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► **To cite this version:**

José Antonio Ortega, Denis Efimov, Andrey Polyakov, Jaime A Moreno, Leonid Fridman. Stabilization of the Cart Pendulum using a Homogeneous Controller. Proc 17th Int. Workshop on Variable Structure Systems and Sliding Mode Control (VSS), Oct 2024, Abu Dhab, United Arab Emirates. pp.170 - 175, 10.1109/vss61690.2024.10753400 . hal-04895361

HAL Id: hal-04895361

<https://inria.hal.science/hal-04895361v1>

Submitted on 21 Jan 2025

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Stabilization of the Cart Pendulum using a Homogeneous Controller

José Antonio Ortega^{1,3}, Denis Efimov², Andrey Polyakov², Jaime A. Moreno³, Leonid Fridman¹

Abstract—This manuscript addresses the problem of stabilization of a cart pendulum using a homogeneous controller. The model of the cart pendulum is composed by a combination of strict-feedback and feedforward parts. The proposed homogeneous controller allows the feedforward nonlinearities to be globally dominated. Emphasizing the simplicity of the controller’s design, the methodology relies on solving a system of linear matrix inequalities. Furthermore, a finite-time bound of convergence is evaluated. The effectiveness of the proposed methodology is validated through experimental results on a real cart pendulum.

I. INTRODUCTION

Underactuated systems constitute a class of mechanical ones, where the number of control inputs is fewer than the degrees of freedom [1]. The dynamics of such systems can feature non-minimum phase zero-dynamics, nonholonomic constraints, and feedforward nonlinearities.

Indeed, because of the underactuation, these systems cannot be fully linearizable, and there is no output with relative degree equal to the order of the system. Consequently, a zero-dynamics will emerge [2]. The work [3] presents sufficient conditions for the existence of a transformation to a normal form and classify the underactuated systems based on the structure of their normal forms.

This manuscript focuses on the underactuated systems referred to as *Class III* in [3]. The nominal model of this type of underactuated systems can be presented in feedforward form (also referred to as upper-triangular form). Consequently, such type of systems suffered the so-called *peaking phenomenon* [4]. That’s why stabilizing feedforward systems poses a significant challenge in control theory.

Two notable methods for tackling the problem are the Lyapunov Forwarding Procedure and Nested Saturations. The Lyapunov Forwarding Procedure, proposed in [5], focuses on constructing an appropriate Lyapunov function that ensures stability and convergence by designing a forward controller that injects additional dynamics into the system.

On the other hand, stabilization using nested saturations, as developed in [6] employs a hierarchical structure of

saturation functions to shape the control inputs and attain stability. By implementing successive saturation operations, the control inputs are progressively constrained. However, a significant drawback of this approach is that the nested saturation requires small gain of the closed loop dynamics, which slows the performance of the controller.

The sliding-mode control is a widely recognized methodology for robust regulator design, offering desirable features such as bounded disturbance rejection and finite-time convergence. A first-order sliding-mode based algorithm for stabilization of the cart pendulum was presented in [7]. However, this algorithm guarantees only semiglobal stability.

In the present work a methodology based on a homogeneous controller design for the global finite time stabilization of the cart pendulum is developed, which is capable to dominate the feedforward nonlinearities. The gains of proposed algorithm can be found solving a system of linear matrix inequalities, just setting one design parameter ρ .

The structure of this paper is as follows. In Section II some preliminaries about the linear generalized homogeneity are presented. The model and description of the cart pendulum are presented in Section III, where also the system is transformed into a LTI form while some feedforward nonlinearities are considered as disturbances. Then in Section IV a homogeneous controller is designed stabilizing the cart pendulum in finite-time. To prove the finite-time stability of the cart pendulum equilibrium, in Section V, the homogeneous canonical norm is used as Lyapunov function and also an estimation of the settling time is presented. For the sake to corroborate the results, some experiments in a real cart pendulum plant are shown in Section VI.

II. GENERALIZED HOMOGENEITY

Homogeneity is a symmetry with respect to a group of transformations called dilations [8], [9], [10]. Dilations in finite-dimensional vector spaces are characterized in [10].

