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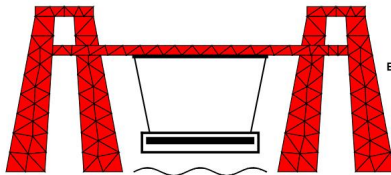
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XXVIII
CONGRESS OF DIFFERENTIAL
EQUATIONS AND APPLICATIONS

XVIII
CONGRESS OF
APPLIED MATHEMATICS

BILBAO # JUNE 24-28, 2024

High order ImEx method for the shallow water model

M. Kazolea, R. Lteif, M. Parisot

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CEDYA 2024

Bilbao - June 24-28 2024

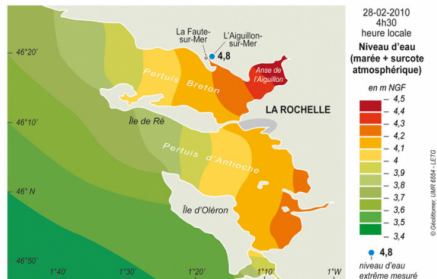
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Climate change is expected to increase coastal flooding hazard in years to come.



La Faute-sur-mer, western France
after Xynthia storm, February 28
2010

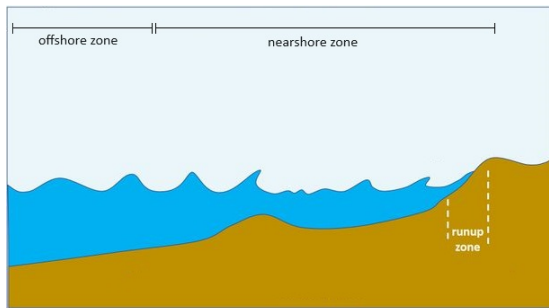


Sources : carte de simulation des niveaux d'eau établie par le BRGM (mars 2010, BRGM/MP-58261-FR, p36.)
niveau d'eau extrême mesuré dans le bourg de l'Aiguillon selon rapport CGEDD (16 sept. 2010, Mission n°007336-01, p21.)

Water level simulation map established by BRGM

Recent storm events (Xynthia [2010], Klaus [2009], Lothar [1999]) have shown the very destructive effects that flooding can have in terms of human victims and economic losses.

Objective: Design a multi-scale tool capable of simulating different features of oceanic flows, from large scale linear waves offshore to small scale non-linear flows in coastal areas.



Zone division

Offshore, tidal waves and atmospheric surges are very linear processes under the influence of Earth rotation.

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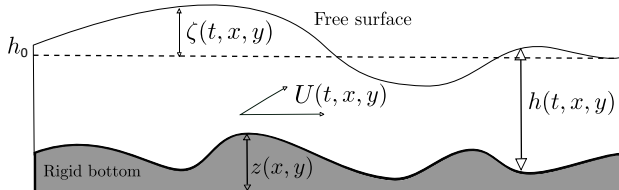
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$$\begin{cases} \partial_t h + \nabla \cdot (hU) = 0, \\ \partial_t (hU) + \nabla \cdot (hU \otimes U) + \nabla \left(\frac{g}{2} h^2 \right) = -gh \nabla z. \end{cases} \quad (\text{SW})$$



Consider the following rescaling:

$$\tilde{x} = \frac{\mathbf{x}}{l^*}, \quad \tilde{h} = \frac{h}{h^*}, \quad \tilde{U} = \frac{U}{U^*}, \quad \tilde{t} = \frac{t}{t^*}, \quad \tilde{z} = \frac{z}{h^*}$$

System (SW) becomes (after dropping the tildes):

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with the characteristic Froude number

$$Fr := \frac{U^*}{\sqrt{gh^*}} = \frac{\text{flow velocity}}{\text{velocity of gravity waves}}.$$

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- We focus in particular on the variation of the Froude number Fr that goes from 1 at the coastline to two or three orders less offshore.

$$\left. \begin{array}{l} \blacktriangleright \text{mean depth } h^* > 3000m \\ \blacktriangleright \text{current speed } U^* \approx 1m/s \end{array} \right\} \Rightarrow Fr \approx 10^{-3}$$

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- Limiting system in the limit $Fr \rightarrow 0$ satisfy the “lake equations”.

