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State estimation for a class of nonlinear time-varying uncertain systems under multiharmonic disturbance

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Abstract

The paper considers the observer synthesis for nonlinear, time-varying plants with uncertain parameters under multiharmonic disturbance. It is assumed that the relative degree of the plant is known, the regressor linearly depends on the state vector and may have a nonlinear relationship with the output signal. The proposed solution consists of three steps. Initially, an unknown input state observer is synthesized. This observer, however, necessitates the measurement of output derivatives equal to the plant's relative degree. To relax this limitation, an alternative representation of the observer is introduced. Further, based on this observer, the unknown parameters and disturbances are reconstructed using an autoregression model and the dynamic regressor extension and mixing (DREM) approach. This approach allows the estimates to be obtained in a finite time. Finally, based on these estimates, an observer has been constructed that does not require measurements of the output derivatives. The effectiveness and efficiency of this solution are demonstrated through a computer simulation.

Keywords: unknown input observer, disturbance, time-varying plant, nonlinear systems

1. Introduction

To enhance the effectiveness of automatic control systems, it is crucial to implement state feedback, necessitating the measurement of state vector

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variables. However, achieving this in real-world applications poses significant challenges. The installation of sensors for measuring all necessary variables can be prohibitively expensive and sometimes technologically unfeasible. To address this issue, control theory has introduced the concept of observers. These tools provide the estimation of state variables without direct measurement, offering a practical solution to the limitations posed by sensor implementation (see, for example, [1, 2, 3, 4]).

In 1964, Luenberger pioneered the theory of state observer synthesis for linear systems [4]. However, as science and technology have advanced, linear dynamic models with constant parameters often fail to meet practical demands. Real-world systems are frequently affected by various unknown external perturbations, presenting challenges in developing algorithms for constructing state observers for nonlinear, time-varying systems under such disturbances. To tackle these challenges, several classic methods have been proposed.

One of the most popular approaches consists in (piecewise) approximation of nonlinearities by linear functions [5, 6, 7, 8, 9]. This method, illustrated in various studies, includes the use of exponential functions [8], linear-quadratic objective function minimization [6], and Taylor series expansion combined with the least squares method, utilizing Moore-Penrose pseudo-inverse transformation [9]. This approach offers ease in parameter selection and practical application for specific technical systems. It is notably effective where nonlinear filters are impractical or unfeasible. However, it faces challenges and limitations in systems with complex mathematical models or when precise perturbation distributions are required, as inaccuracies in linearization can sometimes result in system instability.

Methods employing the extended Kalman filter [10, 11, 12] are popular in various practical applications. The key advantage of this approach is its suitability for nonlinear systems and its capacity to estimate both the system state and the parameters of disturbances, described by probabilistic models. However, this method also requires model linearization and knowledge of perturbation distributions, whose absence may lead to estimation errors.

In [13, 14], a sliding mode-based observer synthesis method for state vector estimation is introduced. The primary advantage of this method is its robustness to parametric disturbances, owing to its invariance properties. However, the occurrence of sliding modes introduces challenges, notably oscillations and high-frequency switching in the control channel, which are significant concerns in the application of this technique.

An alternative approach involves transforming the original system into a linear regression form with an unknown parameter vector [15, 16]. This transformation allows the application of various identification techniques, such as the least squares method [17], gradient descent [17], dynamic regressor extension and mixing method [18, 19, 20], etc. The primary advantage of this approach is the independence of the identification block from the control law, allowing the use of diverse control methods without altering the identification algorithm. However, a limitation of such identification techniques is the necessity to fulfill the persistent excitation condition for the regressor. Additionally, there is no universally applicable method for different plant classes; each system requires a unique parameterization solution. A recent development in this area focuses on constructing observers for uncertain plants with output-dependent unknown parameters [21].

A specific class of solutions in the field of observer synthesis provides a state estimation in the presence of unmeasured input signals. This class of solutions is called Unknown Input Observers (UIOs) [22, 23, 24]. These algorithms ensure effective functioning even in the presence of disturbances and other factors that affect the signals in the control channel. They are designed to estimate state vector despite the presence of external influencing factors. Unknown input observers were first introduced by N. Kobayashi [25] in 1982. This result has been further developed for discrete-time systems [26, 27], various nonlinear systems [26, 27, 28], and time-varying systems [28, 29].

For time-varying systems a common approach is to transform the original plant into a polytopic form with weight functions that depend on parameters. To construct the observer and ensure its stability, Lyapunov function framework and solution of a system of matrix inequalities are typically used. For example, in [30] an unknown input observer was developed for Takagi-Sugeno continuous and discrete-time systems which are analogous to polytopic non-stationary systems. Further, this method was extended to a class of nonlinear systems in [28].

Generally, constructing unknown input observers is feasible primarily for systems where the relative degree between unknown inputs and the output signal is one. For plants with a higher relative degree, stringent conditions must be met to build an observer. For instance, researches like [31, 32] necessitate measuring the derivatives of the system's output signal. In works such as [28, 33], the system's original dynamic equation must be separable to isolate the observable components in the output.

