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Two Choice Behavioral Game Dynamics with Myopic-Rational and Herding Players*

Khushboo Agarwal^{#†}, Konstantin Avrachenkov[#], Raghupati Vyas[§], Veeraruna Kavitha[§]

[#]Inria Sophia Antipolis, 2004 Route des Lucioles, Valbonne 06902, France,

[§]Indian Institute of Technology Bombay, Powai, Mumbai 40076, India

{khushboo.agarwal, k.avrachenkov}@inria.fr, {raghupati.vyas, vkavitha}@iitb.ac.in

Abstract

In classical game theory, the players are assumed to be rational and intelligent, which is often contradictory to the reality. We consider more realistic behavioral game dynamics where the players choose actions in a turn-by-turn manner and exhibit two prominent behavioral traits — α -fraction of them are myopic who strategically choose optimal actions against the empirical distribution of the previous plays, while the rest exhibit herding behavior by choosing the most popular action till then. The utilities are realised for all, at the end of the game, and each player gets to play only once. Our analysis focuses on scenarios when players encounter two possible choices, common in applications like participation games (e.g., crowd-sourcing) or minority games.

To begin with, we derive the almost sure mean-field limits of such dynamics. The proof is constructive and progressively narrows down the potential limit set and finally establishes the existence of a unique limit for almost all sample paths. We argue that the dynamics at the limit is captured by a differential inclusion (and not the usual ordinary differential equation) due to the discontinuities arising from the switching behavioral choices. It is noteworthy that our methodology can be easily modified to analyse the avoid-the-crowd behavior, in place of herding behavior.

We conclude with two interesting examples, named participation game and routing game, which encapsulates several real-life scenarios. Interestingly, for the first game, we observe that the game designer can induce a higher level of participation in an activity with smaller reward, by leveraging upon the presence of herding players.

Keywords: behavioral game theory, irrationality, herding, game dynamics, population games.

1 Introduction

Game theory is the science of strategic interactions, where players are assumed to be intelligent and rational [1]. Each player has a set of choices/actions, and it chooses the optimal action based on some utility function; the utility of a player depends on the action it chooses and the actions chosen by the others.

An important branch of game theory focuses on population games involving large population sizes, where a single player has insignificant influence on the outcome of the game. Thus, the impact of others' actions is not considered individually, but is accounted via the empirical distribution of players choosing

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[†]Corresponding author: Khushboo Agarwal (khushboo.agarwal@inria.fr).

different actions [2]. The game progresses in discrete/continuous time, where players sometimes get to update their actions in view of some myopic (short-sighted) rules known as revision protocols (see e.g., [2]). At the time of updation, the players can imitate the action of the opponent(s) with some probability (leading to imitation dynamics like replicator dynamics [3]), or choose the action which yields better utility in comparison to the average utility of the population (excess payoff dynamics [2]), or choose the optimal action in response to the current state of the system (best response dynamics [2]), etc. *An important feature of all such dynamics is that players get multiple chances to learn better actions or improve their choices. We deviate precisely at this point.*

In this paper, we consider a special kind of dynamics for population games, namely turn-by-turn dynamics, where the players choose their actions one after the other and only once. Such dynamics are less studied (see [4] for one such example), but are prevalent in many applications. For instance, the individuals may decide to participate in crowd-funding, or invest in some shares/cryptocurrency, or choose a route to reach a destination, etc., one after the other. In such scenarios, the utility of the players may also depend upon the decisions of the future players. Note that even though population game dynamics study many realistic variants of myopic rational players (as explained above), in reality, players with different behavioral traits exist and interact with each other. We aim to enhance the study of the population games (in this direction), *by considering two distinct types of irrational players.*

For some $\alpha \in (0, 1]$, we consider that α -fraction of the population is composed of myopic rational players (briefly referred to as myopic) who choose their optimal actions only against the empirical distribution of the previous plays, without accounting for/anticipating the choices of the future players. Observe that the myopic players in turn-by-turn dynamics, behave like the myopic players in the best response dynamics (or replicator dynamics etc.) in terms of making their choices based on the current status. However, they do not learn like in the latter dynamics, by repeatedly modifying/improving their actions; they instead make their choices only once. The rest of the population ($(1 - \alpha)$ -fraction) consists of herding players whose choices are swayed by the prevailing trends. That is, a herding player chooses an action, which has been chosen by the majority of the players before its turn. Clearly, the latter behavior is common, as we often observe individuals choosing a hyped restaurant, buying a rumoured (or well-discussed) product, or participating in a viral trend, etc. (see, e.g., [5, 6]).

We assume that both the fraction α and the utility function are common knowledge among the players. Further, it is assumed that the utilities are realised for all the players at the end of the game.

Our interest lies in analysing the above-defined behavioral game dynamics when the population size grows to infinity. The study focuses on scenarios where players have two actions to choose from, like going to restaurant x or y, buying one of the two trending products, choosing one of the two suitable career options, participating or not in a campaign, etc.

Now, before moving ahead, it is important to discuss another related strand of literature on social-learning (see, e.g., [7, 8]), and highlight the key distinctions between our model and the said literature. First, note that all players are myopic rational in the latter setup, while we also consider herding players. Second, the utility function is not explicitly known to the players in the latter setup as it depends on the true, but unobservable, state of the system, in addition to being dependent on the action chosen by the player; in contrast, we consider that the utility is known to all (as said before). Importantly, due to the unobservable state, the players attempt to learn the true state in the latter by taking an optimal action by utilizing their private signals and the observations of all/some of the previous players' actions and/or private signals. While in our case, players simply take an action with no intent of learning based only on the empirical distribution of the choices made by the previous players. Now, observe that due to the lack of information in the latter, irrationality¹ is inherently imposed on the players, while irrationality (myopic-rationality and herding) is a behavioral trait for the players of our interest.

Thus, to the best of our knowledge, there is no study considering myopic players like in this paper, and the interactions among myopic and herding players.

¹Note that for us, the term 'herding' refers to a behavioral trait of the players, and it should not be confused with the same term used in social learning based literature to refer to the phenomenon where, beyond a time point, all future players rationally choose the same action.

In our analysis, the almost sure limits (referred to as α -rational attractors) of the behavioral choices of the entire population and that of the individual population types (myopic and herding) are derived. We investigate the effect of the presence of herding players on the eventual outcome of the dynamics. Interestingly, the set of limits does not change with the fraction α , when the herding players are in the minority (i.e., when $\alpha \geq 1/2$). However, the limits change once the fraction of herding players is larger (i.e., when $\alpha < 1/2$) — new limits like α or $(1 - \alpha)$ can arise as the fraction of players choosing action 1 (say) at the limit. Likewise, the limits that exist when $\alpha \geq 1/2$, no longer appear when herding players dominate the population.

For the analysis, we begin by describing the proportion of players choosing action 1, as a stochastic approximation (SA) scheme. In an attempt to apply the ordinary differential equation (ODE)-based SA analysis [9, 10], we derive an appropriate ODE for the scheme. However, the right-hand side of the resulting ODE is discontinuous due to the switching behavioral choices² of the players; as a result, one can not apply the regular SA-based results. In fact, the solution for the resulting ODE can only be defined in the Filippov sense by constructing an appropriate differential inclusion (DI) [11]. Thus, we study the limiting behavior of the dynamics using rigorous probabilistic arguments, that progressively narrow down the potential set of the limits; we finally establish the existence of a unique limit in almost all the sample paths. Further, we establish that the set of α -rational attractors equals the set of asymptotically stable points of the DI corresponding to the SA scheme. We also provide a simpler characterisation of the set of limits using the zeros and other properties of a ‘utility difference’ function. Additionally, comparisons across various levels and types of rationality are provided. All of the said results can be easily reproduced using our methodology for yet another commonly seen behavior, namely avoid-the-crowd.

Toward the end, we study a participation game where players decide to participate or not in a turn-by-turn manner, and the game designer attempts to motivate the players to participate through a reward-based scheme. Our analysis shows that the game designer can increase participation with lesser reward by leveraging upon the irrationality of the herding players. Further, for a routing game, we numerically demonstrate the convergence of the switching dynamics around a particular α -rational attractor arising due to the inherent discontinuities in the system.

2 Turn-by-turn dynamics among myopic rational and herding players

In this section, we study the ‘interactions’ among ‘irrational players with different degrees of irrationality’. In particular, we consider a system with n players, where the players sequentially make decisions (i.e., one after the other) and only once. Thus, we call such dynamics as *turn-by-turn dynamics*. Our interest lies in exploring the limiting analysis of such dynamics as the population size (n) grows to ∞ .

Each player has to choose an action between two possible choices from the set $\mathcal{A} := \{1, 2\}$. The player receives a certain utility based on the action chosen. In particular, let $u : \mathcal{A} \times [0, 1] \rightarrow \mathbb{R}$ be the utility function, which depends on the action a and the fraction of players who chose action 1 (denoted by z). Due to the dependency on the latter (i.e., z), the resultant interaction leads to a game. Basically, we assume that the utility function is the same for all the players and that it depends on the empirical distribution $(z, 1 - z)$ of the actions chosen by the population as in mean-field games (for example [12]). The utility function is a common knowledge to the players. Let z_k represent the proportion of players choosing action 1, among the first k -players. We assume the status of the system z_k is known to the $(k + 1)$ -th player, making the choice. The players receive utility once all the players have chosen their actions and it depends upon the *eventual outcome of the game measured (approximately) by $z_\infty := \lim_{k \rightarrow \infty} z_k$ (if the limit exists)* — thus, a player choosing action a receives $u(a, z_\infty)$ at the end.

²In our study, the switching behavioral choices of the herding players occur because of the change in the maximally chosen action, which is a resultant of the choices of the preceding (myopic) rational players. In contrast to this, the switching in social learning arise from the changes in the beliefs about predecessors’ actions.

The game begins with the first player randomly choosing an action from the action set, \mathcal{A} . Let the player who chooses its action at the k -th round/turn be referred to as the k -th player for $k \geq 1$. Contrary to the classical game theory, we consider two kinds of players, none of which are perfectly rational: (i) myopic rational players (referred to as myopic players, in short) and (ii) herding players. The former kind comprises the α -fraction of the population, for some $\alpha \in (0, 1]$, which is assumed to be common knowledge among the players. Next, we describe the players and explain how they make their action choices.

Myopic players:

The notion of myopic players is well known in the population games-based literature (e.g., see [2]). These players are not far-sighted and, instead, choose their actions based only on the current state of the system. Let R_k be the flag indicating that k -th player is myopic, for any $k \geq 1$. Further, let F_k and G_k be the indicators that the k -th player chooses action 1 if it is myopic or herding player, respectively. Then, the state z_k of the system at the end of the k -th round is given by:

$$z_k = \frac{\sum_{i=1}^k (R_i F_i + (1 - R_i) G_i)}{k}. \quad (1)$$

If $(k + 1)$ -th player is myopic, then based on z_k , it computes the utility $u(a, z_k)$ for each $a \in \mathcal{A}$ and chooses the action $i \in \mathcal{A}$ which maximizes $u(\cdot, z_k)$, i.e.,

$$\begin{aligned} \mathbb{P}(F_{k+1} = 1) &= 1_{\{1=a^R(z_k)\}}, \text{ where,} \\ a^R(z) &:= \min \{i : i \in \text{Arg max}_{a \in \mathcal{A}} u(a, z)\}. \end{aligned} \quad (2)$$

In the above, we assume that action 1 is chosen in case of a tie (i.e., if $u(1, z_k) = u(2, z_k)$), as is usually considered in game theory literature.

Herding players:

These players choose the most popular action till then. Such a choice is made with no acknowledgment of the utility function, and thus, herding behavior should not be confused with free-riding behavior; the latter is the result of the rational choice to leverage upon the actions chosen by the others. Thus, if the $(k + 1)$ -th player is a herding player, it chooses action $a = 1$ if at least 50% of the previous (k) players chose action 1:

$$\mathbb{P}(G_{k+1} = 1 | R_{k+1} = 0) = 1_{\{z_k \geq \frac{1}{2}\}}. \quad (3)$$

The above discussion completes the description of the dynamics of interest.

Remarks: (i) It is evident from (3) that the herding players are different from the perfectly rational players considered in the classical literature. It is equally important to understand how even the myopic players considered above share no similarity with the perfectly rational players. To see this, note that a perfectly rational player follows a rigorous procedure to choose its action: (a) it observes the status of the system (e.g., z_k), (b) it anticipates the actions of all future players to evaluate z_∞ , (c) it calculates the anticipated utility $u(\cdot, z_\infty)$, and then finally (d) it chooses the optimal action with respect to the anticipated utility. On the contrary, the myopic players do not perform such high-level computations. Instead, they simply take the optimal action ($a^R(z_k)$) by evaluating $u(\cdot, z_k)$, see (2).

