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Towards the Computation of Stabilizing Controllers of Multidimensional Systems

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Abstract: In this paper, we further study the effective computation of stabilizing controllers of multidimensional systems. Within the algebraic analysis approach, the stabilization problem can be characterized by the fact that a certain finitely presented \mathcal{A} -module \mathcal{M} , naturally associated with the multidimensional system, is projective, where \mathcal{A} denotes the ring of multivariate rational functions without poles in the closed complex unit polydisc $\overline{\mathbb{D}}^n$. This condition can be reduced to the existence of an element s of a polynomial ideal \mathcal{I} which has no zero in $\overline{\mathbb{D}}^n$. According to the Polydisc Nullstellensatz, the latter condition is equivalent to the fact that no complex zero of the elements of \mathcal{I} belongs to $\overline{\mathbb{D}}^n$. If this condition is satisfied, using cyclic resultants and linear programming, we then propose a method to compute such a polynomial s . Finally, using computer algebra methods for effectively handling basic operations on $\mathcal{R}[s^{-1}]$ -modules, where \mathcal{R} is a polynomial ring, we show how to compute stabilizing controllers.

Keywords: n D systems, stabilizability, stabilizing controllers, Polydisc Nullstellensatz

1. INTRODUCTION

Let \mathbb{K} denote a field of characteristic 0 (e.g., \mathbb{Q} , \mathbb{R} , \mathbb{C}), $\mathcal{R} = \mathbb{K}[z_1, \dots, z_n]$ the polynomial ring in z_1, \dots, z_n with coefficients in \mathbb{K} , and $\mathcal{K} = \mathbb{K}(z_1, \dots, z_n)$ the field of fractions of \mathcal{R} , i.e., the field of rational functions in z_1, \dots, z_n with coefficients in \mathbb{K} . If $r_1, \dots, r_m \in \mathcal{R}$, we denote by $\langle r_1, \dots, r_m \rangle = \sum_{i=1}^m \mathcal{R} r_i$ the ideal of \mathcal{R} generated by the r_i 's. If \mathcal{I} is an ideal of \mathcal{R} , we can define

$$\mathcal{V}(\mathcal{I}) = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid \forall P \in \mathcal{I}, P(z) = 0\}.$$

Let $\overline{\mathbb{D}}^n$ be the closed unit polydisc of \mathbb{C}^n , namely

$$\overline{\mathbb{D}}^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_i| \leq 1, i = 1, \dots, n\}.$$

We can consider the integral domain of the single-input single-output *structurally stable* n D systems defined by

$$\mathcal{A} = \left\{ \frac{n}{d} \in \mathcal{K} \mid 0 \neq d, n \in \mathcal{R}, \gcd(d, n) = 1, \mathcal{V}(\langle d \rangle) \cap \overline{\mathbb{D}}^n = \emptyset \right\}.$$

Within the fractional representation approach to synthesis problems, a linear n D system is defined by a transfer matrix $P \in \mathcal{K}^{q \times r}$, where q (resp., r) is the number of outputs (resp., inputs). This system, also called plant, is *structurally stable* if $P \in \mathcal{A}^{q \times r}$. If P is unstable, i.e., $P \notin \mathcal{A}^{q \times r}$, then an important problem is to study the existence of a *controller* $C \in \mathcal{K}^{r \times q}$ which stabilizes P .

Let $P \in \mathcal{K}^{q \times r}$, $C \in \mathcal{K}^{r \times q}$, and $p = q + r$. The closed-loop system – defined in Figure 1 – is said to be *well-posed* if $\det(I_q - PC) \neq 0$ or $\det(I_r - CP) \neq 0$. Then, we have

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = H(P, C) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad H(P, C) := \begin{pmatrix} I_q & P \\ C & I_r \end{pmatrix}^{-1} \in \mathcal{K}^{p \times p}.$$

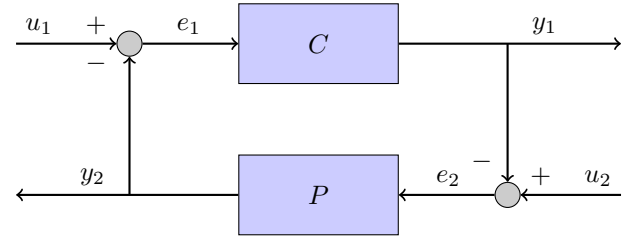


Fig. 1. Closed-loop system

Definition 1. The plant $P \in \mathcal{K}^{q \times r}$ is *internally stabilized* by the controller $C \in \mathcal{K}^{r \times q}$ if $H(P, C) \in \mathcal{A}^{p \times p}$. Then, the matrix P is said to be *internally stabilizable* or simply *stabilizable*, and C is a *stabilizing controller* of P .

Using computer algebra methods and implementations, the goal of this paper is to further develop the effective computation of stabilizing controllers for an internally stabilizable plant P initiated in Bouzidi et al. (2017, 2019).

2. CHARACTERIZATION OF STABILIZABILITY

2.1 Construction of the ring \mathcal{A} as a localization of \mathcal{R}

Using the fact that \mathcal{K} is the field of fractions of \mathcal{A} , we can write $P \in \mathcal{K}^{q \times r}$ as $P = D^{-1}N$, where $D \in \mathcal{A}^{q \times q}$, $\det(D) \neq 0$, $N \in \mathcal{A}^{q \times r}$. Such a representation $P = D^{-1}N$ is called a *left fractional representation* of P .

Note that $\mathcal{R} \subset \mathcal{A}$ and $\mathcal{S} = \{r \in \mathcal{R} \mid \mathcal{V}(\langle r \rangle) \cap \overline{\mathbb{D}}^n = \emptyset\}$ is a *multiplicatively closed subset* of \mathcal{R} , namely, $1 \in \mathcal{S}$ and $s_1, s_2 \in \mathcal{S}$ yields $s_1 s_2 \in \mathcal{S}$. Furthermore, \mathcal{S} is *saturated*,

i.e., if $r_1 r_2 \in \mathcal{S}$, then $r_1 \in \mathcal{S}$ and $r_2 \in \mathcal{S}$. We can define the ring $\mathcal{S}^{-1} \mathcal{R} := \{r/s \mid r \in \mathcal{R}, s \in \mathcal{S}\}$ (Eisenbud (1995); Rotman (2009)). Clearly, we have $\mathcal{A} = \mathcal{S}^{-1} \mathcal{R}$, i.e., \mathcal{A} is the *localization* of \mathcal{R} with respect to \mathcal{S} .

The quotient field $Q(\mathcal{R}) = \{n/d \mid 0 \neq d, n \in \mathcal{R}\}$ of \mathcal{R} is $\mathcal{K} = \mathbb{K}(z_1, \dots, z_n)$, the latter also being the quotient field $Q(\mathcal{A}) = \{n/d \mid 0 \neq d, n \in \mathcal{A}\}$ of \mathcal{A} . Hence, if $P \in \mathcal{K}^{q \times r}$, then there is $D \in \mathcal{R}^{q \times q}$, $\det(D) \neq 0$, and $N \in \mathcal{R}^{q \times r}$ satisfying $P = D^{-1} N$. For instance, we can consider $D = d I_q$, where d is the product (or the least common multiple) of all the denominators of the entries of P written as elements of $Q(\mathcal{R})$, and $N = d P \in \mathcal{R}^{q \times r}$.

Let us state a few standard definitions of module theory. These definitions hold for any integral domain \mathcal{A} with a quotient field \mathcal{K} (e.g., \mathcal{A} can be substituted by \mathcal{R}).