In recent studies various methods have been developed for estimating state vectors in nonlinear systems. However, these solutions face significant constraints. For example, paper [21] presents a state vector estimation method for nonlinear systems with unknown parameters, where the regressor is dependent on the output. The dependence of regressor on unknown state vectors adds complexity to the problem. Research [34] addresses this challenge. However, proposed solution is applicable only to a class of stationary systems, and it is difficult to regulate the estimation error convergence rates.

In this paper, we combine two methodologies to develop a novel, comprehensive solution for nonlinear time-varying systems based on the unknown input and the identification-based observer synthesis. This integrated approach aims to address the limitations inherent in each individual method, facilitating efficient operation across a broader spectrum of systems providing the estimation of the state, with the reconstruction of the parameters and the unknown input in a finite time.

Traditional approaches to observer design often focus on specific aspects, such as state estimation or disturbance rejection, but rarely address the simultaneous estimation of states, parameters, and external disturbances. Many prior methods rely on restrictive assumptions, such as the time-invariant parameters, bounded disturbances, or systems with linear time-invariant dynamics. This study addresses these challenges by proposing a method that simultaneously observes the state vector, nonstationary parameters, and external multiharmonic disturbances. Unlike prior works, the proposed approach ensures finite-time convergence of parameter and disturbance estimation while achieving exponential convergence of the state estimation error. The main contribution of this study lies in the development of a unified observer that can handle nonlinear systems with arbitrary relative degrees and time-varying parameters. The method ensures finite-time convergence, significantly enhancing its practical applicability. Furthermore, the proposed solution is not limited to systems with unit relative degree but extends to systems with arbitrary relative degrees and some types of nonlinearity. This advancement broadens the range of systems to which the solution can be applied, making it suitable for a variety of nonlinear systems.

The structure of the paper is outlined as follows: Section 2 presents the problem statement. Section 3 proposes the unknown input observer synthesis algorithm, first, assuming the availability of the derivatives of the measured output signal, and second, relaxing this constraint. Section 4 introduces an algorithm to estimate the unknown parameters and external harmonic dis-

turbances, and state observer based on these estimates that does not require output derivatives. Finally, the paper concludes with simulation results that demonstrate the efficiency of the proposed method.

2. Problem Statement

Consider a class of single input single output plants described by equations

$$\begin{cases} \dot{x}(t) = Ax(t) + B[u(t) + \varphi^T(x(t), y(t), t)\theta(t) + f(t)], \\ y(t) = Cx(t), \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is an unmeasured state vector, $u(t) \in \mathbb{R}$ is a known input signal, $\varphi(x, y, t)$ is a known vector function, $\theta(t) \in \mathbb{R}^w$ is an unknown vector of time-varying parameters, $y(t)$ is a measured output signal, $f(t) \in \mathbb{R}$ is an unknown multiharmonic disturbance; A, B, C are known matrices with appropriate dimensions; plant has relative degree between the input and output $r \leq n$ that means $CB = CA^jB = 0$ for $j = 1, \dots, r - 2$. Recall that the relative degree of a system is a key concept in control theory, defined as the smallest integer r for which the r -th derivative of the output explicitly depends on the input. Importantly, while the relative degree characterizes the dynamics of the system, it does not restrict the applicability of the solution proposed below. The system under consideration is time-varying, containing nonlinear functions of the output signal and the product of unknown variable parameters and an unmeasured state vector. The last factor does not allow the use of such well-known solutions as the extended Kalman filter, the internal model principle, etc. This class of models describes the dynamics of such technical systems as the bicycles [35], surface ships [36] and many others.

We will need the following assumptions:

1. The external disturbance $f(t)$ is represented as $f(t) = \sum_{i=1}^q R_i \sin(\omega_i t + \phi_i)$, where R_i, ω_i, ϕ_i are unknown amplitudes, frequencies and phases, respectively, q is a known number of harmonics.
2. Pair (A, C) is observable.
3. Matrix B has full column rank and matrix C has full row rank.
4. Time-varying parameters vector $\theta(t)$ is an output of a linear generator:

$$\begin{aligned} \theta(t) &= H\xi(t), \\ \dot{\xi}(t) &= \Gamma\xi(t), \end{aligned}$$

where $\xi(t) \in \mathbb{R}^m$ is a generator state vector with unknown initial conditions $\xi(0)$. Constant matrices H and Γ are known.

5. The regressor is a linear function of the state weighted by a continuous nonlinear function of the output and the time:

$$\varphi(x, y, t) = \alpha(y, t)x,$$

where $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}$ is known.

The goal is synthesis of an adaptive observer for the plant (1) under the introduced restrictions, which guarantees realization of the following relations:

$$\lim_{t \rightarrow +\infty} |\hat{x}(t) - x(t)| = 0$$

and $\forall t \geq \tau$

$$\hat{f}(t) = f(t), \hat{\theta}(t) = \theta(t),$$

where $\hat{x}(t)$ denotes the estimated state vector, $\hat{f}(t)$ represents the estimated external disturbance, $\hat{\theta}(t)$ is an estimate for $\theta(t)$, and $\tau > 0$ is a finite time of estimation.