(ii) The system with two choices may appear restrictive, however, it is still an important case study, as demonstrated by the two examples in the immediate next. Further, due to the inherent discontinuities (see the ODE of Section 3), the analysis for this case study itself is significantly complex. The analysis with a general action set would be further complicated due to two or higher-dimensional systems with discontinuities and can be a good problem for future research.

2.1 Motivating examples

2.1.1 Participation game

To participate or not in an activity is a question that one often encounters, and many a times, the answer is influenced by others' choices. Thus, a variety of practical scenarios, like crowd-sourcing (for example, [13, 14, 15, 16]), choosing a restaurant [17], participating in a campaign, etc., can be modelled in terms of, what we call as, participation game with two possible choices of participation or non-participation. Importantly, in such scenarios, only some individuals make strategic choices, while herding behavior is prominently exhibited by the individuals. Further, it is unlikely that two individuals will make their choices simultaneously. Therefore, the turn-by-turn dynamics described in (1)-(3) are important to study such games, as we will see next.

Consider an entity (for example, game designer, controller, or planner) that seeks input from the players to achieve a specific goal. For example, researchers conduct surveys to collect the data for their research. The players can participate (denoted by $a = 1$) or not ($a = 2$) in the activity proposed by the game designer. Assume each player chooses their action in a turn-by-turn manner and only once (for example, to fill out a survey). Further, suppose that each non-participant perceives a utility equal to 1 (for example, if the player prefers to relax instead of filling out the survey). On the other hand, the participant perceives a utility $C > 0$, due to the efforts in contributing to the activity (for example, investing time while filling out the survey, and feeling satisfied to have participated in the survey). In many scenarios, the players are not self-motivated to participate, which can be captured by a $C < 1$. We consider such scenarios, and assume that the platform implements the following two strategies:

- (i) it publicly displays/advertises the fraction of participants (denoted by z), expecting the herding behavior to show the ripple effect, and,
- (ii) it introduces a reward-based scheme, as explained below.

The platform announces that a reward $R > 0$ will be equally distributed among the participants before the start of the game. The reward is distributed at the end of the game, based on the eventual participation and the 'success' of the participation game; we now explain the 'success' of the game. The game designer intends to collect the inputs in pursuit of a goal, and the chances of achieving this goal improve with higher participation. Let $f(z)$ represent the probability of success of the activity of interest, where $f : [0, 1] \rightarrow [0, 1]$ is an appropriate continuous increasing function such that $f(0) = 0$ and $f(1) = 1$. Then, for a given participation level z , each participant gets an (extra) average reward $Rf(z)/z$. Further, suppose that all the players involved in the game are affected by the outcome of the game, and thus, all players incur an average cost $P(1 - f(z))$, where $P > 0$. Thus, given z , the utility function for a player is given by:

$$u(a, z) = \begin{cases} C + \frac{R}{z}f(z) - P(1 - f(z)), & \text{if } a = 1, \\ 1 - P(1 - f(z)), & \text{if } a = 2. \end{cases} \quad (4)$$

It is now easy to see that the above utility function induces a participation mean-field game among the players. It is unrealistic to assume that the players decide rationally. Some players (say α -fraction) may observe the (current) participation level (z_k , at the k -th round) and make short-sighted choices (like myopic players, see (2)), while others may exhibit herding behavior (as in (3)). Thus, it would be realistic to study such a game in the presence of myopic and herding players, using our analysis for the turn-by-turn dynamics (1)-(3).

2.1.2 Routing game

Everyone faces the dilemma of choosing a route, or mode of transport, or kind of facility (say, luxury or regular) to be used, to move from a source (S) to a destination (D). Now, some players choose strategically by carefully weighing the options, while many others simply follow others. To see the

latter behavior, consider for instance the scenario where a route has been chosen by many individuals, then it is not uncommon to think that such a route may be safer or well-constructed (and thus, more suitable to be opted). In such a case, players may herd towards such a route. Or, if luxury option is opted often, then players may perceive it to have greater value (say in terms of experience) and thus, be attracted towards this option. Further, as in participation game, it is unlikely that two individuals decide at the same time. Thus, the routing decisions can be formulated as turn-by-turn dynamics (1)-(3), see details below.

Consider that there are two routes between S and D, denoted by $a = 1$ and $a = 2$. Assume that individuals (players) decide their route one after the other, and no player chooses more than once. Further, assume that each player has an estimate of the occupancy of the two routes, i.e., the fraction of players who chose $a = 1$, denoted by z , is assumed to be (approximately) known to all. Now, suppose that it takes γz hours to travel via route 1 (denoted by $a = 1$), for some $\gamma \in (1, \infty)$, and 1 hour to travel via route 2 (say, $a = 2$). Thus, the utility function for a player is given by the following, for a given value of z :

$$u(a, z) = \begin{cases} -\gamma z, & \text{if } a = 1, \\ -1, & \text{if } a = 2. \end{cases} \quad (5)$$

Due to the dependency of $u(\cdot, z)$ on z , the resultant interaction among players is a mean-field game. As discussed above, some players (α -fraction) make myopic choices as per (2) based only on the history (captured via z_k), while the rest ($1 - \alpha$)-fraction imitates majority, as per (3). We shall see in section 4.2 how one can study the above game using the coming analysis of turn-by-turn-dynamics.

We next present the main results of this paper.

2.2 Main results

Our first main objective is to derive the limiting analysis for the two-choice behavioral game dynamics (1)-(3) among a large population of myopic and herding players. Towards this, *we will derive the almost sure limits of the (z_k) -sequence in Theorem 1* and provide a simpler representation of the same in Theorem 4; both of them are provided in section 3. The next objective is to analyse the effect of the presence of herding players on the outcome of the dynamics, using Theorem 4. In particular, we will draw comparison between the three scenarios, with $\alpha < 1$ (non-zero fraction of herding players), $\alpha = 1$ (all players are myopic rational), and the one with all perfectly rational players. Along the way, we would also study the participation and routing games in detail, and investigate the influence of the fraction of herding players, $(1 - \alpha)$, on the outcome of the game.

At this point, we would like to draw the attention of the reader to an interesting aspect of the limits of (1). Observe from (1) that the iterates for z_k can be re-written as below, using the indicators defined at the beginning of this section:

$$z_{k+1} = z_k + \frac{1}{k+1} g(R_{k+1}, F_{k+1}, G_{k+1}, z_k), \text{ where } g(R, F, G, z) := RF + (1 - R)G - z. \quad (6)$$

The above iterative equation resembles a stochastic approximation (SA) scheme (e.g., [9, 10]), and typically such iterates converge to the attractors of the ODE, $\dot{z} = M(z)$, derived using the conditional expectation of function $g(\cdot)$ as given below:

$$M(z) = E[g(R_{k+1}, F_{k+1}, G_{k+1}, z_k) | z_k = z] = \alpha 1_{\{u(1,z) \geq u(2,z)\}} + (1 - \alpha) 1_{\{z \geq 1/2\}} - z, \text{ almost surely.}$$

Clearly, the conditional expectation function, $M(\cdot)$, is not continuous. The solution of ODEs with this kind of discontinuities can be defined using the solution of the differential inclusion (DI) (see e.g., [11]). For this reason, the proof of Theorem 1 can not be constructed using the regular ODE approximation results (e.g., as in [10, 9]), and we instead obtain the same using some rigorous probabilistic arguments. *Interestingly, we will establish in Theorem 8 that the limits of the (z_k) -sequence exactly match with the attractors or asymptotically stable points of an appropriate DI.*

3 Analysis

In this section, we analyse the behavioral dynamics (1)-(3) involving myopic and herding players. The game involves a large number of players and therefore, the interest will be on deriving and characterizing the limit(s) of the (z_k) -process, as $k \rightarrow \infty$, i.e., to analyse $\lim_{k \rightarrow \infty} z_k$, as in mean-field game framework. In particular, we would like to derive almost sure limits.

Towards this, we assume the following structure for the function $h(z) := u(1, z) - u(2, z)$, which dictates the actions chosen by the myopic players (see Figure 1):

- (A) Define the function $h(z) := u(1, z) - u(2, z)$ for $z \in [0, 1]$, and assume it to be continuous on $[0, 1]$. Let the set of zeroes of h defined as $\Phi := \{z : h(z) = 0\}$ be finite (i.e., $|\Phi| < \infty$) and such that $\Phi = \{z : \text{sign}(h(z + \epsilon)) \neq \text{sign}(h(z - \epsilon))\}$, for some $\epsilon > 0$.

The continuity is a typical assumption used in the literature (e.g., see [9]); in fact, the majority of the literature works with Lipschitz continuous functions. By the last assumption on the set of zeros Φ , we require that the function h should have opposite signs on either side of its zeros. This might look restrictive, however, we believe that one can easily extend the proof to other configurations. This particular choice is mainly considered for the ease of explanation.

3.1 Asymptotic analysis of the population

Recall that the proportion z_k is simply the fraction of players who choose action 1 and its updates can be re-written in the form of a SA scheme as shown in (6). It is well known in SA-based literature ([9, 10]) that the SA-scheme (and thus, z_k -iterates) typically converges to some specific ‘zeroes’ (say $z_\infty \in [0, 1]$) of the conditional expectation of the function $g(\cdot)$, see (6):

$$E[g(R_{k+1}, F_{k+1}, G_{k+1}, z_k) | \mathcal{H}_k] = M(z_k), \quad (7)$$

where \mathcal{H}_k is the natural sigma-algebra with the information for all $i \leq k$ rounds, and the function M is given by (recall the definition of h from the assumption (A), and see (2), (3), (6)):

$$M(z) := \alpha 1_{\{u(1,z) \geq u(2,z)\}} + (1 - \alpha) 1_{\{z \geq \frac{1}{2}\}} - z = \alpha 1_{\{h(z) \geq 0\}} + (1 - \alpha) 1_{\{z \geq \frac{1}{2}\}} - z. \quad (8)$$

Notice that the conditional expectation $M(\cdot)$ is discontinuous. Nonetheless, one can easily identify the potential zeroes/attractors of the ODE among $\{0, 1, \alpha, (1 - \alpha)\}$, by inspection³ — observe here that the indicators, representing the behavioral choices made by the players, are piece-wise constant functions.

Additional ‘zeros’ or ‘attractors’: Due to the discontinuities in M arising from these behavioral choices, some other ‘zeroes’ (and thus, limit points) are also possible; these are not zeros of M in a strict sense, but behave like one. To explain the same, we consider an example in the immediate next. For $\alpha > 1/2$, consider a $z_\infty \in (0, 1/2)$, which further satisfies the following for some $\epsilon > 0$ (choose $\epsilon > 0$ such that $z_\infty + \epsilon < 1/2$):

$$h(z) = u(1, z) - u(2, z) \begin{cases} > 0, & \text{for all } z \in N_\epsilon^-(z_\infty) := (z_\infty - \epsilon, z_\infty) \\ = 0, & \text{if } z = z_\infty, \\ < 0, & \text{for all } z \in N_\epsilon^+(z_\infty) := (z_\infty, z_\infty + \epsilon). \end{cases} \quad (9)$$

Now consider that for some k_0 , we have $z_{k_0} \in N_\epsilon^+(z_\infty)$. Then $h(z_{k_0}) < 0$ as well as $z_{k_0} < 1/2$, implying the (future) myopic as well as the herding players choose action 2, see (1)-(3). This in turn implies z_{k_0+l} decreases with l and will potentially cross z_∞ at some instance — say $k_1 (> k_0)$ is the instance at which $z_{k_1} \in N_\epsilon^-(z_\infty)$. After k_1 , the myopic players choose action 1 while the herding players continue to choose action 2. From (8), the dynamics now have an average drift, $M(z) = \alpha - z$, and this is strictly

³For example, observe α is a zero if $\alpha < 1/2$ and if $h(\alpha) = u(1, z = \alpha) - u(2, z = \alpha) > 0$, see also Theorem 4. Similarly 0 is an attractor if $h(0) < 0$, etc.

positive because of the choice of ϵ . Then there is a possibility that the dynamics surpasses z_∞ again to enter $N_\epsilon^+(z_\infty)$. Once (and if) this happens, both the types of players choose action 2, and then the dynamics certainly enters $N_\epsilon^-(z_\infty)$ at a latter point (some $k_2 > k_1$). This can continue forever, with the dynamics crossing z_∞ continuously. It is not difficult to guess that the overshoots/undershoots around z_∞ decrease, as time (k) progresses because $1/k$ decreases to zero (see (6)). In view of these observations, one can anticipate that any z_∞ satisfying,

$$u(1, z_\infty) = u(2, z_\infty) \text{ and } z_\infty = \alpha y_\infty + (1 - \alpha)1_{\{z_\infty \geq 1/2\}}, \quad (10)$$

can form a potential attractor of the discontinuous dynamics (1)-(3) with the help of an additional variable y_∞ — here y_∞ is an appropriate value that represents the time-asymptotic average of the myopic players choosing action 1, by virtue of which the time-asymptotic fraction of players choosing action 1, irrespective of their type (i.e., myopic or herding), becomes z_∞ . Note here that by comparing with $M(z_\infty)$ given in (8), the value y_∞ should have been representative of the conditional expectation, $E[F_\infty|z_\infty]$, as in the typical ODE-approximation based results. However, this is not the case due to the discontinuities, and we have two important repercussions:

- (i) to study the limits of one-dimensional dynamics (1)-(3), one requires the analysis of an additional quantity: the fraction of the myopic players choosing action 1, among the first k -players,

$$y_k = \frac{\sum_{i=1}^k R_i F_i}{\sum_{i=1}^k R_i}. \quad (11)$$

- (ii) The limits of the dynamics need not just be the zeros of $M(\cdot)$, they would also include points like z_∞ of the above example. Using similar logic, we conjecture the limits to satisfy the following equation:

$$\text{support}(\mu_\infty^R) \subseteq \text{Arg max}_{a \in \mathcal{A}} u(a, z_\infty), \text{ and} \quad (12)$$

$$z_\infty = \alpha y_\infty + (1 - \alpha)1_{\{z_\infty \geq 1/2\}}, \quad (13)$$

where $\mu_\infty^R := (y_\infty, 1 - y_\infty)$ and $\text{support}(\mu_\infty^R) := \{a \in \mathcal{A} : \mu_\infty^R(a) > 0\}$.

