(\mu))$ the solution of (3.7) with source term $\psi=\partial_\nu w_0$. From the linearity of those solutions with respect to $\frac{1}{\mu}$ (see decomposition (3.8)), one can obtain the following expansion:

$$w(\mu)-w_0 = (\frac{1}{\mu}-\frac{1}{\mu_0})w_N^0,\tag{4.19}$$

where w_N^0 does not depend on μ and is solution of (3.8) with $\psi=\partial_\nu w_0$.

Proposition 4.7. *Let μ_0 be a $\mathcal{B}-$eigenvalue associated with w_0. For $\mu\in\mathbb{R}^*$, define $(w(\mu),w^s(\mu))$ the solution of (3.7) with source term $\psi=\partial_\nu w_0$ and parameter μ. Then :*

$$\langle T_{\mathcal{B}}(\mu)\partial_\nu w_0,\partial_\nu w_0\rangle = (\frac{1}{\mu}-\frac{1}{\mu_0})\langle\mathcal{B}(\partial_\nu w_0),\partial_\nu w_0\rangle + (\frac{1}{\mu}-\frac{1}{\mu_0})^2\langle\mathcal{B}(\partial_\nu w_N^0),\partial_\nu w_0\rangle.\tag{4.20}$$

Proof. According to the definition of $T_{\mathcal{B}}(\mu)$, we get:

$$\langle T_{\mathcal{B}}(\mu)\partial_\nu w_0,\partial_\nu w_0\rangle = \left\langle w(\mu)+\frac{1}{\mu}\mathcal{B}(\partial_\nu w(\mu)),\partial_\nu w_0\right\rangle.\tag{4.21}$$

Since $-\langle T_{\mathcal{B}}(\mu_0)\partial_\nu w_0, \partial_\nu w_0\rangle = 0$, we can add it to the previous equality and by rearranging the terms, one has

$$\langle T_{\mathcal{B}}(\mu)\partial_\nu w_0, \partial_\nu w_0\rangle = \left\langle w(\mu) - w_0 + \frac{1}{\mu}\mathcal{B}(\partial_\nu w(\mu) - \partial_\nu w_0), \partial_\nu w_0 \right\rangle + (\frac{1}{\mu} - \frac{1}{\mu_0})\,\langle \mathcal{B}(\partial_\nu w_0), \partial_\nu w_0\rangle\,.$$

The previous equality can be rewritten using (4.19) as:

$$\langle T_{\mathcal{B}}(\mu)\partial_\nu w_0, \partial_\nu w_0\rangle = (\frac{1}{\mu} - \frac{1}{\mu_0})\left(\left\langle w_N^0 + \frac{1}{\mu}\mathcal{B}(\partial_\nu w_N^0), \partial_\nu w_0 \right\rangle + \langle \mathcal{B}(\partial_\nu w_0), \partial_\nu w_0\rangle\right). \tag{4.22}$$

Using Green's theorem, we have:

$$\begin{aligned} 0 &= -\int_{D_b}(\Delta w_N^0 + k^2 n w_N^0)\overline{w_0}dx = \int_{\partial D_b}(\partial_\nu \overline{w_0} w_N^0 - \partial_\nu w_N^0 \overline{w_0})ds \\ &= \left\langle w_N^0 + \tfrac{1}{\mu_0}\mathcal{B}(\partial_\nu w_N^0), \partial_\nu w_0 \right\rangle. \end{aligned}$$

Expression (4.20) is obtained by substituting $\left\langle w_N^0, \partial_\nu w_0\right\rangle$ in (4.22). ∎

We possess all the ingredients to prove the following necessary condition.

Proposition 4.8. *Assume that μ_0 is a $\mathcal{B}-$eigenvalue and k^2 is not an eigenvalue of (2.6). Then $\delta_*(\mu) \to 2\pi$ as $\mu \to \mu_0$ with $\mu > \mu_0$.*

Proof. The proof uses the positiveness of the $\mathcal{B}$ operator. Let us first show that the first term in the decomposition (4.20) is non zero. Assume by contradiction that $\langle \mathcal{B}(\partial_\nu w_0), \partial_\nu w_0\rangle = 0$. Using the form of $\mathcal{B}$ in (2.3), one has that $\langle \partial_\nu w_0, e_i\rangle = 0$ for all $i \leq M$ which implies $\mathcal{B}(\partial_\nu w_0) = 0$. Using that $\mu_0 \neq 0$, one obtains thanks to the boundary condition that $w_0|_{\partial D_b} = 0$ which contradicts the assumption on k^2. Hence, we have $\langle \mathcal{B}(\partial_\nu w_0), \partial_\nu w_0\rangle > 0$.
Let μ_0 be a $\mathcal{B}-$eigenvalue associated with the eigenvector w_0. Then we have for μ in a neighborhood of μ_0:

$$\cot(\delta_*(\mu)/2) = \inf_{\psi \in H^{-\frac{1}{2}}(\partial D_b)} \frac{\Re\,\langle T_{\mathcal{B}}(\mu)\psi, \psi\rangle}{\Im m\,\langle T_{\mathcal{B}}(\mu)\psi, \psi\rangle} \leq \frac{\Re\,\langle T_{\mathcal{B}}(\mu)\partial_\nu w_0, \partial_\nu w_0\rangle}{\Im m\,\langle T_{\mathcal{B}}(\mu)\partial_\nu w_0, \partial_\nu w_0\rangle} \tag{4.23}$$

with $\Im m\,\langle T_{\mathcal{B}}(\mu)\partial_\nu w_0, \partial_\nu w_0\rangle > 0$ thanks to Lemma 4.2. From the previous Lemma, we conclude that:

$$\frac{\Re\,\langle T_{\mathcal{B}}(\mu)\partial_\nu w_0, \partial_\nu w_0\rangle}{\Im m\,\langle T_{\mathcal{B}}(\mu)\partial_\nu w_0, \partial_\nu w_0\rangle} = \frac{\langle \mathcal{B}(\partial_\nu w_0), \partial_\nu w_0\rangle + (\frac{1}{\mu} - \frac{1}{\mu_0})\Re e(\left\langle \mathcal{B}(\partial_\nu w_N^0), \partial_\nu w_0\right\rangle)}{(\frac{1}{\mu} - \frac{1}{\mu_0})\Im m\,\langle T_{\mathcal{B}}(\mu)\partial_\nu w_0, \partial_\nu w_0\rangle} \to -\infty,$$

as $\mu \to \mu_0$ with $\mu > \mu_0$. Combined with (4.23), this proves the claim of the proposition. ∎

We end this section by a result that indicates how one can also recover an eigenvector w_0 associated with μ. The proof is similar to the one in [3].