Definition 2. (Eisenbud (1995); Rotman (2009)). • Let \mathcal{M} and \mathcal{M}' be two \mathcal{A} -modules and f a homomorphism (i.e., an \mathcal{A} -linear map) from \mathcal{M} to \mathcal{M}' , denoted by $f \in \text{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{M}')$. If f is bijective, then f is an *isomorphism* and \mathcal{M} and \mathcal{M}' are said to be *isomorphic*, a fact denoted by $\mathcal{M} \cong \mathcal{M}'$.

- An \mathcal{A} -module \mathcal{M} is *finitely generated (f.g.)* if there are $r \in \mathbb{N}$ and $m_k \in \mathcal{M}$, $k = 1, \dots, r$, such that

$$\mathcal{M} = \sum_{k=1}^r \mathcal{A} m_k := \left\{ \sum_{k=1}^r a_i m_i \mid a_i \in \mathcal{A}, i = 1, \dots, r \right\}.$$

- If $s \in \mathcal{A} \setminus \{0\}$, then the *localization* of \mathcal{A} with respect to the multiplicatively closed subset $\mathcal{S} = \{s^i\}_{i \in \mathbb{N}}$ of \mathcal{A} is the ring $\mathcal{A}_s = \mathcal{S}^{-1} \mathcal{A} = \{a/s \mid s \in \mathcal{S}, a \in \mathcal{A}\}$. In particular, we have the ring inclusion $\mathcal{A} \subset \mathcal{A}_s$.
- If \mathcal{B} is a ring containing \mathcal{A} and \mathcal{M} an \mathcal{A} -module, then we can define the \mathcal{B} -module $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}$ obtained by extending the coefficients of \mathcal{M} from \mathcal{A} to \mathcal{B} . An element $m \in \mathcal{M}$ is identified with $1 \otimes m$ in $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}$, $b(1 \otimes m) = b \otimes m$ for all $b \in \mathcal{B}$ and $m \in \mathcal{M}$, and

$$\forall a \in \mathcal{A}, m \in \mathcal{M}, a(1 \otimes m) = a \otimes m = 1 \otimes a m.$$

- The *rank* of a f.g. \mathcal{A} -module \mathcal{M} is defined by $\text{rank}_{\mathcal{K}}(\mathcal{M}) := \dim_{\mathcal{K}}(\mathcal{K} \otimes_{\mathcal{A}} \mathcal{M})$, i.e., by the dimension of the finite-dimensional \mathcal{K} -vector space $\mathcal{K} \otimes_{\mathcal{A}} \mathcal{M}$.
- The *torsion submodule* $t(\mathcal{M})$ of \mathcal{M} is defined by

$$t(\mathcal{M}) = \{m \in \mathcal{M} \mid \exists a \in \mathcal{A} \setminus \{0\} : a m = 0\}.$$

- \mathcal{M} is a f.g. *free* \mathcal{A} -module if $\mathcal{M} \cong \mathcal{A}^r$ for $r \in \mathbb{N}$.
- \mathcal{M} is a f.g. *projective* \mathcal{A} -module if there exist an \mathcal{A} -module \mathcal{M}' and $r \in \mathbb{N}$ such that $\mathcal{M} \oplus \mathcal{M}' \cong \mathcal{A}^r$.
- A *sequence* of $f_i \in \text{Hom}_{\mathcal{A}}(\mathcal{M}_i, \mathcal{M}_{i-1})$ for $i \in \mathbb{Z}$ is called a *complex* if $f_i \circ f_{i+1} = 0$, i.e., $\text{im } f_{i+1} \subseteq \ker f_i$ for all $i \in \mathbb{Z}$. It is *exact* if $\ker f_i = \text{im } f_{i+1}$ for all $i \in \mathbb{Z}$.

In what follows, we shall consider three localizations: $\mathcal{K} = (\mathcal{A}^*)^{-1} \mathcal{A}$, where $\mathcal{A}^* = \mathcal{A} \setminus \{0\}$, $\mathcal{A} = \mathcal{S}^{-1} \mathcal{R}$, where $\mathcal{S} = \{r \in \mathcal{R} \mid \mathcal{V}(\langle r \rangle) \cap \mathbb{D}^n = \emptyset\}$, and $\mathcal{R}_s = \mathcal{T}^{-1} \mathcal{R}$, where $s \in \mathcal{R} \setminus \{0\}$ and $\mathcal{T} = \{s^i\}_{i \in \mathbb{N}}$. If $s \in \mathcal{S}$, note that $\mathcal{R}_s \subset \mathcal{A}$.

Theorem 1. (Eisenbud (1995); Rotman (2009)). If \mathcal{B} is a localization of a ring \mathcal{A} , then \mathcal{B} is a *flat* \mathcal{A} -module, namely, the functor $\mathcal{B} \otimes_{\mathcal{A}} \cdot$ is *exact*, i.e., transforms exact sequences of \mathcal{A} -modules $\dots \xrightarrow{f_{i+2}} \mathcal{M}_{i+1} \xrightarrow{f_{i+1}} \mathcal{M}_i \xrightarrow{f_i} \mathcal{M}_{i-1} \xrightarrow{f_{i-1}} \dots$ into exact sequences of \mathcal{B} -modules

$$\dots \xrightarrow{g_{i+2}} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}_{i+1} \xrightarrow{g_{i+1}} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}_i \xrightarrow{g_i} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}_{i-1} \xrightarrow{g_{i-1}} \dots$$

where $g_i = \text{id}_{\mathcal{B}} \otimes f_i \in \text{Hom}_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}_i, \mathcal{B} \otimes_{\mathcal{A}} \mathcal{M}_{i-1})$ is defined by $g_i(b \otimes m) = b \otimes f_i(m)$ for all $b \in \mathcal{B}$ and $m \in \mathcal{M}_i$.

Elimination theory is effective for the polynomial ring \mathcal{R} in the case of $\mathbb{K} = \mathbb{Q}$ (using, e.g., *Gröbner basis methods* (Chyzak et al. (2005); Eisenbud (1995))). Theorem 1 will allow us to lift computations from the ring \mathcal{R} to \mathcal{R}_s or \mathcal{A} .

2.2 A necessary and sufficient condition for stabilizability

As explained in Section 2.1, if $P \in \mathcal{K}^{q \times r}$ and $p = q + r$, we can always consider a left fractional representation $P = D^{-1} N$ of P , where $R = (D \quad -N) \in \mathcal{R}^{q \times p}$. Within the *algebraic analysis approach* to synthesis problems (Quadrat (2003)), we can define the *factor* \mathcal{R} -module $\mathcal{L} = \mathcal{R}^{1 \times p} / \text{im}_{\mathcal{R}}(\cdot R)$, where $\text{im}_{\mathcal{R}}(\cdot R) := \{\mu R \mid \mu \in \mathcal{R}^{1 \times q}\}$, formed by the *residue classes* $\pi(\lambda)$ of the elements of $\mathcal{R}^{1 \times p}$ (i.e., the row vectors of length p with entries in \mathcal{R}) modulo the \mathcal{R} -submodule of *relations* $\text{im}_{\mathcal{R}}(\cdot R)$. In other words, we have $\pi(\lambda') = \pi(\lambda)$ for $\lambda, \lambda' \in \mathcal{R}^{1 \times p}$ if there exists $\nu \in \mathcal{R}^{1 \times q}$ such that $\lambda' = \lambda + \nu R$, and we further have $r_1 \pi(\lambda_1) + r_2 \pi(\lambda_2) := \pi(r_1 \lambda_1 + r_2 \lambda_2)$ for all $r_1, r_2 \in \mathcal{R}$ and $\lambda_1, \lambda_2 \in \mathcal{R}^{1 \times p}$. If we now consider the *standard basis* $\{e_i\}_{i=1, \dots, p}$ of $\mathcal{R}^{1 \times p}$ (i.e., e_i is the row vector with 1 at the i^{th} position and 0 elsewhere), $y_i = \pi(e_i)$ for $i = 1, \dots, q$, and $u_i = \pi(e_i)$ for $i = q + 1, \dots, p$, then we can show that \mathcal{L} is finitely generated by the *generators* $\{y_1, \dots, y_q, u_1, \dots, u_r\}$ which satisfy the *linear relations* $D y - N u = 0$, where $y = (y_1, \dots, y_q)^T$ and $u = (u_1, \dots, u_r)^T$ (Quadrat (2003); Chyzak et al. (2005)). We have $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{L} \cong \mathcal{A}^{1 \times p} / (\mathcal{A}^{1 \times q} R) = \mathcal{M}$. The modules \mathcal{L} and \mathcal{M} are said to be *finitely presented (f.p.)* because they are defined by a finite set of generators satisfying a finite generating set of relations (Rotman (2009)).