Definition 1: A one-parameter family of transformations $\mathbf{d}(s) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with parameter $s \in \mathbb{R}$ is said to be a dilation in \mathbb{R}^n if it satisfies

- **Group property:** $\mathbf{d}(0) = I_n$ and $\mathbf{d}(t + s) = \mathbf{d}(t) \circ \mathbf{d}(s), \forall s, t \in \mathbb{R};$
- **Limit property:** $\lim_{s \rightarrow \pm\infty} \|\mathbf{d}(s)x\| = e^{\pm\infty}, \forall x \neq 0.$

The dilation is continuous if the function $s \mapsto \mathbf{d}(s)x$ is continuous for any $x \in \mathbb{R}^n$. The dilation is linear (see, e.g., [11]) if $\mathbf{d}(s)$ is a linear mapping (i.e., $\mathbf{d}(s) \in \mathbb{R}^{n \times n}$). Any linear continuous group admits the representation (e.g., [12])

$$\mathbf{d}(s) = e^{sG\mathbf{a}} = \sum_{i=0}^{\infty} \frac{(sG\mathbf{a})^i}{i!}, \quad s \in \mathbb{R}, \quad (1)$$

This work was supported by PAPIIT-UNAM under Project IN106323. The authors thank the financial support of the project ANR SLIMDISC and the support of CONAHACYT, CVU 958158.

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where $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$. To be a dilation in \mathbb{R}^n , the group (1) must have an anti-Hurwitz matrix¹ $G_{\mathbf{d}}$ being a generator of linear dilation [11]. The weighted dilation [8], [13], [14], [15] corresponds to a diagonal generator $G_{\mathbf{d}}$. For weighted dilations some examples of the so-called homogeneous norms in \mathbb{R}^n can be found in [16], [14], [17], [15]. The linear dilation introduces an alternative norm topology in \mathbb{R}^n by means of the canonical homogeneous norm.

Definition 2: ([11]) The function $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \rightarrow \mathbb{R}_+$ given by $\|x\|_{\mathbf{d}} = 0$ for $x = \mathbf{0}$ and

$$\|x\|_{\mathbf{d}} = e^{s_x}, \quad \text{where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, \quad x \neq \mathbf{0} \quad (2)$$

is called the canonical homogeneous norm in \mathbb{R}^n , where \mathbf{d} is a linear continuous monotone dilation².

Since $\|\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x\| = 1$ then the mapping $x \mapsto \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x$ is the \mathbf{d} -homogeneous projector of x on the unit sphere [18, page 159]. For the standard dilation $\mathbf{d}(s) = e^s I_n, s \in \mathbb{R}$ (being a multiplication by a scalar) we have $\|\cdot\| = \|\cdot\|_{\mathbf{d}}$ and $\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x = \frac{x}{\|x\|}$ is the standard homogeneous projector of x on the unit sphere.

Theorem 1 ([11]): If $\|x\| = \sqrt{x^T P x}$ with a symmetric matrix $P \in \mathbb{R}^{n \times n}$ satisfying $P G_{\mathbf{d}} + G_{\mathbf{d}}^T P \succ 0, P \succ 0$ then \mathbf{d} is monotone and the canonical homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is well-defined and continuous on \mathbb{R}^n , infinitely smooth on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ and

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \frac{\|x\|_{\mathbf{d}} x^T \mathbf{d}^T(-\ln \|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x}{x^T \mathbf{d}^T(-\ln \|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x}, \quad \forall x \neq \mathbf{0}. \quad (3)$$

The canonical homogeneous norm is utilized below as an implicit Lyapunov function for a control design and stability analysis.

Definition 3 ([10], [19]): A vector field $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ (resp., a function $h : \mathbb{R}^n \mapsto \mathbb{R}$) is said to be \mathbf{d} -homogeneous of degree $\mu \in \mathbb{R}$ if $f(\mathbf{d}(s)x) = e^{\mu s} \mathbf{d}(s) f(x)$ (resp. $h(\mathbf{d}(s)x) = e^{\mu s} h(x)$), for all $x \in \mathbb{R}^n, s \in \mathbb{R}$.

In the following section the cart pendulum model is presented and also we design a controller based on homogeneity to make stable the equilibrium of the cart pendulum.

