



**HAL**  
open science

# Flux Reconstruction Type Auxiliary Solvers for a Trefftz Domain Decomposition Method Dealing with Time-Harmonic Electromagnetism

Sébastien Pernet, Matthias Rivet, Sébastien Tordeux

► **To cite this version:**

Sébastien Pernet, Matthias Rivet, Sébastien Tordeux. Flux Reconstruction Type Auxiliary Solvers for a Trefftz Domain Decomposition Method Dealing with Time-Harmonic Electromagnetism. ECCO-MAS 2024 - 9th European Congress on Computational Methods in Applied Sciences and Engineering, Jun 2024, Lisbonne, Portugal. hal-04877044

**HAL Id: hal-04877044**

**<https://inria.hal.science/hal-04877044v1>**

Submitted on 9 Jan 2025

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

# Flux Reconstruction Type Auxiliary Solvers for a Trefftz Domain Decomposition Method Dealing with Time-Harmonic Electromagnetism

Sébastien Pernet<sup>1</sup>, Matthias Rivet<sup>1,2</sup>, Sébastien Tordeux<sup>2</sup>

<sup>1</sup>DTIS, ONERA, Université de Toulouse, 31000, Toulouse, France

<sup>2</sup>EPI Makutu, INRIA, Université de Pau et des Pays de l'Adour, TotalEnergies, CNRS UMR 5142

6th June 2024



## Context of the work

- Increasing need of 3D simulations of time-harmonic electromagnetic waves: **high-frequency**, **heterogeneous** environments, **wide** domains (in terms of wavelengths)...

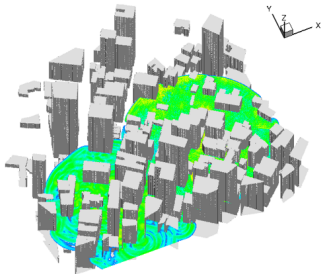


Figure: FDTD simulation on a Manhattan mesh of  $700 \times 600 \times 300\text{m}$ , Thibault Volpert (DEMR ONERA)

Example of computation challenge:  
field radiated by two antennas in Manhattan

- **Large simulation domain.**
- **Wide range of frequencies** (1-18 GHz).

⇒ Need of **efficient methods** to solve such ambitious cases!

## Plan of the talk

1. 3D Maxwell: limitations and Trefftz method
2. FR for wave propagation problems
3. Quasi-Trefftz approach

# The 3D Maxwell problem: current limitations and Trefftz method interests

## Current issues: memory and iterative resolution

**Time-harmonic Maxwell equations** for the electromagnetic field  
 $\mathbf{Y} := (\mathbf{E}, \mathbf{H}) \in [\mathbf{H}(\text{curl}, \Omega)]^6$ :

$$i\kappa\mathbf{M}\mathbf{Y} + \sum_{j=1}^3 \frac{\partial \mathbf{F}^j \mathbf{Y}}{\partial x_j} = \mathbf{0} \text{ in } \Omega, \quad (1)$$

with **impedance Boundary Conditions** (BCs)

$$(\mathbf{n}_{\partial\Omega} \times \mathbf{E}) \times \mathbf{n}_{\partial\Omega} + Z_{\partial\Omega} \mathbf{n}_{\partial\Omega} \times \mathbf{H} = \mathbf{g} \text{ on } \partial\Omega. \quad (2)$$

Many classic methods as Finite Differences [Yee 1966], Finite Elements [Nédélec 1980], Discontinuous Galerkin [Fezoui et al. 2005], but limitations for current applications:

- Wide domains lead to **very large linear systems**.
- Traditional direct solvers induce a **prohibitive memory consumption**.
- Usual numerical methods are **not well adapted to iterative resolution**.

## Considered numerical method: the Trefftz approach

1. Belongs to the Discontinuous Galerkin methods, with the Galerkin space  $\mathbb{X} := \prod_{T \in \mathcal{T}_h} \mathbb{X}_T$  made up of **local solutions** (i.e. in each mesh cell  $T \in \mathcal{T}_h$ ) of the Maxwell equations.

2. Use of the **reciprocity formula** verified in each cell

$$\forall T \in \mathcal{T}_h, \int_{\partial T} \gamma_{\times}^T \mathbf{H}^T \cdot \overline{\gamma_t \mathbf{E}'^T} + \gamma_t \mathbf{E}^T \cdot \overline{\gamma_{\times}^T \mathbf{H}'^T} = 0, \quad (3)$$

in addition to the introduction of **numerical traces** [Sirdey 2022]:

Find  $\mathbf{Y} = (\mathbf{E}, \mathbf{H}) \in \mathbb{X}$  such that  $\forall \mathbf{Y}' = (\mathbf{E}', \mathbf{H}') \in \mathbb{X}$ ,

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \widehat{\gamma_{\times}^T \mathbf{H}^T} \cdot \overline{\gamma_t \mathbf{E}'^T} + \widehat{\gamma_t \mathbf{E}^T} \cdot \overline{\gamma_{\times}^T \mathbf{H}'^T} = \ell(\mathbf{E}'),$$

where  $\widehat{\cdot}$  stands for *upwind* (or Riemann) **numerical traces** and with

$$\gamma_t \mathbf{u}^T = \mathbf{u}^T - (\mathbf{u}^T \cdot \mathbf{n}_T) \mathbf{n}_T \text{ and } \gamma_{\times}^T \mathbf{u}^T = \mathbf{n}_T \times \mathbf{u}^T.$$

## The Trefftz approach: properties and classic limits

1. The formulation is posed on the **mesh skeleton** and adaptable to an **iterative resolution (contraction property)** [Cessenat, Després 1998].
2. A relevant choice of the numerical traces naturally implies formulation **coercivity**.
3. Basis functions have a **physical meaning**: leads to a **reduced numerical pollution** [Ihlenburg, Babuška 1995].