We have the following characterization of stabilizability.

Theorem 2. (Quadrat (2003)). Let $P \in \mathcal{K}^{q \times r}$ be a plant and $P = D^{-1} N$ a fractional representation of P , where $R = (D \quad -N) \in \mathcal{A}^{q \times p}$. Let $\mathcal{M} = \mathcal{A}^{1 \times p} / (\mathcal{A}^{1 \times q} R)$ be the \mathcal{A} -module finitely presented by R . Then, P is stabilizable if and only if the \mathcal{A} -module $\mathcal{M}/t(\mathcal{M})$ is projective of rank r .

Theorem 2 yields the following practical issues:

- (1) Compute a presentation matrix $R' \in \mathcal{R}^{q' \times p}$ for $\mathcal{M}/t(\mathcal{M})$, i.e., such that $\mathcal{M}/t(\mathcal{M}) = \mathcal{A}^{1 \times p} / (\mathcal{A}^{1 \times q'} R')$.
- (2) Test whether or not the f.p. \mathcal{A} -module $\mathcal{M}/t(\mathcal{M})$ is projective of rank r .

The next result is a direct consequence of the fact that \mathcal{A} is a flat \mathcal{R} -module (see Theorem 1).

Lemma 3. (Bouzidi et al. (2019)). Let $P = D^{-1} N$ be a left fractional representation of $P \in \mathcal{K}^{q \times r}$, where $R = (D \quad -N) \in \mathcal{R}^{q \times p}$ (see Section 2.1), $\mathcal{L} = \mathcal{R}^{1 \times p} / (\mathcal{R}^{1 \times q} R)$ the \mathcal{R} -module finitely presented by R , and the \mathcal{A} -module $\mathcal{M} = \mathcal{A}^{1 \times p} / (\mathcal{A}^{1 \times q} R)$. Furthermore, let $Q \in \mathcal{R}^{p \times m}$ and $R' \in \mathcal{R}^{q' \times p}$ be respectively defined by

$$\begin{aligned} \ker_{\mathcal{R}}(\cdot R^T) &:= \{\lambda \in \mathcal{R}^{1 \times p} \mid \lambda R^T = 0\} = \text{im}_{\mathcal{R}}(\cdot Q^T), \\ \ker_{\mathcal{R}}(\cdot Q) &= \text{im}_{\mathcal{R}}(\cdot R'). \end{aligned}$$

Then, we have

$$t(\mathcal{L}) = (\mathcal{R}^{1 \times q'} R') / (\mathcal{R}^{1 \times q} R), \quad \mathcal{L}/t(\mathcal{L}) = \mathcal{R}^{1 \times p} / (\mathcal{R}^{1 \times q'} R'),$$

which yields $t(\mathcal{M}) = (\mathcal{A}^{1 \times q'} R') / (\mathcal{A}^{1 \times q} R)$ and

$$\mathcal{M}/t(\mathcal{M}) = \mathcal{A}^{1 \times p} / (\mathcal{A}^{1 \times q'} R').$$

The matrices Q and R' can be computed using polynomial elimination methods (e.g., Gröbner basis methods). See OREMODULES package (Chyzak et al. (2007)).

Example 1. Let us consider the MIMO plant defined by

$$P = \begin{pmatrix} z_3 - 2 & z_1 \\ z_1 & z_2 + 3 \end{pmatrix}^T \in \mathbb{Q}(z_1, z_2, z_3)^{2 \times 1}.$$

Let us first compute a left fractional representation of P . Let $d = \text{lcm}(z_1, z_2 + 3) = z_1(z_2 + 3)$, $D = dI_2$, $N = DP$. Let $\mathcal{L} = \mathcal{R}^{1 \times 3} / (\mathcal{R}^{1 \times 2} R)$ be the $\mathcal{R} = \mathbb{Q}[z_1, z_2, z_3]$ -module finitely presented by $R = \begin{pmatrix} D & -N \end{pmatrix} \in \mathcal{R}^{2 \times 3}$. Using the OREMODULES package (Chyzak et al. (2007)), we obtain

$t(\mathcal{L}) = (\mathcal{R}^{1 \times 3} R') / (\mathcal{R}^{1 \times 2} R)$, $\mathcal{L}/t(\mathcal{L}) = R^{1 \times 3} / (\mathcal{R}^{1 \times 2} R')$, where $R' = \begin{pmatrix} D' & -N' \end{pmatrix}$ and

$$D' = \begin{pmatrix} z_1 & 0 \\ 0 & -z_2 - 3 \end{pmatrix}, \quad N' = \begin{pmatrix} z_3 - 2 \\ -z_1 \end{pmatrix}.$$

Using Theorem 2 and Lemma 3, the plant $P \in \mathcal{K}^{q \times r}$ is stabilizable if and only if the f.p. \mathcal{A} -module $\mathcal{M}/t(\mathcal{M}) = \mathcal{A}^{1 \times p} / (\mathcal{A}^{1 \times q'} R')$ is projective of rank r . To characterize projectivity, we introduce the concept of *Fitting ideals* of f.p. modules over any commutative ring \mathcal{A} .

Definition 3. (Eisenbud (1995)). Let $R' \in \mathcal{A}^{q' \times p}$ and $\mathcal{M}' = \mathcal{A}^{1 \times p} / (\mathcal{A}^{1 \times q'} R')$. Then, the i^{th} *Fitting ideal* of \mathcal{M}' , denoted by $\text{Fitt}_i(\mathcal{M}')$, is defined by

$$\text{Fitt}_i(\mathcal{M}') = \begin{cases} 0, & \text{if } p - i > \min(q', p), \\ \langle a_1, \dots, a_i \rangle, & \text{if } 0 < p - i \leq \min(q', p), \\ \mathcal{A}, & \text{if } p - i \leq 0, \end{cases}$$

where the a_i 's denote the $p - i$ minors of R' .

The next result holds for any integral domain \mathcal{A} .

Theorem 4. (Eisenbud (1995)). Let $\mathcal{M}' = \mathcal{A}^{1 \times p} / (\mathcal{A}^{1 \times q'} R')$ be the \mathcal{A} -module finitely presented by $R' \in \mathcal{A}^{q' \times p}$. Then, \mathcal{M}' is a projective \mathcal{A} -module of rank r if and only if $\text{Fitt}_i(\mathcal{M}') = 0$ for $i = 0, \dots, r - 1$ and $\text{Fitt}_r(\mathcal{M}') = \mathcal{A}$.