Remark 1: For simplification of the mathematical representation and for the reader's convenience, we assume $u(t) \equiv 0$ in the sequel.

3. Unknown Input Observer

In this section observer synthesis algorithm is given. First, an exponentially converging state UIO is introduced that describes the basic structure of the estimator. Next, an alternative realization of this observer is derived that reduces the number of needed derivatives to $r - 1$ and that provides a finite-time estimation of the state.

Consider plant (1) with relative degree $r \leq n$. The unknown input observer is formulated as

$$\dot{\hat{x}}(t) = F\hat{x}(t) + Ly(t) + Gy^{(r)}(t), \quad (2)$$

where matrices F, L and G satisfy the following conditions:

$$B - GCA^{r-1}B = 0, \quad (3)$$

$$M = A - GCA^r, \quad (4)$$

$$F = M - LC. \quad (5)$$

To demonstrate that (2) is an observer, consider the error $\tilde{x}(t) = x(t) - \hat{x}(t)$. By differentiating and incorporating (1) and (2), the dynamic model of $\tilde{x}(t)$ is derived as follows:

$$\begin{aligned} \dot{\tilde{x}}(t) &= Ax(t) + B[\varphi^T(x, y, t)\theta(t) + f(t)] - F\hat{x}(t) - Ly(t) - Gy^{(r)}(t) = \\ &= Ax(t) + B[\varphi^T(x, y, t)\theta(t) + f(t)] - F\hat{x}(t) - Ly(t) - GCA^r x(t) - \\ &\quad - GCA^{r-1}B[\varphi^T(x, y, t)\theta(t) + f(t)]. \end{aligned}$$

Applying (4) and (5), we get:

$$\dot{\tilde{x}}(t) = F\tilde{x}(t) + (B - GCA^{r-1}B)(\varphi^T(x, y, t)\theta(t) + f(t)). \quad (6)$$

Substituting (3) into (6) results in:

$$\dot{\tilde{x}}(t) = F\tilde{x}(t). \quad (7)$$

Obviously, if the matrix F is Hurwitz, then $\tilde{x}(t)$ exponentially converges to zero.

Constructing the observer as defined in (2) necessitates solving the system of equations given by (3), (4) and (5). This system is solvable under the following condition [23]:

- $\text{rank}(CA^{r-1}B) = \text{rank}(B)$,
- (C, M) is a detectable pair.

Introduce the following auxiliary variables to relax the problem of utilization of the unmeasured output derivatives (hereinafter omit the time dependence for brevity):

$$\begin{aligned} z_1 &= \hat{x} - Gy^{(r-1)}, \\ \dot{z}_1 &= F\hat{x} + Ly + Gy^{(r)} - Gy^{(r)} = F(z_1 + Gy^{(r-1)}) + Ly, \\ z_2 &= z_1 - FGy^{(r-2)}, \\ \dot{z}_2 &= F(z_1 + Gy^{(r-1)}) + Ly - FGy^{(r-1)} = F(z_2 + FGy^{(r-2)}) + Ly, \\ &\vdots \\ z_r &= z_{r-1} - F^{r-1}Gy, \\ \dot{z}_r &= F(z_r + F^{r-1}Gy) + Ly. \end{aligned} \quad (8)$$

Therefore, the state vector estimate is given by

$$\hat{x} = z_1 + Gy^{(r-1)} = z_r + F^{r-1}Gy + \dots + Gy^{(r-1)}. \quad (9)$$

If the matrix L is chosen to satisfy $F^rG + L = 0$, then $\dot{z}_r(t) = Fz_r(t)$. In this case, $z_r(t)$ is an exponentially decaying function and state vector estimate takes the form

$$\hat{x} = F^{r-1}Gy + \dots + Gy^{(r-1)} + e^{Ft}z_r(0). \quad (10)$$

Theorem 1. Let $r - 1$ derivatives of the output signal y are available for measurement and the matrices M, L, G are chosen to satisfy conditions (3) – (5) with F being Hurwitz, then the observer (9) with the auxiliary variables (8) provides estimation of the state vector $x(t)$ with exponential convergence. If additionally

$$L + F^rG = 0, \quad (11)$$

then (10) provides an immediate reconstruction of the state of (1) for all $t \geq 0$.

The proof follows the calculations (2) – (9).

The convergence speed of $\hat{x}(t)$ to $x(t)$ depends on the eigenvalues of the matrix F . Therefore, we can choose the matrix F using pole placement procedure to satisfy desired transients.

Remark 2. Direct computations show that in the case $F^rG + L = 0$ we get $\tilde{x}(0) = -z_r(0)$.

4. Estimation of unknown parameters and state vector

The observer proposed in the previous section provides a finite-time estimation of the state vector benefiting from the derivatives of the output. In the current section we will demonstrate how this observer can be used to estimate unknown parameters and disturbances, as well as to further estimate the state vector without using derivatives of the output signal.