Proposition 4.9. *Let μ_0 be a $\mathcal{B}-$eigenvalue. Take the sequence*

$$\psi_j = \frac{H(\mu_j)g_j}{\|H(\mu_j)g_j\|_{H^{-\frac{1}{2}}(\partial D_b)}}, \tag{4.24}$$

and (w_j, w_j^s) *the associated solution of (3.7) with source term* ψ_j*. Then,* ψ_j *admits a subsequence which converges strongly in* $H^{-\frac{1}{2}}(\partial D_b)$ *to* $\partial_\nu w_0$*, the normal trace of an eigenvector* w_0 *of (2.5). Here,* g_j *is the normalised eigenvector of* $\mathbf{F}(\mu_j)$ *associated with* $\lambda_*(\mu_j)$*.*

Proof. Since the sequence $(\psi_j)_j$ is bounded, it weakly converges (up to a subsequence) in $H^{-\frac{1}{2}}(\partial D_b)$ to some ψ_0 (and w_j, w_j^s converge weakly to some w_0, w_0^s in $H^1(D_b) \times H^1(B_R \backslash D_b)$. Observe that:

$$\langle T_{\mathcal{B}}(\mu_j)\psi_j, \psi_j\rangle = \theta_j \frac{\lambda_*(\mu_j)}{|\lambda_*(\mu_j)|}, \tag{4.25}$$

where $\theta_j := |\lambda_*(\mu_j)|/\|H(\mu_j)g_j\|^2_{H^{-\frac{1}{2}}(\partial D_b)}$ is real and $\frac{\lambda_*(\mu_j)}{|\lambda_*(\mu_j)|}$ converges to -1 by Theorem 4.1. Using similar arguments as in Proposition 4.6, one can prove from equation (4.25) that $\Im m\, \langle T_{\mathcal{B}}(\mu_0)\psi_0, \psi_0\rangle = 0$ and therefore w_0 satisfies (2.5). Let us show that $(\psi_j)_j$ converges strongly to ψ_0 and that ψ_0 is non trivial.
Using the fact that $\mu \to T_{\mathcal{B}}(\mu)$ is continuous, up to a subsequence, one can show that the sequence $(\theta_j)_j$ converges to some $\theta_0 \geq 0$. From the decomposition in Lemma 4.4, we have:

$$\langle T_0(\mu_j)\psi_j, \psi_j\rangle = \langle T_{\mathcal{B}}(\mu_j)\psi_j, \psi_j\rangle - \langle K(\mu_j)\psi_j, \psi_j\rangle\,, \tag{4.26}$$

where $\langle T_{\mathcal{B}}(\mu_j)\psi_j, \psi_j\rangle$ converges to some negative number and $\langle K(\mu_j)\psi_j, \psi_j\rangle$ converges to $\langle K(\mu_0)\psi_0, \psi_0\rangle$ (see proof of Proposition 4.6). We deduce that $\lim_{j\to+\infty} \langle T_0(\mu_j)\psi_j, \psi_j\rangle$ exists and that

$$\lim_{j\to+\infty} \langle T_0(\mu_j)\psi_j, \psi_j\rangle \leq -\langle K(\mu_0)\psi_0, \psi_0\rangle\,. \tag{4.27}$$

One can observe using the definition of $K(\mu_0)$ that $\langle K(\mu_0)\psi_0, \psi_0\rangle = -\|w_0\|^2_{H^1(D_b)}$ and, using the coercivity of T_0, show that

$$\lim_{j\to+\infty} \left(\|w_j\|^2_{H^1(D_b)} + \|w_j^s\|^2_{H^1(B_R\backslash D_b)}\right) \leq \|w_0\|^2_{H^1(D_b)}, \tag{4.28}$$

which is enough to prove the strong convergence of w_j toward w_0 in $H^1(D_b)$ and the strong convergence of w_j^s in $H^1(B_R\backslash D_b)$ toward 0. Those convergences are sufficient to demonstrate that $\psi_j = \partial_\nu w_j - \partial_\nu w_j^s$ strongly converges to $\psi_0 = \partial_\nu w_0$. Since for all $j \in \mathbb{N}$, $\|\psi_j\|_{H^{-\frac{1}{2}}(\partial D_b)} = 1$, we have $\psi_0 \neq 0$. ∎

Remark 4.10. In the context of imaging algorithms, the assumption $D \subset D_b$, while practical for the presentation of results, is both restrictive and inappropriate, as obstacles may exist outside the region D_b. To properly formalize this scenario while maintaining the structure of the proofs, particularly to keep an eigenvalue problem within D_b, we consider the cases where D does not intersect ∂D_b (as depicted in Figure 2) and the transmission eigenvalues [11] k^2 associated with D. The contribution of the obstacles $\Omega_D := D\backslash D_b$ (the component of D outside of D_b) leads to the following modifications of the operators $\mathcal{H}_{\mathcal{B}} : L^2(\mathbb{S}) \to H^{-\frac{1}{2}}(\partial D_b) \times L^2(\Omega_D)$ and $T_{\mathcal{B}} : H^{-\frac{1}{2}}(\partial D_b) \times \overline{\mathcal{R}(\mathcal{H}_{\mathcal{B}})} \to H^{\frac{1}{2}}(\partial D_b) \times L^2(\Omega_D)$:

$$\begin{cases} \mathcal{H}_{\mathcal{B}} g = (\partial_\nu u_{b,g}|_{\partial D_b}, u_{b,g}|_{\Omega_D}), \\ \\ T_{\mathcal{B}}(\psi, u) = (w + \frac{1}{\mu}\mathcal{B}(\partial_\nu w), -k^2(1-n)(u + w^s)|_{\Omega_D}), \end{cases} \tag{4.29}$$

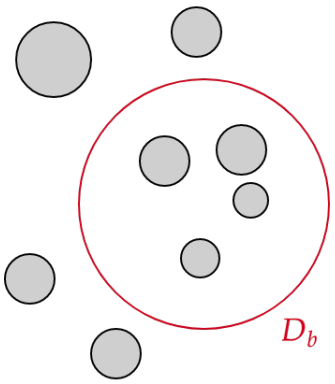

Figure 2. *Example of a configuration where* $D \cap \partial D_b = \emptyset$

where (w, w^s) are solution of:

$$\begin{cases} \Delta w + k^2 n w = 0 \text{ in } D_b, \\ \Delta w^s + k^2 w^s = k^2(1-n)(u+w^s)|_{\Omega_D} \text{ in } \mathbb{R}^m \backslash D_b, \\ w^s + \frac{1}{\mu}\mathcal{B}(\partial_\nu w^s) = w + \frac{1}{\mu}\mathcal{B}(\partial_\nu w) \text{ on } \partial D_b, \\ \partial_\nu w - \partial_\nu w^s = \psi \text{ on } \partial D_b, \\ \lim\limits_{r\to+\infty} \int_{|x|=r} |\frac{\partial w^s}{\partial r} - ikw^s|^2 = 0, \end{cases} \tag{4.30}$$

and $\overline{\mathcal{R}(\mathcal{H}_\mathcal{B})}$ denotes the closure of the range of $\mathcal{H}_\mathcal{B}$ in $L^2(\Omega_D)$. The decomposition in Lemma 4.4 still holds provided that $n > 1$ in Ω_D.

5. Numerical validation of the inside-outside duality method. The goal of this section is to provide a numerical validation for the inside-outside duality method inspired by Theorem 4.1. Our numerical experiments will be conducted in dimension 2, meaning for $m = 2$.

5.1. The case of a disc. Consider the simple situation where $D = D_b = B_\rho$ is a disk of radius $\rho > 0$ centered at the origin. We assume that n is constant inside B_ρ and $n = 1$ outside. For $\mathcal{Q}$ a subset of $\mathbb{N}$, we use as operator $\mathcal{B}$ in the background problem (2.2) the one defined by

$$\mathcal{B}\psi = \sum_{q\in\mathcal{Q}} \langle \psi, e_q \rangle \, e_q, \tag{5.1}$$

where $e_q(\theta) := e^{iq\theta}$, $\theta \in [0, 2\pi)$. In that case, the $\mathcal{B}-$eigenvalues, denoted $\mu_{\text{ref,q}}$ assume the following analytical expression for $q \in \mathcal{Q}$:

$$\mu_{\text{ref,q}}(n) = -2\pi k\rho\sqrt{n}\frac{J_q'(k\rho\sqrt{n})}{J_q(k\rho\sqrt{n})}, \tag{5.2}$$

where J_j denotes the Bessel function of the first kind of order j.
The far field pattern Fg due to a Herglotz wave function of the form (3.2) as an incident field with density

$$g(\theta) = \sum_{m\in\mathbb{Z}} g_m e^{im\theta}, \tag{5.3}$$

takes the form

$$Fg(\theta) = \sum_{m\in\mathbb{Z}} \alpha_m g_m e^{im\theta}, \tag{5.4}$$

where

$$\alpha_m = i^m \sqrt{\frac{8\pi}{k}} e^{-i\frac{\pi}{4}} \frac{J'_m(k\rho)J_m(k\rho\sqrt{n}) - \sqrt{n}J_m(k\rho)J'_m(k\rho\sqrt{n})}{-J_m(k\rho\sqrt{n})H'^1_m(k\rho) + \sqrt{n}J'_m(k\sqrt{n}\rho)H^1_m(k\rho)}.$$