III. DESCRIPTION OF THE PLANT

Consider the cart pendulum shown in Fig III. This figure describes a pendulum rotating in a vertical plane and its pivot pin is mounted in a cart. The cart can move along a horizontal rail, lying in the plane of rotation. The vector of generalized coordinates is $q = [q_x \ q_s]^T \in \mathbb{R}^2$, where the cart position is q_x [m], and the pendulum angular position is q_s [rad/s] measured counter clockwise ($q_s = 0$ for the upright position of the pendulum). The equations of motion of the cart pendulum are

$$\begin{aligned} m_{xx} \ddot{q}_x + m_{sx}(q_s) \ddot{q}_s - m_s l_s \sin(q_s) \dot{q}_s^2 &= \tau \\ m_{sx}(q_s) \ddot{q}_x + m_{ss} \ddot{q}_s - m_s l_s g \sin(q_s) &= 0 \end{aligned} \quad (4)$$

with

$$\begin{aligned} m_{xx} &:= m_s + m_x, & m_{sx}(q_s) &:= m_s l_s \cos(q_s), \\ m_{ss} &:= m_s l_s + I_s, \end{aligned}$$

¹A matrix $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$ is anti-Hurwitz if $-G_{\mathbf{d}}$ is Hurwitz.

²A dilation in \mathbb{R}^n is monotone if for any $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ the function $s \mapsto \|\mathbf{d}(s)x\|, s \in \mathbb{R}$ is strictly increasing.

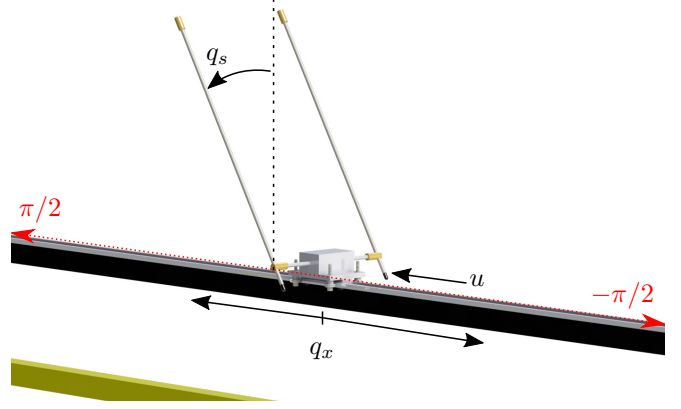


Fig. 1. Cart-pendulum system: Schematic figure.

where m_x is the mass of the cart, m_s is the mass of the pole, l_s is the length of the pole, I_s is the moment of Inertia of the pole, g is the gravity force, and $\tau \in \mathbb{R}$ is the control input.

The potential energy of the system is

$$V(q_s) = m_s l_s g \cos(q_s).$$

Note that the external variable q_x is actuated, the shape variable q_s is unactuated. The m_{xx} is constant, and the potential energy only depends of the shape variable q_s . A safe set can be defined as

$$U = \left\{ q_s \in \mathbb{R} \mid q_s \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right\},$$

such that $\cos(q_s) \neq 0 \quad \forall q_s \in U$.

One of the main challenges to stabilize the cart pendulum is the null effect of the control input τ when $q_s = \pm \frac{\pi}{2}$ in the dynamics of the pole, i.e., when the pendulum position is on the horizontal plane. Note that considering U avoids the points where the control input does not affect the dynamics of the pole.

Consider the following change of coordinates for the cart pole system (4),

$$\begin{aligned} z_1 &= q_x + \gamma(q_s), & z_2 &= \dot{q}_x + m_{sx}^{-1}(q_s) m_{ss} \dot{q}_s, \\ \gamma(q_s) &= \int_0^{q_s} m_{sx}^{-1}(s) m_{ss} ds, \end{aligned}$$

and the noncollocated partial feedback linearization:

$$\tau = \frac{m_{xx}(m_s l_s g \sin q_s) - \Delta(q_s) u}{m_{sx}(q_s)} - m_s l_s \sin q_s p_s^2,$$

with $\Delta(q_s) = m_{xx} m_{ss} - m_{sx}^2(q_s) > 0$ and $p_s := \dot{q}_s$, transforming the system into the cascade form:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= g\psi(q_s) + \Pi(q_s) p_s^2 \\ \dot{q}_s &= p_s \\ \dot{p}_s &= u \end{aligned} \quad (5)$$

$$\psi(q_s) = \tan(q_s), \quad \Pi(q_s) = \frac{m_{ss} \sin(q_s)}{m_s l_s \cos^2(q_s)}.$$