By Theorem 4, $\mathcal{M}/t(\mathcal{M}) = \mathcal{A}^{1 \times p} / (\mathcal{A}^{1 \times q'} R')$ is a projective \mathcal{A} -module of rank r if and only if we have

$$\begin{cases} \text{Fitt}_i(\mathcal{M}/t(\mathcal{M})) = 0, & i = 0, \dots, r - 1, \\ \text{Fitt}_r(\mathcal{M}/t(\mathcal{M})) = \mathcal{A}. \end{cases} \quad (1)$$

We state again that $R' \in \mathcal{R}^{q' \times p}$, and thus, the minors of R' belong to $\mathcal{R} \subset \mathcal{A}$. More precisely, if $\{g_{i,j}\}_{j=1, \dots, d_i}$ is a set of generators of $\text{Fitt}_i(\mathcal{L}/t(\mathcal{L}))$, then we have

$$\text{Fitt}_i(\mathcal{M}/t(\mathcal{M})) = \sum_{j=1}^{d_i} \mathcal{A} g_{i,j} = \mathcal{A} \text{Fitt}_i(\mathcal{L}/t(\mathcal{L})), \quad (2)$$

where $\mathcal{A} \text{Fitt}_i(\mathcal{L}/t(\mathcal{L}))$ denotes the ideal of \mathcal{A} obtained by extending the coefficients of $\text{Fitt}_i(\mathcal{L}/t(\mathcal{L}))$ from \mathcal{R} to \mathcal{A} .

Proposition 5. With the above notations, the following assertions are equivalent:

- (1) $\text{Fitt}_r(\mathcal{M}/t(\mathcal{M})) = \mathcal{A}$.
- (2) There exist $c_1, \dots, c_{d_r} \in \mathcal{R}$ satisfying

$$\sum_{j=1}^{d_r} c_j g_{r,j} \in \mathcal{S} = \{s \in \mathcal{R} \mid \mathcal{V}(\langle s \rangle) \cap \overline{\mathbb{D}}^n = \emptyset\}, \quad (3)$$

where $\text{Fitt}_r(\mathcal{L}/t(\mathcal{L})) = \langle g_{r,1}, \dots, g_{r,d_r} \rangle_{\mathcal{R}}$, or equivalently, we have $\text{Fitt}_r(\mathcal{L}/t(\mathcal{L})) \cap \mathcal{S} \neq \emptyset$.

Proof. $\text{Fitt}_r(\mathcal{M}/t(\mathcal{M})) = \mathcal{A}$ is equivalent to the existence of $a_{r,j} = b_{r,j}/s_{r,j} \in \mathcal{A}$, where $b_{r,j} \in \mathcal{R}$ and $s_{r,j} \in \mathcal{S}$ for $j = 1, \dots, d_r$, satisfying $\sum_{j=1}^{d_r} a_{r,j} g_{r,j} = 1$, i.e.,

$$\sum_{j=1}^{d_r} \left(\left(\prod_{1 \leq i \neq j \leq d_r} s_{r,i} \right) b_{r,j} \right) g_{r,j} = \prod_{j=1}^{d_r} s_{r,j}. \quad (4)$$

Hence, the necessity of (3) is a direct consequence of (4). Now, (3) is also sufficient because $t := \sum_{j=1}^{d_r} c_j g_{r,j} \in \mathcal{S}$ yields $\sum_{j=1}^{d_r} (c_j/t) g_{r,j} = 1$, where $c_{r,j}/t \in \mathcal{A}$, $j = 1, \dots, d_r$.

Example 2. Let us continue Example 1. We first have $\text{Fitt}_0(\mathcal{L}/t(\mathcal{L})) = \langle 0 \rangle_{\mathcal{R}}$ and $\text{Fitt}_1(\mathcal{L}/t(\mathcal{L})) = \langle g_1, g_2, g_3 \rangle_{\mathcal{R}}$, where $g_1 = (z_2 + 3)(z_3 - 2)$, $g_2 = z_1(z_2 + 3)$, and $g_3 = z_1^2$. Note that $g_1 \in \mathcal{S}$, which shows that (3) holds with $c_1 = 1$ and $c_2 = c_3 = 0$, and proves that P is stabilizable.

3. POLYDISC NULLSTELLENSATZ

As explained in Section 2.2, P is stabilizable if and only if $\text{Fitt}_i(\mathcal{M}/t(\mathcal{M})) = 0$, for $i = 0, \dots, r - 1$ and there exists a polynomial combination $s = \sum_{i=1}^m r_i g_i$ of polynomial generators g_1, \dots, g_m of $\mathcal{I} := \text{Fitt}_r(\mathcal{L}/t(\mathcal{L}))$ such that s has no complex zeros in $\overline{\mathbb{D}}^n$.

As noticed in Bouzidi et al. (2019), this condition is exactly the so-called *Polydisc Nullstellensatz* for the ideal \mathcal{I} .

Theorem 6. (Bridges et al. (2004)). Let $\mathcal{I} = \langle g_1, \dots, g_m \rangle$ be an ideal of \mathcal{R} generated by $g_i \in \mathcal{R}$ for $i = 1, \dots, m$. Then, the following conditions are equivalent:

- (1) $\mathcal{V}(\mathcal{I}) \cap \overline{\mathbb{D}}^n = \emptyset$.
- (2) There exists $s \in \mathcal{I}$, i.e., $s = \sum_{i=1}^m r_i g_i$ for certain $r_i \in \mathcal{R}$, for $i = 1, \dots, m$, such that

$$\mathcal{V}(\langle s \rangle) \cap \overline{\mathbb{D}}^n = \emptyset. \quad (5)$$

Corollary 7. With the above notations, P is internally stabilizable if and only if $\mathcal{V}(\text{Fitt}_r(\mathcal{L}/t(\mathcal{L}))) \cap \overline{\mathbb{D}}^n = \emptyset$.

Let $\mathcal{I} = \text{Fitt}_r(\mathcal{L}/t(\mathcal{L})) = \langle g_1, \dots, g_m \rangle$, where the g_i 's form a set of generators of \mathcal{I} . Writing $z_j = x_j + i y_j$, where $x_j, y_j \in \mathbb{R}$, and noting $v = (x_1, \dots, x_n, y_1, \dots, y_n)$, $X_j(v)$ (resp., $Y_j(v)$) the real (resp., imaginary) part of $g_j(z)$, then testing $\mathcal{V}(\mathcal{I}) \cap \overline{\mathbb{D}}^n = \emptyset$ reduces to testing the emptiness of the following *semi-algebraic set* of \mathbb{R}^{2n}

$$\{v \in \mathbb{R}^{2n} \mid X_j(v) = 0, Y_j(v) = 0, j = 1, \dots, m, \\ x_i^2 + y_i^2 \leq 1, i = 1, \dots, n\}.$$

This problem can be studied using *quantifier elimination* (e.g., *Cylindric Algebraic Decomposition* (CAD)) (Collins (1975)). In practice, it is usually inefficient for $n \geq 3$.

Using computer algebra methods (*Rational Univariate Representation*), in Bouzidi et al. (2017), it was shown how to effectively test if $\mathcal{V}(\mathcal{I})$ intersects $\overline{\mathbb{D}}^n$ in the case where \mathcal{I} is a *0-dimensional ideal*, namely, $\mathcal{V}(\mathcal{I})$ consists in a finite number of complex points. Furthermore, using *critical point methods*, the case of a *principal ideal* $\mathcal{I} = \langle g_1 \rangle$ (i.e., $\mathcal{V}(\mathcal{I})$ is a hypersurface of \mathbb{C}^n) was made effective (Bouzidi et al. (2015)). The corresponding algorithms were implemented in `Maple` (see Bouzidi et al. (2015, 2017)).