Limits of the classic Plane Waves (PWs) choice:

- Numerical dependence phenomena: **ill-conditioned basis** [Congreve et al. 2019].
- **No adaptability of the basis** to the local properties: expected singularities, complex interference phenomena...

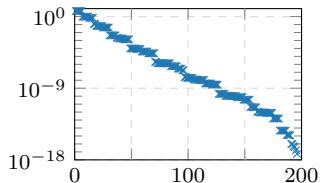


Figure: Eigenvalues of the mass matrix for 196 PWs.

⇒ Possible algebraic adaptations [Barucq et al. 2021].

⇒ Introduction of **well-conditioned basis** by parameterising the space of local solutions by **polynomial boundary conditions**.



## Quasi-Trefftz approach

### Definition of the local problem

A local solution  $\mathbf{Y}^T$  in  $T$  is parameterised by a tangential field  $\mathbf{g}^T \in \mathbb{V}$  such that

$$\gamma_t \mathbf{E}^T + Z_{\partial T} \gamma_{\times}^T \mathbf{H}^T = \mathbf{g}^T \text{ on } \partial T. \quad (4)$$

$\implies$  Consider a finite-dimensional subspace  $\mathbb{V}_h \subset \mathbb{V}$  and the associated Maxwell solutions [Fure et al. 2020].

Construction of  $\mathbb{V}_h$ :

- Consider a cell  $T \in \mathcal{T}_h$ .
- Define a mesh of  $\mathcal{T}(\partial T)$ .
- Consider a piecewise polynomial  $\mathbf{g}^T$  of degree  $k_{QT}$  on  $\mathcal{T}(\partial T)$ .

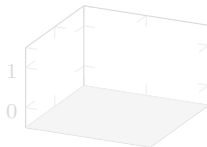


Figure: 2D example of the mesh of  $\partial T$  for  $k_{QT} = 1$ .

$\implies$  **Associated Maxwell solutions are unknown:** need of an **auxiliary solver** to compute approximations!

$\implies$  **Other Quasi-Trefftz approaches:** generalised PWs [Imbert-Gérard, Després 2013], polynomial Quasi-Trefftz functions [Imbert-Gérard, Sylvand 2023], Embedded Trefftz DG [Lehrenfeld, Stocker 2023], BEM local solver [Barucq et al. 2017]...

## Quasi-Trefftz approach

### Definition of the local problem

A local solution  $\mathbf{Y}^T$  in  $T$  is parameterised by a tangential field  $\mathbf{g}^T \in \mathbb{V}$  such that

$$\gamma_t \mathbf{E}^T + Z_{\partial T} \gamma_{\times}^T \mathbf{H}^T = \mathbf{g}^T \text{ on } \partial T. \quad (4)$$

$\implies$  Consider a finite-dimensional subspace  $\mathbb{V}_h \subset \mathbb{V}$  and the associated Maxwell solutions [Fure et al. 2020].

Construction of  $\mathbb{V}_h$ :

- Consider a cell  $T \in \mathcal{T}_h$ .
- Define a mesh of  $\mathcal{T}(\partial T)$ .
- Consider a piecewise polynomial  $\mathbf{g}^T$  of degree  $k_{QT}$  on  $\mathcal{T}(\partial T)$ .

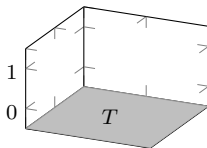


Figure: 2D example of the mesh of  $\partial T$  for  $k_{QT} = 1$ .

$\implies$  Associated Maxwell solutions are unknown: need of an auxiliary solver to compute approximations!

$\implies$  Other Quasi-Trefftz approaches: generalised PWs [Imbert-Gérard, Després 2013], polynomial Quasi-Trefftz functions [Imbert-Gérard, Sylvand 2023], Embedded Trefftz DG [Lehrenfeld, Stocker 2023], BEM local solver [Barucq et al. 2017]...

## Quasi-Trefftz approach

### Definition of the local problem

A local solution  $\mathbf{Y}^T$  in  $T$  is parameterised by a tangential field  $\mathbf{g}^T \in \mathbb{V}$  such that

$$\gamma_t \mathbf{E}^T + Z_{\partial T} \gamma_{\times}^T \mathbf{H}^T = \mathbf{g}^T \text{ on } \partial T. \quad (4)$$

$\implies$  Consider a finite-dimensional subspace  $\mathbb{V}_h \subset \mathbb{V}$  and the associated Maxwell solutions [Fure et al. 2020].

