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# A new class of symmetry-preserving observers

Danilo R. Lima<sup>1</sup>, Rosane Ushirobira<sup>1</sup>, Denis Efimov<sup>1</sup>

**Abstract**—This paper introduces the concept of  $\theta$ -invariant systems, which can be regarded as a generalization of symmetry-preserving systems. This new property enables the re-scaling of time coordinates while dilating the solutions. We discuss some of the implications of this concept for observer design and adapt the results about the existence of a canonical representation for an observer obtained in the context of invariant systems. Furthermore, we demonstrate how our generalization can be used to construct an observer design framework for homogeneous systems with arbitrary degrees.

## I. INTRODUCTION

An effective control of a system requires a sufficiently accurate knowledge of its state at a given moment in time. This knowledge might be restricted by an absence of sensors, sensor nonlinearity (*i.e.*, saturation), or the presence of measurement noises. The state estimation theory was developed to tackle such restrictions and obtain a valid knowledge of a system state given a measured noisy output in cases achievable by mathematical tools [7].

For continuous-time systems with inputs, whose behavior is described by a differential equation of the form  $\dot{x} = f(x, u)$ , we denote the measured output as  $y = h(x, u)$ . The problem is to determine the state  $x$  at time  $t$  as a function of the outputs  $y$  in time prior to  $t$ , where  $u$  is a known or unknown input, and  $f, h$  are certain maps of proper dimensions and regularity (the so-called *state estimation problem*). Direct formulas can be obtained for particular classes of systems, such as observable linear systems, using an explicit operator that assigns each output function  $y$  on the interval  $[0, t]$  to the unique initial condition  $x(0)$  and input  $u$  on the interval  $[0, t]$  that produced it, then by integrating the system we may obtain the state  $x(t)$  as well. However, being an open-loop process, this has been demonstrated not to be a robust strategy for estimation, as it leads to increasing error in the presence of noise.

An alternative strategy is that of observers, which uses a dynamic output injection, taking into account the actual behavior of the state variables. An observer of a system  $\dot{x} = f(x, u)$  is a second system  $\dot{\hat{x}} = F(\hat{x}, u, y)$  having the property that  $e(t) = x(t) - \hat{x}(t) \rightarrow 0$  as  $t \rightarrow \infty$ , for no matter which pair of initial conditions  $x(0), \hat{x}(0)$  (for some

applications it is enough to consider close enough pairs). The quantity  $e(t)$  is called the state estimation error and has its dynamics, written as  $\dot{e} = f(e + \hat{x}, u) - F(\hat{x}, u, y)$ . The dependence on the external variables  $\hat{x}, u$ , and  $y$  can make the error dynamics challenging to analyze, so it is often desirable to reduce such a dependence on these variables to get a clearer dynamical analysis and to simplify the observer design. This reduction may be achieved by a proper design of  $F$ , changing the definition of state estimation error, or finding special symmetries of the original system and then designing the observers accordingly. Preserving or introducing symmetries in the dynamics of the estimation error guarantees that the observer inherits valuable features of the estimated process (as non-negativity of the state, for instance) or demonstrates desired performances (as accelerated rates of convergence for homogeneous systems [6]).

Multiple techniques have been developed in this direction. In [1], [5], the system's homogeneity properties have been proposed to explore for the design and stability analysis. More recently, a canonical representation of a symmetry-preserving observer was found [2], considering any general symmetry that the corresponding system possesses. This canonical representation can also be utilized for stability analysis and observer gain design. Since it has been known that any system possesses symmetries of some sort (see, for instance, [3]), the problem of finding the right symmetries and designing observers that make explicit use of them is a topic of research that has much potential. Even though the method proposed in [2] does deal with guarantees on the convergence of the error for a general observable system, one can still benefit from a complete characterization of the equations describing the dynamics of any admissible candidate to be an observer of a system – also called *pre-observer* – provided that the observer and the system share certain symmetries. Those pre-observers have, together with a suitable definition of the state estimation error, a useful property that the error dynamics is also invariant, and it depends on the state and input variables only through an invariant function, which reduces the degree of freedom for the design and simplifies the convergence analysis.