Observe here that for any z_∞ satisfying (10), the Arg max in (12) equals $\{1, 2\} = \mathcal{A}$ and hence the above equation is trivially satisfied using an appropriate y_∞ . The conjecture in (ii) is indeed true and we prove the same in Theorem 3. Prior to that we make some important definitions.

α -rational ‘zeros’ and ‘attractors’: Taking motivation from the above discussion, we define α -rational zeroes, each of which will form the candidates for the limits of the (z_k) -sequence:

Definition 1. A point $z_\infty \in [0, 1]$ is called an α -rational zero if for some $\epsilon > 0$, $\text{sign}\{(z - z_\infty)M(z)\}$ is either positive or negative for all $z \in N_\epsilon(z_\infty) - \{z_\infty\}$, where $N_\epsilon(z_\infty)$ is the ϵ -neighborhood⁴ of z_∞ , i.e., $N_\epsilon(z_\infty) = (z_\infty - \epsilon, z_\infty + \epsilon)$.

Let \mathcal{Z} be the set of α -rational zeroes. Now, recall from the example in (9) that if the drift around $M(z_\infty)$ is appropriate ($M(z) = \alpha - z > 0$ for z in some left-neighborhood, and $M(z) = -z < 0$ for z is some right-neighborhood), then such z_∞ is a potential limit point. This observation is indeed true, as we will prove below in our first main result that the dynamics converge to some of the α -rational zeroes, referred to as α -rational attractors which are defined below:

Definition 2. An α -rational zero $z_\infty \in [0, 1]$ is called an α -rational attractor if for some $\epsilon > 0$, $\text{sign}\{(z - z_\infty)M(z)\}$ is negative for all $z \in N_\epsilon(z_\infty) - \{z_\infty\}$.

⁴For $z_\infty = 0$, we only need the right neighborhood; thus, with slight abuse of notation, we consider $N_\epsilon(0) = (0, \epsilon)$. Similarly, $N_\epsilon(1) = (1 - \epsilon, 1)$.

Differential inclusion (DI): Before proceeding ahead, we would like to discuss an important observation about α -rational zeroes and attractors defined above. Recall that the iterates z_k can be re-written in a SA-based scheme (see (6)). A common approach to derive the asymptotic analysis of SA schemes is to first derive an appropriate ODE using conditional expectation ($M(\cdot)$ in our case), and then to show that the stable time-asymptotic limits of the ODE are the limits of the iterative scheme, with a certain probability (see, for example, [9, Chapter 2]). Due to the discontinuities in $M(\cdot)$, the analysis via ODE is not possible in certain cases, and thus, one can construct an appropriate DI to derive the analysis (a detailed discussion and construction of the DI is provided in Appendix A). Interestingly, we show that the set of α -rational zeroes is a subset of the stationary points of the DI, and further, the set of α -rational attractors exactly matches with the asymptotically stable points of the DI (see Theorems 7, 8 in Appendix A).

Convergence of dynamics: We now prove the convergence of the dynamics in almost sure sense.

Theorem 1. *Consider the behavioral dynamics described in (1)-(3). Under the assumption (A), $z_k \rightarrow \mathcal{A}_f$ as $k \rightarrow \infty$, almost surely, where \mathcal{A}_f is the set of α -rational attractors.*

Proof. The proof of this theorem consists of three major steps, which are given by Lemmas 1-3 stated in the coming. At first, we provide the required mathematical framework and then proceed towards the necessary arguments required to complete the proof.

Let $\Phi^D := \Phi \cap [0, 1/2)$ and $\Phi^U := \Phi \cap (1/2, 1]$. Thus, $\Phi - \{1/2\} = \Phi^U \cup \Phi^D$. Also, assume Φ^D and Φ^U are two ordered sets, and let z_i^D, z_i^U be typical elements of these sets respectively. In particular, let $z_i^D > z_{i+1}^D$ for any i , with $1 \leq i < d$ and $z_i^U < z_{i+1}^U$ for any i , with $1 \leq i < u$, where $u := |\Phi^U|$ and $d := |\Phi^D|$.

Now, observe that there are three possibilities for h : either $h(1/2) > 0$ or < 0 or $= 0$. Thus, due to the continuity of h assumed in (A), either $h(z) > 0$ or $h(z) < 0$ for all $z \in (z_1^D, z_1^U)$ or $\text{sign}(h(1/2 - \epsilon)) \neq \text{sign}(h(1/2 + \epsilon))$ for some $\epsilon > 0$, respectively. We will prove the result for the former case (which is represented in Figure 1), as it can be proved analogously for the other cases.

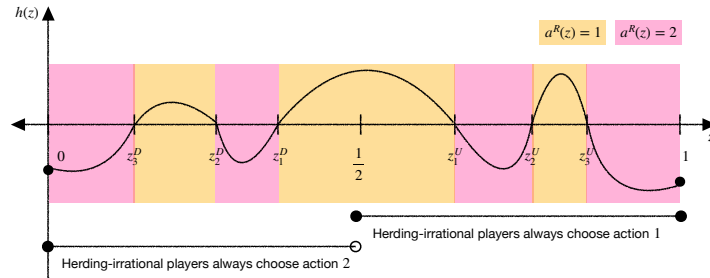


Figure 1: Depiction of function h , when $h(z) > 0$ for all $z \in (z_1^D, z_1^U)$

As the first step, let us partition the interval $(\frac{1}{2}, 1]$ into sub-intervals as follows:

$$R_1^U := \left(\frac{1}{2}, z_2^U\right), R_2^U := [z_2^U, z_4^U), \dots, R_p^U := \begin{cases} [z_u^U, 1], & \text{if } u \text{ is even,} \\ [z_{u-1}^U, 1], & \text{otherwise,} \end{cases} \quad (14)$$

where

$$p := \begin{cases} \frac{u}{2} + 1, & \text{if } u \text{ is even,} \\ \frac{u+1}{2}, & \text{otherwise.} \end{cases}$$

Along the same lines, we partition the interval $[0, z_1^D)$ as follows:

$$R_1^D := [z_3^D, z_1^D), R_2^D := [z_5^D, z_3^D), \dots, R_q^D := \begin{cases} [0, z_{d-1}^D), & \text{if } d \text{ is even,} \\ [0, z_d^D), & \text{otherwise,} \end{cases} \quad (15)$$

where

$$q := \begin{cases} \frac{d}{2}, & \text{if } d \text{ is even,} \\ \frac{d+1}{2}, & \text{otherwise.} \end{cases}$$

With respect to the function h illustrated in Figure 1, the above partition is depicted in Figure 2.

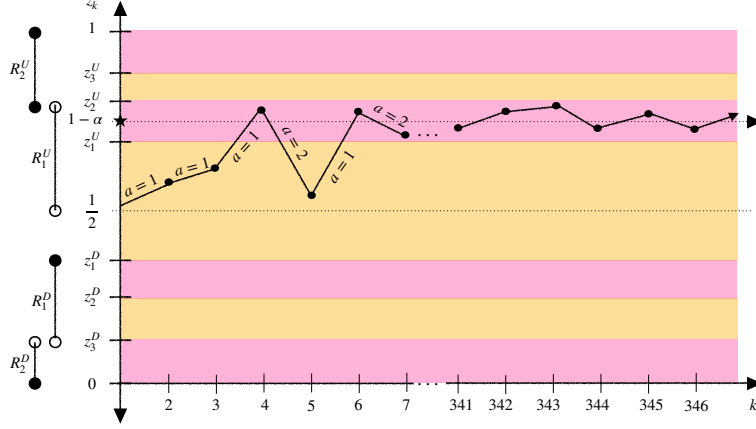


Figure 2: Partition of the interval $(1/2, 1]$ and $[0, z_1^D]$, and evolution of z_k

Further, define $\bar{R}_l^U := \cup_{1 \leq j \leq u} R_j^U$ for all $1 \leq l \leq u$ and $\bar{R}_l^D := \cup_{1 \leq j \leq d} R_j^D$ for all $1 \leq l \leq d$. Let $m \in \{U, D\}$.

Now, for iterate value large enough, i.e., for $\tilde{K} := \lceil \frac{1}{\max\{a, b\}} \rceil + 1$, where:

$$a := \max \left\{ \max\{z_{j+1}^U - z_j^U : j \in \{1, \dots, u-1\}\}, z_1^U - \frac{1}{2}, 1 - z_u^U \right\} \text{ and}$$

$$b := \max \left\{ \max\{z_{j-1}^D - z_j^D : j \in \{2, \dots, d\}\}, \frac{1}{2} - z_1^D, z_d^D \right\},$$

we show that if $z_k > 1/2$ and z_k lies within one of the sub-intervals, then the tail $(z_i)_{i \geq k}$ either gets trapped in that sub-interval or moves to a ‘higher’ sub-interval. Similar argument holds symmetrically for the complementary case. We prove the following formal statement in Appendix B:

Lemma 1. *If $z_k \in R_l^m$ for some $k \geq \tilde{K}$, then $z_i \in \bar{R}_l^m$ for all $i \geq k$, almost surely.*

Since the interest is in understanding the limiting behavior of z_k , we henceforth consider $k \geq \tilde{K}$. Now, suppose $z_k \in \bar{R}_j^m$ for some j . Then, denote the first instance when (z_k) sequence enters \bar{R}_{j+1}^m as τ_j^m (with usual convention $\tau_j^m = \infty$ otherwise). Further, for the sample paths with $\tau_j^m = \infty$, define $\tau_{j+1}^m = \infty$. While, for the sample paths with $\tau_j^m < \infty$, define $\tau_{j+1}^m = \inf\{k : z_k \in \bar{R}_{j+2}^m\}$ if the infimum exists and set $\tau_{j+1}^m = \infty$ otherwise. Now, define the potential limit points for the (z_k) -sequence:

$$a_l^U := \max\{1 - \alpha, z_{2l-1}^U\} \text{ for } 1 \leq l \leq p-1, \text{ and } a_p^U := \begin{cases} 1, & \text{if } u \text{ is even,} \\ \max\{1 - \alpha, z_u^U\}, & \text{otherwise.} \end{cases} \quad (16)$$

$$a_l^D := \min\{\alpha, z_{2l}^D\} \text{ for } 1 \leq l \leq q-1, \text{ and } a_q^D := \begin{cases} \min\{\alpha, z_d^D\}, & \text{if } d \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

In the next result (proved in Appendix B), we prove that if the tail gets trapped in some sub-interval (say, $\tau_l^m = \infty$ for some l, m), then the sequence converges to a_l^m , else the process continues and the sequence converges to a_i^m for some $i > l$; to be precise, we prove that:

Lemma 2. *The sequence (z_k) converges to $\{a_j^D : j \in \{1, \dots, d\}\} \cup \{a_j^U : j \in \{1, \dots, u\}\}$ as $k \rightarrow \infty$ almost surely.*

Finally, we establish that the limit set of the SA scheme (z_k) is the set of α -rational attractors (proof is again provided in Appendix B):

Lemma 3. *If $z_k \rightarrow z_\infty$, then $z_\infty \in \mathcal{A}_f$.*

The above three lemmas conclude the proof of the underlying Theorem. \square

Thus, by virtue of Theorem 1, the proportion of (myopic and herding) players who choose action 1 converges to the set of α -rational attractors (\mathcal{A}_f) , with probability 1. In other words, the proportion of chosen actions (z_k) converges to a limit value in \mathcal{A}_f almost surely. Recall that this result holds under **(A)** which only requires the difference $h(z) = u(1, z) - u(2, z)$ to be continuous on $[0, 1]$. Further, by Theorem 3 provided in Appendix A, we can conclude that (z_k) converges to the asymptotically stable points of the DI (24). The limit set \mathcal{A}_f is explicitly characterised later in Theorem 4.