Similarly, one can derive an expression of the background far field as:

$$F_b^\mu g(\theta) = \sum_{m\in\mathbb{Z}} \beta_m g_m e^{im\theta}, \tag{5.5}$$

where

$$\begin{cases} \beta_m = -i^m \sqrt{\frac{8\pi}{k}} e^{-i\frac{\pi}{4}} \frac{\mu J_m(k\rho)+2\pi\rho k J'_m(k\rho)}{\mu H_m^{(1)}(k\rho)+2\pi\rho k H_m'^{(1)}(k\rho)}, & m \in \mathcal{Q} \\ \beta_m = -i^m \sqrt{\frac{8\pi}{k}} e^{-i\frac{\pi}{4}} \frac{J_m(k\rho)}{H_m^{(1)}(k\rho)} \quad \text{otherwise}, \end{cases} \tag{5.6}$$

We clearly observe that the eigenvalues of $\mathcal{F}^\mu$ are $(\alpha_m - \beta_m)$, $m \in \mathbb{Z}$ and they are respectively associated with the eigenvector e_m. Consequently, the eigenvalues of the scattering operator $\mathbf{S}$ are

$$e^{i\delta_m} := 1 + \frac{2ik}{\overline{\gamma}}(1 - \frac{2ik}{\gamma}\overline{\beta_m})(\alpha_m - \beta_m)$$

where $\delta_m \in [0, 2\pi)$ denote their corresponding phases. Observe that only the eigenvalues for $m \in \mathcal{Q}$ depend on μ.

In Figure 3 (left), we display the curves $\mu \to \delta_m(\mu)$ for $m \in \{-10, ...10\}$ and $\mathcal{Q} = \{0\}$. We clearly see that the curves for $m \neq 0$ are horizontal lines, which means that they do not depend on μ. The only non constant curve $\delta_0(\mu)$ has a discontinuity at the expected $\mathcal{B}-$eigenvalue $\mu_{\text{ref},0}(2)$. In addition, we observe that $\delta_0(\mu) \to 2\pi$ as $\mu \to \mu_{\text{ref},0}(2)$ and $\mu > \mu_{\text{ref},0}(2)$, which coincides with Theorem 4.1. In Figure 3 (right), one can draw similar conclusions with $\mathcal{Q} = \{0, 1, 2\}$ and the presence of 3 distinct $\mathcal{B}-$eigenvalues.

5.2. The case of other geometries using synthetic simulated data. We consider a domain D_b shown in Figure 4 is composed of small circular scatterers with $n = 2$ that occupy four different regions in the space (see Figure 4). The operator F cannot be computed analytically in this case. It is numerically generated by solving the scattering problem (2.1) using a finite element method implemented with the FreeFEM [20] package. We use the Perfectly

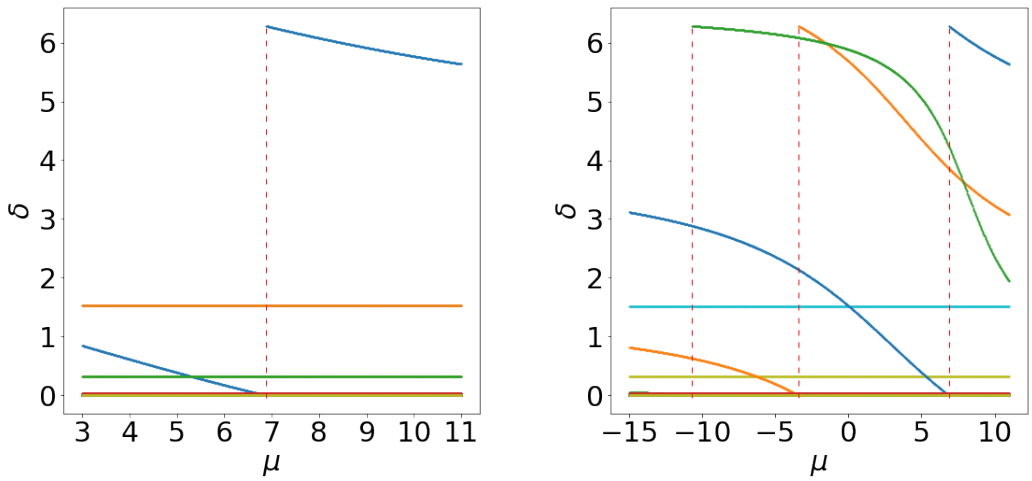

Figure 3. *For $m \in \{-10, ...10\}$, we plot the curves $\mu \to \delta_m(\mu)$ where $D = D_b = B_\rho$ for $\rho = 0.4333$, $k = 3$ and $n = 1$. Each color (other than red) corresponds to a value of m. The red dashed line indicates the $\mathcal{B}-$eigenvalue (5.2) for $\mathcal{Q} = \{0\}$ (left) and $\mathcal{Q} = \{0, 1, 2\}$ (right).*

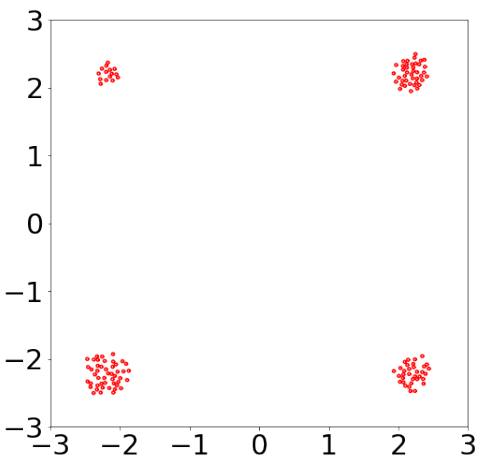

Figure 4. *Left: The domain D (in red) composed of four areas in the four corners that we fill with small circular scatterers of radius 0.02 with $n = 2$ inside*

Matched Layer ([9]) technique to bound the computational domain and model the Sommerfeld radiation condition.

The outcome of our numerical solver is the matrix $\mathbb{F}$ (that plays the role of an numerical approximation of F) with entries

$$\mathbb{F}_{pq} = u^\infty(\hat{x}_p, d_q), \quad 1 \leq p, q \leq N, \tag{5.7}$$

where $\hat{x}_p = d_p = (\cos(\theta_p), \sin(\theta_p))$ with $\theta_p = \frac{p}{N} 2\pi$ and where u^∞ is the numerically computed far field. In this following, we take $N = 40$ and $k = 3$.
We shall test the inside-outside duality for the case where the domain D_b will occupy two different regions as illustrated in Figure 5 (left) and Figure 7 (left). The first one does not contain any inclusion and the second one contains one of the packs of small circles.