Now defining the variables

$$\zeta_1 := \psi(q_s), \quad \zeta_2 := \psi'(q_s) = \sec^2(q_s) p_s$$

and applying the following control

$$u = \cos^2(q_s) (\nu - 2 \sec^2(q_s) \tan(q_s) p_s^2),$$

where $\nu \in \mathbb{R}$ is the new input to be designed, we obtain:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= g\zeta_1 + \Pi_\zeta(\zeta_1)\zeta_2^2 \\ \dot{\zeta}_1 &= \zeta_2 \\ \dot{\zeta}_2 &= \nu \end{aligned}$$

where

$$\Pi_\zeta(\zeta_1) = \frac{m_{ss}\zeta_1}{m_s l_s (\zeta_1^2 + 1)^{\frac{3}{2}}}.$$

Remark 1: It can be directly computed that

$$\max_{\zeta_1 \in \mathbb{R}} |\Pi_\zeta(\zeta_1)| = \beta_\Pi = \frac{2m_{ss}}{3\sqrt{3}m_s l_s}.$$

IV. CONTROL DESIGN

The cascade system (5) can be rewritten in new coordinates as follows,

$$\dot{\chi} = A\chi + B\nu + h(\chi) \quad (6)$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad h(\chi) = \begin{bmatrix} 0 \\ \Pi_\zeta(\zeta_1)\zeta_2^2 \\ 0 \\ 0 \end{bmatrix},$$

where the state-vector $\chi = [z_1, z_2, \zeta_1, \zeta_2]^T$ and the function $h(\chi)$ contains feedforward nonlinearities violating fourth integrator structure of the system.

The following theorem gives conditions to design a homogeneous controller stabilizing the cart pendulum globally and in finite-time despite of feedforward nonlinearities.

Theorem 2: Let $G_d = \text{diag}(4, 3, 2, 1)$ and a linear dilation be defined by (1). Consider the system (6) with the controller

$$\nu = K\pi(\chi), \quad K = YX^{-1}, \quad \pi(\chi) = d(-\ln \|\chi\|_d) \chi. \quad (7)$$

If for some $\rho > 0$, $X \in \mathbb{R}^{4 \times 4}$ and $Y \in \mathbb{R}^{1 \times 4}$ have solution satisfying the following system of linear matrix inequalities:

$$\begin{aligned} \begin{bmatrix} \Phi & \beta_\Pi M X \\ \beta_\Pi X M^T & I \end{bmatrix} &\geq 0, \\ G_d X + X G_d^T &> 0, \\ X &= X^T > I, \end{aligned} \quad (7)$$

where

$$\Phi = -(A + \rho G_d)X - X(A^T + \rho G_d^T) - BY - Y^T B^T,$$

then, the origin is globally finite-time stable.

Moreover, the canonical homogeneous norm $\|\chi\|_d$ induced by $\|\chi\| = \sqrt{\chi^T X^{-1} \chi}$ is a Lyapunov function of the closed loop system (6), (7) satisfying $\frac{d}{dt} \|\chi(t)\|_d \leq -\rho$, and $\chi(t) = 0$, $\forall t \geq t_r$ with a settling time $t_r \leq \frac{\|\chi(0)\|_d}{\rho}$.

V. PROOF OF THEOREM 2

Consider the canonical homogeneous norm as a Lyapunov candidate function. Its time derivative is given by

$$\begin{aligned} \frac{d}{dt} \|\chi\|_d &= \frac{\partial \|\chi\|_d}{\partial \chi} \frac{d}{dt} \chi \\ &= \|\chi\|_d \frac{\pi^T(\chi) X^{-1} d(-\ln \|\chi\|_d)}{\pi^T(\chi) X^{-1} G_d \pi(\chi)} (A\chi + B\nu + h(\chi)) \end{aligned}$$