Construction of  $\mathbb{V}_h$ :

- Consider a cell  $T \in \mathcal{T}_h$ .
- Define a mesh of  $\mathcal{T}(\partial T)$ .
- Consider a piecewise polynomial  $\mathbf{g}^T$  of degree  $k_{QT}$  on  $\mathcal{T}(\partial T)$ .

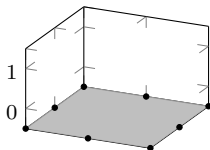


Figure: 2D example of the mesh of  $\partial T$  for  $k_{QT} = 1$ .

$\implies$  Associated Maxwell solutions are unknown: need of an auxiliary solver to compute approximations!

$\implies$  Other Quasi-Trefftz approaches: generalised PWs [Imbert-Gérard, Després 2013], polynomial Quasi-Trefftz functions [Imbert-Gérard, Sylvand 2023], Embedded Trefftz DG [Lehrenfeld, Stocker 2023], BEM local solver [Barucq et al. 2017]...

## Quasi-Trefftz approach

### Definition of the local problem

A local solution  $\mathbf{Y}^T$  in  $T$  is parameterised by a tangential field  $\mathbf{g}^T \in \mathbb{V}$  such that

$$\gamma_t \mathbf{E}^T + Z_{\partial T} \gamma_{\mathbf{x}}^T \mathbf{H}^T = \mathbf{g}^T \text{ on } \partial T. \quad (4)$$

$\implies$  Consider a finite-dimensional subspace  $\mathbb{V}_h \subset \mathbb{V}$  and the associated Maxwell solutions [Fure et al. 2020].

Construction of  $\mathbb{V}_h$ :

- Consider a cell  $T \in \mathcal{T}_h$ .
- Define a mesh of  $\mathcal{T}(\partial T)$ .
- Consider a piecewise polynomial  $\mathbf{g}^T$  of degree  $k_{QT}$  on  $\mathcal{T}(\partial T)$ .

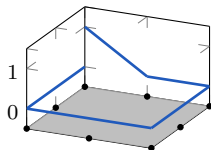


Figure: 2D example of the mesh of  $\partial T$  for  $k_{QT} = 1$ .

$\implies$  Associated Maxwell solutions are unknown: need of an auxiliary solver to compute approximations!

$\implies$  Other Quasi-Trefftz approaches: generalised PWs [Imbert-Gérard, Després 2013], polynomial Quasi-Trefftz functions [Imbert-Gérard, Sylvand 2023], Embedded Trefftz DG [Lehrenfeld, Stocker 2023], BEM local solver [Barucq et al. 2017]...

## Quasi-Trefftz approach

### Definition of the local problem

A local solution  $\mathbf{Y}^T$  in  $T$  is parameterised by a tangential field  $\mathbf{g}^T \in \mathbb{V}$  such that

$$\gamma_t \mathbf{E}^T + Z_{\partial T} \gamma_{\times}^T \mathbf{H}^T = \mathbf{g}^T \text{ on } \partial T. \quad (4)$$

⇒ Consider a finite-dimensional subspace  $\mathbb{V}_h \subset \mathbb{V}$  and the associated Maxwell solutions [Fure et al. 2020].

Construction of  $\mathbb{V}_h$ :

- Consider a cell  $T \in \mathcal{T}_h$ .
- Define a mesh of  $\mathcal{J}(\partial T)$ .
- Consider a piecewise polynomial  $\mathbf{g}^T$  of degree  $k_{QT}$  on  $\mathcal{J}(\partial T)$ .

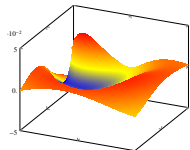


Figure: 2D example of the mesh of  $\partial T$  for  $k_{QT} = 1$ .

⇒ Associated Maxwell solutions are unknown: need of an auxiliary solver to compute approximations!

⇒ Other Quasi-Trefftz approaches: generalised PWs [Imbert-Gérard, Després 2013], polynomial Quasi-Trefftz functions [Imbert-Gérard, Sylvand 2023], Embedded Trefftz DG [Lehrenfeld, Stocker 2023], BEM local solver [Barucq et al. 2017]...

## Quasi-Trefftz approach

### Definition of the local problem

A local solution  $\mathbf{Y}^T$  in  $T$  is parameterised by a tangential field  $\mathbf{g}^T \in \mathbb{V}$  such that

$$\gamma_t \mathbf{E}^T + Z_{\partial T} \gamma_{\times}^T \mathbf{H}^T = \mathbf{g}^T \text{ on } \partial T. \quad (4)$$

$\implies$  Consider a finite-dimensional subspace  $\mathbb{V}_h \subset \mathbb{V}$  and the associated Maxwell solutions [Fure et al. 2020].

Construction of  $\mathbb{V}_h$ :

- Consider a cell  $T \in \mathcal{T}_h$ .
- Define a mesh of  $\mathcal{T}(\partial T)$ .
- Consider a piecewise polynomial  $\mathbf{g}^T$  of degree  $k_{QT}$  on  $\mathcal{T}(\partial T)$ .

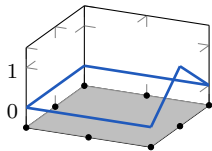


Figure: 2D example of the mesh of  $\partial T$  for  $k_{QT} = 1$ .