Before proceeding further, we would like to mention that our analysis can be easily extended to cater to another combination of irrational players, namely myopic rational (as before) and avoid-the-crowd players. The latter kind of players do the exact opposite of herding players by choosing the action which is least chosen by the previous players. For such a new mix of players, one can get a result similar to Theorem 1 by firstly changing the term (corresponding to herding) $1_{\{z_k \geq \frac{1}{2}\}}$ to $1_{\{z_k \leq \frac{1}{2}\}}$ (e.g., in (3), (6), (8)) and then doing some logical modifications in the proof of Theorem 1 accordingly. Clearly, such an analysis will be helpful to study more realistic market or resource games where a naive strategy for the players is to choose a minority action. Similarly, other behavioral traits like preferential attachment to a particular action or making random choices, etc., can be handled with our framework.

3.2 Asymptotic analysis of individual types

We will now derive the limits of the empirical distribution of the behavioral choices of the myopic and the herding players separately. These results are derived with the help of Theorem 1. The plan is also to show that the limits indeed satisfy (12)-(13). To this end, recall the (y_k) -process from (11) and observe that its limit (when exists) represents the limit of the empirical choices of the myopic players. Define in a similar way, the proportion of the herding players choosing action 1, (among the herding players) till k -th round, i.e.,

$$x_k := \frac{\sum_{i=1}^k (1 - R_i) G_i}{k - \sum_{i=1}^k R_i}. \quad (17)$$

Also, let the proportion of myopic players till k -th round be represented by p_k , i.e.,

$$p_k := \frac{\sum_{i=1}^k R_i}{k}. \quad (18)$$

From (6), observe that z_k can be written in terms of y_k , p_k and x_k as below:

$$z_{k+1} = y_{k+1} p_{k+1} + x_{k+1} (1 - p_{k+1}) \implies y_{k+1} = \frac{z_{k+1} - x_{k+1} (1 - p_{k+1})}{p_{k+1}}. \quad (19)$$

Observe that the above equation resembles the second equation of (10), and (13) (note by law of large numbers, $p_k \rightarrow \alpha$ almost surely). We now prove the result which confirms some of the intuitions discussed in example (9) and paves way towards the representation in (12)-(13).

Theorem 2. *Assume **(A)**. Suppose $z_k \rightarrow z_\infty$ as $k \rightarrow \infty$ for some $z_\infty \in [0, 1]$ with probability $\delta > 0$. Then, the following statements are true with probability δ (more precisely, almost surely on the set $\{z_k \rightarrow z_\infty\}$):*

(i) *the proportion of myopic players choosing action 1,*

$$y_k \rightarrow y_\infty, \text{ as } k \rightarrow \infty, \text{ where } y_\infty := \begin{cases} \frac{z_\infty}{\alpha}, & \text{if } z_\infty < \frac{1}{2}, \\ 1 - \frac{1-z_\infty}{\alpha}, & \text{if } z_\infty \geq \frac{1}{2}. \end{cases} \quad (20)$$

(ii) the proportion of herding players choosing action 1, i.e., $x_k \rightarrow 1_{\{z_\infty \geq 1/2\}}$ as $k \rightarrow \infty$.

Proof. The proof is provided in Appendix B. □

Remarks: (i) To begin with, observe that y_∞ can be written as a function of z_∞ , however, does not always equal the conditional expected value, $E[F_\infty | z_\infty] = 1_{\{h(z_\infty) \geq 0\}}$.

(ii) Whenever $z_\infty = \alpha < 1/2$ or $z_\infty = 1 - \alpha$ with $\alpha \leq 1/2$, all the myopic players chose action 1 or 2 respectively.

(iii) In the remaining cases, y_∞ is an interior point — basically, y_∞ fraction of the group choose action 1 while the others choose action 2.

(iv) In fact, by (20) we have that y_∞ satisfies the second equation of (10); and this is true irrespective of the value of z_∞ and irrespective of whether it satisfies the first equation of (10). Thus the conjecture related to the second equation in (10) or more aptly, the equation (13) is true for all the limits z_∞ .

(v) On the other hand, by part (ii), the herding players always choose one of the actions at limit.

We finally prove that the limits of the dynamics (as well as the zeros of Definition 1) satisfy both (12)-(13) in the following:

Theorem 3. Assume (A) and let $\alpha \in (0, 1]$. Then, we have the following:

(i) if $z_\infty \in \mathcal{Z}$ and y_∞ is given by equation (20), then that pair $(\mu_\infty, \mu_\infty^R)$ satisfies (12)-(13).

(ii) the limits of (z_k) defined in (1)-(3) almost surely satisfy (12)-(13).

Proof. The proof is provided in Appendix B. □

We now obtain a simpler characterisation of the limits, which is considered in the immediate next, before moving on to the analysis of the motivating examples.

3.3 Characterisation of the limits

In the previous sections, we have derived the limiting empirical distribution of the actions corresponding to the entire population ($\mu_\infty := (z_\infty, 1 - z_\infty)$) and that of the myopic players ($\mu_\infty^R := (y_\infty, 1 - y_\infty)$), in terms of the α -rational attractors (\mathcal{A}_f). In this section, we will explicitly derive the set \mathcal{A}_f , i.e., provide simpler expressions to identify the α -rational attractors \mathcal{A}_f or the limits of the turn-by-turn dynamics.

Theorem 4. Assume (A). Let $O_{a,b} := \{z : a < z < b\}$ represent the open interval, and define the following set:

$$\mathcal{S} = \{z_\infty \in [0, 1] : \text{sign}\{(z - z_\infty)h(z)\} \text{ is negative for all } z \in N_\epsilon(z_\infty) - \{z_\infty\}, \text{ for some } \epsilon > 0\}, \quad (21)$$

where the open neighbourhood $N_\epsilon(z) := O_{z-\epsilon, z+\epsilon} \cap [0, 1]$. Then, the following statements are true:

(i) If $\alpha \geq 1/2$, then $\mathcal{A}_f = \mathcal{S}$,

(ii) Now consider $\alpha < 1/2$. Then $\mathcal{A}_f \subseteq \mathcal{S} \setminus O_{\alpha, 1-\alpha} \cup \{\alpha, 1 - \alpha\}$. Further, we have:

(a) $\mathcal{S} \setminus O_{\alpha, 1-\alpha} \subseteq \mathcal{A}_f$,

(b) $\alpha \in \mathcal{A}_f$ only if $h(\alpha) > 0$, and,

(c) $1 - \alpha \in \mathcal{A}_f$ only if $h(1 - \alpha) < 0$.

Proof. The proof is provided in Appendix B. □

Note that the above result facilitates the identification of the set \mathcal{A}_f using simpler computation involving the utility difference function h . In particular, the set Φ of the zeros of h has an important role, see assumption (A). Thus, in summary, the following is the procedure to identify the α -rational attractors or the potential limits of the dynamics:

- (i) find the zeros of function h , i.e., the set Φ ,
- (ii) add 0 or 1 respectively if $h(0) < 0$ and $h(1) > 0$,
- (iii) pick those zeros $z_* \in \Phi$ for which $h(z) > 0$ for $z \in N_\epsilon^-(z_*)$ and $h(z) < 0$ for $z \in N_\epsilon^+(z_*)$ for some $\epsilon > 0$. This is the set of the potential limits (\mathcal{A}_f) when $\alpha \geq 1/2$.
- (iv) to obtain the set for $\alpha < 1/2$, subtract the zeros in the open interval $O_{\alpha, 1-\alpha}$, add α if $h(\alpha) > 0$, and add $(1 - \alpha)$ if $h(1 - \alpha) < 0$.

Next, consider the game with perfectly rational players. Then, from the classical theory of mean-field games, the outcome of the game or the Nash equilibrium $\mu^* := (z^*, 1 - z^*)$ satisfies the following [12]:

$$\text{support}(\mu^*) \subseteq \text{Arg max}_{a \in \mathcal{A}} u(a, z^*). \quad (22)$$

In other words, the perfectly rational players choose the action which maximizes their utility. Interestingly, observe that if $\alpha = 1$ in turn-by-turn dynamics, i.e., if all the players are myopic, then $y_\infty = z_\infty$, and thus, (12) (same as (22) in this case) holds. *Importantly, this implies that the myopic players reach at the same limit (in fact, equilibrium), as perfectly rational players do, if there are no herding players.* However, *when $\alpha < 1$, the conditions (12)-(13) appear to extend the concept of NE to account for the presence of herding players.* In fact, as a part of future research, we plan to investigate (12)-(13) in more depth, with an aim to understand if its solution(s) can form the equilibrium of some interesting variant of a generic game. An initial study on the equilibrium (or stationary solution) is considered in [18], where for example, we analyse whether strategic (myopic) players gain any advantage over herding players at equilibrium.

At this point, the reader may recall that some limits of the replicator dynamics (or similar dynamics) involving all myopic players also satisfy the mean-field NE of the perfectly rational players (see e.g., [2]).

4 Analysis of motivating examples

4.1 For participation game

Let us first analyse the participation game introduced in section 2.1.1. Recall that the game designer invites players to participate in an activity to achieve a certain goal for a task. Here, we set $f(z) = z^2$ as the probability of successfully achieving that goal. For such f , the utility difference function h , defined in **(A)** is given by $h(z) = C + Rz - 1$, with $\Phi = \{(1-C)/R\}$. Below, we state the following result for the limits (α -rational attractors) of the participation game dynamics:

Corollary 5. *Consider the participation game. Suppose the utility function for each player is given by (4) with $f(z) = z^2$. Then, the following statements are true:*

(i) *if $R > 1 - C$, then $\mathcal{A}_f = \{0, 1\}$ for any $\alpha \in (0, 1]$,*

(ii) *if $R \leq 1 - C$, then:*

$$\mathcal{A}_f = \begin{cases} \{0\}, & \text{if } \alpha \in [1/2, 1], \\ \{0, 1 - \alpha\}, & \text{if } \alpha \in (0, 1/2). \end{cases}$$

Proof. The proof follows immediately using Theorem 4 and simple algebra. □

Thus, if the reward is higher ($R > 1 - C$), then by Theorem 1 and the above Corollary, almost surely, only two possibilities arise for any $\alpha \in (0, 1]$: either all the players participate ($z_\infty = 1$) or zero fraction participates ($z_\infty = 0$). It is interesting to see how these limits can emerge. Suppose the first player is myopic and chooses $a = 1$. Then, $z_1 = 1$. Then, no matter what the type of the second player

is, $a = 1$ will be chosen: a myopic player will observe $h(z_1) = C + R - 1 > 0$ and act as per (2), while the herding player will observe $z_1 = 1$ and use (3). This implies, $z_2 = 1$. The process can continue forever to yield $z_\infty = 1$. Similarly, one can construct a sample path where $z_\infty = 0$ by starting with a herding player who chooses $a = 2$. This situation may be undesirable for the designer as the chances of zero participation are 50%.

The dynamics, and thus the outcome of the game, become more interesting when the reward declared by the game designer is lesser, i.e., when $R \leq 1 - C$. Here, if the population has more myopic players (i.e., $\alpha \geq 1/2$), then the eventual participation is $z_\infty = 0$ almost surely; thus, the game designer can not succeed. However, in the reverse situation, the game designer can induce $(1 - \alpha)$ -level (which is more than 50%) of participation by exploiting the inherent herding behavior of the irrationals. There is still a risk of 0-level of participation as the other attractor. Since the game designer is a rational entity, it can still use viral marketing techniques (like, involving a celebrity to advertise, etc.) to reach $(1 - \alpha)$ -level of participation. This is because the final outcome is sensitive to the initial choices, which can be influenced heavily by the external factors not considered in the current study; these aspects can be explored in the future.

In all, the game designer can announce lesser reward ($R \leq 1 - C$) to the players and possibly reap higher value, under a more realistic setting of $\alpha < 1/2$ (where the fraction of herding players is larger).