In this paper, we propose the class of  $\theta$ -invariant systems, whose symmetries allow for time rescaling, and cover the important example of homogeneous systems of arbitrary degrees. We prove an analog of the result on the characterization of invariant pre-observers adapted for  $\theta$ -invariant systems and provide an example of what these pre-observers look like for general weighted homogeneous systems of a given degree.

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This note is organized as follows: Section II recalls basic definitions and results from the  $G$ -invariant systems theory. In Section III, we define  $\theta$ -invariant systems and state some of its properties, and prove a result of the characterization of  $\theta$ -invariant pre-observers.

All proofs are excluded due to space limitations.

## II. PRELIMINARIES

We begin by recalling some basic definitions needed in this paper.

**Definition II.1.** Given a differentiable function  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $x \in \mathbb{R}^n$ , we denote by  $D\phi(x)$  its differential at  $x$ , that is, the linear transformation associated to the matrix  $D\phi(x) = \left( \frac{\partial \phi_i}{\partial x_j} \right)_{1 \leq i, j \leq n} \in \mathbb{R}^{n \times n}$ .

**Definition II.2.** Let  $G$  be a group. We say that  $G$  is a Lie group if it is endowed with the structure of a smooth manifold and the group operations  $(a, b) \mapsto ab$  and  $a \mapsto a^{-1}$  are smooth (their differential maps exist in every point and vary continuously).

In what follows, for a dynamical system, we assume the state space to be  $\mathcal{X} \subset \mathbb{R}^n$ , the input space to be  $\mathcal{U} \subset \mathbb{R}^m$ , and the output space to be  $\mathcal{Y} \subset \mathbb{R}^p$ . Furthermore, we assume  $G$  as a Lie group of dimension  $r$ .

The systems we analyze have a general form (we occasionally omit the dependence on time to facilitate the reading):

$$\begin{aligned} \dot{x} &= f(x, u), \\ y &= h(x, u), \end{aligned} \quad (1)$$

with  $x \in \mathcal{X}$ ,  $u \in \mathcal{U}$ , and  $y \in \mathcal{Y}$ ,  $f : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{X}$  is a (locally Lipschitz) continuous function and  $h : \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{Y}$  is a continuously differentiable function.

The following two definitions can be found in [2].

**Definition II.3.** Let us consider two groups actions  $\phi : G \times \mathcal{X} \rightarrow \mathcal{X}$  and  $\psi : G \times \mathcal{U} \rightarrow \mathcal{U}$ . For  $g \in G$ , denote  $\phi_g(x) = \phi(g, x)$ , for all  $x \in \mathcal{X}$ , and similarly for  $\psi_g$ . The system (1) is said to be invariant (or  $G$ -invariant) if

$$D\phi_g(x) [f(x, u)] = f(\phi_g(x), \psi_g(u)), \quad (2)$$

for all  $x \in \mathcal{X}$ ,  $u \in \mathcal{U}$  and  $g \in G$ . The output  $y = h(x, u)$  is said to be equivariant (or  $G$ -compatible) if there exists an action  $\rho : G \times \mathcal{Y} \rightarrow \mathcal{Y}$  such that

$$\rho_g(h(x, u)) = h(\phi_g(x), \psi_g(u)), \quad (3)$$

for all  $x \in \mathcal{X}$ ,  $u \in \mathcal{U}$  and  $g \in G$ , with  $\rho_g = \rho(g, \cdot)$  as above.