The system (SW_{Fr}) is a hyperbolic system of conservation laws:

$$\frac{\partial U}{\partial t} + \nabla \cdot F(U) = S(U, z)$$

$$U = \begin{pmatrix} h \\ hU \end{pmatrix}, F(U) = \begin{pmatrix} hU^T \\ hU \otimes U + h^2 \mathbf{1}_2 / (2Fr^2) \end{pmatrix}, S(U, z) = \begin{pmatrix} 0 \\ -\frac{h}{Fr^2} \nabla z \end{pmatrix}$$

Its eigenvalues in the direction \mathbf{n} are $\lambda_j = (U \cdot \mathbf{n}) + j \frac{c}{Fr}$, $j \in \{-1, 0, 1\}$ with $c = \sqrt{h}$ being the scaled speed of sound.

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Drawback: The time step constraint of an explicit time discretization satisfies

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- D3 high gravity wave speed require small time steps \Rightarrow does not allow long time simulations (the simulation of tide-surge), even with high performance computing. 

Specialized numerical schemes for low-Froude number flows found in literature:

- The first family of schemes respond to $D2 \Rightarrow$ asymptotic consistency when Fr tends to zero 👍. **[Bruel et al.'18, Dellacherie et al.'16]**

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The CPR (centered-potential regularization) ImEx scheme [**Parisot, Vila'16**] is a good first order candidate.

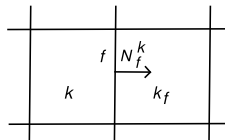
- A fully diagonal segregated method.
- Avoid resolution of large linear systems and limits the number of linear systems to be solved.

Numerical strategy: Cell-centered FV method on uniform Cartesian grid

in cell:
$$\psi_k = \frac{1}{|V_k|} \int_{V_k} \psi dx$$

at the face:
$$2(\psi)_f = \psi_k + \psi_{k_f} \quad \text{and} \quad 2[\psi]_k^k = \psi_{k_f} - \psi_k$$

parameters:
$$l_k = \frac{|V_k|}{|\partial V_k|} \quad \text{and} \quad \mu_f^k = \frac{|f|}{|\partial V_k|}$$

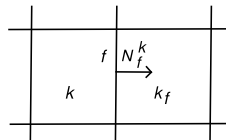


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Step 1:

→ Use an **AUSM** (Advection Upstream Splitting Method) scheme [Liou, Steffen'93]

Splitting of the equation into the advection part and the potential forces

$$\partial_t \begin{pmatrix} h_k \\ h_k U_k \end{pmatrix} + \frac{1}{l_k} \sum_{f \in \mathbb{F}_k} \begin{pmatrix} \mathcal{F}_f^h \\ \mathcal{F}_f^{hu} \end{pmatrix} \cdot N_f^k \mu_f^k = \begin{pmatrix} 0 \\ \mathcal{D}_k \end{pmatrix}$$

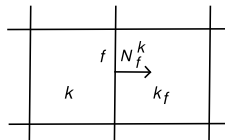
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→ Advect the velocity with an **up-wind** scheme

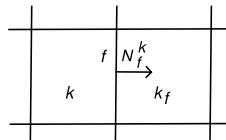
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- Ensure the **dissipation of the discrete kinetic energy**.