Note that A is a nilpotent matrix and the vector field $x \rightarrow Ax$ is homogeneous of any degree μ , then

$$A d(s) = e^{\mu s} d(s)A.$$

Therefore, the following identity holds

$$d(-\ln \|\chi\|_d)A = \|\chi\|_d^\mu A d(-\ln \|\chi\|_d). \quad (8)$$

Since $G_d B = B$,

$$d(-\ln \|\chi\|_d)B = \|\chi\|_d^{-1} B. \quad (9)$$

Consider $\mu = -1$. Using (8), (9) the time derivative can be represented as

$$\begin{aligned} \frac{d}{dt} \|\chi\|_d &= \frac{\pi^T(\chi) X^{-1} (A + BK) \pi(\chi)}{\pi^T(\chi) X^{-1} G_d \pi(\chi)} + \\ &+ \|\chi\|_d \frac{\pi^T(\chi) X^{-1} d(-\ln \|\chi\|_d) h(\chi)}{\pi^T(\chi) X^{-1} G_d \pi(\chi)}. \quad (10) \end{aligned}$$

Lemma 1: The following inequality

$$\pi^T(\chi) X^{-1} d(-\ln \|\chi\|_d) h(\chi) \leq \pi^T(\chi) \beta_\Pi M^T M \pi(\chi),$$

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

is satisfied provided that the condition

$$X > I \quad (11)$$

is fulfilled.

Proof: The proof is in the appendix A. \blacksquare

Then, using Lemma 1 it is possible to bound (10) as

$$\frac{d}{dt} \|\chi\|_d \leq \frac{\pi^T(\chi) (X^{-1} (A + BK) + \beta_\Pi M^T M) \pi(\chi)}{\pi^T(\chi) X^{-1} G_d \pi(\chi)}.$$

Consider that $K = YX^{-1}$ such that,

$$\begin{aligned} \frac{d}{dt} \|\chi\|_d &\leq \frac{\pi^T(\chi) X^{-1} \Theta X^{-1} \pi(\chi)}{\pi^T(\chi) X^{-1} (G_d X + X G_d^T) X^{-1} \pi(\chi)}, \\ \Theta &= AX + BY + XA^T + Y^T B^T + 2\beta_\Pi X M^T M X \end{aligned}$$

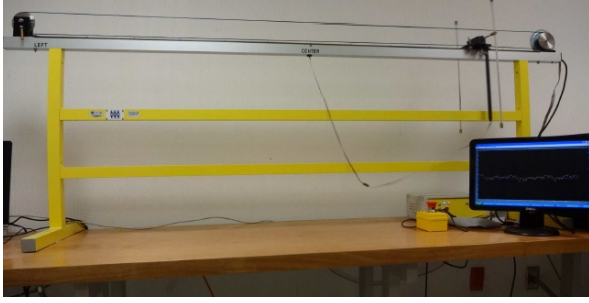


Fig. 2. Cart Pendulum Inteco.

If the following inequality holds

$$\Theta \leq -\rho (G_d X + X G_d^T), \quad (12)$$

the required upper estimate on the derivative of $\|\chi\|_d$ is obtained. The LMI for (12) can be written using Schur complement as

$$\begin{bmatrix} \Phi & \beta_{\Pi} M X \\ \beta_{\Pi} X M^T & I \end{bmatrix} \geq 0,$$

which is given in the formulation of the theorem. Thus in order to prove that $\|\chi\|_d$ is a Lyapunov function of the closed-loop system the following linear matrix inequalities have to be satisfied.

$$G_d X + X G_d^T > 0, \quad X = X^T > I, \quad (13)$$

which are also required by the theorem conditions.

Consequently,

$$\frac{d}{dt} \|\chi\| \leq -\rho,$$

and, integrating over the time interval $[0, t)$,

$$\|\chi(t)\|_d - \|\chi(0)\|_d \leq -\rho t.$$

Therefore, $\|\chi(t)\|_d$ reaches zero in a finite time

$$t_r \leq \frac{\|\chi(0)\|}{\rho}.$$

Thus, we prove that the closed loop system (6), (7) it is stable in the sense of Lyapunov and it is finite time attractive, so the equilibrium is *finite-time stable*. Note that the condition from Lemma 1, $X > I$, is contained in (13).

This concludes the proof of Theorem 2.

Remark 2: It is worth to remark that for a sufficiently small ρ , the algebraic system (13) is always feasible since the linear part of the system is in the canonical controllable form.

VI. NUMERICAL RESULTS

For the sake to corroborate the theoretical results some simulations are performed in Matlab with the Euler method to solve the ODE with the parameters given in Table I.

Fig. 3 presents the trajectories of the cart pole model with the controller (7). The upper figure illustrates how the

TABLE I

TABLE OF THE PARAMETERS OF THE CART PENDULUM SYSTEM.