**Definition II.4.** A system  $\hat{x} = F(\hat{x}, u, y)$  is called a pre-observer of (1) if for all  $x \in \mathcal{X}$  and  $u \in \mathcal{U}$ ,

$$F(x, u, h(x, u)) = f(x, u). \quad (4)$$

**Definition II.5.** [2] Let (1) be an invariant system with an equivariant output, with respect to group actions  $\phi : G \times$

$\mathcal{X} \rightarrow \mathcal{X}$ ,  $\psi : G \times \mathcal{U} \rightarrow \mathcal{U}$  and  $\rho : G \times \mathcal{Y} \rightarrow \mathcal{Y}$ . A pre-observer  $\hat{x} = F(\hat{x}, u, y)$  is called an invariant pre-observer for (1) if for all  $x \in \mathcal{X}$ ,  $u \in \mathcal{U}$ ,  $y \in \mathcal{Y}$ , and  $g \in G$ ,

$$D\phi_g(x) [F(x, u, y)] = F(\phi_g(x), \psi_g(u), \rho_g(y)). \quad (5)$$

Since  $G$  is a smooth manifold, up to composition with a chart, we can regard any function  $\phi : G \times \mathcal{X} \rightarrow \mathcal{X}$  as a smooth function between open sets of  $\mathbb{R}^{r+n}$  and  $\mathbb{R}^n$  for a suitably defined integer  $r \geq 1$ , written  $\phi(g, x) = \phi(g_1, \dots, g_r, x_1, \dots, x_n)$ . We can then consider  $D_g\phi(g, x)$  the transformation associated to the  $r \times n$ -matrix

$$\left( \frac{\partial \phi_i}{\partial g_j} \right)_{1 \leq i \leq n, 1 \leq j \leq r} \in \mathbb{R}^{n \times r} \quad (6)$$

which is a submatrix of  $D\phi(g, x) \in \mathbb{R}^{(r+n) \times r}$ . We say that the application  $g \mapsto \phi_g(x)$  is full-rank in  $x_0$  if the matrix  $D_g\phi(e, x_0)$  has full rank, where  $e \in G$  is the identity.

The next concept is essential for building all invariant elements appearing in the final invariant pre-observer canonical form.

**Lemma II.6.** Let  $\phi : G \times \mathcal{X} \rightarrow \mathcal{X}$  be a group action and suppose  $r \leq n$ . Let  $x_0 \in \mathcal{X}$  be such that  $g \mapsto \phi_g(x)$  has full-rank in  $x_0$ . Then there exists a neighborhood  $\mathcal{O}$  of  $x_0$  and a function  $\gamma : \mathcal{O} \rightarrow G$  such that

$$\gamma(\phi_g(x))g = \gamma(x) \quad (7)$$

for every  $g \in G$  and  $x \in \mathcal{O}$ .

The function  $\gamma$  is called the *moving frame* of the invariant system. In what follows, we restate the definitions of invariant functions, invariant output error, and invariant frame and how the moving frame can be used to compute them explicitly. For a more detailed exposition, we refer to [2] and the proofs of each statement.

**Definition II.7.** We say that  $J : \mathcal{X} \rightarrow \mathbb{R}^n$  is an invariant function for an action  $\phi : G \times \mathcal{X} \rightarrow \mathcal{X}$  if

$$J(\phi_g(x)) = J(x)$$

for every  $x \in \mathcal{X}$  and  $g \in G$ . If  $J$  has the additional property that any other invariant function  $I$  can be written as

$$I(x) = \mathcal{F}(J(x)),$$

for some function  $\mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , then  $J$  is called a complete set of invariant functions.

**Proposition II.8.** [2] Let  $x_0 \in \mathcal{X}$  be such that  $\text{rank}(D_g\phi(e, x_0)) = r = \dim(G)$ . Then the function  $J(x) = \phi_{\gamma(x)}^b(x)$ , where  $\phi_g^b$  is as in (??), is a complete set of invariant functions in a neighborhood of  $x_0$ .

**Definition II.9.** An invariant output error is a function  $E : \mathcal{X} \times \mathcal{U} \times \mathcal{Y} \rightarrow \mathbb{R}^p$  such that

- 1) The map  $y \mapsto E(\hat{x}, u, y)$  is invertible for every  $\hat{x} \in \mathcal{X}$  and every  $u \in \mathcal{U}$ ,

- 2)  $E(\hat{x}, u, h(\hat{x}, u)) = 0$ , for every  $\hat{x} \in \mathcal{X}$  and every  $u \in \mathcal{U}$ ,
- 3)  $E(\phi_g(\hat{x}), \psi_g(u), \rho_g(y)) = E(\hat{x}, u, y)$ , for every  $\hat{x} \in \mathcal{X}$ ,  $u \in \mathcal{U}$  and  $y \in \mathcal{Y}$ .