$\implies$  Associated Maxwell solutions are unknown: need of an auxiliary solver to compute approximations!

$\implies$  Other Quasi-Trefftz approaches: generalised PWs [Imbert-Gérard, Després 2013], polynomial Quasi-Trefftz functions [Imbert-Gérard, Sylvand 2023], Embedded Trefftz DG [Lehrenfeld, Stocker 2023], BEM local solver [Barucq et al. 2017]...

## Quasi-Trefftz approach

### Definition of the local problem

A local solution  $\mathbf{Y}^T$  in  $T$  is parameterised by a tangential field  $\mathbf{g}^T \in \mathbb{V}$  such that

$$\gamma_t \mathbf{E}^T + Z_{\partial T} \gamma_{\times}^T \mathbf{H}^T = \mathbf{g}^T \text{ on } \partial T. \quad (4)$$

⇒ Consider a finite-dimensional subspace  $\mathbb{V}_h \subset \mathbb{V}$  and the associated Maxwell solutions [Fure et al. 2020].

Construction of  $\mathbb{V}_h$ :

- Consider a cell  $T \in \mathcal{T}_h$ .
- Define a mesh of  $\mathcal{J}(\partial T)$ .
- Consider a piecewise polynomial  $\mathbf{g}^T$  of degree  $k_{QT}$  on  $\mathcal{J}(\partial T)$ .

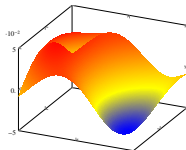


Figure: 2D example of the mesh of  $\partial T$  for  $k_{QT} = 1$ .

⇒ Associated Maxwell solutions are unknown: need of an auxiliary solver to compute approximations!

⇒ Other Quasi-Trefftz approaches: generalised PWs [Imbert-Gérard, Després 2013], polynomial Quasi-Trefftz functions [Imbert-Gérard, Sylvand 2023], Embedded Trefftz DG [Lehrenfeld, Stocker 2023], BEM local solver [Barucq et al. 2017]...

## Quasi-Trefftz approach

### Definition of the local problem

A local solution  $\mathbf{Y}^T$  in  $T$  is parameterised by a tangential field  $\mathbf{g}^T \in \mathbb{V}$  such that

$$\gamma_t \mathbf{E}^T + Z_{\partial T} \gamma_{\times}^T \mathbf{H}^T = \mathbf{g}^T \text{ on } \partial T. \quad (4)$$

⇒ Consider a finite-dimensional subspace  $\mathbb{V}_h \subset \mathbb{V}$  and the associated Maxwell solutions [Fure et al. 2020].

Construction of  $\mathbb{V}_h$ :

- Consider a cell  $T \in \mathcal{T}_h$ .
- Define a mesh of  $\mathcal{T}(\partial T)$ .
- Consider a piecewise polynomial  $\mathbf{g}^T$  of degree  $k_{QT}$  on  $\mathcal{T}(\partial T)$ .

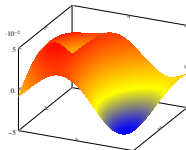


Figure: 2D example of the mesh of  $\partial T$  for  $k_{QT} = 1$ .

⇒ **Associated Maxwell solutions are unknown:** need of an **auxiliary solver** to compute approximations!

⇒ **Other Quasi-Trefftz approaches:** generalised PWs [Imbert-Gérard, Després 2013], polynomial Quasi-Trefftz functions [Imbert-Gérard, Sylvand 2023], Embedded Trefftz DG [Lehrenfeld, Stocker 2023], BEM local solver [Barucq et al. 2017]...



# The Flux Reconstruction method for wave propagation problems

## FR principle

Consider the homogeneous **1D model system** on  $\Omega = [0, L]$ :  
Find  $\mathbf{y} = (u, v) : \Omega \rightarrow \mathbb{C}^2$  such that

$$\begin{cases} i\kappa\mathbf{y} + \frac{d\phi}{dx} = \mathbf{0}, \\ \phi = \mathbf{F}\mathbf{y}, \end{cases} \quad \text{in } \Omega \text{ with } \mathbf{F} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad (5)$$

and the BCs

$$\begin{cases} u(0) - Z_1 v(0) = g_1, \\ u(L) + Z_2 v(L) = g_2. \end{cases} \quad (6)$$



Presentation for a **linear** equation!

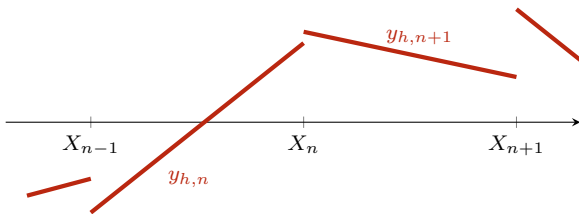
$\implies$  Copy this equation at the discrete level on each cell of the mesh [Huynh 2007]:

$$\forall n \in \llbracket 1, N \rrbracket, i\kappa\mathbf{y}_h + \frac{d\tilde{\phi}_h}{dx} = \mathbf{0} \text{ in } [X_{n-1}, X_n]. \quad (7)$$

## FR construction steps and properties

$$\forall n \in \llbracket 1, N \rrbracket, i\kappa \mathbf{y}_h + \frac{d\tilde{\phi}_h}{dx} = \mathbf{0} \text{ in } [X_{n-1}, X_n]. \quad (8)$$