4.1. Estimation of unknown parameters

Let us show how the vector of unknown parameters can be evaluated together with the harmonic disturbance. For brevity we perform the main computations for the case $\varphi(t, x, y) = x(t)$ only. The other cases can be

resolved in a completely analogous manner. Assuming that the condition (11) is satisfied, substitute the estimation $\hat{x}(t)$ into (1):

$$\dot{\hat{x}} = A\hat{x} + B[u + \hat{x}^T\theta + f].$$

Considering that in the case under study $u(t) \equiv 0$ and $\theta(t)$ is the output of a linear generator, we have:

$$\dot{\hat{x}} = A\hat{x} + B[\hat{x}^T H e^{\Gamma t} \xi(0) + f].$$

By substituting the expression (10) for the estimation $\hat{x}(t)$, we obtain

$$\begin{aligned} \dot{z}_r + F^{r-1}G\dot{y} + \dots + Gy^{(r)} &= A(z_r + F^{r-1}Gy + \dots + Gy^{(r-1)}) + \\ &+ B(z_r + F^{r-1}Gy + \dots + Gy^{(r-1)})^T H e^{\Gamma t} \xi(0) + Bf(t). \end{aligned} \quad (12)$$

Equation (12) involves unmeasured derivatives of the output signal, an unknown external disturbance $f(t)$, and the initial conditions $\xi(0)$. Let us apply r -th order linear filters and the swapping lemma [37] which will allow us to eliminate the terms containing unmeasured derivatives in the regressor for $\xi(0)$. To illustrate this approach, and simplifying the writing, consider the case when the relative degree is equal two and the disturbance is represented by a harmonic signal. Equation (12) takes the form

$$\begin{aligned} \dot{z}_2 + FG\dot{y} + G\ddot{y} &= A(z_2 + FGy + G\dot{y}) + \\ &+ B(z_2 + FGy + G\dot{y})^T H e^{\Gamma t} \xi(0) + Bf. \end{aligned} \quad (13)$$

Apply to (13) linear filter $\frac{\lambda_2^2}{(p+\lambda_2)^2}$ and swapping lemma

$$\begin{aligned} \frac{\lambda_2}{p+\lambda_2} [yG^T H e^{\Gamma t} \xi(0)] &= G^T H e^{\Gamma t} \xi(0) \frac{\lambda_2 p}{p+\lambda_2} [y] - \\ \frac{1}{p+\lambda_2} \left[G^T H \Gamma e^{\Gamma t} \xi(0) \frac{\lambda_2 p}{p+\lambda_2} [y] \right] &= \left[\frac{\lambda_2 p}{p+\lambda_2} [y] G^T H e^{\Gamma t} - \right. \\ \left. \frac{1}{p+\lambda_2} \left[\frac{\lambda_2 p}{p+\lambda_2} [y] G^T H \Gamma e^{\Gamma t} \right] \right] &\xi(0) = S_1(t) \xi(0), \end{aligned} \quad (14)$$

where

$$S_1(t) = \frac{\lambda_2 p}{p+\lambda_2} [y] G^T H e^{\Gamma t} - \frac{1}{p+\lambda_2} \left[\frac{\lambda_2 p}{p+\lambda_2} [y] G^T H \Gamma e^{\Gamma t} \right].$$

Rewrite the equation (13) as follows

$$q_2(t) = B \left(S_0(t) + \frac{\lambda_2}{p + \lambda_2} [S_1(t)] \right) \xi(0) + B\bar{f}(t),$$

where

$$q_2(t) = \frac{\lambda_2^2}{(p + \lambda_2)^2} [(\dot{z}_2 + FG\dot{y} + G\ddot{y}) - A(z_2 + FGy + G\dot{y})],$$

$$S_0(t) = \frac{\lambda_2^2}{(p + \lambda_2)^2} [(z_2 + FGy)^T H e^{\Gamma t}], \quad \bar{f}(t) = \frac{\lambda_2^2}{(p + \lambda_2)^2} [f(t)].$$

Thus, by applying a filter to (12), and after a series of transformations, we obtain the following expression for a system with an arbitrary relative degree:

$$q_r(t) = B \left[S_0(t) + \frac{\lambda^{r-1}}{(p + \lambda)^{r-1}} [S_1(t)] + \dots + S_{r-1}(t) \right] \xi(0) + B\bar{f}(t), \quad (15)$$

where

$$\begin{aligned} q_r(t) &= \frac{\lambda^r}{(p + \lambda)^r} [(\dot{\bar{z}}_r + F^{r-1}G\dot{y} + \dots + Gy^{(r)}) - \\ &A(\bar{z}_r + F^{r-1}Gy + \dots + Gy^{(r-1)})] = \\ &= \frac{\lambda^r p}{(p + \lambda)^r} \bar{z}_r + F^{r-1}G \frac{\lambda^r p}{(p + \lambda)^r} y + \dots + G \frac{(\lambda p)^r}{(p + \lambda)^r} y - \\ &- A \left(\frac{\lambda^r}{(p + \lambda)^r} \bar{z}_r + F^{r-1}G \frac{\lambda^r}{(p + \lambda)^r} y + \dots + G \frac{\lambda^r p^{r-1}}{(p + \lambda)^r} y \right) \\ S_0(t) &= \frac{\lambda^r}{(p + \lambda)^r} [(\bar{z}_r + F^{r-1}Gy)^T H e^{\Gamma t}], \quad \bar{f}(t) = \frac{\lambda^r}{(p + \lambda)^r} [f(t)], \\ S_k(t) &= G^T H \left(I + \frac{1}{\lambda} \Gamma \right)^{k-1} e^{\Gamma t} \frac{\lambda^k p^k}{(p + \lambda)^k} [y(t)] - \\ &\frac{1}{p + \lambda} \left[G^T H \Gamma \left(I + \frac{1}{\lambda} \Gamma \right)^{k-1} e^{\Gamma t} \frac{\lambda^k p^k}{(p + \lambda)^k} [y(t)] \right], \quad k = 1, \dots, r - 1. \end{aligned}$$