**Proposition II.10.** [2] *The function*

$$E(\hat{x}, u, y) = \rho_{\gamma(\hat{x})}(h(\hat{x}, u)) - \rho_{\gamma(\hat{x})}(y)$$

is an invariant output error. Furthermore, every invariant output error  $\tilde{E}$  can be written as

$$\tilde{E}(\hat{x}, u, y) = \mathcal{L}(I(\hat{x}, u), E(\hat{x}, u, y)),$$

where  $I$  is an invariant function of the actions  $\phi$  and  $\psi$  and  $\mathcal{L}$  is a smooth function such that  $\mathcal{L}(I, 0) = 0$ .

**Definition II.11.** A vector field  $w : \mathcal{X} \rightarrow T\mathcal{X}$  is said to be invariant if  $D\phi_g(x)[w(x)] = w(\phi_g(x))$ , for all  $x \in \mathcal{X}$  and  $g \in G$ , where  $T\mathcal{X}$  denotes the tangent bundle of  $\mathcal{X}$ .

An invariant frame in  $\mathcal{X}$  is a set  $\{w_1, \dots, w_n\}$  of invariant vector fields such that  $\{w_1(x), \dots, w_n(x)\}$  forms a basis of the tangent space  $T_x\mathcal{X}$ , for every  $x \in \mathcal{X}$ .

**Proposition II.12.** [4] (Theorem 2.84) Let  $\mathcal{X}$  be an open subset of  $\mathbb{R}^n$  and  $G$  a Lie group of dimension  $r \leq n$ . Given an action  $\phi : G \times \mathcal{X} \rightarrow \mathcal{X}$  that is full-rank at every point, there always exists an invariant frame  $\{w_1, \dots, w_n\}$ , which is given by the columns of the matrix  $(D\phi_{\gamma(x)}(x))^{-1}$ .

All the above-defined concepts are united to yield a result that classifies all possible invariant pre-observers, given an invariant system with equivariant output.

**Theorem II.13.** [2] Suppose that (1) is invariant with an equivariant output. Assume that for each  $x \in \mathcal{X}$ , the mapping  $g \rightarrow \phi_g(x)$  is full-rank. Then,  $\hat{x} = F(\hat{x}, u, y)$  is an invariant pre-observer if and only if

$$F(\hat{x}, u, y) = f(\hat{x}, u) + \sum_{i=1}^n \mathcal{L}_i(I(\hat{x}, u), E(\hat{x}, u, y)) w_i(\hat{x}), \quad (8)$$

where  $I$  is an invariant function (that is,  $I(\phi_g(x), \psi_g(u)) = I(x, u)$ , for all  $x \in \mathcal{X}$ ,  $u \in \mathcal{U}$  and  $g \in G$ ),  $E$  is an output error, the  $\mathcal{L}_i$  are smooth functions such that for all  $\hat{x} \in \mathcal{X}$  and  $u \in \mathcal{U}$ ,  $\mathcal{L}_i(I(\hat{x}, u), 0) = 0$ , and  $\{w_1, \dots, w_n\}$  is an invariant frame.

This canonical form is particularly useful because by combining it with a more suitable definition of the state estimation error one can obtain an error dynamic equation with fewer variables, making it simpler for analysis.

**Theorem II.14.** [2] *The invariant state error dynamics*

$$\eta(x, \hat{x}) = \phi_{\gamma(x)}(\hat{x}) - \phi_{\gamma(x)}(x)$$

can be written as

$$\dot{\eta} = \Upsilon(\eta, I(x, u))$$

where  $\Upsilon$  is a smooth function and  $I$  is an invariant function.