Example 3. Let $P \in \mathcal{K}$ be a SISO plant, $P = N/D$ a fractional representation of P , where $0 \neq D, N \in \mathcal{R}$, and $R = (D \quad -N) \in \mathcal{R}^{1 \times 2}$. Let $G = \gcd(D, N)$ be the greatest common divisor of D and N , and $D', N' \in \mathcal{R}$ such that $D = G D'$ and $N = G N'$. Now, $(\lambda_1 \ \lambda_2) \in \ker_{\mathcal{R}}(.R^T)$ yields $G(\lambda_1 D' - \lambda_2 N') = 0$, i.e., $\lambda_1 D' = \lambda_2 N'$ because $G \neq 0$ and \mathcal{R} is an integral domain. Hence, D' divides $\lambda_2 N'$, and thus, λ_2 because $\gcd(D', N') = 1$. Thus, let $\mu \in \mathcal{R}$ be such that $\lambda_2 = \mu D'$. Then, $\lambda_1 D' = \mu D' N'$, and thus, $\lambda_1 = \mu N'$ because $D' \neq 0$ and \mathcal{R} is an integral domain. If we note $Q = (N' \ D')^T \in \mathcal{R}^{2 \times 1}$, then $(\lambda_1 \ \lambda_2) = \mu(N' \ D') \in \text{im}_{\mathcal{R}}(.Q^T)$. Moreover, we can check that $\text{im}_{\mathcal{R}}(.Q^T) \subseteq \ker_{\mathcal{R}}(.R^T)$, which yields $\ker_{\mathcal{R}}(.R^T) = \text{im}_{\mathcal{R}}(.Q^T)$. Similarly, we can show that $\ker_{\mathcal{R}}(.Q) = \text{im}_{\mathcal{R}}(.R')$, where $R' = (D' \quad -N')$, which proves that the f.p. \mathcal{R} -module $\mathcal{L} = \mathcal{R}^{1 \times 2}/(\mathcal{R}R)$ is such that $t(\mathcal{L}) = (\mathcal{R}R')/(\mathcal{R}R)$ and $\mathcal{L}/t(\mathcal{L}) = \mathcal{R}^{1 \times 2}/(\mathcal{R}R')$. Hence, we have $\text{Fitt}_0(\mathcal{L}/t(\mathcal{L})) = \langle 0 \rangle$ and $\mathcal{I} = \text{Fitt}_1(\mathcal{L}) = \langle D', N' \rangle$. Using Theorem 2, Theorem 6, and Proposition 5, $P = N'/D'$ is stabilizable if and only if there exist $X, Y \in \mathcal{R}$ satisfying $D'X - N'Y \in \mathcal{S}$, i.e., if and only if $\mathcal{V}(\mathcal{I}) \cap \overline{\mathbb{D}}^n = \emptyset$. If $n = 2$, then $\mathcal{V}(\mathcal{I})$ is 0-dimensional, and thus, $\mathcal{V}(\mathcal{I}) \cap \overline{\mathbb{D}}^2 = \emptyset$ can be tested (Bouzidi et al. (2017)). Finally, in Bouzidi et al. (2017), it is shown how to compute $X, Y \in \mathcal{R}$ satisfying $D'X - N'Y \in \mathcal{S}$.

We point out that Bridges et al.'s proof (Bridges et al. (2004)) of (1) \Rightarrow (2) is not effective. Thus, an algorithmic proof of the Polydisc Nullstellensatz must be developed to get a stabilizability test for general n D systems.

In the rest of this section, we shall suppose that (1) of Theorem 6 is satisfied. We then propose a new method to compute $s \in \mathcal{I}$ satisfying (2) of Theorem 6 (independently of the dimension of $\mathcal{V}(\mathcal{I})$). In particular, we shall introduce a criterion that is simpler to test (see Corollary 12 below). We shall use it to construct $s \in \mathcal{R}$ that does not vanish on $\overline{\mathbb{D}}^n$ by reducing the problem to a *linear program*.

Definition 4. Let $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$, $x^\mu = x_1^{\mu_1} \dots x_n^{\mu_n}$, $s \in \mathcal{R}$, and be the *support* $\text{supp}(s)$ of s defined by the finite subset of \mathbb{N}^n which is such that the coefficients $a_\mu \in \mathbb{K}$ of s satisfy $a_\mu \neq 0$ for all $\mu \in \text{supp}(s)$, i.e., $s = \sum_{\mu \in \text{supp}(s)} a_\mu x^\mu \in \mathcal{R}$. Then, we say that s has a *dominant constant* if the coefficient of s associated to the monomial of exponent $\mu = (0, \dots, 0)$, namely $a_{(0, \dots, 0)}$, has an absolute value greater than the sum of the absolute values of the other coefficients of s , i.e.,

$$|a_{(0, \dots, 0)}| > \sum_{\mu \in \text{supp}(s) \setminus \{(0, \dots, 0)\}} |a_\mu|.$$

Lemma 8. Let $s \in \mathcal{R}$ has a dominant constant. Then, we have $s \in \mathcal{S}$, i.e., $\mathcal{V}(\langle s \rangle) \cap \overline{\mathbb{D}}^n = \emptyset$.

Proof. We suppose that s has a zero z in $\overline{\mathbb{D}}^n$. Using $a_{(0, \dots, 0)} = -\sum_{\mu \in \text{supp}(s) \setminus \{(0, \dots, 0)\}} a_\mu z^\mu$, the triangular inequality then yields $|a_{(0, \dots, 0)}| \leq \sum_{\mu \in \text{supp}(s) \setminus \{(0, \dots, 0)\}} |a_\mu|$, which shows that s has no dominant constant term.

3.1 Cyclic resultant

The method to compute $s \in \mathcal{I}$ satisfying (2) of Theorem 6 that we now propose is based on the notion of *cyclic resultants* (Purbhoo, 2008; Forsgård et al., 2019). It has

been recently used to test the structural stability of multidimensional systems (Bossoto et al., 2019). Using cyclic resultants, we show how to reduce the problem of finding a polynomial that does not vanish in $\overline{\mathbb{D}}^n$ to the problem of finding a polynomial with a dominant constant.

Definition 9. (Purbhoo (2008)). Let $s \in \mathcal{R}$ and $d \in \mathbb{N}^*$. The *cyclic resultant of s of order d* is defined by

$$\text{CycRes}(s, d) := \prod_{k_1=0}^{d-1} \dots \prod_{k_n=0}^{d-1} s \left(e^{2\pi i k_1/d} z_1, \dots, e^{2\pi i k_n/d} z_n \right).$$

Remark 10. A direct consequence of the above Poisson formula is that s divides $\text{CycRes}(s, d)$ for any positive integer d . Moreover, grouping the factors by pairs of conjugate polynomials, we can see that if s has real coefficients, then $\text{CycRes}(s, d)$ also has real coefficients.

Given a polynomial $s \in \mathcal{R}$ satisfying (5), the next theorem shows how the cyclic resultant can be used to construct a polynomial with a dominant constant.

Theorem 11. (Bossoto et al., 2019, Theorem 4 and Theorem 5). If s is a polynomial that does not vanish on $\overline{\mathbb{D}}^n$, then there is an integer $k \in \mathbb{N}^*$ such that for all $d \geq k$, the polynomial $\text{CycRes}(s, d)$ has a dominant constant.

Corollary 12. Let \mathcal{I} be an ideal of \mathcal{R} . Then the following conditions are equivalent:

- (1) $\mathcal{V}(\mathcal{I}) \cap \overline{\mathbb{D}}^n = \emptyset$.
- (2) There exists $s \in \mathcal{I}$ having a dominant constant.

Proof. First, let $s \in \mathcal{I}$ has a dominant constant. By Lemma 8, (5) holds. Now, the inclusion $\mathcal{V}(\mathcal{I}) \subset \mathcal{V}(\langle s \rangle)$ yields $\mathcal{V}(\mathcal{I}) \cap \overline{\mathbb{D}}^n = \emptyset$. Conversely, if $\mathcal{V}(\mathcal{I}) \cap \overline{\mathbb{D}}^n = \emptyset$, then, using Theorem 7, there exists $s \in \mathcal{I}$ satisfying (5). Using Theorem 11, there is an integer k such that $\text{CycRes}(s, k)$ has a dominant constant. Finally, using Remark 10, s divides $\text{CycRes}(s, k)$, and thus, $\text{CycRes}(s, k) \in \mathcal{I}$.