4.2 For routing game

Here, we study the routing game characterised by (5) in section 2.1.2. It is easy to see that $h(z) = 1 - \gamma z$ has the set of zeroes, $\Phi = \{\Delta\}$, for $\Delta := 1/\gamma$. Now, the set of α -rational attractors (limits for the underlying game dynamics) can be evaluated to be the following, easily using Theorem 4 and simple computations:

Corollary 6. *The following statements hold for the routing game:*

(i) if $\Delta \leq \frac{1}{2}$:

$$\mathcal{A}_f = \begin{cases} \{\alpha, 1 - \alpha\}, & \text{if } \alpha \leq \Delta, \\ \{\Delta, 1 - \alpha\}, & \text{if } \Delta < \alpha \leq \frac{1}{2}, \\ \{\Delta\}, & \text{if } \alpha > \frac{1}{2}; \end{cases}$$

(ii) if $\Delta > \frac{1}{2}$:

$$\mathcal{A}_f = \begin{cases} \{\alpha, 1 - \alpha\}, & \text{if } \alpha \leq 1 - \Delta, \\ \{\Delta, \alpha\}, & \text{if } 1 - \Delta < \alpha < \frac{1}{2}, \\ \{\Delta\}, & \text{if } \alpha \geq \frac{1}{2}. \end{cases} \quad \square$$

Observe that unlike participation game, it is not possible that everyone goes to a single route; further, Δ emerges as a new limit in addition to α and $1 - \alpha$. In particular, when $\alpha > 1/2$, the fraction of players choosing route 1 is almost surely given by Δ , irrespective of the value of Δ . However, when $\alpha < 1/2$, the limits are given above, and depend on the comparisons between α and Δ . Interestingly, in this situation (with $\alpha < 1/2$), if route 1 is more expensive with $\Delta \leq 1/2$, the myopic-rational players can choose smartly by inducing $z_\infty = 1 - \alpha$, simply by going to the route 2. Similarly, the myopic players can experience lesser congestion, even when $\Delta > 1/2$, with $z_\infty = \alpha$.

At this point, we would like to stress about how the dynamics behave differently around Δ and other attractors. To understand this, recall our discussion about the conditional expectation $M(\cdot)$ being discontinuous, and having potentially $\{0, 1, \alpha, 1 - \alpha\}$ and other additional zeroes/attractors. From Corollary 6, Δ is the additional attractor for the routing game, which satisfies (9). In Figure 3, we illustrate the convergence for the case $1 - \Delta < \alpha < 1/2$ for few sample paths (see (6)), with common

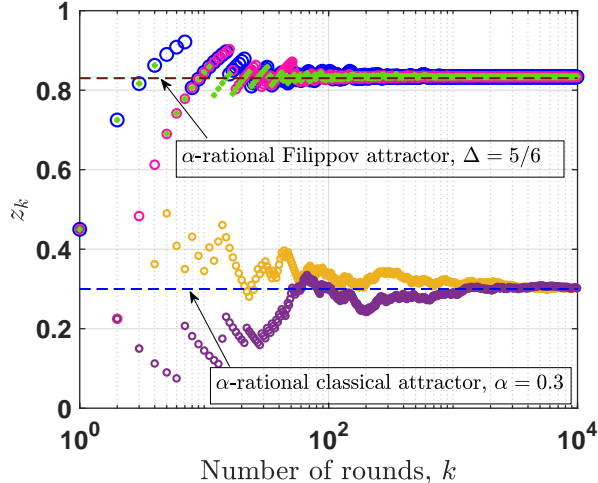


Figure 3: Illustration of convergence for routing game

$z_0 = 0.45$. Evidently, some trajectories⁵ converge around Δ and others around α . Importantly, the trajectories converging to Δ observe the diminishing overshoots/undershoots around Δ , as discussed with respect to (9).

5 Conclusions

In this paper, we study a special kind of behavioral dynamics, which differs from the classical population game dynamics in two ways. Firstly, the players play in a turn-by-turn manner and only once. Each player receives a utility at the end of the game depending upon its action and the eventual empirical distribution of the choices made by the players. Secondly, unlike the classical population games with all rational players, in our setting, only α -fraction of the population consists of myopic rational players, while, the rest are herding players (who choose the most popular action till their turn); here $\alpha \in (0, 1]$ is a common knowledge.

We derived the limiting analysis of the game when players had only two choices to choose from. To this end, we identified the almost sure limit proportion of the players choosing an action. Additionally, we derived the limit of the proportion of myopic players choosing an action.

For the analysis, when the underlying dynamics are written in the form of a stochastic approximation (SA) scheme, we observed that classical ODE-based SA analysis can not be employed, due to the discontinuities arising from the switching behavioral choices of the players. Thus, we constructed the proof using non-trivial probabilistic arguments driven by the behavior of the players. Nonetheless, we proved that the limit set of the dynamics coincides with the asymptotically stable points of an appropriate differential inclusion corresponding to the SA scheme. The convergence of the switching dynamics to the limits is demonstrated numerically for a routing game. We provide a detailed characterisation of the limits and examined the effect of the presence of herding players on the limits.

A generic participation game is also studied using our analysis, where a controller attempts to incentivise the players to participate in an activity by some reward-based scheme. Interestingly, we show that higher participation can be induced with lesser reward, by leveraging upon the presence of herding players.

⁵We refer to Δ and α differently due to the connection of $M(\cdot)$ with ODE/DI theory established in Appendix A, where we show that the set \mathcal{A}_f corresponds to the asymptotically stable points of some ODE or DI introduced by Filippov in [11]. Based on the behavior of $M(\cdot)$ around $z_\infty \in \mathcal{A}_f$, we call z_∞ as an α -rational classical attractor if $M(z)$ is continuous in some ϵ -neighborhood of z_∞ . And, $z_\infty \in \mathcal{A}_f$ is called an α -rational Filippov attractor if $M(z)$ is discontinuous in some ϵ -neighborhood of z_∞ .

Future Work: In our analysis, we showed that the limits of the turn-by-turn dynamics satisfy some conditions, which resemble the conditions for Nash equilibrium in mean-field games. In future, we plan to investigate this aspect in detail. Further, a natural progression of this research is to include multiple actions. We briefly indicated how our analysis can be extended for another important behavioral pattern, namely avoid-the-crowd. In future, it would also be interesting to study many more behaviors and their combinations.

Acknowledgments

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A Appendix: Relation between limits of dynamics and a differential inclusion

Our aim in this appendix is to show the correspondence between the limits derived in Theorem 1 for the turn-by-turn dynamics and the limits of an appropriate approximating deterministic system for the SA scheme (6), as is usually done in SA-based literature (see [10], [9]).

In an attempt to do so, recall the discontinuous conditional expectation $M(\cdot)$ from (7)-(8). Had it been the simple case of $M(z)$ being continuous on $[0, 1]$, then by standard results in SA theory ([9, 10]), the following ODE could have facilitated the analysis of the SA scheme described in (6):

$$\dot{z} = M(z). \tag{23}$$

However, for our framework, under the assumption **(A)**, the set of discontinuity points of M , denoted by \mathcal{D} equals $\Phi \cup \{1/2\}$ and is finite; here, $1/2$ is explicitly written to take into account the discontinuity arising from the behavioral choices of the herding players. Now, there are two possibilities: either $M(z) = 0$ for all $z \in \mathcal{D}$ or $M(z) \neq 0$ only for some $z \in \mathcal{D}$; each case is discussed next in detail.

(a) if $M(z) = 0$ for all discontinuity points $z \in \mathcal{D}$

Here, the (autonomous) ODE (23) can still provide the required analysis, however, the ODE solution is defined in extended sense⁶. *Importantly, from Definition 1, note that the set of α -rational zeroes (\mathcal{Z}) is a subset of the equilibrium points of the ODE (23) (defined, for example, in [19, 9, 10]). Further, a special subset of \mathcal{Z} , namely the set of α -rational attractors (\mathcal{A}_f), is equals to the set of attractors (limit points) for the ODE (23) in [19, Theorem 4.2].*

⁶Consider a function $M : \mathbb{R} \rightarrow \mathbb{R}$ with finitely many discontinuities such that $M(z) = 0$ at all discontinuity points $z \in \mathbb{R}$. A function $z(\cdot)$ is said to be a generalised (or extended) solution of the ODE $\dot{z} = M(z)$ if it is absolutely continuous and satisfies the equation for almost all $t \geq 0$.

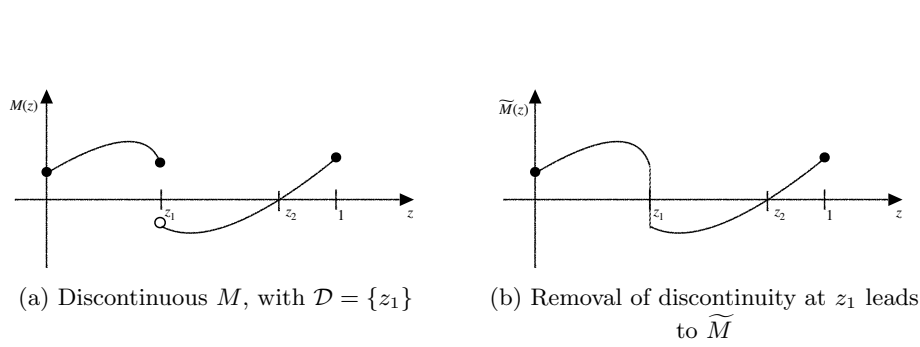


Figure 4: Depiction of $M(\cdot)$ under case (ii): ODE (23) does not have even an extended solution at z_1 ; need differential inclusion (24)

(b) if $M(z) = 0$ for some $z \in \mathcal{D}$, while $M(z) \neq 0$ for other $z \in \mathcal{D}$

This case is illustrated in Figure 4(a) with $\mathcal{D} = \{z_1\}$. Observe that if one considers the ODE (23) here, then for $z(0) < z_1$, $z(t)$ increases till z_1 and thereafter the (classical or even extended) solution is not well-defined as there is no clarity⁷ about how the trajectory $z(\cdot)$ moves beyond z_1 . Thus, in this case, a solution concept is provided in terms of differential inclusion (DI) in [11].

For the construction of the DI, we follow the procedure prescribed in [11]: if $z \notin \mathcal{D}$, the right hand side of the DI is given exactly by the value $M(z)$; however, if $z \in \mathcal{D}$, then the right hand side of the DI is the convex hull of the left and right limits of the function $M(\cdot)$, with respect to z . Since the function $M(\cdot)$ is one-dimensional, therefore, the convex hull becomes an interval joining the said left and right hand limits. Thus, the DI for (6) is:

$$\dot{z} \in \widetilde{M}(z), \text{ where the set } \widetilde{M}(z) := \begin{cases} \{M(z)\}, & \text{if } z \notin \mathcal{D}, \\ [\lim_{a \uparrow z} M(a), \lim_{a \downarrow z} M(a)], & \text{if } z \in \mathcal{D}. \end{cases} \quad (24)$$

See Figure 4(b) for an example of \widetilde{M} , corresponding to the function M given in Figure 4(a). Observe that if $\mathcal{D} = \emptyset$, then the above DI reduces to the ODE (23). With our objective in mind, we first define the solution of the DI, as in [11]:

Definition 3. Let \widetilde{M} be a non-empty convex, compact set-valued map in \mathbb{R} . Consider the differential inclusion (DI) $\dot{z} \in \widetilde{M}(z)$ with initial point $z(0) = z_0$. Then, an absolutely continuous mapping $z : \mathbb{R} \rightarrow \mathbb{R}$ is called the solution of the DI if $z(0) = z^*$ and $\dot{z}(t) \in \widetilde{M}(z(t))$ for almost every $t \in \mathbb{R}$.

For ODEs, the potential candidates for limit points are the zeroes of the right-hand side of the ODE (known as the equilibrium/stationary points). Along similar lines, the stationary points for a one-dimensional DI are defined below, as in [11, Section 12]:

Definition 4. Let \widetilde{M} be a non-empty convex, compact set-valued map in \mathbb{R} . Consider the differential inclusion (DI) $\dot{z} \in \widetilde{M}(z)$ with initial point $z(0) = z_0$. Then, a point $z_\infty \in \mathbb{R}$ is called a stationary point for the DI if $0 \in \widetilde{M}(z_\infty)$.

Let \mathcal{Z}^* be the set of stationary points for the DI (24). Observe from above that if for a DI, the initial condition $z(0) = z_\infty \in \mathcal{Z}^*$, then $z(t) = z_\infty$ is a well-defined solution of the DI. Thus, all stationary points are solutions of the DI. At this juncture, we establish the first connection between our probabilistic analysis of the SA scheme and the DI-based SA analysis:

Theorem 7. The set of α -rational zeroes is a subset of the stationary points of the DI (24), i.e., $\mathcal{Z} \subseteq \mathcal{Z}^*$.