We remark that the previous definition of  $G$ -invariance [2] fails to include the important class of homogeneous systems and weighted homogeneous systems, to which a large body of the theory on observer design and stabilization techniques have been proposed [5], [8].

A. Extension of invariance property

In order to contour this problem, we have proposed a slightly more general concept, that of  $\theta$ -invariance:

**Definition III.1.** Let  $\mathcal{X} \subset \mathbb{R}^n$  and  $\mathcal{U} \subset \mathbb{R}^m$  be open sets and  $G$  be a Lie group, with differentiable actions  $\phi : G \times \mathcal{X} \rightarrow \mathcal{X}$  and  $\psi : G \times \mathcal{U} \rightarrow \mathcal{U}$ . Let  $\theta : G \rightarrow \mathbb{R}_{>0}$  be a differentiable function. The system (1) is said to be  $\theta$ -invariant with respect to  $G$  and the actions  $\phi$  and  $\psi$  if the following holds:

$$D\phi_g(x)[f(x, u)] = \theta(g)f(\phi_g(x), \psi_g(u)), \quad (9)$$

for every  $x \in \mathcal{X}$ ,  $u \in \mathcal{U}$  and  $g \in G$ .

We remark that there are some restrictions on the function  $\theta : G \rightarrow \mathbb{R}_{>0}$  so that (9) is satisfied:

**Proposition III.2.** Suppose that (9) holds for every  $x \in \mathcal{X}$ ,  $u \in \mathcal{U}$ ,  $g \in G$ , and that  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is not identically zero. Then  $\theta : G \rightarrow \mathbb{R}_{>0}$  is a group homomorphism. In particular, the mapping  $(t, g) \mapsto \theta(g)t$  is an action of  $G$  over the set of strictly positive real numbers.

The interest behind  $G$ -invariance is that the actions  $\phi$  and  $\psi$  permute solutions of the system. That is also the case for  $\theta$ -invariant systems, with the addition of a rescaling of the time coordinate. In what follows, we specify what we mean by that. To avoid repetition, we make the convention that  $\phi$  always denotes an action of  $G$  over the state space  $\mathcal{X} \subset \mathbb{R}^n$  and  $\psi$  denotes an action over the input space  $\mathcal{U} \subset \mathbb{R}^m$ .

**Definition III.3.** A trajectory of the system (1) is a curve  $\varrho : \mathbb{R}_{\geq 0} \rightarrow \mathcal{X} \times \mathcal{U}$  given by  $\varrho(t) = (x(t), u(t)) \in \mathbb{R}^n \times \mathbb{R}^m$  such that  $\dot{x}(t) = f(x(t), u(t))$ .

**Proposition III.4.** The system (1) is  $\theta$ -invariant with respect to the actions  $\phi$  and  $\psi$  and the homomorphism  $\theta$  if, and only if, for every trajectory  $t \mapsto (x(t), u(t))$  of (1), the curve  $t \mapsto (\phi_g(x(\theta(g)t)), \psi_g(u(\theta(g)t)))$  is also a trajectory.

One class of systems of particular interest that falls into the definition of  $\theta$ -invariance is that of  $\mathbf{r}$ -homogeneous systems.

**Example III.5.** Let  $\mathbf{r} = (r_1 \dots r_n)^\top \in \mathbb{R}_{>0}^n$ . For  $g > 0$ , define  $\Lambda_g^r = \text{diag}(g^{r_1}, \dots, g^{r_n})$ . We say that  $\dot{x} = f(x, u)$  is an  $\mathbf{r}$ -homogeneous system of degree  $\mu$  if

$$f(\Lambda_g^r x, u) = g^\mu \Lambda_g^r f(x, u)$$

for every  $x \in \mathcal{X}$ ,  $u \in \mathcal{U}$ .