Since in the following examples and in the imaging algorithm introduced later, we need to compute the background far field operator for different positions of the circular domain D_b. We therefore briefly explain how one quickly evaluates this far field operator.
Denote by B_ρ^y the disk of center $y \in \mathbb{R}^2$ and radius $\rho > 0$. The far field pattern $u_{b,y}^\infty$ associated with the background problem (2.2) with $D_b = B_\rho^y$ can be expressed as

$$u_{b,y}^\infty(\hat{x}, d) = e^{iky.(d-\hat{x})} u_b^\infty(\hat{x}, d), \tag{5.8}$$

where u_b^∞ is the far field associated with $D_b = B_\rho^0$ (i.e the disk of radius ρ centered at the origin). The far field pattern assumes the following analytic expression:

$$u_{b,0}^\infty(\hat{x}, d) = \sum_{m \in \mathbb{Z}} \beta_m e^{im(\theta_{\hat{x}} - \theta_d)}, \tag{5.9}$$

with $\hat{x} = (\cos(\theta_{\hat{x}}), \sin(\theta_{\hat{x}}))$, $d = (\cos(\theta_d), \sin(\theta_d))$ and β_m are given by (5.6).
The numerical approximation of the background far field operator $F_{b,y}^\mu$ is the background far field matrix $\mathbb{F}_{b,y}^\mu$ which entries are:

$$(\mathbb{F}_{b,y}^\mu)_{pq} = u_{b,y}^\infty(\hat{x}_p, d_q), \quad 1 \leq p, q \leq N, \tag{5.10}$$

where $u_{b,y}^\infty$ is evaluated using (5.8) and where u_b^∞ is approximated by a truncation of the sum in (5.9), keeping the indices $j \in \mathbb{Z}$ such that $|j| < J$. In the following, we use $J = 10$.

We shall illustrate how the $\mathcal{B}-$eigenvalues depend on the position of the domain D_b on the synthetic configuration depicted in Figure 4.
In Figure 5 (right), we display (for $\mathcal{Q} = \{0\}$), the curves $\mu \to \delta_m(\mu)$ associated with the configuration in Figure 5 (left) where D_b is a disk of radius $\rho = \frac{1.3}{k}$ centered at the origin with $k = 3$. Since there are no scatterers inside D_b, we expect the $\mathcal{B}-$eigenvalue to be equal to $\mu_{\text{ref,q}}(1)$ for $q \in \mathcal{Q}$. In line with the theory, we observe that the discontinuity for one of the curves occurs at the predicted eigenvalue (represented by the red dashed line) and a behavior similar to Figure 3 (left). In the case $\mathcal{Q} = \{0, 1, 2\}$, we have the same conclusions as we obtain results represented in Figure 6 (right) that are similar to the one in Figure 3 (right). The only difference between the analytical case and this one is that the curves not associated with $\mathcal{B}-$eigenvalues are no longer independent from μ.

In Figure 7 and 8, the domain D_b is the same disk as before but centered at the point $(2.2, 2.2)$. The reference $\mathcal{B}-$eigenvalues cannot be analytically computed for this configuration. We numerically evaluate them by solving (2.7) using the finite element methods implemented in FreeFEM. The expression of these eigenvalues is given by (2.8). These computed values

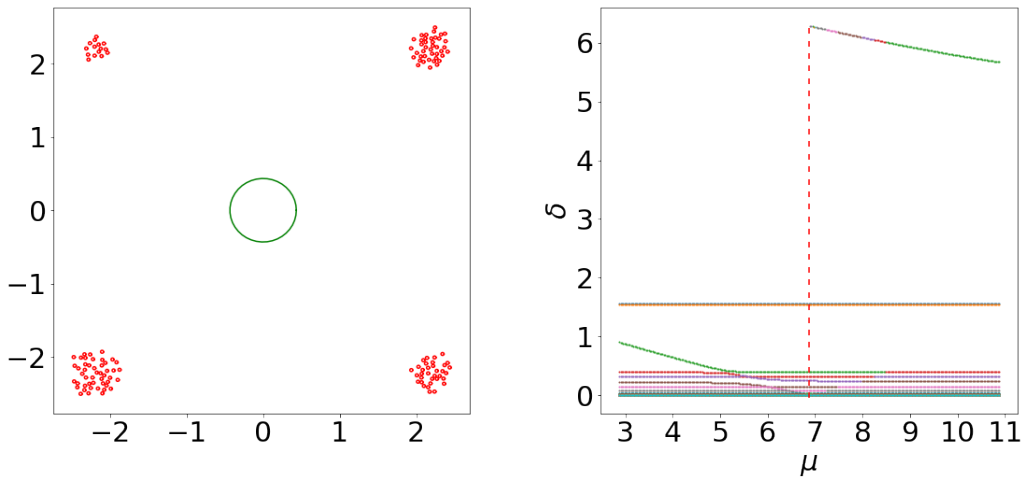

Figure 5. *Left: The domain D (in red) and D_b, the disk of radius $\rho = 0.433$ (in green). Right: Plot of the curves $\mu \to \delta_m(\mu)$ for $k = 3$ and $n = 2$ inside the defects. The red dashed line indicates the analytical $\mathcal{B}-$eigenvalue (5.2) for constant $n = 1$ and $\mathcal{Q} = \{0\}$.*

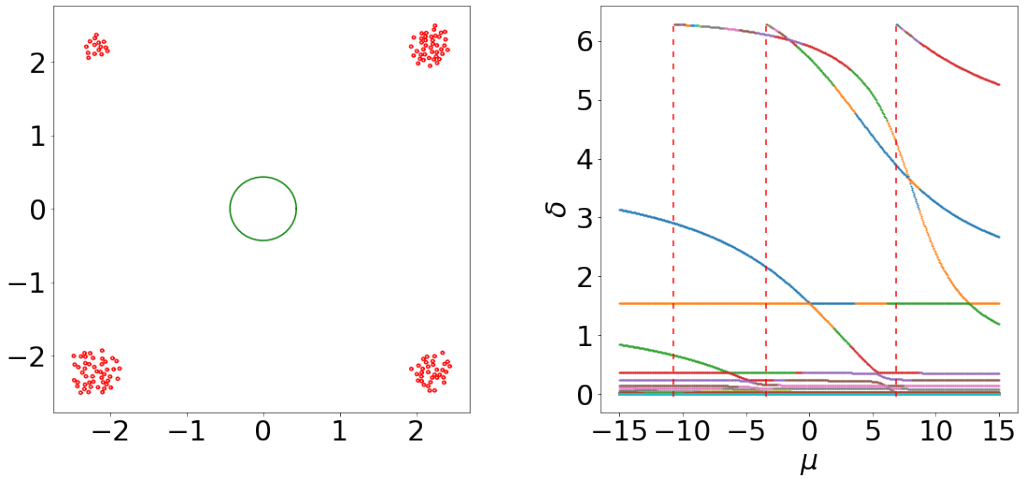

Figure 6. *Left: The domain D (in red) and D_b, the disk of radius $\rho = 0.433$ (in green). Right: Plot of the curves $\mu \to \delta_m(\mu)$ for $k = 3$ and $n = 2$ inside the defects. The red dashed line indicates the analytical $\mathcal{B}-$eigenvalue (5.2) for constant $n = 1$ and $\mathcal{Q} = \{0, 1, 2\}$.*

are indicated in the figures by a red straight dashed vertical line. The results for $\mathcal{Q} = \{0\}$ in Figure 7 and $\mathcal{Q} = \{0, 1, 2\}$ in Figure 8 show that the inside-outside duality is capable of correctly identifying the $\mathcal{B}-$eigenvalues for this configuration.