The resulting equation in a simplified form is as follows:

$$q_r(t) = B\bar{S}_r(t)\xi(0) + B\bar{f}(t), \quad (16)$$

where

$$\bar{S}_r(t) = S_0(t) + \frac{\lambda^{r-1}}{(p + \lambda)^{r-1}} [S_1(t)] + \dots + S_{r-1}(t).$$

Next, we apply the properties of a sinusoidal signal:

$$p^2 \bar{f}(t) = -\omega^2 \bar{f}(t),$$

where ω is an unknown frequency. Rewrite equation (16)

$$\begin{aligned} p^2 [q_r(t) - B\bar{S}_r(t)\xi(0)] &= p^2 B\bar{f}(t) = -\omega^2 B\bar{f}(t) = \\ &= -\omega^2 [q_r(t) - B\bar{S}_r(t)\xi(0)]. \end{aligned} \quad (17)$$

Multiply (17) by row-vector \bar{B} such that $\bar{B}B = 1$. Then, apply a second order filter $\frac{\lambda_r^2}{(p+\lambda_r)^2}$ ($\lambda_r > 0$) and group the unknown terms to obtain a regression

$$q_r^*(t) = m_r^T(t)k_r, \quad (18)$$

where $q_r^*(t) = \frac{\lambda_r^2 p^2}{(p+\lambda_r)^2} [\bar{B}q_r(t)]$,

$$\begin{aligned} m_r^T(t) &= \left[\frac{\lambda_r^2 p^2}{(p + \lambda_r)^2} [\bar{S}_r(t)], \frac{\lambda_r^2}{(p + \lambda_r)^2} [-\bar{B}q_r(t)], \frac{\lambda_r^2}{(p + \lambda_r)^2} [\bar{S}_r(t)] \right], \\ k_r &= [\xi(0); \omega^2; \omega^2 \xi(0)]. \end{aligned}$$

If the signal $\bar{f}(t)$ includes several harmonics, then the system (16) can also be transformed to a linear regression [38].

Equation (18) has the form of linear regression and enables the estimation of the initial conditions for the vector $\xi(0)$ and disturbance frequency $\hat{\omega}$. Transients can exhibit both asymptotic (using the gradient descent method) and finite-time convergence. We propose to use the DREM method based on Kreisselmeier's regressor extension scheme described in [37] (which does not destroy the excitation in the system [39]). Let us introduce the vector $\Phi_r(t)$ and the matrix $Y_r(t)$ as solutions of the differential equations

$$\begin{aligned} \dot{\Phi}_r(t) &= -h_r \Phi_r(t) + m_r(t) m_r^T(t), \Phi_r(t_0) = 0, \\ \dot{Y}_r(t) &= -h_r Y_r(t) + m_r(t) q_r^*(t), Y_r(t_0) = 0, \end{aligned} \quad (19)$$

where $h_r > 0$ is a tuning coefficient. Let us denote the adjoint matrix of $\Phi_r(t)$ as $\text{Adj}(\Phi_r(t))$. After applying DREM, we obtain

$$\Upsilon_r(t) = \Delta_r(t)k_r,$$

where $\Delta_r(t) = \det(\Phi_r(t))$, $\Upsilon_r(t) = \text{Adj}(\Phi_r(t))Y_r(t) = [\Upsilon_{r1}(t) \dots \Upsilon_{r, \dim k_r}(t)]^\top$. Therefore, we can determine each element of the vector k_r using the following equations [40]:

$$\hat{k}_{ri}^{dir}(t) = \frac{\Upsilon_{ri}(t)}{\max\{\Delta_r(t), \varepsilon_r\}}, i = \overline{1, \dim k_r}, \quad (20)$$

where $\varepsilon_r > 0$ is a small constant incorporated to guarantee the practicality of executing the direct estimation for the initial transients.

After applying conventional noise filters to equation (20) allows us to benefit the new measurements to improve the accuracy of estimation [40] in the presence of disturbances. We have the following result.

Proposition 1. If $\Delta_r \notin L_2$, then the algorithm (18)-(20) provides an estimation of the initial conditions $\xi(0)$ and the frequency of the external disturbance ω in a finite time.

Proof follows the above calculations (14) – (20).