Parameter	Value
t_s	$1e-5$
ρ	0.057
m_s	0.872
m_x	0.08
l_s	0.011
g	9.81
I_s	0.0034

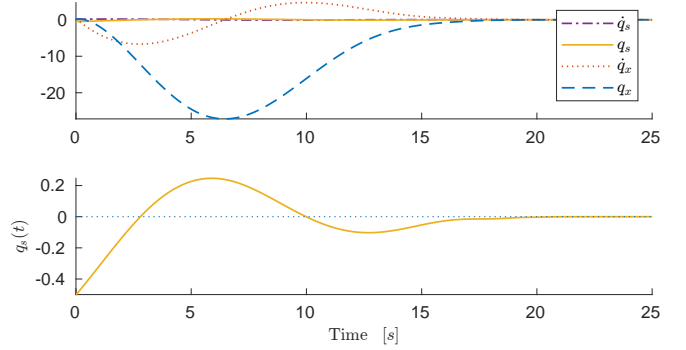


Fig. 3. Trajectories of the cart pendulum along the simulation.

trajectories converge to zero. The above figure shows with more detail the pendulum position trajectory.

In Fig. 4, the trajectories of the state $\chi = [z_1, z_2, \zeta_1, \zeta_2]^T$ along the simulation are shown.

In Fig. 5, the homogeneous canonical norm $\|\chi\|_d$ and the weighted norm $\|\chi\|$ are plotted. The control input applied to the cart pendulum τ and the controller (7) are shown in Fig. 6.

VII. EXPERIMENTAL RESULTS

This section presents experiments in the cart pendulum showed in Fig. 2. The parameters are given in the Table I but the sampling time is $t_s = 1e-3$ [s]. The positions of the cart and pendulum are measured by encoder and the velocities are estimated using the Levant differentiator [20], with gains $L_s = 30$ for the velocity of pendulum and $L_x = 5$ for cart.

A swing-up controller based on energy is used to drive the pendulum into the upright zone, then the homogeneous

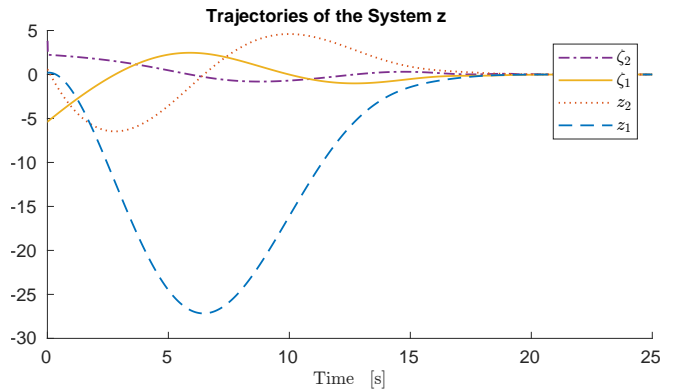


Fig. 4. χ Trajectories along the simulation.

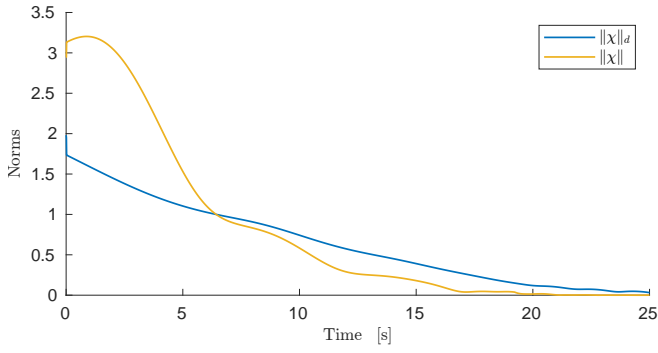


Fig. 5. Trajectories of weighted norm and homogeneous canonical norm along the simulation.

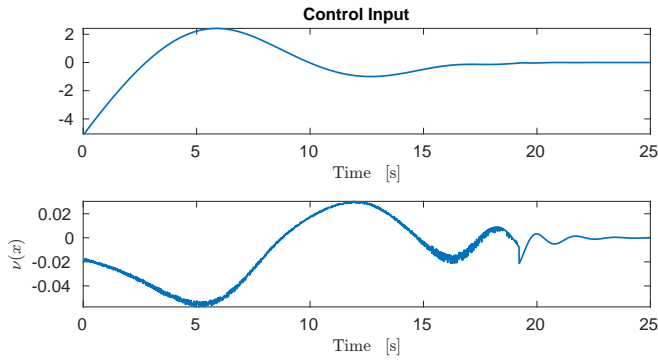


Fig. 6. Control input signal along the simulation.

controller drive the trajectories into the equilibrium.