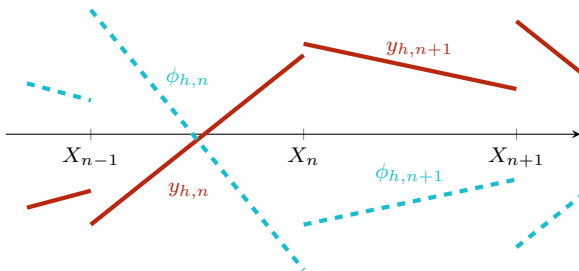
1. Approximation  $\mathbf{y}_h$  as piecewise polynomial of degree  $k_{FR}$ .
2. Natural flux approximation  $\phi_h = \mathbf{F}\mathbf{y}_h$ .
3. Numerical fluxes  $\hat{\phi}_n$  as a mean of  $\phi_n^-$  and  $\phi_n^+$ .
4. Correction polynomials  $P_n^{\rightarrow}$  and  $P_n^{\leftarrow}$  of degree  $k_{FR} + 1$ .
5. Correction  $\tilde{\phi}_{h,n} = \phi_{h,n} + \delta_{n-1}^{\rightarrow} P_n^{\rightarrow} + \delta_n^{\leftarrow} P_n^{\leftarrow}$ .



## FR construction steps and properties

$$\forall n \in \llbracket 1, N \rrbracket, i\kappa \mathbf{y}_h + \frac{d\tilde{\phi}_h}{dx} = \mathbf{0} \text{ in } [X_{n-1}, X_n]. \quad (8)$$

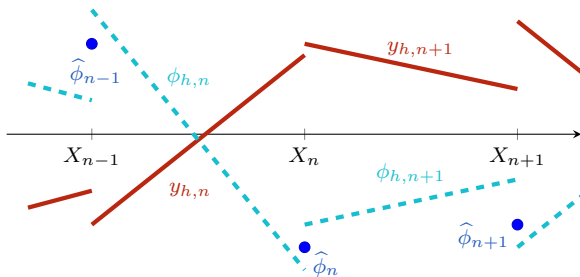
1. Approximation  $\mathbf{y}_h$  as piecewise polynomial of degree  $k_{FR}$ .
2. Natural flux approximation  $\phi_h = \mathbf{F}\mathbf{y}_h$ .
3. Numerical fluxes  $\hat{\phi}_n$  as a mean of  $\phi_n^-$  and  $\phi_n^+$ .
4. Correction polynomials  $P_n^{\rightarrow}$  and  $P_n^{\leftarrow}$  of degree  $k_{FR} + 1$ .
5. Correction  $\tilde{\phi}_{h,n} = \phi_{h,n} + \delta_{n-1}^{\rightarrow} P_n^{\rightarrow} + \delta_n^{\leftarrow} P_n^{\leftarrow}$ .



## FR construction steps and properties

$$\forall n \in \llbracket 1, N \rrbracket, i\kappa \mathbf{y}_h + \frac{d\tilde{\phi}_h}{dx} = \mathbf{0} \text{ in } [X_{n-1}, X_n]. \quad (8)$$

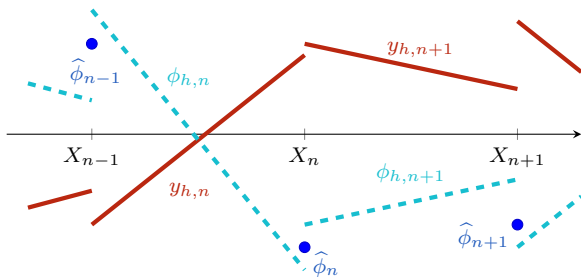
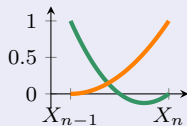
1. Approximation  $\mathbf{y}_h$  as piecewise polynomial of degree  $k_{FR}$ .
2. Natural flux approximation  $\phi_h = \mathbf{F}\mathbf{y}_h$ .
3. Numerical fluxes  $\hat{\phi}_n$  as a mean of  $\phi_n^-$  and  $\phi_n^+$ .
4. Correction polynomials  $P_n^{\rightarrow}$  and  $P_n^{\leftarrow}$  of degree  $k_{FR} + 1$ .
5. Correction  $\tilde{\phi}_{h,n} = \phi_{h,n} + \delta_{n-1}^{\rightarrow} P_n^{\rightarrow} + \delta_n^{\leftarrow} P_n^{\leftarrow}$ .