⁷Observe that the positive drift at z_1 implies that the solution goes to the right of z_1 , but the negative drift after z_1 implies that the solution could not have progressed to the right of z_1 .

Proof. The claim immediately follows by Definition 2. \square

Next, we proceed to show that even the α -rational zeroes correspond to the asymptotically stable points for the DI (24), as we did in case (a). Towards this, first we define the notion of asymptotic stability for a DI, as in [11, Section 15]:

Definition 5. Let \widetilde{M} be a non-empty convex, compact set-valued map in \mathbb{R} . Consider the differential inclusion (DI) $\dot{z} \in \widetilde{M}(z)$. Then, a solution $z(t)$, for $t \geq 0$ is a asymptotically stable if it satisfies two conditions:

(i) if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\widetilde{z}(t) - z(t)| < \epsilon, \text{ for all } t \geq 0, \text{ if } |\widetilde{z}_0 - z(0)| < \delta, \quad (25)$$

for each solution $\widetilde{z}(t)$ of the DI with the initial condition $\widetilde{z}(0) = \widetilde{z}_0$.

(ii) if $|\widetilde{z}(t) - z(t)| \rightarrow 0$, as $t \rightarrow \infty$.

Observe from above that some of the stationary points of a DI are asymptotically stable: if $z(0) = z_\infty \in \mathcal{Z}^*$ and say, $\dot{z} > 0$ in $N_\epsilon^-(z_\infty)$ and $\dot{z} < 0$ in $N_\epsilon^+(z_\infty)$ for some $\epsilon > 0$, then, due to appropriate drifts in the two directions, it is easy to see that $z(t) = z_\infty$ for all $t \geq 0$ is asymptotically stable. Thus, the notion of stability in DIs is similar to that in ODEs. With this understanding, we present the second result highlighting the relation between \mathcal{A}_f and the asymptotically stable points of DI (24) (denoted by \mathcal{A}_f^*):

Theorem 8. Under assumption (A), the set of α -rational attractors equals the set of asymptotically stable points of DI (24), i.e., $\mathcal{A}_f = \mathcal{A}_f^*$.

Proof. Consider $z_\infty \in \mathcal{A}_f$. We claim that $z(t) = z_\infty$ for all $t \geq 0$ is asymptotically stable. By Definition 2, the following holds:

$$M(z) \begin{cases} > 0, & \text{for all } z \in (z_\infty - \epsilon, z_\infty), \\ < 0, & \text{for all } z \in (z_\infty, z_\infty + \epsilon). \end{cases} \quad (26)$$

Thus, clearly, $0 \in \widetilde{M}(z_\infty)$; this implies that either $M(z_\infty) = 0$ or $0 \in [\lim_{a \uparrow z_\infty} M(a), \lim_{a \downarrow z_\infty} M(a)]$. In the former case, the DI (24) becomes $\dot{z} = M(z)$ for $z \in N_\epsilon(z_\infty)$. Now, by ODE result in [19], the claim holds. Further, in the latter case, $\dot{z} > 0$, for all $z \in (z_\infty - \epsilon, z_\infty)$ and $\dot{z} < 0$, for all $z \in (z_\infty, z_\infty + \epsilon)$. Then, it is easy to argue that $z(t) = z_\infty$ for all $t \geq 0$ is a solution for the DI (24), if $z(0) = z_\infty$ and also, the claim holds. This concludes that $\mathcal{A}_f \subseteq \mathcal{A}_f^*$.

For the converse, first observe that $\mathcal{A}_f^* \subseteq \mathcal{Z}^*$. Thus, the only possible solutions in \mathcal{A}_f^* are those $z(t) = z_\infty$ for all $t \geq 0$. Consider one such solution. We next prove that $z_\infty \in \mathcal{A}_f$. If z_∞ is a continuity point for $M(\cdot)$, then again the DI (24) becomes $\dot{z} = M(z)$ for $z \in N_\epsilon(z_\infty)$. It is easy to see that such $z_\infty \in \mathcal{A}_f^*$ only if $z_\infty \in \mathcal{A}_f$ (observe that if $z_\infty \notin \mathcal{A}_f$, then z_∞ is either a repeller or a saddle point, see [19]). Similarly, if $z_\infty \in \mathcal{D}$, then again $z_\infty \in \mathcal{A}_f^*$ only if $z_\infty \in \mathcal{A}_f$. Thus, $\mathcal{A}_f^* \subseteq \mathcal{A}_f$. In all, we get $\mathcal{A}_f = \mathcal{A}_f^*$. \square

Thus, by virtue of Theorem 1 and Theorem 8, we get that the turn-by-turn dynamics z_k converges to the asymptotically stable points of the DI (24), almost surely, as $k \rightarrow \infty$. Importantly, the parallels drawn in Theorem 7 and Theorem 8 suggests that one can analyse our turn-by-turn dynamics with DI-based SA-analysis, as in [20]; we leave this line of thinking and research for future study.

B Appendix: Details of the proofs

Note: In the coming, to complete the proof of Theorem 1, we prove the Lemmas 1-3 for the case when $h(1/2) > 0$, as shown in Figure 1. As said before, the proof follows analogously when $h(1/2) < 0$ or $= 0$.

Proof of Lemma 1: Suppose u is odd. Let $z_{\bar{K}} \in R_l^U$ for some l , for some $\bar{K} \geq \tilde{K}$. Then, only two possibilities arise:

- $z_{\bar{K}} \in (z_l^U, z_{l+1}^U)$ for some $l \geq 0$ which is even, with $z_0^U := \frac{1}{2}$. Recall from the above description of the dynamics that for such a given z_k , any player chooses action 1 at $(k+1)$ -th round. Therefore, we have (again see Figure 2):

$$z_{k+1} = z_k + \frac{1}{k+1}(1 - z_k), \text{ for all } k < e_l^U := \inf\{t \geq \bar{K} : z_t \notin (z_l^U, z_{l+1}^U)\}. \quad (27)$$

The above implies that $z_{k+1} > z_k$ for all $\bar{K} < k < e_l^U$. This implies, $z_{e_l^U} \in [z_{l+1}^U, z_{l+2}^U] \cup \bar{R}_{l+1}^U$, i.e., $z_{e_l^U} \in \bar{R}_l^U$.

- $z_{\bar{K}} \in [z_{l+1}^U, z_{l+2}^U]$ for some $l \geq 0$ which is even. In this case, $R_k F_k + (1 - R_k)G_k$ in (6) need not be 1. Nonetheless, we have:

$$|z_{k+1} - z_k| = \frac{1}{k+1} |R_k F_k + (1 - R_k)G_k - z_k| \leq \frac{1}{k+1} \leq \frac{1}{\bar{K}}.$$

Thus, z_{k+1} either lies in (z_l^U, z_{l+1}^U) or in $[z_{l+1}^U, z_{l+2}^U]$ or in \bar{R}_{l+1}^U , by definition of \tilde{K} .

In all, above two arguments implies that $z_k \in \bar{R}_l^U$ for all $k \geq \bar{K}$, when $m = U$ and u is odd. The claim can be proved analogously for $m = U$ when u is even, or when $m = D$.

Proof of Lemma 2: We begin with the case when $z_{\tilde{K}} \in (1/2, 1]$. In particular, say $z_{\tilde{K}} \in R_l^U$ for some l . Then, by Lemma 1, $z_k \in \bar{R}_l^U$ for all $k \geq \tilde{K}$.

(a) Suppose $1 - \alpha \in R_l^U$. Then, either z_k stay in R_l^U for all $k \geq \tilde{K}$ (i.e., $\tau_l^U = \infty$), or it escapes R_l^U ($\tau_l^U < \infty$).

When $\tau_l^U = \infty$, z_k is exactly same as the fictitious process, say θ_k , defined in Theorem 9 with $\theta_{\tilde{K}} := z_{\tilde{K}}$. By Theorem 9, $z_k = \theta_k \rightarrow a_l^U$ almost surely on the set $\{z_{\tilde{K}} \in R_l^U, \tau_l^U = \infty\}$.

When $\tau_l^U < \infty$, $z_k \in \bar{R}_{l+1}^U$ for all $k \geq \tau_l^U$. If $z_k \in R_s^U$ for some $s > l$, then again either $\tau_s^U = \infty$ or $\tau_s^U < \infty$. Proceeding as before, either $z_k \rightarrow a_s^U$ almost surely on the set $\{z_{\tilde{K}} \in R_l^U, \tau_l^U < \infty, \tau_s^U = \infty\}$, or, one can proceed iteratively to finally conclude that $z_k \rightarrow a_j^U$ for some $j \geq l$, $\tau_j^U = \infty$ and $\tau_s^U < \infty$ for all $l \leq s < j$ on the set $\{z_{\tilde{K}} \in R_l^U\}$.

(b) Suppose $1 - \alpha > \sup\{x : x \in R_l^U\}$. Here, we claim that with probability 1, $\tau_l^U < \infty$, i.e., $z_k \notin R_l^U$ for all $k \geq \tilde{K}$. To prove this, suppose on the contrary that with probability 1, $z_k \in R_l^U$ for all $k \geq \tilde{K}$. Then, as above, $z_k = \theta_k$ for all $k \geq \tilde{K}$ where θ_k is the process defined in Theorem 9. Since $\theta_k \rightarrow 1 - \alpha$ (as $a_l^U = 1 - \alpha$), therefore, $z_k \rightarrow 1 - \alpha$. This implies that for every $\epsilon > 0$, there exists a K_ϵ such that $z_k > 1 - \alpha - \epsilon$ for all $k \geq K_\epsilon$. In particular, consider $\epsilon \in (0, 1 - \alpha - \sup\{x : x \in R_l^U\})$. Then, $z_k > \sup\{x : x \in R_l^U\}$ for all $k \geq K_\epsilon$, which is a contradiction to the assumption. This proves the claim, and thereby implying that $\tau_l^U < \infty$ and $z_k \not\rightarrow a_l^U$. However, one can still show as above that $z_k \rightarrow a_j^U$ for some $j > l$, $\tau_j^U = \infty$ and $\tau_s^U < \infty$ for all $l \leq s < j$.

Following the exact same steps as above for $m = D$, one can show that $z_k \rightarrow \{a_j^D : j = 1, \dots, d\}$, if $z_{\tilde{K}} \in [0, z_1^D)$.

Now, suppose $z_{\tilde{K}} \in (z_1^D, 1/2)$. If $\nu := \inf\{t \geq \tilde{K} : z_t \notin (z_1^D, 1/2)\} = \infty$, then for all $k \geq \tilde{K}$, we have (note, $G_i = 0$ for all $i \geq \tilde{K}$):

$$z_{k+1} = \frac{\sum_{i=1}^{k+1} (R_i \cdot F_i + (1 - R_i)G_i)}{k+1} = \frac{\sum_{i=1}^{\tilde{K}-1} (R_i \cdot F_i + (1 - R_i)G_i) + \sum_{i=\tilde{K}}^{k+1} R_i F_i}{k+1}.$$

By strong law of large numbers, $z_k \rightarrow \alpha$ almost surely on the set $\{z_{\tilde{K}} \in (z_1^D, 1/2), \nu = \infty\}$. If $\nu < \infty$, then $\tau^m < \infty$ for some $m \in \{U, D\}$; as a result, $z_\nu \in \overline{R}_1^m$ for such m . Thereafter, as shown above, one can argue that $z_k \rightarrow a_j^m$ for some j , almost surely on the set $\{z_{\tilde{K}} \in (z_1^D, \frac{1}{2}), \nu < \infty\}$.

Now, suppose $z_{\tilde{K}} = z_1^D$ or $z_{\tilde{K}} = \frac{1}{2}$. In the former case, either $z_{\tilde{K}} \in \overline{R}_1^D$ or in $(z_1^D, \frac{1}{2})$ or in \overline{R}_1^U . While in the latter case, $z_{\tilde{K}} \in \overline{R}_1^U$. Then, using above arguments as above, one can show that $z_k \rightarrow a_j^m$ for some m and some j .