Remark 3. In the case of $\varphi(x, y, t)$ is a nonlinear function of the form

$$\varphi(x, y, t) = \alpha(y)x(t),$$

where $\alpha(y)$ is a known function, satisfying condition $\dot{\alpha}(y) = \frac{\partial \alpha}{\partial y} \dot{y} = \beta \dot{y}$ with a constant β , instead of (21) we obtain

$$\begin{aligned} \dot{\bar{z}}_r + F^{r-1}G\dot{y} + \dots + Gy^{(r)} = \\ A(\bar{z}_r + F^{r-1}Gy + \dots + Gy^{(r-1)}) + Bf(t) \\ + B\alpha(y)(\bar{z}_r + F^{r-1}Gy + \dots + Gy^{(r-1)})^T H e^{\Gamma t} \xi(0). \end{aligned}$$

Application of swapping lemma allows to solve problem entirely analogous to the case $\varphi = x(t)$ as follows:

$$\begin{aligned} \frac{\lambda}{p+\lambda} [\alpha(y)\dot{y}G^T H e^{\Gamma t} \xi(0)] = \alpha(y)G^T H e^{\Gamma t} \xi(0) \frac{p\lambda}{p+\lambda} [y] - \\ \frac{1}{p+\lambda} \left[(\beta y G^T H e^{\Gamma t} + \alpha(y)G^T H \Gamma e^{\Gamma t}) \xi(0) \frac{p\lambda}{p+\lambda} [y] \right]. \end{aligned}$$

From equation (16), taking into account Proposition 1, we obtain the following

$$\bar{f}(t) = \bar{B}q_r(t) - \bar{S}_r(t)\xi(0) = a_1 \sin(\omega t) + a_2 \cos(\omega t) = \psi^T(t)a, \quad (21)$$

where $\psi^T(t) = [\sin(\omega t), \cos(\omega t)]$, $a = [a_1; a_2]$.

Then the linear regression equation (21) can be solved using the DREM method based on Kreisselmeier's regressor extension scheme to find the amplitude of the disturbance. Similar to the adaptive algorithm (18)-(20) the amplitude is determined as follows

$$\Upsilon_a(t) = \Delta_a(t)a, \quad (22)$$

$$\hat{a}_i^{dir}(t) = \frac{\Upsilon_{ai}(t)}{\max\{\Delta_a(t), \varepsilon_a\}}, i = \overline{1, \dim a}, \quad (23)$$

where $\Delta_a(t) = \det(\Phi_a(t))$, $\Upsilon_a(t) = \text{Adj}(\Phi_a(t))Y_a(t) = [\Upsilon_{a1}(t) \dots \Upsilon_{a, \dim a}(t)]^\top$, $\Phi_a(t)$ and $Y_a(t)$ are solutions of the differential equations

$$\begin{aligned} \dot{\Phi}_a(t) &= -h_a\Phi_a(t) + \psi(t)\psi^T(t), \Phi_a(t_0) = 0, \\ \dot{Y}_a(t) &= -h_aY_a(t) + \psi(t)\bar{f}(t), Y_a(t_0) = 0, \end{aligned}$$

where $\varepsilon_a > 0$ and $h_a > 0$ are tuning parameters

On the other hand, considering the determined value of the amplitude in (22)-(23) together with (15), it is evident that the external disturbance $f(t)$ can be reconstructed as follows

$$f(t) = \frac{(p + \lambda)^r}{\lambda^r} [\bar{f}(t)] \quad (24)$$

since with known parameters of $\bar{f}(t)$ its derivatives can be calculated due to the harmonic nature of the signal.

Proposition 2. If $\Delta_a \notin L_2$, then the algorithm (21) – (24) provides an estimation of the disturbance $f(t)$ in a finite time.

Remark 4. Proposed algorithm allows the parameter vector to be estimated even in the case when the regressor depends on the $x(t)$. It is the main difference between the solution proposed in this work and approach [21] that is applicable if the function φ depends only on the output signal $y(t)$.

4.2. State estimation

Estimates of all unknown parameters allow us to construct an arbitrary observer of the state vector for (1). Introduce the following state vector observer that does not require derivatives of the output signal

$$\dot{\hat{x}} = A\hat{x} + B[u + \alpha(y, t)\hat{x}^T\hat{\theta} + \hat{f}] + Ky - KC\hat{x}, \quad (25)$$

where K is a design matrix. Taking into account $x^T\theta = (\bar{x} + \tilde{x})^T(\hat{\theta} + \tilde{\theta})$, obtain observation error dynamic model

$$\begin{aligned}\dot{\tilde{x}} &= (A - KC)\tilde{x} + B\tilde{f} + \alpha(y, t)Bx^T\tilde{\theta} + \alpha(y, t)B\tilde{x}^T\hat{\theta} = \\ &= (A - KC)\tilde{x} + \alpha(y, t)B\hat{\theta}^T\tilde{x} + B\delta,\end{aligned}\quad (26)$$

where $\delta = \tilde{f} + \alpha(y, t)x^T\tilde{\theta}$ vanishes in a finite time. Choose $K = NC^T$, where $N = N^T > 0$ is a solution of Riccati equation

$$\dot{N}(t) = 2\gamma(t)N(t) + N(t)A^T + AN(t) - 2N(t)C^T CN(t) + \mu(t)BB^T, \quad (27)$$

where γ and μ are positive functions of time defined below.