## FR construction steps and properties

$$\forall n \in \llbracket 1, N \rrbracket, i\kappa \mathbf{y}_h + \frac{d\tilde{\phi}_h}{dx} = \mathbf{0} \text{ in } [X_{n-1}, X_n]. \quad (8)$$

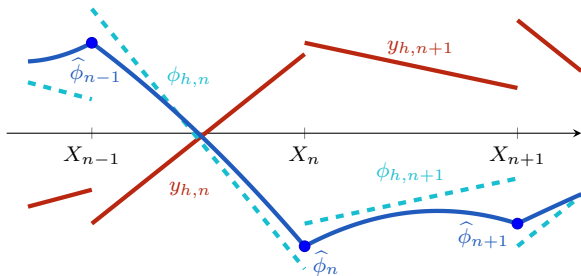
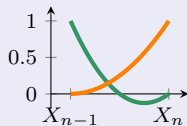
1. Approximation  $\mathbf{y}_h$  as piecewise polynomial of degree  $k_{FR}$ .
2. Natural flux approximation  $\phi_h = \mathbf{F}\mathbf{y}_h$ .
3. Numerical fluxes  $\hat{\phi}_n$  as a mean of  $\phi_n^-$  and  $\phi_n^+$ .
4. Correction polynomials  $P_n^{\rightarrow}$  and  $P_n^{\leftarrow}$  of degree  $k_{FR} + 1$ .
5. Correction  $\tilde{\phi}_{h,n} = \phi_{h,n} + \delta_{n-1}^{\rightarrow} P_n^{\rightarrow} + \delta_n^{\leftarrow} P_n^{\leftarrow}$ .



## FR construction steps and properties

$$\forall n \in \llbracket 1, N \rrbracket, i\kappa \mathbf{y}_h + \frac{d\tilde{\phi}_h}{dx} = \mathbf{0} \text{ in } [X_{n-1}, X_n]. \quad (8)$$

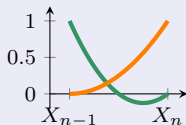
1. Approximation  $\mathbf{y}_h$  as piecewise polynomial of degree  $k_{FR}$ .
2. Natural flux approximation  $\phi_h = \mathbf{F}\mathbf{y}_h$ .
3. Numerical fluxes  $\hat{\phi}_n$  as a mean of  $\phi_n^-$  and  $\phi_n^+$ .
4. Correction polynomials  $P_n^{\rightarrow}$  and  $P_n^{\leftarrow}$  of degree  $k_{FR} + 1$ .
5. Correction  $\tilde{\phi}_{h,n} = \phi_{h,n} + \delta_{n-1}^{\rightarrow} P_n^{\rightarrow} + \delta_n^{\leftarrow} P_n^{\leftarrow}$ .



## FR construction steps and properties

$$\forall n \in \llbracket 1, N \rrbracket, i\kappa \mathbf{y}_h + \frac{d\tilde{\phi}_h}{dx} = \mathbf{0} \text{ in } [X_{n-1}, X_n]. \quad (8)$$

1. Approximation  $\mathbf{y}_h$  as piecewise polynomial of degree  $k_{FR}$ .
2. Natural flux approximation  $\phi_h = \mathbf{F}\mathbf{y}_h$ .
3. Numerical fluxes  $\hat{\phi}_n$  as a mean of  $\phi_n^-$  and  $\phi_n^+$ .
4. Correction polynomials  $P_n^{\rightarrow}$  and  $P_n^{\leftarrow}$  of degree  $k_{FR} + 1$ .
5. Correction  $\tilde{\phi}_{h,n} = \phi_{h,n} + \delta_{n-1}^{\rightarrow} P_n^{\rightarrow} + \delta_n^{\leftarrow} P_n^{\leftarrow}$ .



- **Strong formulation:** no dependence on a quadrature rule.
- Choice of correction polynomials allows to **retrieve usual methods**: classic choice of Radau polynomials (nodal DG).
- Natural **generalisation to 3D** and adaptable to **unstructured meshes**.



## Numerical results

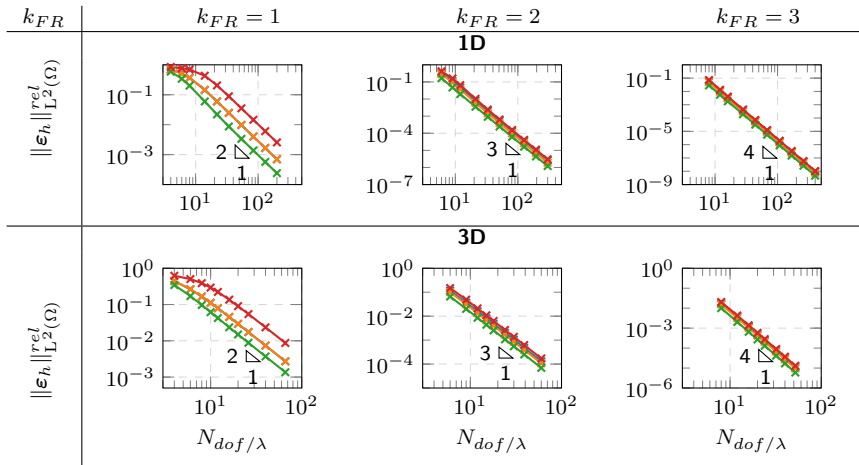


Table:  $h$ -convergence of the FR method in 1D and 3D for different correction polynomials (SD\_CLo (— $\times$ —), SD\_IG (— $\times$ —), FR\_Rad (— $\times$ —) and FR\_G2 (— $\times$ —)).

$\Rightarrow$  **Dependence on the choice** of the correction polynomials: what is the **best choice**?