A direct computation shows that the multiplicative group of strictly positive real numbers  $(\mathbb{R}_{>0}, \cdot)$  together with the actions  $\phi_g(x) = \Lambda_g^r x$ ,  $\psi_g(u) = u$  and the homomorphism  $\theta(g) = g^{-\mu}$  makes the system invariant:

$$f(\phi_g(x), \psi_g(u)) = f(\Lambda_g^r x, u) = g^\mu \Lambda_g^r f(x, u)$$

$$= g^\mu D\phi_g(x)[f(x, u)] = \theta(g)^{-1} D\phi_g(x)[f(x, u)].$$

Therefore, by Proposition III.4, for every trajectory  $x(t)$ , the curve  $\Lambda_g^r x(g^\mu t)$  is also a trajectory.

The orbits of the action  $\phi$  are the homogeneous rays  $R_x = \{\Lambda_\varepsilon^r x : \varepsilon > 0\}$ . Therefore, to understand the system dynamics, it suffices to know the trajectories starting outside the homogeneous unit sphere

$$\mathbb{S}_r = \{x \in \mathbb{R}^n : \|x\|_r = 1\},$$

where  $\|x\|_r = (|x_1|^{\kappa/r_1} + \dots + |x_n|^{\kappa/r_n})^{1/\kappa}$  with  $\kappa \geq \max_{1 \leq i \leq n} \{r_i\}$  is the  $\mathbf{r}$ -homogeneous norm.

Notice that a homogeneous system does not fall into the regular definition of invariance unless the defining vector field has degree zero. So far, the formulations of invariance seen consider  $\theta$  as the constant homomorphism  $\theta \equiv 1$ .

**Example III.6.** Consider the system

$$\begin{aligned} \dot{x} &= ue^{-\alpha} \cos \alpha \\ \dot{y} &= ue^{-\alpha} \sin \alpha \\ \dot{\alpha} &= e^{-\alpha} v \end{aligned} \quad (10)$$

where  $X = (x, y, \alpha) \in \mathbb{R}^2 \times \mathbb{S}^1 \simeq SE(2)$  are the state variables and  $U = (u, v) \in \mathbb{R}^2$  are the input variables. This is a modified version of the Unicycle Mobile Robot model, which is written without the exponential term. The original version was shown in [2] to be  $G$ -invariant under the group  $G = SE(2)$  and the actions

$$\begin{aligned} \phi_g \begin{pmatrix} x \\ y \\ \alpha \end{pmatrix} &= \begin{pmatrix} x \cos(\alpha_g) - y \sin(\alpha_g) + x \\ x \sin(\alpha_g) + y \cos(\alpha_g) + y \\ \alpha + \alpha_g \end{pmatrix} \text{ and} \\ \psi_g \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} u \\ v \end{pmatrix} \end{aligned}$$

Denoting

$$f(X, U) = \begin{pmatrix} ue^{-\alpha} \cos \alpha \\ ue^{-\alpha} \sin \alpha \\ ve^{-\alpha} \end{pmatrix},$$

a direct computation shows that

$$\begin{aligned} D\phi_g(X)f(X, U) &= \begin{pmatrix} \cos \alpha_g & -\sin \alpha_g & 0 \\ \sin \alpha_g & \cos \alpha_g & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ue^{-\alpha} \cos \alpha \\ ue^{-\alpha} \sin \alpha \\ ve^{-\alpha} \end{pmatrix} \\ &= \begin{pmatrix} ue^{-\alpha} \cos(\alpha + \alpha_g) \\ ue^{-\alpha} \sin(\alpha + \alpha_g) \\ ve^{-\alpha} \end{pmatrix} \\ &= e^{\alpha_g} \begin{pmatrix} ue^{-(\alpha + \alpha_g)} \cos(\alpha + \alpha_g) \\ ue^{-(\alpha + \alpha_g)} \sin(\alpha + \alpha_g) \\ ve^{-(\alpha + \alpha_g)} \end{pmatrix} \\ &= e^{\alpha_g} f(\phi_g(X), \psi_g(U)) \end{aligned}$$

showing that system (10) is  $\theta$ -invariant for the homomorphism  $\theta(g) = e^{\alpha_g}$ .