Introduce Lyapunov function $V = \tilde{x}^T P \tilde{x}$, where $P = N^{-1}$, and consider its derivative

$$\begin{aligned}\dot{V} &= (\delta B^T - \tilde{x}^T C^T K^T + \alpha(y, t)\tilde{x}^T \hat{\theta} B^T + \tilde{x}^T A^T)P\tilde{x} + \tilde{x}^T \dot{P}\tilde{x} + \\ &+ \tilde{x}^T P(A\tilde{x} + \alpha(y, t)B\hat{\theta}^T\tilde{x} - KC\tilde{x} + B\delta).\end{aligned}$$

Since $PN = I$, $\dot{P}N + P\dot{N} = 0$ we obtain $\dot{P} = -P\dot{N}P$, then

$$\begin{aligned}\dot{V} &= -\tilde{x}^T P\dot{N}P\tilde{x} + \tilde{x}^T (A^T P + PA - 2CC^T)\tilde{x} + 2\delta B^T P\tilde{x} + 2\alpha(y, t)\tilde{x}^T PB\hat{\theta}^T\tilde{x} = \\ &= -\tilde{x}^T P(2\gamma N + NA^T + AN - 2NC^T CN + \mu BB^T)P\tilde{x} + 2\delta B^T P\tilde{x} + \\ &+ \tilde{x}^T P(NA^T + AN - 2NC^T CN)P\tilde{x} + 2\alpha(y, t)\tilde{x}^T PB\hat{\theta}^T\tilde{x} = -2\gamma\tilde{x}^T P\tilde{x} - \\ &- (\tilde{x}^T C^T)^2 - \mu\tilde{x}^T PBB^T P\tilde{x} + 2\delta B^T P\tilde{x} + 2\alpha(y, t)\tilde{x}^T PB\hat{\theta}^T\tilde{x} \\ &\leq -2\gamma\tilde{x}^T P\tilde{x} - \mu(\tilde{x}^T PB)^2 + \delta^2 + (\tilde{x}^T PB)^2 + k(\hat{\theta}^T N \hat{\theta})(\tilde{x}^T PB)^2 + \frac{1}{k}\alpha^2(y, t)\tilde{x}^T P\tilde{x},\end{aligned}$$

where $k > 0$ and the Young's inequality was used on the last step.

Choose $\mu \geq 1 + k(\hat{\theta}^T N \hat{\theta})$ and $\gamma \geq \frac{\alpha^2(y, t)}{k}$. Then the derivative of the Lyapunov function is bounded by the inequality

$$\dot{V} \leq -\gamma V + \delta^2.$$

Since $\delta = 0$ in a finite time, the observer (25) provides convergence of the \tilde{x} to zero. Consequently, we can formulate our main result, whose proof is given above:

Theorem 2. Under Assumptions 1–5, the observers of propositions 1 and 2 guarantee the estimation of $\theta(t)$ and $f(t)$ in a finite time, while the time-varying observer (25) asymptotically reconstructs the state.

5. Simulation

To demonstrate the functionality and effectiveness of the proposed approach, let us consider a second-order dynamic system with a relative degree $r = 2$, given by

$$\begin{cases} \dot{x}_1(t) = x_2(t) + x_1(t)\theta_1(t), \\ \dot{x}_2(t) = -x_1(t) - 2x_2(t) + u(t) + x_2(t)\theta_2(t) + f(t), \\ y(t) = x_1(t) + \zeta(t), \end{cases} \quad (28)$$

with $f(t) = 5 \sin(2t)$, $x(0) = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, parameter $\theta(t)$ is generated by the output of the linear generator with matrices $H = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$, $\Gamma = \begin{bmatrix} 0 & 1 \\ -36 & 0 \end{bmatrix}$ with initial conditions $\xi(0) = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ and $\zeta(t)$ is measurement noise with normal distribution, mean 0.01, variance 0.001, which is introduced to demonstrate robustness of the proposed approach.

Let us rewrite the equation (28) in the form (1):

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u(t) + \begin{bmatrix} \theta_1(t) & 0 \\ 0 & \theta_2(t) \end{bmatrix} x(t) + f(t)] \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$

then the state observer (9) can be applied. Matrix F is chosen to be Hurwitz,

$$G = B[(CAB)^T CAB]^{-1}(CAB)^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$M = A - GCA^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$L = \text{place}(M^T, C^T, [-15 \quad -10])^T,$$

$$F = M - LC = \begin{bmatrix} -25 & 1 \\ -150 & 0 \end{bmatrix},$$

with initial conditions of the observer $z_2(0) = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}$. The matrix \bar{B} is chosen as $\begin{bmatrix} 1 & 1 \end{bmatrix}$.