Step 2:

→ Use a **centered discretization** of the potential for any $Fr > 0$. **[Dellacherie'10]**

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Step 3:

→ **Regularization** using the potential jump. [**Grenier et al.'13, Parisot and Vila'15**]

$$\mathcal{F}_f^h = (hU)_f - \gamma \Delta t \left(\frac{h}{l}\right)_f \frac{[\phi]_k^{k_f}}{\text{Fr}^2} N_f^k, \quad (\Delta t : \text{time scale}, \quad \gamma : \text{regu. param.})$$

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Step 1:

→ **water level: implicit** scheme of type **non-linear** advection-diffusion

$$h_k^{n+1} - h_k^n + \frac{\Delta t}{l_k} \sum_{f \in \mathbb{F}_k} \underbrace{\left((h^{n+1} U^n)_f \cdot N_f^k - \gamma \Delta t \left(\frac{h^{n+1}}{l} \right)_f \frac{[\phi^{n+1}]_k^{k_f}}{\text{Fr}^2} \right)}_{\mathcal{F}_f^{n+1} \cdot N_f^k} \mu_f^k = 0$$

Solving a **non-linear** system by using the Newton-Raphson method, transforming the nonlinear system into a series of linear systems.

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Step 1:

→ **water level: implicit** scheme of type **non-linear** advection-diffusion

$$h_k^{n+1} - h_k^n + \frac{\Delta t}{l_k} \sum_{f \in \mathbb{F}_k} \underbrace{\left((h^{n+1} U^n)_{f \cdot} N_f^k - \gamma \Delta t \left(\frac{h^{n+1}}{l} \right)_f \frac{[\phi^{n+1}]_k^{k_f}}{\text{Fr}^2} \right)}_{\mathcal{F}_f^{n+1} \cdot N_f^k} \mu_f^k = 0$$

Solving a **non-linear** system by using the Newton-Raphson method, transforming the nonlinear system into a series of linear systems.

Step 2:

→ **velocity: explicit** up-wind scheme with source term.

$$\begin{aligned} h_k^{n+1} U_k^{n+1} - h_k^n U_k^n + \frac{\Delta t}{l_k} \sum_{f \in \mathbb{F}_k} \left(U_k^n \left(\mathcal{F}_f^{n+1} \cdot N_f^k \right)_+ - U_{k_f}^n \left(\mathcal{F}_f^{n+1} \cdot N_f^k \right)_- \right) \mu_f^k \\ = - \frac{\Delta t}{l_k} h_k^{n+1} \sum_{f \in \mathbb{F}_k} \frac{[\phi^{n+1}]_k^{k_f}}{\text{Fr}^2} N_f^k \mu_f^k \end{aligned}$$

Entropy dissipation [Parisot and Vila '16]

Let $\gamma \geq 1$ and assume the following CFL-like condition is satisfied

$$\left(|(h^{n+1} U^n)_f \cdot N_f^k| + (h_k^{n+1})_f \sqrt{\frac{\gamma |\phi^{n+1}|_k^{k_f}}{2 \text{Fr}^2}} \right) \Delta t \leq \frac{\min(l_k, l_{k_f}) \min(h_k^{n+1}, h_{k_f}^{n+1})}{2}$$

then the discrete mechanic energy is **decreasing**.

The efficient high order time integration is more challenging in the context of ImEx schemes.

Several Runge-Kutta schemes can be found in the literature:

- **[Dimarco et al.'17, Boscheri et al.'20]** second order schemes based on several multi-stage or step ImEx formulations.
- **[Dimarco et al.'18, Michel-Dansac, Thomann'21]** higher order TVD ImEx schemes.

Large number of linear systems to solve.

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Step 1:

→ **water level: implicit + extrapolation**

$$h_k^{n+1} - h_k^n + \frac{\Delta t}{I_k} \sum_{f \in \mathbb{F}_k} \underbrace{\left((h^n U^n)_f \cdot N_f^k - \frac{\Delta t}{2} \left(\frac{h^{n+\frac{1}{2}}}{2I} \right)_f \frac{[\phi^{n+\frac{1}{2}}]_k^{k_f}}{Fr^2} - \frac{\Delta t}{2} \left(\nabla(\tilde{h}|\tilde{U} \cdot N_f^k|^2)^{n+\frac{1}{2}} \cdot N_f^k \right)_f - \frac{\Delta t}{2} \left(\nabla^\perp(\tilde{h}uv)^{n+\frac{1}{2}} \cdot N_f^k \right)_f \right)}_{\mathcal{F}_f^{n+\frac{1}{2}} \cdot N_f^k} \mu_f^k = 0$$