More generally, one can show that if the system  $\dot{x} = f(x, u)$  is invariant for actions  $\phi$  and  $\psi$ , has a moving frame

$\gamma : \mathcal{X} \rightarrow G$  and  $\theta : G \rightarrow \mathbb{R}_+^*$  is a homomorphism, then the system  $\dot{x} = \theta(\gamma(x))f(x, u)$  is  $\theta$ -invariant for the same actions together with the homomorphism  $\theta : G \rightarrow \mathbb{R}_+^*$ .

## B. Characterization of pre-observers

Next, we aim to obtain a result similar to Theorem II.13, in which we characterize all possible pre-observers that obey the system symmetries. In order to do that, we have to define  $\theta$ -invariant pre-observers and  $\theta$ -invariant frames.

**Lemma III.7.** Let  $\mathcal{X}$  be an open subset of  $\mathbb{R}^n$  and  $G$  a Lie group of dimension  $r \leq n$ . Given an action  $\phi : G \times \mathcal{X} \rightarrow \mathcal{X}$  that is full-rank at every point, there always exists a  $\theta$ -invariant frame  $\{\tilde{w}_1, \dots, \tilde{w}_n\}$ .

**Definition III.8.** Let  $\dot{x} = f(x, u)$  be a  $\theta$ -invariant system with an equivariant output  $y = h(x, u)$  for actions  $\phi : G \times \mathcal{X} \rightarrow \mathcal{X}, \psi : G \times \mathcal{U} \rightarrow \mathcal{U}, \rho : G \times \mathcal{Y} \rightarrow \mathcal{Y}$  and a homomorphism  $\theta : G \rightarrow \mathbb{R}_{>0}$ . Let  $\hat{x} = F(\hat{x}, u, y)$  be a pre-observer. We say that this pre-observer is  $\theta$ -invariant if it satisfies

$$D\phi_g(x)[F(x, u, y)] = \theta(g)F(\phi_g(x), \psi_g(u), \rho_g(y)), \quad (11)$$

for every  $x \in \mathcal{X}, u \in \mathcal{U}, y \in \mathcal{Y}$  and  $g \in G$ .

One of the main contributions of this note is stated in the next theorem:

**Theorem III.9.** Suppose the system (1) is  $\theta$ -invariant with a  $G$ -compatible output. Assume that for each  $x \in \mathcal{X}$ , the mapping  $g \rightarrow \phi_g(x)$  is full-rank. Then,  $\hat{x} = F(\hat{x}, u, y)$  is a  $\theta$ -invariant pre-observer if and only if

$$F(\hat{x}, u, y) = f(\hat{x}, u) + \sum_{i=1}^n \mathcal{L}_i(I(\hat{x}, u), E(\hat{x}, u, y))w_i(\hat{x}), \quad (12)$$

where  $I$  is an invariant function,  $E$  is an invariant output error, the  $\mathcal{L}_i$ 's are smooth functions such that for all  $\hat{x} \in \mathbb{R}^n$ ,  $\mathcal{L}_i(I(\hat{x}), 0) = 0$ , and  $(w_1, \dots, w_n)$  is a  $\theta$ -invariant frame.

Therefore, the shape of pre-observers proposed in Proposition II.10 keeps its meaning for  $\theta$ -invariant systems.

## C. Application to homogeneous systems

We now explicitly construct all possible  $\theta$ -invariant pre-observers for a homogeneous system.

**Example III.10.** Suppose that system (1) is  $\mathbf{r}$ -homogeneous of degree  $\mu$ . Example III.5 shows that the group of symmetries  $G = \mathbb{R}_{>0}$  with actions  $\phi_g(x) = gx, \psi_g(u) = u, \rho_g(y) = gy$  and  $\theta(g) = g^{-\mu}$  make the system  $\theta$ -invariant.

The condition that  $g \mapsto \phi_g(x)$  is full-rank is met for every  $x \neq 0$ , so Theorem III.9 applies in  $\mathbb{R}^n \setminus \{0\}$ .