In the imaging algorithm introduced below, we shall use $\mathcal{Q} = \{0\}$ to minimize the number of $\mathcal{B}-$eigenvalues that has to be determined. Figure 9 indicates the dependence of the inside-

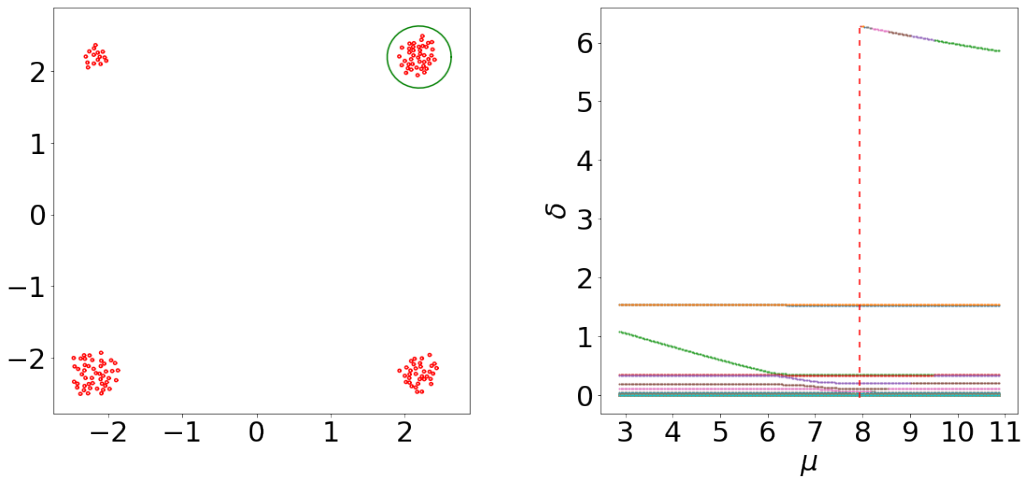

Figure 7. *Left: The domain D (in red) and D_b, the ball of radius $\rho = 0.433$ centered at $(2.2, 2.2)$ (in green). Right: Plot of the curves $\mu \to \delta_m(\mu)$ for $k = 3$ and $n = 2$ inside the defects. The red dashed line indicates the approximated $\mathcal{B}-$eigenvalue computed with the relation (2.8) for $\mathcal{Q} = \{0\}$.*

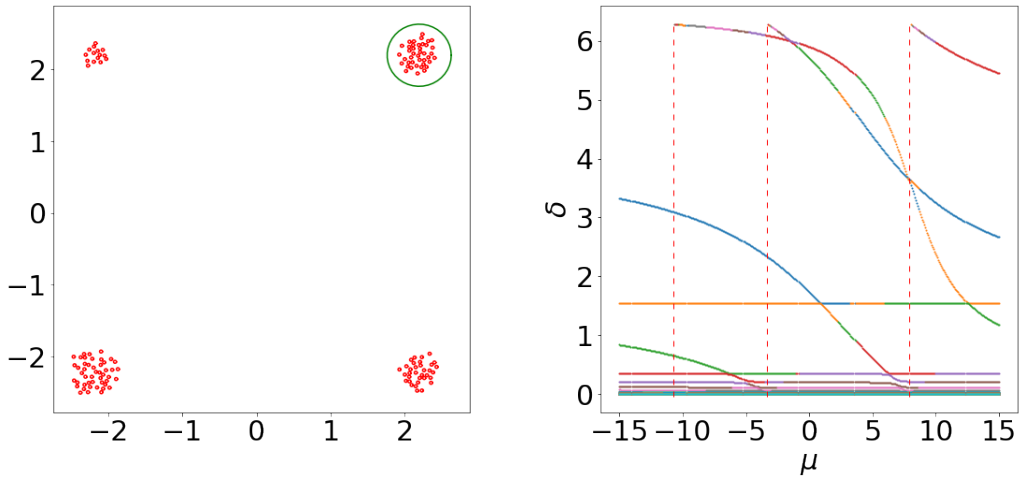

Figure 8. *Left: The domain D (in red) and D_b, the ball of radius $\rho = 0.433$ centered at $(2.2, 2.2)$ (in green). Right: Plot of the curves $\mu \to \delta_m(\mu)$ for $k = 3$ and $n = 2$ inside the defects. The red dashed line indicates the approximated $\mathcal{B}-$eigenvalue computed with the relation (2.8) for $\mathcal{Q} = \{0, 1, 2\}$.*

outside duality outcome with respect to noise in the data F. In order to simulate noise in the data, we change the values of the synthetic data $\mathbb{F}$ by adding random noise of level δ to construct the noisy far field matrix $\mathbb{F}^\delta = \mathbb{F} \cdot (1 + \delta(A + iB))$ where the entries of the matrices A and B are uniformly distributed real values in $[-1, 1]$ and where $\cdot$ denotes the element-wise product of matrices. Figure 7 clearly shows that the location of the eigenvalues is very

robust with respect to the noise level. This is one of the main motivations for using these $\mathcal{B}-$eigenvalues.

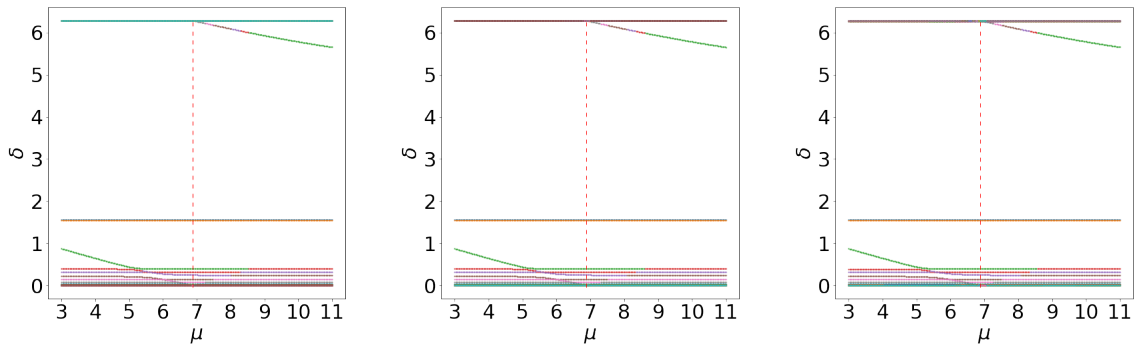

Figure 9. *Plot of the curves $\mu \to \delta_m(\mu)$ for $k = 3$ and $n = 2$ inside the defects for the configuration shown in Figure 7 (left), D_b being a ball of radius $\rho = 0.4333$ centered at the origin and $\mathcal{Q} = \{0\}$ Left: $\delta = 1\%$, Middle: $\delta = 10\%$, Right: $\delta = 50\%$. The red dashed line indicates the analytical $\mathcal{B}-$eigenvalue (5.2) for constant $n = 1$ and $\mathcal{Q} = \{0\}$.*

5.3. Application to the reconstruction of averaged values of the refractive index n.

Several works in the literature have proposed inversion algorithms that exploit special forms of $\mathcal{B}-$eigenvalues or so-called transmissions eigenvalues [8, 4, 7]. In order to determine the $\mathcal{B}-$eigenvalues, these contributions employ a method based on the GLSM method ([5]). We here revisit the method proposed in [7] for reconstructing the averaged values of the refractive index by replacing the GLSM with the inside-outside duality method. Let us first outline this imaging algorithm. The method uses the operator $\mathcal{B}$ in (5.1) with $\mathcal{Q} = \{0\}$. In this case, the boundary condition on ∂D_b is of the form:

$$\mu u + \int_{\partial D_b} \partial_\nu u ds = 0 \text{ on } \partial D_b. \tag{5.11}$$

In this case, there is only one $\mathcal{B}-$eigenvalue and the reference eigenvalue for constant index n is given by $\mu_{\text{ref},0}(n)$ in (5.2). The function $\mu_{\text{ref},0}$ is a bijection between $[0, j_0/(k\rho)[$ and $\mathbb{R}^+$, where j_0 is the first zero of J_0.

The algorithm can be described with the following steps:

1. Let $\rho > 0$ be a given parameter. Choose D_b to be the ball B_ρ^y of radius ρ and center $y \in \mathcal{Y}$, a grid of points sampling the region of interest.
2. Evaluate the $\mathcal{B}-$eigenvalue $\mu(y, n)$ from the measurements $\mathbb{F}$ and the analytically computed $\mathbb{F}_{b,y}^{\mu}$ using the inside-outside method (see *Method 1* and *Method 2* below).
3. Compute $n^*(y) \in [0, j_0/(k\rho)[$ such that $\mu(y, n) = \mu_{\text{ref},0}(n^*(y))$.
4. Plot the function $y \to n^*(y)$

The key step in this algorithm is (ii), that is how to automatically recover $\mu(y, n)$ from graphics as in Figure 7 (right). We indicate two possible methods.