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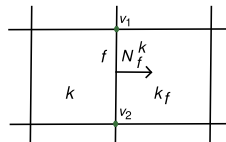
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with $(\nabla \phi \cdot N_f^k)_f = \frac{\phi_{k_f} - \phi_k}{\Delta \cdot N_f^k}$ and $(\nabla^\perp \phi \cdot N_f^k)_f = \frac{\phi_{v_1} - \phi_{v_2}}{|f|}$



$$\psi^{n+1/2} := \frac{\psi^n + \psi^{n+1}}{2}, \quad \tilde{\psi}^{n+1/2} := \frac{\psi^n + \tilde{\psi}^{n+1}}{2} \quad \text{and} \quad \tilde{\psi}^{n+1} := 2\psi^n - \psi^{n-1}.$$

Step 2:

→ **velocity: explicit + extrapolation**, up-wind scheme with source term.

$$\begin{aligned}
 h_k^{n+1} U_k^{n+1} - h_k^n U_k^n + \frac{\Delta t}{l_k} \sum_{f \in \mathbb{F}_k} \left(\widetilde{U}_f^{n+\frac{1}{2},+} \left(\mathcal{F}_f^{n+\frac{1}{2}} \cdot N_f^k \right)_+ - \widetilde{U}_f^{n+\frac{1}{2},-} \left(\mathcal{F}_f^{n+\frac{1}{2}} \cdot N_f^k \right)_- \right) \mu_f^k \\
 = -\frac{\Delta t}{l_k} h_k^{n+\frac{1}{2}} \sum_{f \in \mathbb{F}_k} \frac{[\phi^{n+\frac{1}{2}}]_k^{k_f}}{\text{Fr}^2} N_f^k \mu_f^k
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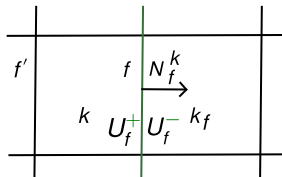
Order 2 in space: **MUSCL** reconstruction of the velocity (with an unlimited slope).

- Edge f with neighboring cells k and k_f

$$\nabla_{ij} U = \left(\frac{U_{i+1,j} - U_{i,j}}{\Delta x}, \frac{U_{i,j+1} - U_{i,j}}{\Delta y} \right), \quad \delta_f U = \frac{U_{k_f} - U_k}{\text{dist}(k, k_f)}$$

$$U_f^+ = U_k + \text{dist}(k, f) \times \frac{1}{2} \left(\nabla_k U \cdot N_f^k + \delta_{f'} U \right)$$

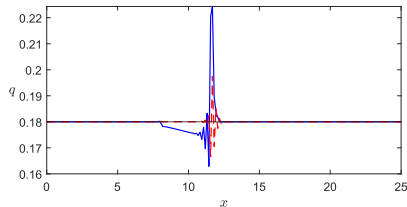
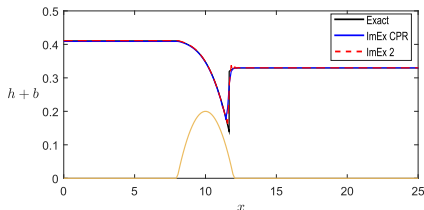
$$U_f^- = U_{k_f} - \text{dist}(f, k_f) \times \frac{1}{2} \left(\nabla_{k_f} U \cdot N_f^k + \delta_f U \right)$$



Transcritical flow

Initial data: $h(x, 0) + z(x) = 0.33$, $h(x, 0)u(x, 0) = 0$.