3.2 Linear programming

The advantage of Corollary 12 is that if $\mathcal{V}(\mathcal{I}) \cap \overline{\mathbb{D}}^n = \emptyset$, then we can use *linear programming* to find a polynomial with a dominant constant that does not vanish on $\overline{\mathbb{D}}^n$. Let $\mathcal{I} = \langle g_1, \dots, g_m \rangle$ be an ideal of \mathcal{R} generated by some elements $g_i \in \mathcal{R}$ of degrees bounded by δ . Moreover, assume that there exists a polynomial $s \in \mathcal{I}$ such that s has a dominant constant, and let $k \geq \delta$ be an integer such that $s = \sum_{i=1}^m r_i g_i$ where the r_i 's are polynomials of degree bounded by $k - \delta$. By construction, the constant monomial of s is not zero and, without loss of generality, we can assume that it is 1. We show how to write a linear program to find a polynomial of \mathcal{I} with a dominant constant.

Let $S_j = \{ \mu = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n \mid |\mu| = \sum_{i=1}^n \mu_i \leq j \}$ be the set of exponents of length less than or equal to j , and $c_{i,\alpha}$ a variable that represents the coefficient of r_i of degree α , for $i = 1, \dots, m$ and $\alpha \in S_{k-\delta}$, i.e., $r_i = \sum_{\alpha \in S_{k-\delta}} c_{i,\alpha} z^\alpha$. Then, the coefficients of s are linear combinations of $c = (c_{i,\alpha})_{\substack{1 \leq i \leq m \\ \alpha \in S_{k-\delta}}}$. For $\beta \in S_k$, let ℓ_β be the

linear function from $\mathbb{R}^{|S_{k-\delta}|}$ to \mathbb{R} such that $\ell_\beta(c)$ is the coefficient of s of degree β . If we denote by b_β the variable that bounds the absolute value of the coefficient of degree

β of s , we can now write the linear program that returns a polynomial $s \in \mathcal{I}$ that satisfies (5).

$$\begin{aligned} & \text{Minimize} && \sum_{\beta \in S_k \setminus (0, \dots, 0)} b_\beta \\ & \text{subject to} && \ell_{(0, \dots, 0)}(c) = 1, \\ & && \ell_\beta(c) \leq b_\beta, \quad \forall \beta \in S_k \setminus (0, \dots, 0), \\ & && -\ell_\beta(c) \leq b_\beta, \quad \forall \beta \in S_k \setminus (0, \dots, 0). \end{aligned} \quad (P_k)$$

In the general case, if the polynomials $g_1, \dots, g_m \in \mathcal{R}$ are given and $I = \langle g_1, \dots, g_m \rangle$ is such that $\mathcal{V}(\mathcal{I}) \cap \overline{\mathbb{D}}^n = \emptyset$, Corollary 12 ensures that there is a polynomial $s \in \mathcal{I}$ that has a dominant constant. However we do not know its degree. In this case, we can start by solving the linear program (P_k) for $k = 2$ and then we can, e.g., double k until the objective function has a value less than 1.

Example 4. Let us continue Example 2. We note that we have $\text{Fitt}_1(\mathcal{L}/t(\mathcal{L})) = \langle g_1, g_2, g_3 \rangle = \langle g_1 + g_2, g_2, g_3 \rangle$, where $g_1 + g_2 = (z_2 + 3)(z_1 + z_3 - 2) \notin \mathcal{S}$. Each generator of this second set of generators of $\text{Fitt}_1(\mathcal{L}/t(\mathcal{L}))$ vanishes on $\overline{\mathbb{D}}^n$. In this case, let us show how to use the above method to obtain $s \in \mathcal{S}$, that is a polynomial combination of $g'_1 = g_1 + g_2$, g_2 , and g_3 . Using (P_3) , we find that

$$\begin{aligned} s = & -z_1^3 - 2z_1^2 z_2 - z_1^2 z_3 - z_1 z_2^2 - 2z_1 z_2 z_3 + 55z_2^2 z_3 \\ & - z_2 z_3^2 - 5z_1^2 - 2z_1 z_2 - 6z_1 z_3 - 110z_2^2 - 3z_3^2 + 3z_1 \\ & + 4z_2 - 495z_3 + 1002 \in \text{Fitt}_1(\mathcal{L}/t(\mathcal{L})) \end{aligned}$$

does not vanish on $\overline{\mathbb{D}}^n$, and cofactors associated to g'_1, g_2, g_3 are defined by $r_1 = -167 - z_3 + 55z_2 - 49z_1$, $r_2 = 70 + 48z_3 - 56z_2 + 43z_1$, and $r_3 = 13 - z_3 + 4z_2 - z_1$.

In Example 4, we introduced g'_1 to show that our method works even if each input generator vanishes on $\overline{\mathbb{D}}^n$. Our approach returns a polynomial s of slightly higher degree since it requires to satisfy the stronger criterion that s has a dominant constant (not just in \mathcal{S}). Preliminary experiments suggest that our approach works well when $\mathcal{V}(\mathcal{I})$ is far from $\overline{\mathbb{D}}^n$. In the zero-dimensional case, compared to the effective method based on *resultants* (Bouzidi et al., 2015), our approach also seems to return polynomial with smaller coefficients when $\mathcal{V}(\mathcal{I})$ is far from $\overline{\mathbb{D}}^n$.

4. COMPUTING STABILIZING CONTROLLERS

Let us show how stabilizing controllers can be computed.

Theorem 13. (Bouzidi et al. (2019)). With the above notations, the plant P is stabilizable if and only if there is $Z \in \mathcal{A}^{p \times q'}$ such that $S := ZD'D^{-1}$ satisfies $RS = I_q$. In particular, if R' has a generalized inverse $S' \in \mathcal{A}^{p \times q'}$, namely, $R'S'R' = R'$, then the matrix Z can be chosen to be S' . Writing $S = (X^T \ Y^T)^T$, where $X \in \mathcal{K}^{q \times q}$ and $Y \in \mathcal{K}^{r \times q}$, if $\det(X) \neq 0$, then $C = YX^{-1}$ stabilizes P .

Theorem 14. The following assertions are equivalent:

- (1) The plant P is stabilizable.
- (2) There exists $s \in \text{Fitt}_r(\mathcal{L}/t(\mathcal{L})) \cap \mathcal{S} \neq \emptyset$ such that the f.p. $\mathcal{R}_s = \{r/s \mid r \in \mathcal{R}, s \in \mathcal{S}\} = \mathcal{R}[s^{-1}]$ -module $\mathcal{Q} = \mathcal{R}_s^{1 \times p} / (\mathcal{R}_s^{1 \times q'} R')$ is projective of rank r .

Proof. By Theorem 2, P is internally stabilizable if and only if $\mathcal{M}/t(\mathcal{M})$ is a projective \mathcal{A} -module of rank r , which, using Theorem 4, is equivalent to Equation (1).