Method 1. Let Λ be the interval of values where $\mu(y,n)$ is supposed to belong. This interval should contains $\mu_{\mathrm{ref},0}(1)$. Fix a parameter small $\varepsilon > 0$ and a threshold $\sigma \in [0, 2\pi - \varepsilon)$ not too close to 0. For each μ in Λ, we compute the eigenvalues $e^{i\delta_m}$ of the operator $\mathbf{S}(\mu)$. Then we count the number of phases δ_m that belong to $[\sigma, 2\pi - \varepsilon)$. For carefully chosen ε and σ, this number is constant for $\mu < \mu(y,n)$ in Λ and increases by 1 for the first value in Λ that exceeds $\mu(y,n)$. This allows us to identify $\mu(y,n)$.

Method 2. This method exploits the observation that all the eigenvalues $e^{i\delta_m}$ slowly vary with respect to μ expect one. Denote by $\widehat{\delta}_m \in [-\pi, \pi)$ such that $e^{i\delta_m} = e^{i\widehat{\delta}_m}$. We then expect the function $\mathcal{I}(\mu) = \sum_m |\widehat{\delta}_m|$ to have a minimum at $\mu(y,n)$ (see illustration in Figure 10).

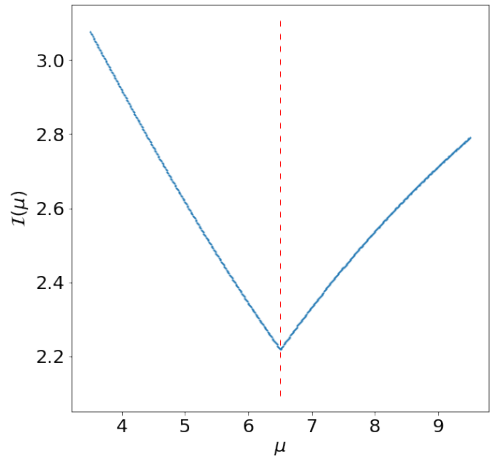

Figure 10. *Plot of the curve $\mu \to \mathcal{I}(\mu)$ for $D = D_b = B_\rho$ the ball centered at the origin for $\rho = 0.3$, $k = 3$ and $n = 2$ in D. The red dashed line indicates $\mu_{\mathrm{ref},0}(2)$.*

In the following, we use method 1 with $\varepsilon = 10^{-2}$ and $\sigma = 2\pi - 1$. The second method also works fine and gives similar results. Before commenting the numerical tests, we briefly explain why $n^*(y)$ is expected to be an approximation of the averaged value:

$$\overline{n}(y) = \frac{1}{|B_\rho^y|} \int_{B_\rho^y} n(x)dx.$$

Consider a sequence of refractive index $(n_\varepsilon)_\varepsilon$ that converges weakly$-*$ in $L^\infty(D_b)$ to some $\widehat{n}$ as $\varepsilon \to 0$ (and denote by μ_ε its associated $\mathcal{B}-$eigenvalue). One can show from (2.8) that μ_ε converges to $\mu(\widehat{n})$ as ε goes to 0. For highly oscillating medium, $\widehat{n}$ can be seen as an approximation of the mean value $\overline{n}$ of n as defined above.
We shall test the algorithms described above for two configurations of the media as indicated in Figure 11. The configuration on the left indicates clustering of small disks with refractive index $n = 2$ and radius 0.02 with different densities in five distinct regions. The second

configuration on the right is formed by 4 circular inhomogeneities with different refractive index $n = 0.25$ (bottom left), $n = 0.5$ (top left), $n = 1.5$ (bottom right) and $n = 2$ (upper right). The reconstruction associated with configuration in Figure 11 (left) is shown in Figure 12. In Figure 12 (left), we display the function $y \to \overline{n}(y)$ and in Figure 9 (right), we display the function $y \to n^*(y)$ resulting from the inside-outside duality with $k = 4$, $\rho = 0.33$ and the noise level $\delta = 1\%$. We clearly observe very good agreement between the two functions. Similar conclusions can be drawn from the results associated with the configuration in Figure 11 (right). These results are indicated in Figure 13, where again the function $y \to \overline{n}(y)$ (left) and $y \to n^*(y)$ (right) are displayed. In this case, we used $k = 4$, $\rho = 0.3$ and the noise level $\delta = 1\%$. Observe that we chose $\rho = 0.3$ to ensure that D_b is not strictly included inside D. In fact, in this case, $\mu(y, n)$ may not be a positive value.

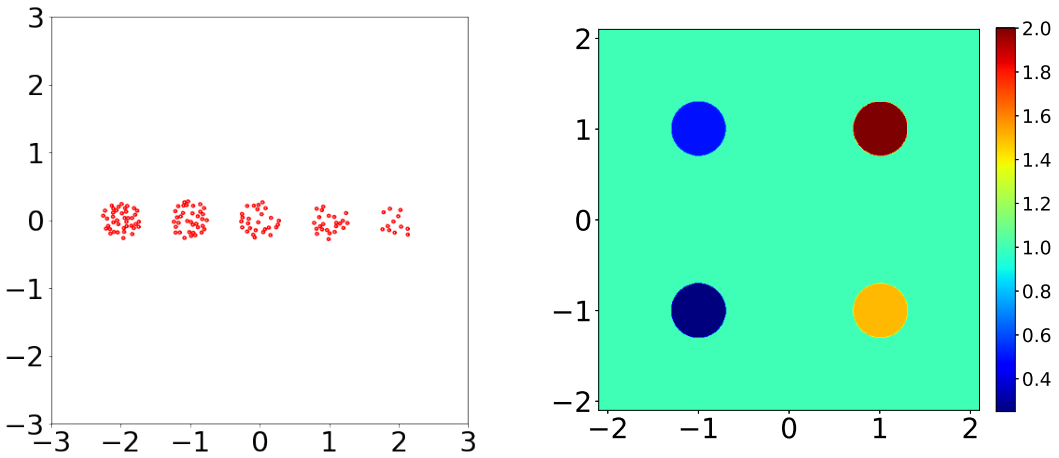

Figure 11. *Left: the domain D constituted by small circles concentrated in four aligned areas. The scatterers have a radius* 0.02 *and a constant index refraction $n = 2$ inside. Right: Four diffracting circles of radius* 0.3 *associated with four different values of the refractive index $n = 0.25$ (bottom left), $n = 0.5$ (top left), $n = 1.5$ (bottom right) and $n = 2$ (upper right).*

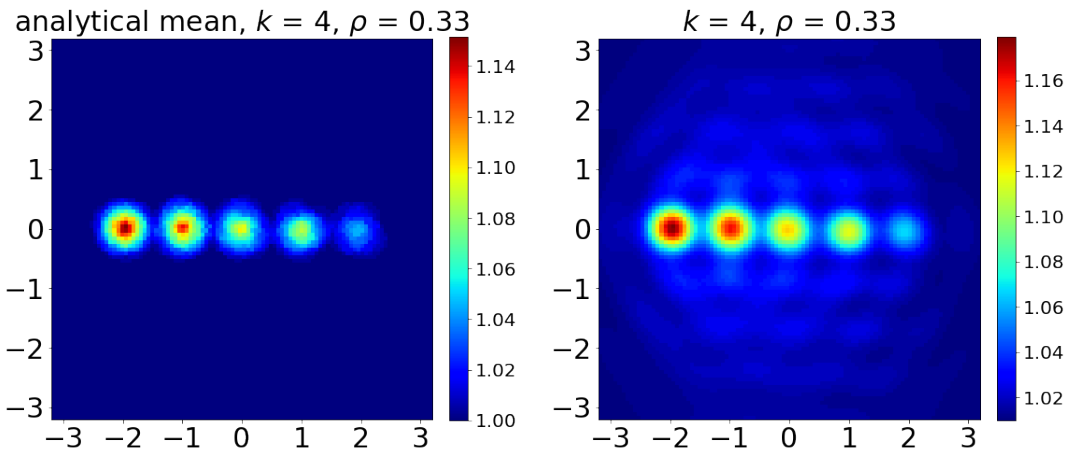

Figure 12. *Results associated with configuration of the domain D in Figure 11 (left). Left: $y \to \overline{n}(y)$ representing the true averaged values of the refractive index. Right: $y \to n^*(y)$ representing the outcome of the inversion algorithm plotted on the 100×100 uniformed grid $\mathcal{Y}$ for $k = 4$, $\rho = \frac{1.3}{k}$ and the noise level $\delta = 1\%$.*