Boundary conditions: $h(0, t)u(0, t) = 0.18$, $h(25, t) = 0.33$.

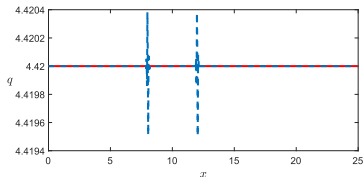
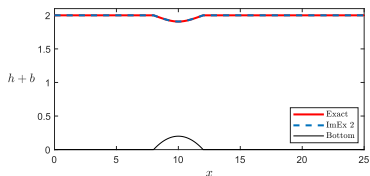


- Good agreement with exact solution. ImEx 2 provides a better solution compared to ImEx CPR. This is clear in the flow discharge figure.

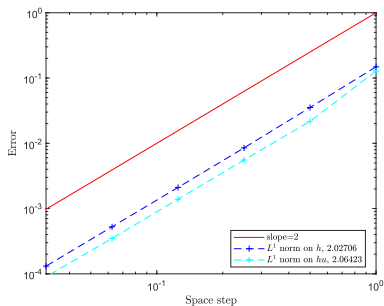
Subcritical flow over a bump

Initial data: $h(x, 0) + z(x) = 2$, $h(x, 0)u(x, 0) = 0$.

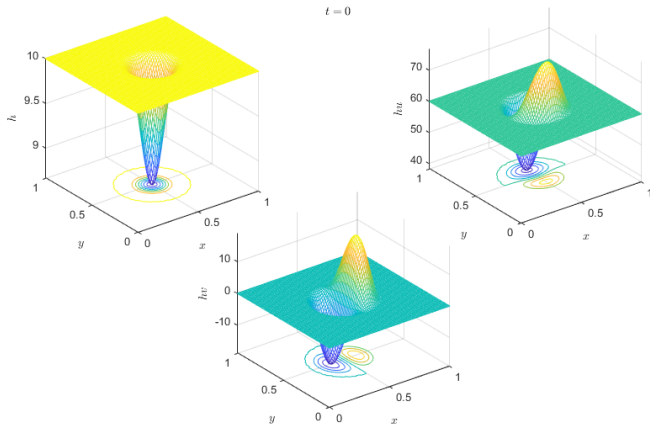
Boundary conditions: $h(0, t)u(0, t) = 4.42$, $h(25, t) = 2$.



a) Subcritical flow

b) L^1 errors for h and hu

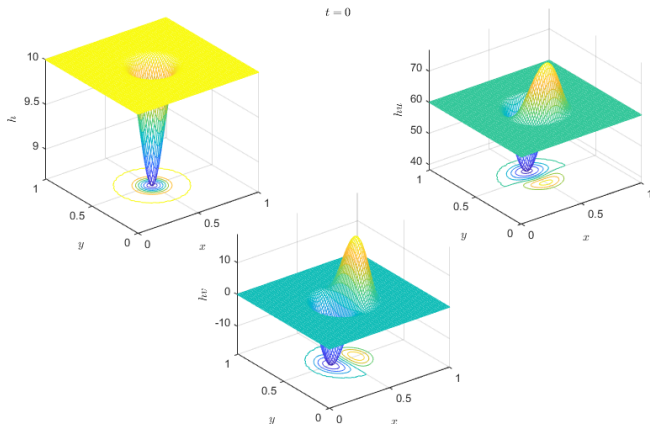
Traveling vortex



Initial condition for the traveling vortex with $Fr = 1$, computed on the 80×80 grid.