We state again that $\text{Fitt}_i(\mathcal{M}/t(\mathcal{M})) = \mathcal{A}\text{Fitt}_i(\mathcal{L}/t(\mathcal{L}))$. Similarly, note that $\text{Fitt}_i(\mathcal{Q})$ is the ideal of \mathcal{R}_s generated by all the $(p-i) \times (p-i)$ -minors of R' , i.e., by the set of generators $\{g_{i,j}\}_{j=1, \dots, d_i}$ of $\text{Fitt}_i(\mathcal{L}/t(\mathcal{L}))$, which yields

$$\text{Fitt}_i(\mathcal{Q}) = \sum_{j=1}^{d_i} \mathcal{R}_s g_{i,j} = \mathcal{R}_s \text{Fitt}_i(\mathcal{L}/t(\mathcal{L})), \quad i \geq 0.$$

Thus, $\text{Fitt}_i(\mathcal{M}/t(\mathcal{M})) = 0$ (resp., $\text{Fitt}_i(\mathcal{Q}) = 0$) if and only if all the $g_{i,j}$'s are zero in \mathcal{A} (resp., \mathcal{R}_s), and thus, all the $g_{i,j}$'s are zero in the integral domain \mathcal{R} , i.e., if and only if $\text{Fitt}_i(\mathcal{L}/t(\mathcal{L})) = 0$. Thus, we have

$\text{Fitt}_i(\mathcal{M}/t(\mathcal{M})) = 0 \Leftrightarrow \text{Fitt}_i(\mathcal{L}/t(\mathcal{L})) = 0 \Leftrightarrow \text{Fitt}_i(\mathcal{Q}) = 0$. Hence, the first part of Equation (1) is equivalent to $\text{Fitt}_i(\mathcal{Q}) = 0$ for $i = 1, \dots, r-1$.

Let us now suppose that $\text{Fitt}_r(\mathcal{M}/t(\mathcal{M})) = \mathcal{A}$ and using Proposition 5, let $s \in \text{Fitt}_r(\mathcal{L}/t(\mathcal{L})) \cap \mathcal{S}$. Then, $s^{-1} \in \mathcal{R}_s$ and $1 = s^{-1}s \in \mathcal{R}_s \text{Fitt}_r(\mathcal{L}/t(\mathcal{L}))$, which shows that $\text{Fitt}_r(\mathcal{Q}) = \mathcal{R}_s \text{Fitt}_r(\mathcal{L}/t(\mathcal{L})) = \mathcal{R}_s$.

Conversely, if $\text{Fitt}_r(\mathcal{Q}) = \mathcal{R}_s \text{Fitt}_r(\mathcal{L}/t(\mathcal{L})) = \mathcal{R}_s$, then there exist $a_{r,j} = b_{r,j}/s^{n_j}$, where $b_{r,j} \in \mathcal{R}$ and $n_j \in \mathbb{N}$ for $j = 1, \dots, d_r$, satisfying $\sum_{j=1}^{d_r} a_{r,j} g_{r,j} = 1$. If we set $n = \max\{n_1, \dots, n_{d_r}\}$, then $\sum_{j=1}^{d_r} (s^{n-n_j} b_{r,j}) g_{r,j} = s^n \in \text{Fitt}_r(\mathcal{L}/t(\mathcal{L}))$, which proves that $\text{Fitt}_r(\mathcal{M}/t(\mathcal{M})) = \mathcal{A}$.

Finally, the result holds using Theorem 4 for either the f.p. \mathcal{R}_s -module \mathcal{Q} or for the f.p. \mathcal{A} -module $\mathcal{M}/t(\mathcal{M})$.

Let us explain how Theorem 14 can be used to compute a stabilizing controller C of P . Set $Q_1 = R'$ and $q_0 = p'$. Consider a *finite free resolution of finite length* of $\mathcal{L}/t(\mathcal{L})$, namely, a long exact sequence of \mathcal{R} -modules of the form

$$\begin{aligned} 0 & \longrightarrow \mathcal{R}^{1 \times q_m} \xrightarrow{\cdot Q_m} \mathcal{R}^{1 \times q_{m-1}} \xrightarrow{\cdot Q_{m-1}} \dots \\ \dots & \xrightarrow{\cdot Q_2} \mathcal{R}^{1 \times q_1} \xrightarrow{\cdot Q_1} \mathcal{R}^{1 \times q_0} \xrightarrow{\kappa} \mathcal{L}/t(\mathcal{L}) \longrightarrow 0. \end{aligned} \quad (6)$$

See Eisenbud (1995); Rotman (2009). See Chyzak et al. (2005) for computational issues and Chyzak et al. (2007) for an implementation in the OREMODULES package.

By Theorem 1, $\mathcal{R}_s = \mathcal{T}^{-1} \mathcal{R} = \mathcal{R}[s^{-1}]$ is a flat \mathcal{R} -module. Tensoring (6) by \mathcal{R}_s and using $\mathcal{Q} \cong \mathcal{R}_s \otimes_{\mathcal{R}} \mathcal{L}/t(\mathcal{L})$, we obtain the following long exact sequence of \mathcal{R}_s -modules

$$\begin{aligned} 0 & \longrightarrow \mathcal{R}_s^{1 \times q_m} \xrightarrow{\cdot Q_m} \mathcal{R}_s^{1 \times q_{m-1}} \xrightarrow{\cdot Q_{m-1}} \dots \\ \dots & \xrightarrow{\cdot Q_2} \mathcal{R}_s^{1 \times q_1} \xrightarrow{\cdot Q_1} \mathcal{R}_s^{1 \times q_0} \xrightarrow{\text{id}_{\mathcal{R}_s} \otimes \kappa} \mathcal{Q} \longrightarrow 0, \end{aligned} \quad (7)$$

which is a finite free resolution of \mathcal{Q} of finite length.

If P is stabilizable, using the fact that \mathcal{Q} is a projective \mathcal{R}_s -module, then the long exact sequence (7) of \mathcal{R}_s -modules *splits* (Eisenbud (1995); Rotman (2009)), namely, there exist matrices $S_i \in \mathcal{R}_s^{q_i-1 \times q_i}$ satisfying the identities

$$Q_i S_i + S_{i+1} Q_{i+1} = I_{q_i}, \quad i = 1, \dots, m-1, \quad (8)$$

where $Q_{m+1} = 0$, i.e., $Q_m S_m = I_{q_m}$ for $i = m-1$. Now, post-multiplying (8) by Q_i and using $Q_{i+1} Q_i = 0$, we obtain $Q_i S_i Q_i + S_{i+1} Q_{i+1} Q_i = Q_i$, i.e., $Q_i S_i Q_i = Q_i$, which shows that S_i is a *generalized inverse* of Q_i .

Let us show how a generalized inverse $S' = S_1$ of $R' = Q_1$ can be computed. We first show how a right inverse S_m of Q_m can be computed. If $\delta : \mathcal{R}[x] \rightarrow \mathcal{R}_s = \mathcal{R}[x]/\langle xs-1 \rangle$

is the canonical projection onto \mathcal{R}_s , then we have the following *commutative exact diagram* of $\mathcal{R}[x]$ -modules

$$(9) \quad \begin{array}{ccc} 0 & & 0 \\ \uparrow & & \uparrow \\ \mathcal{R}_s^{q_m \times 1} & \xleftarrow{Q_m \cdot} & \mathcal{R}_s^{q_{m-1}} \\ \text{id}_{q_m} \otimes \delta \uparrow & & \text{id}_{q_{m-1}} \otimes \delta \uparrow \\ \mathcal{R}[x]^{q_m \times 1} & \xleftarrow{Q_m \cdot} & \mathcal{R}[x]^{q_{m-1} \times 1} \\ (x s - 1) I_{q_m} \cdot \uparrow & & (x s - 1) I_{q_{m-1}} \cdot \uparrow \\ \mathcal{R}[x]^{q_m \times 1} & \xleftarrow{Q_m \cdot} & \mathcal{R}[x]^{q_{m-1} \times 1}. \end{array}$$

Proposition 3.1 of Cluzeau et al. (2008) then shows that

$$\begin{aligned} \text{coker}_{\mathcal{R}_s}(Q_m \cdot) &= \mathcal{R}_s^{q_m \times 1} / \text{im}_{\mathcal{R}_s}(Q_m \cdot) \\ &= \mathcal{R}[x]^{q_m \times 1} / (Q_m \mathcal{R}[x]^{q_{m-1} \times 1} + (x s - 1) I_{q_m} \mathcal{R}[x]^{q_m \times 1}). \end{aligned}$$