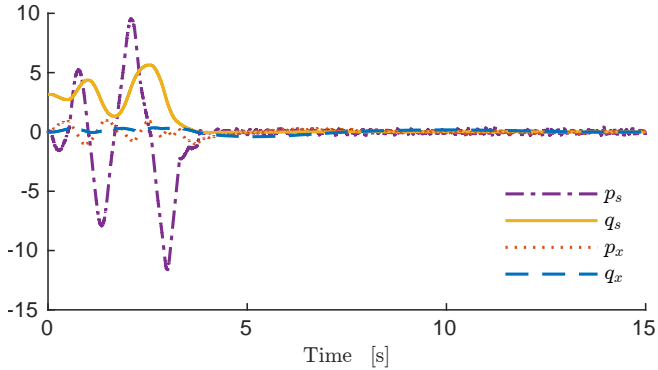


Fig. 7. Trajectories of the cart pendulum along the experiments.

Fig. 7 presents the trajectories of the cart pendulum. The trajectory of the pendulum is plotted in Fig 8. Fig. 9 the state χ is shown. Fig. 10 depicts the homogeneous canonical norm $\|\chi\|_d$ and the weighted norm $\|\chi\|$. The control input applied to the cart pendulum τ is in the Fig. 11.

VIII. CONCLUSIONS

This manuscript aimed to develop a design method for the cart pendulum using a homogeneous controller, with the goal of achieving global finite-time stability.

A significant advantage of this methodology is the straightforward design of the controller. This process involves selecting a design parameter to tune the convergence rate and

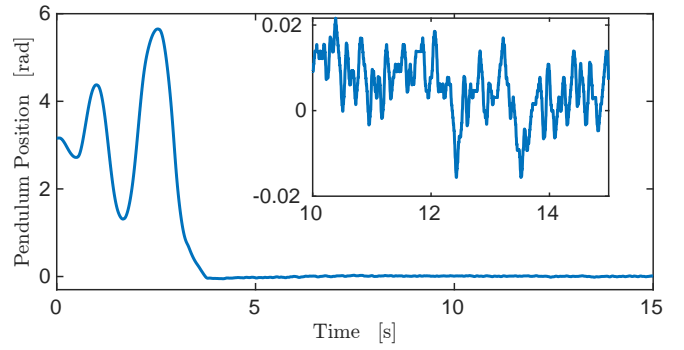


Fig. 8. Close up of the pole position along the experiment.

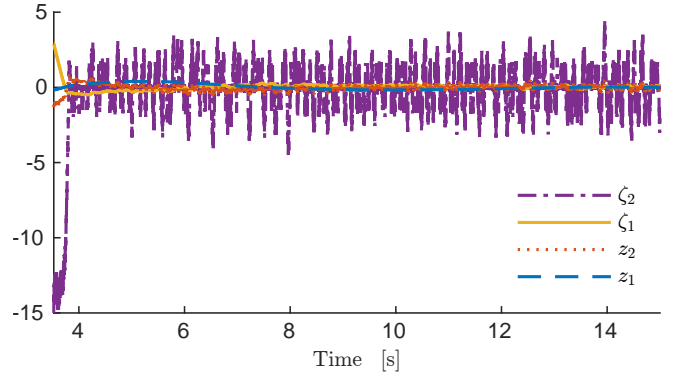


Fig. 9. χ Trajectories along the experiment.

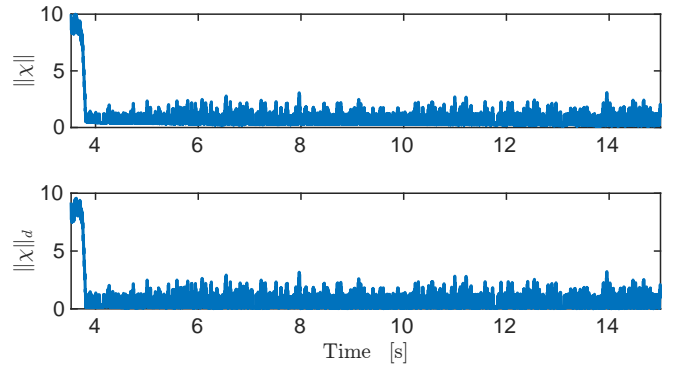


Fig. 10. Weighted norm $\|\chi\|$ and homogeneous canonical norm $\|\chi\|_d$ along the experiment.