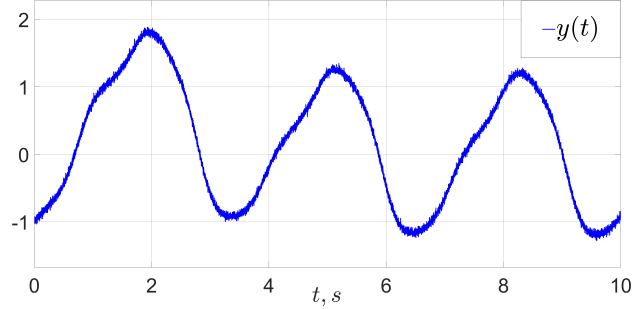


Figure 1: The output signal with measurement noise

We choose the filters as follows

$$\frac{\lambda}{p + \lambda} = \frac{\lambda_1}{p + \lambda_1} = \frac{\lambda_2}{p + \lambda_2} = \frac{\lambda_r}{p + \lambda_r} = \dots = \frac{5}{p + 5}.$$

For the Kreisselmeier's regressor extension scheme, the tuning parameter are selected as

$$h_r = \dots = h_a = 0.5.$$

The direct estimation can be sensitive to noise at the initial time due to the small value of Δ_r, Δ_a . Therefore, the parameters $\varepsilon_r = \varepsilon_a$ is chosen to be 10^{-3} . In the simulation a low-pass filter is applied to (20) and (23) with a parameter $\sigma = 0.7$.

The output signal $y(t)$ measured under the noise $\zeta(t)$ is shown on the Figure 1. Denote $\tilde{x}(t)$ and $\tilde{f}(t) = f(t) - \hat{f}(t)$ the errors of the initial observer error and the external disturbance estimation error, respectively.

The transient processes of simulation of algorithms (18)-(20) and (21)-(23) are shown in the Figures 2, 3.

Remark 6. To reduce the computational complexity it is possible to fix the value of $\hat{x}(0)$ using the modified DREM algorithm [20] as a constant for the predetermined time t_D . Therefore, it is possible to provide convergence with a predetermined time.

To construct a state observer, it is necessary to choose the matrix K to ensure the stability of the closed-loop system (25). In the general case, this requires solving the Ricatti equation (27). In the simple example under

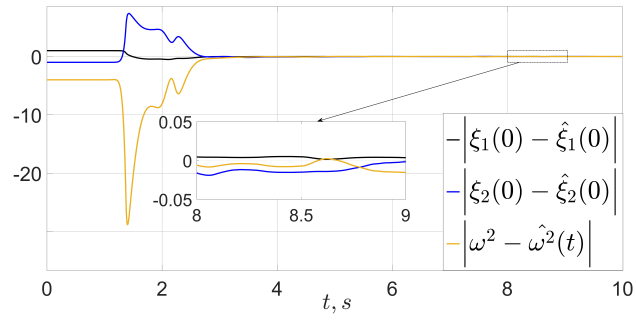


Figure 2: The error in estimating the initial conditions of the time-varying parameter $\xi(0)$ and the external disturbance frequency $\omega(t)$

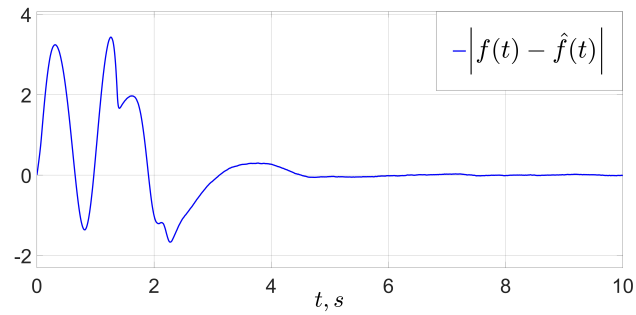


Figure 3: The error in estimating the external disturbance $f(t)$

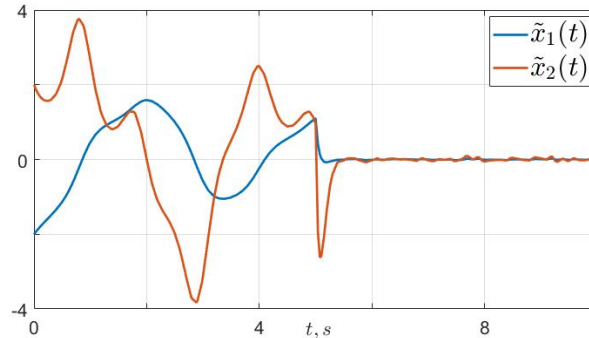


Figure 4: The state observation error $\tilde{x}(t)$

consideration with bounded θ and $\alpha(y)$, we can choose

$$K = \begin{bmatrix} 23 \\ 103 \end{bmatrix}.$$

Figure 4 shows the transient processes of the state vector observation error. The observer is started after finite-time estimation of the unknown parameters and disturbance. Minor deviations of the error from zero are caused by the presence of measurement noise.

6. Conclusion

The paper addressed the problem of unknown input observer synthesis for a class of time-varying uncertain systems under external multiharmonic disturbances. The proposed method provides state vector estimation for systems with arbitrary relative degrees. Additionally, obtained solution allows identification of time-varying parameters and disturbances. All estimates can be derived in a finite time. In the future, the utilization of the proposed unknown input observer with arbitrary relative degrees can be extended to more complex problems, such as systems with unknown nonlinear time varying parameters or nonlinear parameter dependence on the state vector.

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