## Error estimate and optimised correction polynomials

Developed an **explicit a priori error estimate** with incoming BCs, *i.e.*  $Z_1 = Z_2 = 1$  [Rivet et al. 2023]:

$$\|\mathbf{y} - \mathbf{y}_h\|_{L^2(\Omega)}^2 \leq C(\kappa, h, L, P^\rightarrow) (|\mathbf{g}_1|^2 + |\mathbf{g}_2|^2) \quad (9)$$

$\implies$  Choose the correction polynomial  $P^\rightarrow$  as [Rivet et al. 2024]

$$P^\rightarrow = \operatorname{argmin}_{\tilde{P} \in \mathbb{P}^{adm}} C(\kappa, h, L, \tilde{P}). \quad (10)$$

## Error estimate and optimised correction polynomials

⇒ Direction-wise optimisation in the 3D case: **applicable even for general BCs!**

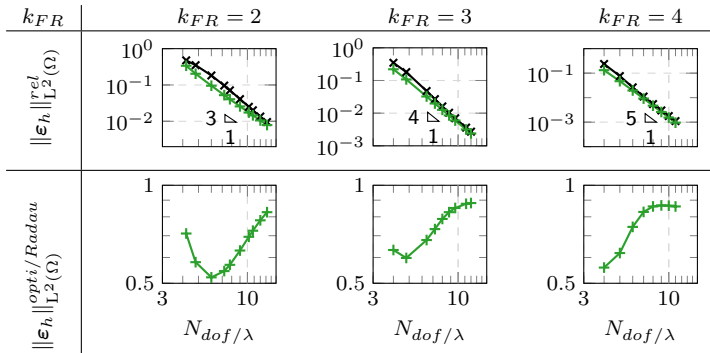


Table: General 3D simulation for Radau (—✖) and optimised (—+) polynomials.

⇒ Possible to find **better polynomials than the classic ones!**

# **A quasi-Trefftz approach based on a Flux Reconstruction auxiliary solver**

## Numerical experiments: basis quality

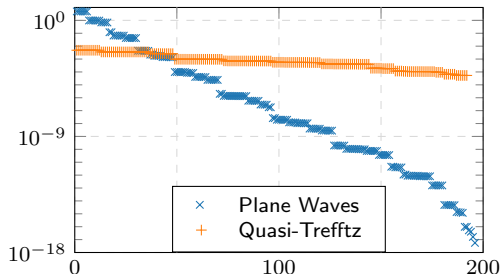


Figure: Eigenvalues of the mass matrix for 196 PWs and 192 Quasi-Trefftz basis functions.

### Basis quality

- ⇒ **Adaptability** to the neighbouring dielectric parameters and BC refinement.
- ⇒ **Avoids spurious numerical modes** and **ill-conditioning** of the mass matrix!

## Numerical experiments: choice of the FR order

Trefftz approach rests on the use of **exact solutions** of the Maxwell equations:

**Is using quasi-solutions enough?  
How to adapt the FR order to the Trefftz one?**

- Uniform Cartesian mesh  $\mathcal{T}_H(\Omega)$  of the domain  $\Omega$ :  $N$  **macro-cells** per direction.
- Uniform Cartesian mesh  $\mathcal{T}_h(\partial T)$  of  $\partial T$ :  $M$  **micro-faces** per direction.
- Piecewise polynomial BCs of degree  $k_{QT} = 1$ .

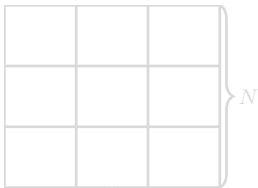


Figure: Mesh  $\mathcal{T}_H(\Omega)$  for  $N = 3$ .



Figure: Mesh  $\mathcal{T}_h(\partial T)$  for  $M = 2$  and  $k_{QT} = 1$ .

$\implies$  No theoretical *a priori* error estimates in  $L^2(\Omega)$ -norm: **numerical convergence in mesh and number of basis functions** [Fure et al. 2020].

## Numerical experiments: choice of the FR order

Trefftz approach rests on the use of **exact solutions** of the Maxwell equations:

**Is using quasi-solutions enough?**  
**How to adapt the FR order to the Trefftz one?**

- Uniform Cartesian mesh  $\mathcal{T}_H(\Omega)$  of the domain  $\Omega$ :  $N$  **macro-cells** per direction.
- Uniform Cartesian mesh  $\mathcal{T}_h(\partial T)$  of  $\partial T$ :  $M$  **micro-faces** per direction.
- Piecewise polynomial BCs of degree  $k_{QT} = 1$ .

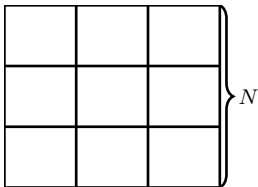


Figure: Mesh  $\mathcal{T}_H(\Omega)$  for  $N = 3$ .

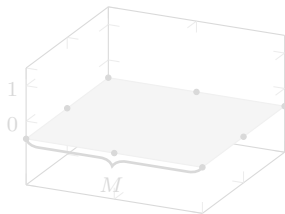


Figure: Mesh  $\mathcal{T}_h(\partial T)$  for  $M = 2$  and  $k_{QT} = 1$ .

⇒ No theoretical *a priori* error estimates in  $L^2(\Omega)$ -norm: **numerical convergence in mesh and number of basis functions** [Fure et al. 2020].