Proof of Lemma 3: Recall, from (8), that the function $M(z)$ is given as:

$$M(z) = \alpha 1_{h(z) \geq 0} + (1 - \alpha) 1_{z \geq \frac{1}{2}} - z. \quad (28)$$

Say $z_\infty = z_j^U$ for some j , then $1 - \alpha < z_j^U$, and further for any $\epsilon > 0$:

$$M(z) = \begin{cases} \alpha + (1 - \alpha) - z > 0 & \text{if } z \in (z_j^U - \epsilon, z_j^U), \\ (1 - \alpha) - z < 0 & \text{if } z \in (z_j^U, z_j^U + \epsilon). \end{cases}$$

However, if $z_\infty = 1 - \alpha$, then $1 - \alpha \in (z_j^U, z_{j+1}^U)$, and thus for some appropriate $\epsilon > 0$:

$$M(z) = (1 - \alpha) - z \begin{cases} > 0 & \text{if } z \in (1 - \alpha - \epsilon, 1 - \alpha), \\ < 0 & \text{if } z \in (1 - \alpha, 1 - \alpha + \epsilon). \end{cases}$$

Thus, if $z_k \rightarrow a_j^U$ for some j , then by Definition 2, this limit is an α -rational attractor. This completes the proof of the underlying Lemma for the case when $m = U$, and the steps can be repeated in a similar manner for $m = D$. \square

Theorem 9. Let $\theta_\infty \in [0, 1] - \{\frac{1}{2}\}$. Consider the fictitious dynamics θ_n such that if the current proportion of players who choose action 1, i.e., $\theta \leq \theta_\infty$, then any myopic player chooses action 1, and it chooses action 2, otherwise. Then, following statements hold almost surely:

- (i) if any herding player always chooses action 1, then $\theta_n \rightarrow \max\{1 - \alpha, \theta_\infty\}$ as $n \rightarrow \infty$, and
- (ii) if any herding player always chooses action 2, then $\theta_n \rightarrow \min\{\alpha, \theta_\infty\}$ as $n \rightarrow \infty$.

Proof. We prove the first part, where any herding player always choose $a = 1$; the proof can be done analogously for part (ii). We split the proof (for part (i)) into two cases.

Case 1: If $1 - \alpha < \theta_\infty$, then we claim that:

- (i) regardless of the initial condition, θ_n crosses θ_∞ with probability 1 in finite time. That is, if $\theta_n < \theta_\infty$ for some $n < \infty$ then there exists $k \in (n, \infty)$ such that $\theta_k > \theta_\infty$; similarly, if $\theta_n > \theta_\infty$ for some $n < \infty$ then (again) there exists some $k \in (n, \infty)$ such that $\theta_k < \theta_\infty$.
- (ii) $|\theta_{n_m} - \theta_\infty| \rightarrow 0$ as $n_m \rightarrow \infty$, for some appropriate sub-sequence (θ_{n_m}) of (θ_n) , which is defined later.

Let us keep the notations F_i , G_i and R_i the same as in the original turn-by-turn dynamics. Then, similar to (1), it is easy to see that (recall, $G_i = 1$ for all i):

$$\theta_{n+m} = \frac{n\theta_n}{n+m} + \frac{\sum_{i=n+1}^{n+m} (R_i F_i + (1 - R_i))}{n+m}, \text{ for all } m \geq 1.$$

Without loss of generality, assume $\theta_1 > \theta_\infty$, and set $\tau_U^{(0)} := 1$. Denote the first time when θ_n goes below θ_∞ as $\tau_D^{(1)}$ and the first time when θ_n goes above θ_∞ as $\tau_U^{(1)}$. In general, one can define the k -th crossing time from above and below as follows:

$$\tau_D^{(k)} := \inf \left\{ n : \theta_{n+\tau_U^{(k-1)}} < \theta_\infty \right\}, \text{ and } \tau_U^{(k)} := \inf \left\{ n : \theta_{n+\tau_D^{(k)}} > \theta_\infty \right\}, \text{ for all } k \geq 1. \quad (29)$$

Furthermore, as is usually done, we set $\tau_D^{(k)} := \infty$ or $\tau_U^{(k)} := \infty$, if the corresponding infimum does not exist.

Now, let us consider a new auxiliary process $D_n^{(k)}$ which starts at $\tau_U^{(k-1)}$ for each $k > 1$ such that rational player always chooses $a = 2$ (irrespective of the actual proportion θ_n) and as before, herding player chooses $a = 1$. Such a process can be formally written as (note $F_i = 0$ for all $i \geq \tau_U^{(k-1)}$):

$$D_n^{(k)} = \frac{\theta_{\tau_U^{(k-1)}} \cdot \tau_U^{(k-1)}}{n} + \frac{\sum_{i=\tau_U^{(k-1)}+1}^n (1 - R_i)}{n}, \text{ for all } n > \tau_U^{(k-1)}. \quad (30)$$

Similarly, define another auxiliary process $U_n^{(k)}$ which starts at $\tau_D^{(k)}$ for each $k \geq 1$ such that any player chooses $a = 1$ (irrespective of the actual proportion θ_n). That is, we have (observe $R_i F_i + (1 - R_i) G_i = 1$ for all $i \geq \tau_D^{(k)}$):

$$U_n^{(k)} = \frac{\theta_{\tau_D^{(k)}} \cdot \tau_D^{(k)}}{n} + \frac{\sum_{i=\tau_D^{(k)}+1}^n (R_i F_i + (1 - R_i) G_i)}{n} = \frac{\theta_{\tau_D^{(k)}} \cdot \tau_D^{(k)}}{n} + \frac{n - \tau_D^{(k)}}{n}, \text{ for all } n > \tau_D^{(k)}. \quad (31)$$

Now, observe that the actual process θ_n exactly matches with the auxiliary process till cross-over times. That is, $\theta_n = D_n^{(k)}$ for $n \in (\tau_U^{(k-1)}, \tau_D^{(k)}]$, $k \geq 1$ and $\theta_n = U_n^{(k)}$ for $n \in (\tau_D^{(k)}, \tau_U^{(k)}]$, $k \geq 1$, for example see Figure 5.

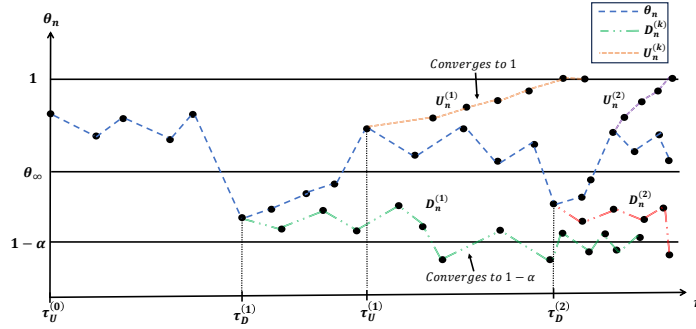


Figure 5: Illustration of auxiliary processes $D_n^{(k)}$ and $U_n^{(k)}$

Now, our aim is to prove the step (i) using auxiliary processes defined above. To be specific, we will show that $D_n^{(k)} \rightarrow 1 - \alpha$, almost surely, as $n \rightarrow \infty$, for each k , and thus, as illustrated in Figure 5, $\tau_D^{(k)} < \infty$ for each k . Similarly, after proving that $U_n^{(k)} \rightarrow 1$ almost surely, as $n \rightarrow \infty$, we will conclude that $\tau_U^{(k)} < \infty$.

Towards this, note that $D_n^{(k)}$ can be re-written as (see (30)):

$$D_n^{(k)} = \frac{z_{\tau_U^{(k-1)}} \cdot \tau_U^{(k-1)}}{n} + \frac{\sum_{i=\tau_U^{(k-1)}+1}^n (1 - R_i)}{n - \tau_U^{(k-1)}} \cdot \frac{n - \tau_U^{(k-1)}}{n} \text{ for all } n \geq \tau_U^{(k-1)} + 1. \quad (32)$$

Then, by Strong Law of Large Numbers, $P\left(\left\{D_n^{(k)} \xrightarrow{n \rightarrow \infty} 1 - \alpha\right\}\right) = 1$. Thus, for any $\epsilon > 0$, there exists $N_\epsilon < \infty$ (dependent on k) such that $|D_n^{(k)} - (1 - \alpha)| < \epsilon$ for all $n \geq N_\epsilon$. Consider any $\epsilon \in (0, \theta_\infty - (1 - \alpha))$. Then:

$$D_n^{(k)} < \epsilon + (1 - \alpha) < \theta_\infty, \text{ for all } n \geq N_\epsilon > \tau_U^{k-1}.$$

This implies that $\Delta := \inf\left\{n \geq \tau_U^{(k-1)} + 1 : D_n^{(k)} < \theta_\infty\right\} < \infty$. By relation between θ_n and $D_n^{(k)}$, observe that $\Delta = \tau_D^{(k)} < \infty$. Thus, for all sample paths where $D_n^{(k)} \rightarrow 1 - \alpha$, we have, $\tau_D^{(k)} < \infty$. This

implies that $P\left(\left\{\tau_D^{(k)} < \infty\right\}\right) = 1$. Observe that $\{\tau_D^{(k+1)} < \infty\} \subseteq \{\tau_D^{(k)} < \infty\}$, and thus (by continuity of probability):

$$P\left(\bigcap_{k \rightarrow \infty} \{\tau_D^{(k)} < \infty\}\right) = \lim_{k \rightarrow \infty} P\left(\tau_D^{(k)} < \infty\right) = 1.$$

Now, it is easy to see from (31) that $U_n^{(k)} \rightarrow 1$ almost surely, as n tends to ∞ . One can follow similar procedure as above to show that $P\left(\bigcap_{k \rightarrow \infty} \{\tau_U^{(k)} < \infty\}\right) = 1$. This completes step (i).

Next, we will show that the difference between iterates θ_n for $n = \tau_U^{(k)}$ and θ_∞ converges to 0 as $k \rightarrow \infty$. Let this difference, $\left|\theta_{\tau_U^{(k)}} - \theta_\infty\right|$, be referred to as ladder height. Similar to (6), for any $m \in \{U, D\}$, we have:

$$\theta_{\tau_m^{(k)}} = \theta_{\tau_m^{(k)} - 1} + \frac{1}{\tau_m^{(k)}} \left(R_{\tau_m^{(k)}} F_{\tau_m^{(k)}} + (1 - R_{\tau_m^{(k)}}) G_{\tau_m^{(k)}} - \theta_{\tau_m^{(k)} - 1} \right).$$

Observe that θ_n requires at least 1 jump from above to below (i.e., $\tau_D^{(k)} - \tau_U^{(k-1)} \geq 1$) and then at least 1 jump from below to above (i.e., $\tau_U^{(k)} - \tau_D^{(k)} \geq 1$). This results in $\tau_m^{(k)} \geq 2k$ for any $m \in \{U, D\}$. Thus, we have:

$$\left| \theta_{\tau_m^{(k)}} - \theta_\infty \right| \leq \left| \theta_{\tau_m^{(k)}} - \theta_{\tau_m^{(k)} - 1} \right| \leq \frac{2}{\tau_m^{(k)}} \leq \frac{1}{k},$$

where the first inequality holds because $\theta_{\tau_D^{(k)} - 1} > \theta_\infty$ and $\theta_{\tau_U^{(k)} - 1} < \theta_\infty$. Finally, from above, we get that the ladder height decreases to 0 as k tends to ∞ .

In all, by virtue of step (i) and (ii), we have proved that $\theta_n \rightarrow \theta_\infty$, when $\theta_\infty > 1 - \alpha$.

Case 2: If $\theta_\infty < 1 - \alpha$. Define a new process where a herding player always chooses action 1, while a rational player always chooses action 2. Further, assume that a herding player plays in the new setup only when a herding player plays in the original game. Define corresponding random variables \tilde{R}_{n+1} , \tilde{F}_{n+1} and \tilde{G}_{n+1} for the new process; observe $\tilde{F}_n = 0$, $\tilde{G}_n = 1$ for all $n \geq 0$ and further, $\tilde{R}_n = R_n$ for all $n \geq 0$. Let the proportion of players who choose action 1 till n -th round be denoted by $\tilde{\theta}_n$. Set $\tilde{\theta}_0 = \theta_0$, and observe:

$$\tilde{\theta}_{n+1} = \tilde{\theta}_n + \frac{1}{n+1} \{(1 - \tilde{R}_{n+1}) - \tilde{\theta}_n\}, \text{ for all } n > 0. \quad (33)$$

From above, for all $n \geq 0$:

$$\tilde{\theta}_{n+m} = \frac{n\tilde{\theta}_n}{n+m} + \frac{\sum_{i=n+1}^{n+m} (1 - \tilde{R}_i)}{n+m}, \text{ for all } m \geq 1.$$

Then, by the strong law of large numbers, $\tilde{\theta}_n \rightarrow 1 - \alpha$ almost surely, as $n \rightarrow \infty$. Thus, for every $\delta \in \Delta := (0, \min\{\alpha, 1 - \alpha - \theta_\infty\})$, there exists $\tilde{N}_\delta > 0$ such that $\tilde{\theta}_n > 1 - \alpha - \delta$ for all $n \geq \tilde{N}_\delta$ and $P(\tilde{N}_\delta < \infty) = 1$.