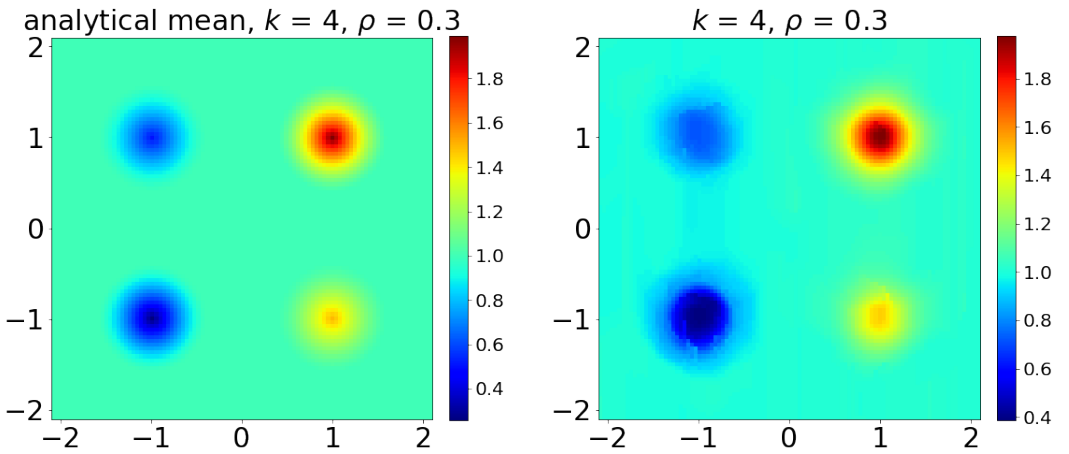

Figure 13. *Results associated with configuration of the domain D in Figure 11 (right). Left: $y \to \overline{n}(y)$ representing the true averaged values of the refractive index. Right: $y \to n^*(y)$ representing the outcome of the inversion algorithm plotted on the 100×100 uniformed grid $\mathcal{Y}$ for $k = 4$, $\rho = 0.3$ and the noise level $\delta = 1\%$.*

Appendix A. Structure of the $\mathcal{B}$ operator.

Theorem A.1. *Consider a bounded linear positive, self-adjoint and compact operator $\mathcal{B} : H^{-\frac{1}{2}}(\partial D_b) \to H^{\frac{1}{2}}(\partial D_b)$. For κ a positive constant, the two following assertions are equivalent.*

1. *For all* $\psi, \phi \in H^{-\frac{1}{2}}(\partial D_b)$,

$$\kappa(\mathcal{B}\psi, \mathcal{B}\phi)_{L^2(\partial D_b)} = \langle \psi, \mathcal{B}\phi \rangle_{H^{-\frac{1}{2}}(\partial D_b), H^{\frac{1}{2}}(\partial D_b)}. \tag{A.1}$$

2. *There exists a family of* M *vectors* $\{e_1, ..., e_M\} \subset H^{\frac{1}{2}}(\partial D_b)$ *which are orthonormal with respect to the* $L^2(\partial D_b)$ *scalar product such that:*

$$\mathcal{B}(\psi) = \kappa \sum_{i=1}^{M} \langle \psi, e_i \rangle_{H^{-\frac{1}{2}}(\partial D_b), H^{\frac{1}{2}}(\partial D_b)} e_i, \quad \forall \psi \in H^{-\frac{1}{2}}(\partial D_b). \tag{A.2}$$

Proof. Assume item *(i)* holds. Let $\mathcal{I} : H^{\frac{1}{2}}(\partial D_b) \to L^2(\partial D_b)$ denote a bijective operator. Applying the spectral theorem to $\mathcal{I}\mathcal{B}\mathcal{I}^* : L^2(\partial D_b) \to L^2(\partial D_b)$, we deduce the existence of a sequence of positive real number $(\lambda_i)_i$ that converges toward 0 and $(b_i)_{i\in\mathbb{N}^*}$ an orthonormal basis of $L^2(\partial D_b)$ such that:

$$\mathcal{I}\mathcal{B}\mathcal{I}^* b_i = \lambda_i b_i,$$

which can be rewritten as

$$\mathcal{B}\mathcal{I}^* b_i = \lambda_i \mathcal{I}^{-1} b_i. \tag{A.3}$$

Every $\psi \in H^{-\frac{1}{2}}(\partial D_b)$ (respectively $\phi \in H^{\frac{1}{2}}(\partial D_b)$) admits an unique element $\widetilde{\psi} \in L^2(\partial D_b)$ such that $\psi = \mathcal{I}^*\widetilde{\psi}$ (respectively $\widetilde{\phi} \in L^2(\partial D_b)$ such that $\phi = \mathcal{I}^{-1}\widetilde{\phi}$).
Setting $a_i := (\widetilde{a}, b_i)_{L^2(\partial D_b)}$, $a \in L^2(\partial D_b)$, one has:

$$\begin{aligned} (\mathcal{B}\psi, \mathcal{B}\phi)_{L^2(\partial D_b)} &= \textstyle\sum_{i,j\in\mathbb{N}^*} \phi_j \psi_i (\mathcal{B}\mathcal{I}^* b_i, \mathcal{B}\mathcal{I}^* b_j)_{L^2(\partial D_b)}, \\ &= \textstyle\sum_{i,j\in\mathbb{N}^*} \phi_j \psi_i \lambda_i \lambda_j (\mathcal{I}^{-1} b_i, \mathcal{I}^{-1} b_j)_{L^2(\partial D_b)}. \end{aligned} \tag{A.4}$$

On the other hand:

$$\kappa \langle \psi, \mathcal{B}\phi \rangle_{H^{-\frac{1}{2}}(\partial D_b), H^{\frac{1}{2}}(\partial D_b)} = \kappa \sum_{i\in\mathbb{N}^*} \phi_i \psi_i \lambda_i. \tag{A.5}$$