We can write  $x = (\omega, \tilde{x})$ , where  $\omega = \|x\|_r$  and  $\tilde{x} = \Lambda_{\omega^{-1}}^r x$ , so that the action  $\phi$  becomes

$$\phi_g(x) = (g\omega, \tilde{x})$$

The invariant frame in this case is the solution to the equation

$$1 = \gamma(x)\omega$$

therefore  $\gamma(x) = \frac{1}{\omega} = \frac{1}{\|x\|_r}$ .

We remark that the only invariant functions  $I : \mathcal{X} \rightarrow \mathbb{R}$  are the constant ones, since for every  $x \in \mathcal{X}$

$$I(x) = \lim_{g \rightarrow 0} I(\Lambda_g^r x) = I(0).$$

The invariant frame is the set of vector fields formed by the columns of the matrix

$$\begin{aligned} w(x) &= (D\phi_{\gamma(x)}(x))^{-1} = \text{diag}(\|x\|_r^{-1}, \dots, \|x\|_r^{-1})^{-1} \\ &= \text{diag}(\|x\|_r, \dots, \|x\|_r) \end{aligned}$$

and, the  $\theta$ -invariant frame is formed by the columns of the matrix

$$\begin{aligned} \theta(\gamma(g))w(x) &= \left( \frac{1}{\|x\|_r} \right)^{-\mu} \text{diag}(\|x\|_r, \dots, \|x\|_r) \\ &= \text{diag}(\|x\|_r^{\mu+1}, \dots, \|x\|_r^{\mu+1}). \end{aligned}$$

The invariant output error is

$$E(\hat{x}, u, y) = \rho_{\gamma(\hat{x})}(y) - \rho_{\gamma(\hat{x})}(h(x)) = \frac{y - C\hat{x}}{\|\hat{x}\|_r}.$$

Therefore, by Theorem III.9 any invariant pre-observer reads

$$\begin{aligned} F(\hat{x}, u, y) &= f(\hat{x}, u) + \sum_{i=1}^n \mathcal{L}_i(I(\hat{x}, u), E(\hat{x}, u, y))\tilde{w}_i(x) \\ &= f(\hat{x}, u) + \text{diag}(\|\hat{x}\|_r^{\mu+1}, \dots, \|\hat{x}\|_r^{\mu+1})L \left( \frac{y - C\hat{x}}{\|\hat{x}\|_r} \right) \\ &= f(\hat{x}, u) + \|\hat{x}\|_r^{\mu} L(y - C\hat{x}) \end{aligned}$$

for some  $L \in \mathbb{R}^{n \times p}$  and some  $\mu \in \mathbb{R}$ .

Linear systems are the simplest example of homogeneous systems. For them, we can explicitly compute the largest possible group of symmetries given by linear actions.

**Theorem III.11.** Consider a linear time-invariant system with output

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad (13)$$

for  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  and  $C \in \mathbb{R}^{p \times n}$ . Suppose that the system is  $G$ -invariant under linear actions  $\phi : G \times \mathcal{X} \rightarrow \mathcal{X}$ ,  $\psi : G \times \mathcal{U} \rightarrow \mathcal{U}$ , and the output is  $G$ -compatible under an action  $\rho : G \times \mathcal{Y} \rightarrow \mathcal{Y}$  where  $G$  is a subgroup of  $GL(n)$ . Denote

$$G_{(A,B,C)} = \{g \in GL(n) : gA = Ag, g(\text{Im } B) \subset \text{Im } B \text{ and } g(\ker C) \subset \ker C\}. \quad (14)$$

Then,  $G$  is a subgroup of  $G_{(A,B,C)}$ .

Similar characterizations of symmetries were considered for systems with no inputs to determine when they are homogeneous [8].

## IV. CONCLUSION

This paper introduced the notion of  $\theta$ -invariant systems, generalizing the class of symmetry-preserving systems. We also characterized all the possible  $\theta$ -invariant pre-observers, thus providing a framework for observer design. Additionally, we have explicitly computed the canonical form of a  $\theta$ -invariant pre-observer for the class of weighted homogeneous systems.

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