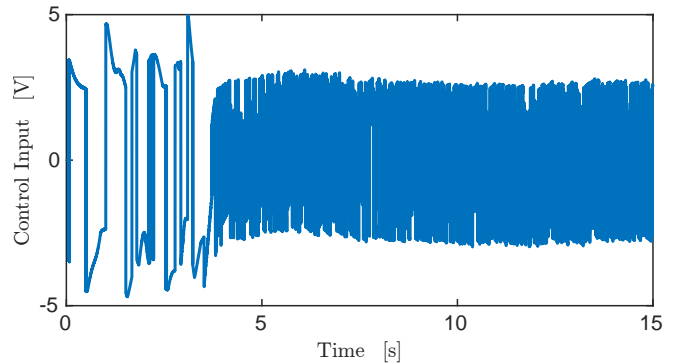


Fig. 11. Control Input applied to the motor of the cart pendulum along the experiment.

then solving a linear matrix inequality to find the gains that guarantee global finite-time stability. Furthermore, an estimate on the settling time, when the trajectories converge to the equilibrium, is presented.

An important remark to note is that for larger values of ρ (used to tune the convergence rate), the linear algebraic system may not be feasible. This limitation is a common drawback in most methodologies for feedforward systems, such as the small gain theorem in the nested saturations methodology.

APPENDIX

A. Proof of Lemma 1

Using the definition of the dilation of the vector $h(\chi)$ with $\mu = -1$,

$$\|\chi\|_{\mathbf{d}} \pi^T(\chi) X^{-1} \mathbf{d}(-\ln \|\chi\|_{\mathbf{d}}) h(\chi) = \|\chi\|_{\mathbf{d}} \pi^T(\chi) X^{-1} \begin{bmatrix} 0 \\ \frac{m_{ss}\zeta_1\zeta_2^2\|\chi\|_{\mathbf{d}}^{-3}}{m_s l_s (\zeta_1^2 + 1)^{\frac{3}{2}}} \\ 0 \\ 0 \end{bmatrix}$$

Let $l = [0 \ 1 \ 0 \ 0]^T$. Using Remark 1 we obtain:

$$\|\chi\|_{\mathbf{d}} \pi^T(\chi) X^{-1} \begin{bmatrix} 0 \\ \frac{m_{ss}\zeta_1\zeta_2^2\|\chi\|_{\mathbf{d}}^{-3}}{m_s l_s (\zeta_1^2 + 1)^{\frac{3}{2}}} \\ 0 \\ 0 \end{bmatrix} \leq \pi^T(\chi) X^{-1} l \beta_{\Pi} \zeta_2^2 \|\chi\|_{\mathbf{d}}^{-2}$$

Since the weighted norm $\|l\| = \sqrt{l^T P l}$ is used, it implies that $\|l\| = \sqrt{P_{22}}$ because of the shape of l . Consequently,

$$\pi^T(\chi) X^{-1} l \beta_{\Pi} \zeta_2^2 \|\chi\|_{\mathbf{d}}^{-2} \leq \|X^{-1}\| \sqrt{P_{22}} \beta_{\Pi} \zeta_2^2 \|\chi\|_{\mathbf{d}}^{-2} \quad (14)$$

Let us assume that

$$X^{-1} < I, \quad X > I$$

Then, it is possible to bound (14) with a quadratic term as

$$\pi^T(\chi) X^{-1} l \beta_{\Pi} \zeta_2^2 \|\chi\|_{\mathbf{d}}^{-2} \leq \pi^T(\chi) \beta_{\Pi} M^T M \pi(\chi).$$

Thus,

$$\|\chi\|_{\mathbf{d}} \pi^T(\chi) X^{-1} \mathbf{d}(-\ln \|\chi\|_{\mathbf{d}}) h(\chi) \leq \pi^T(\chi) \beta_{\Pi} M^T M \pi(\chi).$$

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