Since (A.4) and (A.5) should be equal for every square summable sequences $(\phi_i)_i, (\psi_i)_i$, taking $\phi_k \psi_l = \delta_{ik}\delta_{li}$, we obtain that λ_i is equal to 0 or $\kappa \|\mathcal{I}^{-1} b_i\|^{-2}_{L^2(\partial D_b)}$. Since $\mathcal{I}$ is bounded, the sequence $(\|\mathcal{I}^{-1} b_i\|^2)_i$ remains also bounded. We deduce by compactness of the operator $\mathcal{B}$ that (up to a renumbering) the sequence $(\lambda_i)_i$ is equal to $\kappa \|e_i\|^{-2}_{L^2(\partial D_b)}$ up to a certain rank M $\lambda_i = 0$ for $i > M$.
For $k, l \leq M$, setting $\phi_k \psi_l = \delta_{ik}\delta_{lj}$ enforces $(\mathcal{I}^{-1} b_k, \mathcal{I}^{-1} b_l)_{L^2(\partial D_b)} = 0$ for $k \neq l$ and $k, l \leq M$.
Hence, for all $\psi \in H^{-\frac{1}{2}}(\partial D_b)$, the operator $\mathcal{B}$ can be written as:

$$\begin{aligned} \mathcal{B}\psi &= \mathcal{B}\mathcal{I}^*\mathcal{I}^{*-1}\psi \\ &= \kappa \textstyle\sum_{i=0}^{M} (\mathcal{I}^{*-1}\psi, b_i)_{L^2(\partial D_b)} \mathcal{I}^{-1} b_i \\ &= \kappa \textstyle\sum_{i=0}^{M} \langle \psi, e_i \rangle_{H^{-\frac{1}{2}}(\partial D_b), H^{\frac{1}{2}}(\partial D_b)} e_i, \end{aligned} \tag{A.6}$$

where $e_i = \frac{\mathcal{I}^{-1} b_i}{\|\mathcal{I}^{-1} b_i\|_{L^2(\partial D_b)}}$. This proves that *(i)* implies *(ii)*. The reciprocal is straightforward. ∎

Appendix B. Study of the background problem (2.2). The well posedness of problem (2.2) is established by variational approach. Let B_R be a ball containing D_b for R large enough. We define the Dirichlet to Neumann map $\Lambda : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R)$ by $\Lambda\phi := \partial_\nu v|_{\partial B_R}$, where v is the radiating solution to the Helmholtz equation in $\mathbb{R}^m\backslash B_R$ with $v = \phi$ on ∂B_R, and where ν is the outward unit normal to S_R. Define the closed subspace of $H^1(B_R\backslash D_b)$:

$$H^1_{\mathcal{B}}(B_R\backslash D_b) \quad := \{w \in H^1(B_R\backslash D_b), w|_{\partial D_b} \in \mathcal{B}(H^{-\frac{1}{2}}(\partial D_b))\}. \tag{B.1}$$

Then, by Green's formula and using property (2.4), one can prove that solving the scattering problem (2.2) is equivalent to solving the following variational problem: $u \in H^1_{\mathcal{B}}(B_R\backslash D_b)$,

$$\int_{B_R\backslash D_b} (\nabla u \nabla v - k^2 uv)dx - \int_{\partial B_R} \Lambda uv ds - \mu\kappa \int_{\partial D_b} uv ds = \ell(v) \tag{B.2}$$

for all $v \in H^1_{\mathcal{B}}(B_R\backslash D_b)$, where

$$\ell(v) := \textstyle\int_{\partial B_R} (\partial_\nu u^i - \Lambda u^i)v ds, \tag{B.3}$$

The study of the well-posedness of this problem follows classical schemes based on proving that the problem is of Fredholm type and the uniqueness is a consequence of Rellish's lemma. The details are left to the reader. We then can state the following lemma.

Lemma B.1. *Problem (B.2) is well posed for $\mu \in \mathbb{R}^*$ if $(\partial_\nu u^i - \Lambda u^i) \in H^{-\frac{1}{2}}(\partial B_R)$.*

We prove in the following lemma the normality of the operator F^μ_b.

Proposition B.2. *For $\mu \in \mathbb{R}^*$, F^μ_b is normal and $S_b = I + \frac{2ik}{\overline{\gamma}} F^\mu_b$ is unitary.*

Proof. This proof is similar to the one in [21, Theorem 1.8].
We first show that for all $g, h \in L^2(\mathbb{S})$,

$$2ik(F^\mu_b g, F^\mu_b h)_{L^2(\mathbb{S})} = \gamma(F^\mu_b g, h)_{L^2(\mathbb{S})} - \overline{\gamma}(g, F^\mu_b h)_{L^2(\mathbb{S})}. \tag{B.4}$$

Let us deal with the three terms independently.

1. Let w^s_g and w^s_h radiating solution of problem (2.2) with u^i an Herglotz wave of kernel g and h respectively with far field pattern w^∞_g, w^∞_h. We denote by w^t_g, w^t_h the total fields. Let B_R be a disk of radius $R > 0$ centered at the origin containing D_b. Applying Green second theorem, we get

$$0 = \int_{\partial B_R} (\partial_\nu w^s_g \overline{w^s_h} - \partial_\nu \overline{w^s_h} w^s_g)ds - \int_{\partial D_b} (\partial_\nu w^s_g \overline{w^s_h} - \partial_\nu \overline{w^s_h} w^s_g)ds.$$

Letting $R \to +\infty$ and using the radiation condition, one obtains

$$2ik(F^\mu_b g, F^\mu_b h)_{L^2(\mathbb{S})} = 2ik \int_{\mathbb{S}} w^\infty_g \overline{w^\infty_h} ds = \int_{\partial D_b} (\partial_\nu w^s_g \overline{w^s_h} - \partial_\nu \overline{w^s_h} w^s_g)ds.$$

2. If $u^i = v_h$ an Herglotz wave of kernel $h \in L^2(\mathbb{S})$, then we have

$$\begin{aligned}\int_{\partial D_b}(\partial_\nu w_g^s \overline{v_h} - \partial_\nu \overline{v_h} w_g^s)ds &= \int_{\mathbb{S}} \overline{h}(d) \int_{\partial D_b}(\partial_\nu w_g^s e^{-iky\cdot d} - \partial_\nu e^{-iky\cdot d} w_g^s ds(y)ds(d)\\ &= -\gamma \int_{\mathbb{S}} w_g^\infty(d)\overline{h}(d)ds(d)\\ &= -\gamma(F_b^\mu g, h)_{L^2(\mathbb{S})}.\end{aligned}$$

Similarly, one gets

$$\int_{\partial D_b}(\partial_\nu v_g \overline{w_h^s} - \partial_\nu \overline{w_h^s} v_g)ds = \overline{\gamma}(g, F_b^\mu h)_{L^2(\mathbb{S})}$$

3. We then compute $-2ik(F_b g, F_b h) + \gamma(F_b g, h) - \overline{\gamma}(g, F_b h)$:

$$\begin{aligned}&= -2ik \int_{\mathbb{S}} w_g^\infty \overline{w_g^\infty} ds + \gamma \int_{\mathbb{S}} w_g^\infty(d)\overline{h}(d)ds(d) - \overline{\gamma} \int_{\mathbb{S}} \overline{w_h^\infty}(d) g(d) ds(d)\\ &= \int_{\partial D_b}(w_g^t(y)\partial_{\nu(y)}\overline{w_h^t}(y) - \overline{w_h^t}(y)\partial_{\nu(y)} w_g^t(y))ds(y)\\ &= -\tfrac{1}{\mu}\int_{\partial D_b}(\mathcal{B}(\partial_{\nu(y)} w_g^t(y))\partial_{\nu(y)}\overline{w_h^t}(y) + \tfrac{1}{\overline{\mu}}\mathcal{B}(\partial_{\nu(y)}\overline{w_h^t}(y))\partial_{\nu(y)} w_g^t(y))ds(y)\\ &= (\tfrac{1}{\overline{\mu}} - \tfrac{1}{\mu})\int_{\partial D_b}(\mathcal{B}(\partial_{\nu(y)} w_g^t(y))\partial_{\nu(y)}\overline{w_h^t}(y)ds(y)\\ &= 0,\end{aligned}$$

because $\mu \in \mathbb{R}$.

Therefore, one has:

$$F_b^{\mu *} F_b^\mu = \frac{1}{2ik}(\gamma F_b^\mu - \overline{\gamma} F_b^{\mu *}). \tag{B.5}$$

From this equality, one can easily show that $S_b^* S_b = I$ and, because S_b is a compact perturbation of I, conclude that S_b is unitary. This is enough to prove that F_b^μ is normal. ∎

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