- Exact solution is available. [Ricchiuto, Bollermann'09]

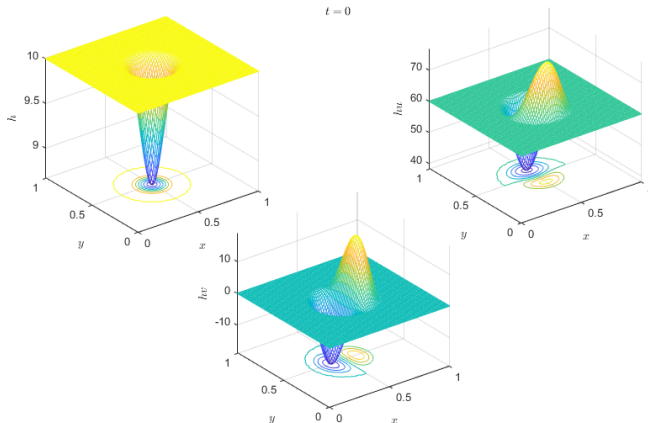
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- $\Omega = [0, 1] \times [0, 1]$. Periodic BC in x , absorbing BC in y .
- A rotating vortex initially located at the domain center $(0.5, 0.5)$ is transported repeatedly from left to right with a period $T = 1/6$ by an x -directional uniform flow with the advection velocity $U_{ref} = (6, 0)$.

Order of accuracy

Table: Traveling vortex test: EOC for the ImEx 2 scheme, CFL = 2, $T = 1/6$ **Fr = 1**

N	L^1 -error in h	EOC	L^1 -error in hu	EOC	L^1 -error in hv	EOC
40	1.262e-02		2.497e-01		4.027e-01	
80	3.0520e-03	2.0474	7.070e-02	1.8206	1.259e-01	1.6771
160	7.002e-04	2.1239	1.827e-02	1.9515	3.507e-02	1.8442
320	1.677e-04	2.0614	4.830e-03	1.9201	9.491e-03	1.8857

Fr = 0.1

N	L^1 -error in h	EOC	L^1 -error in hu	EOC	L^1 -error in hv	EOC
40	1.106e-04		2.614e-01		4.276e-01	
80	2.988e-05	1.8889	7.251e-02	1.8501	1.324e-01	1.6906
160	7.283e-06	2.0367	1.882e-02	1.9457	3.658e-02	1.8565
320	1.808e-06	2.0098	4.964e-03	1.9227	9.859e-03	1.8918

Order of accuracy:

Table: Traveling vortex test: EOC for the ImEx 2 scheme, CFL = 2, $T = 1/6$ **Fr = 0.01**

N	L^1 -error in h	EOC	L^1 -error in hu	EOC	L^1 -error in hv	EOC
40	1.445e-06		2.617e-01		4.274e-01	
80	3.311e-07	2.1258	7.255e-02	1.8509	1.324e-01	1.6901
160	7.295e-08	2.1825	1.884e-02	1.9450	3.659e-02	1.8560
320	1.756e-08	2.0547	4.971e-03	1.9221	9.864e-03	1.8912

Table: Traveling vortex test: EOC for the ImEx 2 scheme, CFL = 0.5, $T = 1/6$ **Fr = 0.001**

N	L^1 -error in h	EOC	L^1 -error in hu	EOC	L^1 -error in hv	EOC
40	1.282e-08		2.564e-01		4.182e-01	
80	2.850e-09	2.1697	7.029e-02	1.8670	1.283e-01	1.7047
160	7.656e-10	1.8962	1.817e-02	1.9514	3.527e-02	1.8629
320	1.701e-10	2.1698	4.784e-03	1.9255	9.490e-03	1.8941

Grid	Fr = 1		Fr = 0.1		Fr = 0.01		Fr = 0.001	
	ImEx2	Explicit	ImEx2	Explicit	ImEx2	Explicit	ImEx2	Explicit
40 × 40	8.65	0.62	5.02	2.29	4.83	13.89	5.14	138.44
80 × 80	38.36	7.54	39.67	31.79	41.59	278.30	41.89	2,052.39
200 × 200	788.57	41.30	714.20	220.91	715.94	2,133.62	746.72	16,718.47

Table: CPU times in seconds consumed by the ImEx 2 and HLL-MUSCL-RK2 explicit schemes on different grids for different values of Fr. The final time is $T = 1/6$.

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Thank you for your attention!