Say $\theta_n = \tilde{\theta}_n$ for some $n < \infty$. Then, we have (by (33)):

$$\tilde{\theta}_{n+1} = \frac{n\tilde{\theta}_n}{n+1} + \frac{1 - \tilde{R}_{n+1}}{n+1} = \frac{n\theta_n}{n+1} + \frac{1 - R_{n+1}}{n+1} \leq \frac{n\theta_n}{n+1} + \frac{1 - R_{n+1} + F_{n+1}R_{n+1}}{n+1} = \theta_{n+1}.$$

One can continue in a similar manner as above and argue by the principle of mathematical induction that $\tilde{\theta}_n \leq \theta_n$ for all $n \geq 0$. Now, define N_δ such that $\theta_n > 1 - \alpha - \delta$ for all $n > N_\delta$ and for every $\delta \in \Delta$. Since we have proved that $\tilde{\theta}_n \leq \theta_n$ for all $n \geq 0$, therefore, $N_\delta \leq \tilde{N}_\delta$ almost surely, for all $\delta \in \Delta$. This implies that $P(N_\delta < \infty) = 1$ for all $\delta \in \Delta$, as we have shown above that $P(\tilde{N}_\delta < \infty) = 1$ for all $\delta \in \Delta$.

Now, for each $\delta \in \Delta$, $\theta_n > \theta_\infty$ for all $n \geq N_\delta$. This implies that (observe $R_n = 0$ for all $n \geq N_\delta$, as now rational players always choose action 2), see (6):

$$\theta_{n+m} = \frac{n\theta_n}{n+m} + \frac{\sum_{i=n+1}^{n+m} 1 - R_i}{n+1}, \text{ for all } n \geq N_\delta.$$

Then, again by strong law of large numbers, $\theta_n \rightarrow 1 - \alpha$ almost surely, as $n \rightarrow \infty$. \square

Proof of Theorem 2: Recall that a player is myopic with probability $\alpha \in (0, 1]$ and thus, a player is herding with probability $1 - \alpha$. This implies that $\lim_{k \rightarrow \infty} p_k = \alpha$. We are given that $\lim_{k \rightarrow \infty} z_k = z_\infty \in [0, 1]$. Thus, taking limit as $k \rightarrow \infty$, in (19), we get:

$$\lim_{k \rightarrow \infty} y_k = \frac{z_\infty - (1 - \alpha) \lim_{k \rightarrow \infty} x_k}{\alpha}. \quad (34)$$

Since $z_k \rightarrow z_\infty$, therefore, for every $\epsilon > 0$, there exists a $K_\epsilon < \infty$ such that:

$$z_\infty - \epsilon \leq z_k \leq z_\infty + \epsilon, \quad \text{for all } k \geq K_\epsilon. \quad (35)$$

Case 1: $z_\infty < \frac{1}{2}$. Consider $\epsilon \in (0, \frac{1}{2} - z_\infty)$. Then, for each such ϵ , there exists $K_\epsilon < \infty$ such that $z_k < \frac{1}{2}$ for all $k \geq K_\epsilon$ by (35). Now, for such ϵ and for all $k \geq K_\epsilon$, if a herding player plays in k -th round, then it will not choose action 1, i.e., $G_k = 0$ when $R_k = 0$. Thus, $\lim_{k \rightarrow \infty} x_k = 0$. Substituting this limit in (34), we get that $\lim_{k \rightarrow \infty} y_k = \frac{z_\infty}{\alpha}$.

Case 2: $z_\infty \geq \frac{1}{2}$. Similar to above, one can show that for all $\epsilon \in (0, z_\infty - \frac{1}{2})$, there exists a $K_\epsilon < \infty$ such that $z_k > \frac{1}{2}$ for all $k \geq K_\epsilon$. This implies that $\lim_{k \rightarrow \infty} x_k = 1$. Thus, by (34), $\lim_{k \rightarrow \infty} y_k = \frac{z_\infty - (1 - \alpha)}{\alpha}$. \square

Proof of Theorem 3: Given $z_\infty \in \mathcal{Z}$ and y_∞ as in (20). Directly, by Theorem 2, one can verify that (13) is satisfied. Now, we will prove that (12) also holds, as claimed. Towards this, notice that $z_\infty \in \mathcal{Z}$ implies that $\text{sign}\{(z - z_\infty)M(z)\} > 0$ or < 0 for all $z \in N_\epsilon(z_\infty) - \{z_\infty\}$, for some $\epsilon > 0$.

Say $M(z) > 0$ for all $z \in (z_\infty - \epsilon, z_\infty)$, and $M(z) < 0$ for all $z \in (z_\infty, z_\infty + \epsilon)$. Then, one possible structure for $M(\cdot)$ in the ϵ -neighborhood of z_∞ is $M(z) = \alpha - z > 0$ for all $z \in (z_\infty - \epsilon, z_\infty)$ and $M(z) = -z < 0$ for all $z \in (z_\infty, z_\infty + \epsilon)$. Observe that such M is possible only if $h(z) \geq 0$ (and < 0) in left (right, respectively) ϵ -neighborhood of z_∞ . By continuity of $h(\cdot)$ assumed in **(A)**, we get that $h(z_\infty) = 0$, and thus, $\text{Arg max}_{a \in \mathcal{A}} u(a, z_\infty) = \{1, 2\}$. Now, irrespective of the value of $\text{support}(\mu_\infty^R)$, the condition (12) holds. These steps can be replicated for any other feasible structure of M , even when $\text{sign}\{(z - z_\infty)M(z)\} > 0$ for all $z \in N_\epsilon(z_\infty) - \{z_\infty\}$; one should observe that any feasible M is such that $\text{sign}\{(z - z_\infty)h(z)\} > 0$ or < 0 for all $z \in N_\epsilon(z_\infty) - \{z_\infty\}$. This completes the proof of part (i). The part (ii) is followed by part (i), Theorem 1 and the Definition 2. \square

Proof of Theorem 4: (i) Given $\alpha \geq 1/2$. To prove that $\mathcal{A}_f = \mathcal{S}$, we begin by showing that $\mathcal{A}_f \subseteq \mathcal{S}$. Consider $z_\infty \in \mathcal{A}_f$. Suppose $z_\infty < 1/2$. Then, for all $\epsilon \in (0, 1/2 - z_\infty)$, we have (see (8)):

$$M(z) = \alpha 1_{\{h(z) \geq 0\}} - z, \quad \text{for } z \in N_\epsilon(z_\infty) - \{z_\infty\}. \quad (36)$$

Clearly, $M(z) < 0$ for all $z \in N_\epsilon^+(z_\infty)$ only if $h(z) < 0$ (recall, $\alpha \geq 1/2$). Further, $M(z) > 0$ for all $z \in N_\epsilon^-(z_\infty)$ only if $h(z) \geq 0$; however, under assumption **(A)**, there can only be finitely many zeroes of the function h , therefore, one can further restrict ϵ in such a way that $h(z) > 0$ for all $z \in N_\epsilon^-(z_\infty)$. This implies that $z_\infty \in \mathcal{S}$. One can follow similar steps to prove the claim for $z_\infty \geq 1/2$ by consider appropriate $\epsilon \in (0, z_\infty - 1/2)$.

For the converse, again consider $z_\infty < 1/2$ from the set \mathcal{S} . Then, by definition of \mathcal{S} (see (21)), the following holds:

$$M(z) = (\alpha - z)1_{z \in N_\epsilon^-(z_\infty)} + (-z)1_{z \in N_\epsilon^+(z_\infty)},$$

which clearly implies that $z_\infty \in \mathcal{A}_f$ (recall $\alpha \geq 1/2$). The converse can again be proven for $z_\infty \geq 1/2$ in a similar manner.

(ii) Here, $\alpha < 1/2$. Firstly, we show that $\mathcal{A}_f \subseteq \mathcal{S} \cup \{\alpha, 1 - \alpha\} - (\alpha, 1 - \alpha)$. Towards this, consider $z_\infty \in \mathcal{A}_f$. Notice that if $z_\infty \leq \alpha$, then, again $M(z)$ is given as in (36); thus by same arguments, one can show that $z_\infty \in \mathcal{S}$.

Following similar procedure as for $z_\infty \leq \alpha$, one can prove the claim for all $z_\infty \geq 1 - \alpha$. Now, consider $z_\infty \in (\alpha, 1 - \alpha)$. Here, we divide the proof into three parts to show that $z_\infty \notin \mathcal{A}_f$:

- if $z_\infty > 1/2$: Observe that $M(z) = (1 - z)1_{h(z) \geq 0} + (1 - \alpha - z)1_{h(z) < 0} > 0$ for all $z \in N_\epsilon^+(z_\infty)$. Thus, irrespective of the value of $h(z)$, in this case, $M(z) \not\leq 0$, as required for z_∞ to be an α -rational attractor.

- if $z_\infty < 1/2$: Observe that $M(z) = (\alpha - z)1_{h(z) \geq 0} + (-z)1_{h(z) < 0} > 0$ for all $z \in N_\epsilon^-(z_\infty)$. Thus, irrespective of the value of $h(z)$, in this case, $M(z) \not\geq 0$, as required for z_∞ to be an α -rational attractor.
- if $z_\infty = 1/2$: one can combine the arguments from the above two cases to prove that $z_\infty \notin \mathcal{S}$.

In all, the above proves that $\mathcal{A}_f \subseteq \mathcal{S} - (\alpha, 1 - \alpha) \subseteq \mathcal{S} \cup \{\alpha, 1 - \alpha\} - (\alpha, 1 - \alpha)$, and thus the claim is proven.

For the converse, we begin with (ii.b). Observe that for $z_\infty = \alpha$, $M(z)$ is given as in (36). If $h(\alpha) > 0$, then by continuity of h assumed in **(A)**, $h(z) > 0$ for all $z \in N_\epsilon(\alpha)$ for some $\epsilon > 0$. Thus, $M(z) = \alpha - z$ which clearly satisfies the requirement for α to be an α -rational attractor. However, if $h(\alpha) < 0$, then $M(z) < 0$ for all $z \in N_\epsilon(\alpha)$, thereby contradicting the definition of α -rational attractor. This completes (ii.b).

The proof for (ii.c) follows as for (ii.b) above.

Now, we are left to prove (ii.a), i.e., $\mathcal{S} - (\alpha, 1 - \alpha) \subseteq \mathcal{A}_f$. Consider $z_\infty \in \mathcal{S}$. Let us consider three separate cases (again see definition of $M(\cdot)$ from (8)):

- if $z_\infty \leq \alpha$, then $M(z) = (\alpha - z)1_{z \in N_\epsilon^-(z_\infty)} + (-z)1_{z \in N_\epsilon^+(z_\infty)}$, for some $\epsilon > 0$. It is easy to see that the conditions of α -rational attractors are satisfied for such z_∞ .
- for $z_\infty \geq 1 - \alpha$, one can again easily show that $z_\infty \in \mathcal{A}_f$ by evaluating $M(z)$.
- if $z_\infty \in (\alpha, 1 - \alpha)$, then (i) $M(z) = \alpha - z < 0$ for all $z < z_\infty < 1/2$, (ii) $M(z) = 1 - \alpha - z > 0$ for all $z > z_\infty > 1/2$, and (iii) $M(z) = (\alpha - z)1_{z < z_\infty} + (1 - \alpha - z)1_{z > z_\infty}$ otherwise. In all three sub-cases, the definition of α -rational attractors is contradicted, and thus, no such z_∞ is in the set \mathcal{A}_f .

The above arguments prove (ii.a). □

C Appendix: Summary of notations

Notation	Description
α	fraction of myopic players
u	utility function
a, \mathcal{A}	action chosen, set of actions
z_k	fraction of players who chose $a = 1$ out of k players
y_k	fraction of myopic players who chose $a = 1$, among myopic players, out of k players
x_k	fraction of herding players who chose $a = 1$, among herding players, out of k players
R_k	indicator that k -th player is myopic
F_k, G_k	indicator that k -th players chooses action 1, if it is a myopic (respectively, herding) player
$a^R(z)$	action chosen by a myopic player, when empirical distribution is $(z, 1 - z)$
$M(z)$	conditional expectation of function g arising in the SA-scheme (6)
$h(z)$	difference in utility functions for actions 1 and 2
Φ	set of zeroes of the function h
z_∞, y_∞	limits of the sequences $(z_k)_k$ and $(y_k)_k$, respectively
$\mathcal{Z}, \mathcal{A}_f$	set of α -rational zeroes and attractors, respectively