Therefore, $\text{coker}_{\mathcal{R}_s}(Q_m \cdot) = 0$ is equivalent to $\text{im}_{\mathcal{R}_s}(Q_m \cdot) = \mathcal{R}_s^{q_m \times 1}$, i.e., to $(Q_m \ (x s - 1) I_{q_m}) \in \mathcal{R}[x]^{q_m \times (q_{m-1} + q_m)}$ has a right inverse $T \in \mathcal{R}[x]^{(q_{m-1} + q_m) \times q_m}$. Right inverses T can be computed using standard Gröbner basis methods and, e.g., the OREMODULES package. Writing $T = (T_1^T \ T_2^T)^T$, $T_1 \in \mathcal{R}[x]^{q_{m-1} \times q_m}$, $T_2 \in \mathcal{R}[x]^{q_m \times q_m}$, $Q_m T_1(x) + (x s - 1) T_2(x) = I_{q_m}$ yields $Q_m T_1(s^{-1}) = I_{q_m}$, which shows that S_m obtained by taking the residue classes of the entries of T_1 in \mathcal{R}_s is a right inverse of Q_m .

Let us explain how to obtain $S_{m-1} \in \mathcal{R}_s^{q_{m-2} \times q_{m-1}}$ satisfying (8) for $i = m - 1$, i.e., $Q_{m-1} S_{m-1} + S_m Q_m = I_{q_{m-1}}$. This identity shows that S_{m-1} is a right factor of the matrix $\Pi_{m-1} := I_{q_{m-1}} - S_m Q_m \in \mathcal{R}_s^{q_{m-1} \times q_{m-1}}$ by Q_{m-1} , i.e., that the columns of Π_{m-1} belong to $\text{im}_{\mathcal{R}_s}(Q_{m-1} \cdot)$.

Let us explain how such a factorization can be computed. Let $\Gamma_{m-1} \in \mathcal{R}[x]^{q_{m-1} \times q_{m-1}}$ be a matrix such that the residue class of its entries in $\mathcal{R}_s = \mathcal{R}[x]/\langle s x - 1 \rangle$ is Π_{m-1} . For instance, consider Γ_{m-1} obtained by substituting s^{-1} by x into Π_{m-1} . Using the fact that $Q_{m-1} \in \mathcal{R}^{q_{m-1} \times q_{m-2}}$, Proposition 3.1 of Cluzeau et al. (2008) shows that

$$\text{im}_{\mathcal{R}_s}(Q_{m-1} \cdot) = (Q_{m-1} \ (s x - 1) I_{q_{m-1}}) \mathcal{R}[x]^{(q_{m-2} + q_{m-1}) \times 1} / ((s x - 1) \mathcal{R}[x]^{q_{m-1} \times 1}).$$

Thus, $v \in \text{im}_{\mathcal{R}_s}(Q_{m-1} \cdot)$ is formed by the residue classes in \mathcal{R}_s of the entries of $w \in \text{im}_{\mathcal{R}[x]}((Q_{m-1} \ (s x - 1) I_{q_{m-1}}) \cdot)$. Hence, S_{m-1} is the matrix formed by the residue classes in \mathcal{R}_s of a matrix $T_{m-1} \in \mathcal{R}[x]^{q_{m-2} \times q_{m-1}}$ satisfying

$$\Gamma_{m-1} = (Q_{m-1} \ (s x - 1) I_{q_{m-1}}) \begin{pmatrix} T_{m-1} \\ U_{m-1} \end{pmatrix},$$

for a certain matrix $U_{m-1} \in \mathcal{R}[x]^{q_{m-1} \times q_{m-1}}$. Such a right factorization can be obtained using standard Gröbner basis methods and, e.g., the OREMODULES package.

Inductively, the matrices S_i 's can be computed to obtain S_1 , and thus, a stabilizing controller C of P by Theorem 13. Such computations can be done by the RANKFACTORIZATIONPROBLEM package (Dagher et al. (2023)).

We finally explain how to compute $\ker_{\mathcal{R}_s}(Q_m \cdot)$ (see Example 5 below for an explanation of its usefulness). Considering (9), Proposition 3.1 of Cluzeau et al. (2008) yields $\ker_{\mathcal{R}_s}(Q_m \cdot) = (V \mathcal{R}[x]^{\alpha \times 1}) / ((x s - 1) I_{q_m} \mathcal{R}[x]^{q_{m-1} \times 1})$, where $V \in \mathcal{R}[x]^{q_{m-1} \times \alpha}$ is such that

$$\ker_{\mathcal{R}[x]}((Q_m \ (s x - 1) I_{q_m}) \cdot) = \text{im}_{\mathcal{R}_s} \left((V^T \ -W^T)^T \cdot \right)$$

for a certain $W \in \mathcal{R}[x]^{q_m \times \beta}$. The matrices V and W can be computed by Gröbner basis methods and the OREMODULES package. The matrix $K \in \mathcal{R}_s^{q_{m-1} \times \alpha}$, obtained by considering the residue class of the entries of V in \mathcal{R}_s , satisfies $\ker_{\mathcal{R}_s}(Q_m \cdot) = \text{im}_{\mathcal{R}_s}(K \cdot)$. This computation can be done by the RANKFACTORIZATIONPROBLEM package.

Example 5. We continue Example 1. Set $s = g_1 \in \mathcal{S}$. Using the RANKFACTORIZATIONPROBLEM package (Dagher et al. (2023)), R' has the following right inverse

$$S' = \begin{pmatrix} 0 & 0 \\ -z_1 x & -z_3 x + 2x \\ -z_2 x - 3x & 0 \end{pmatrix}$$

over $\mathcal{R}_s = \mathcal{R}[x]/\langle x s - 1 \rangle = \mathcal{R}[s^{-1}]$, i.e., x corresponds to s^{-1} in \mathcal{R}_s . The determinant of the matrix formed by the first two rows of $S = S' D' D^{-1}$ is 0. We thus have to consider another right inverse of R' . To do that, using the above comment on the computation of $\ker_{\mathcal{R}_s}(Q_m \cdot)$ for R' , we get $\ker_{\mathcal{R}_s}(R' \cdot) = \text{im}_{\mathcal{R}_s}(K \cdot)$, where

$$K = (z_2 z_3 - 2 z_2 + 3 z_3 - 6 \quad z_1^2 \quad z_1 z_2 + 3 z_1)^T \in \mathcal{R}_s^{3 \times 1}.$$

Then, note that $S'(Q) = S' + K Q$ is a right inverse of R' for all $Q \in \mathcal{R}_s^{1 \times 2}$. Considering, for instance, $Q_\star = (1 \ 0)$,

$$S'(Q_\star) = \begin{pmatrix} z_2 z_3 - 2 z_2 + 3 z_3 - 6 & 0 \\ z_1^2 - z_1 x & -z_3 x + 2x \\ z_1 z_2 - z_2 x + 3 z_1 - 3x & 0 \end{pmatrix}$$

is another right inverse of R' . Then, if we define

$$S_\star = S'(Q_\star) D' D^{-1} = \begin{pmatrix} z_3 - 2 & 0 \\ z_1 (z_1 - x) & x (z_3 - 2) \\ z_2 + 3 & z_1 \\ z_1 - x & 0 \end{pmatrix},$$

then we can check that $R S_\star = I_2$, which yields the stabilizing controller $C_\star = (C_1 \ C_2)$ of P , where $C_2 = 0$ and $C_1 = (z_1 (z_2 + 3) (z_3 - 2) - 1) / ((z_3 - 2)^2 (z_2 + 3))$.

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