## Numerical experiments: choice of the FR order

Trefftz approach rests on the use of **exact solutions** of the Maxwell equations:

**Is using quasi-solutions enough?  
How to adapt the FR order to the Trefftz one?**

- Uniform Cartesian mesh  $\mathcal{T}_H(\Omega)$  of the domain  $\Omega$ :  $N$  **macro-cells** per direction.
- Uniform Cartesian mesh  $\mathcal{T}_h(\partial T)$  of  $\partial T$ :  $M$  **micro-faces** per direction.
- Piecewise polynomial BCs of degree  $k_{QT} = 1$ .

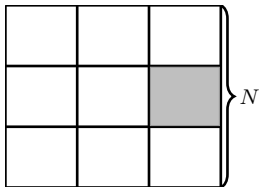


Figure: Mesh  $\mathcal{T}_H(\Omega)$  for  $N = 3$ .

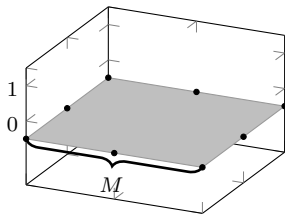


Figure: Mesh  $\mathcal{T}_h(\partial T)$  for  $M = 2$  and  $k_{QT} = 1$ .

⇒ No theoretical *a priori* error estimates in  $L^2(\Omega)$ -norm: **numerical convergence in mesh and number of basis functions** [Fure et al. 2020].



## Numerical experiments: choice of the FR order

Trefftz approach rests on the use of **exact solutions** of the Maxwell equations:

**Is using quasi-solutions enough?  
How to adapt the FR order to the Trefftz one?**

- Uniform Cartesian mesh  $\mathcal{T}_H(\Omega)$  of the domain  $\Omega$ :  $N$  **macro-cells** per direction.
- Uniform Cartesian mesh  $\mathcal{T}_h(\partial T)$  of  $\partial T$ :  $M$  **micro-faces** per direction.
- Piecewise polynomial BCs of degree  $k_{QT} = 1$ .

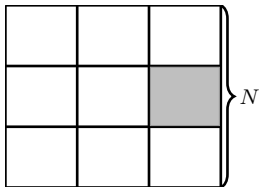


Figure: Mesh  $\mathcal{T}_H(\Omega)$  for  $N = 3$ .

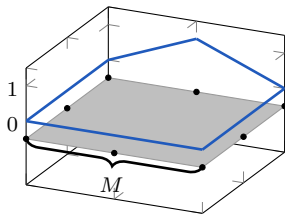


Figure: Mesh  $\mathcal{T}_h(\partial T)$  for  $M = 2$  and  $k_{QT} = 1$ .

⇒ No theoretical *a priori* error estimates in  $L^2(\Omega)$ -norm: **numerical convergence in mesh and number of basis functions** [Fure et al. 2020].

## Numerical experiments: choice of the FR order

Trefftz approach rests on the use of **exact solutions** of the Maxwell equations:

**Is using quasi-solutions enough?  
How to adapt the FR order to the Trefftz one?**

- Uniform Cartesian mesh  $\mathcal{T}_H(\Omega)$  of the domain  $\Omega$ :  $N$  **macro-cells** per direction.
- Uniform Cartesian mesh  $\mathcal{T}_h(\partial T)$  of  $\partial T$ :  $M$  **micro-faces** per direction.
- Piecewise polynomial BCs of degree  $k_{QT} = 1$ .

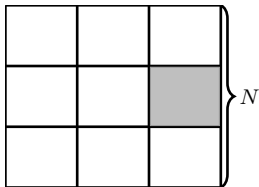


Figure: Mesh  $\mathcal{T}_H(\Omega)$  for  $N = 3$ .

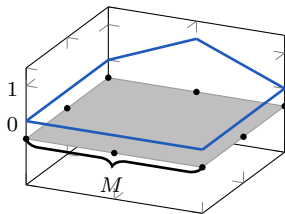


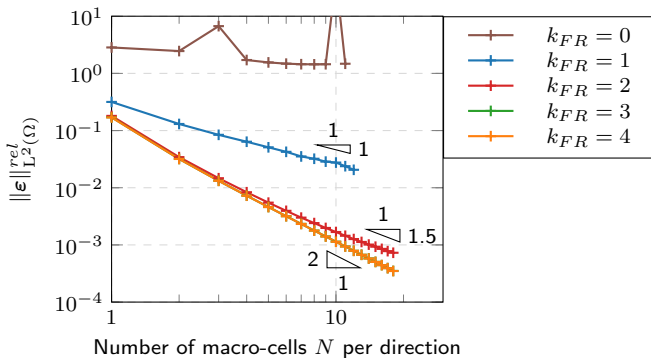
Figure: Mesh  $\mathcal{T}_h(\partial T)$  for  $M = 2$  and  $k_{QT} = 1$ .

$\implies$  No theoretical *a priori* error estimates in  $L^2(\Omega)$ -norm: **numerical convergence in mesh and number of basis functions** [Fure et al. 2020].

## Numerical experiments: mesh-convergence

Is using quasi-solutions enough?  
How to adapt the FR order to the Trefftz one?

⇒ Mesh-convergence: refinement of the macro-mesh  $\mathcal{T}_H(\Omega)$ .

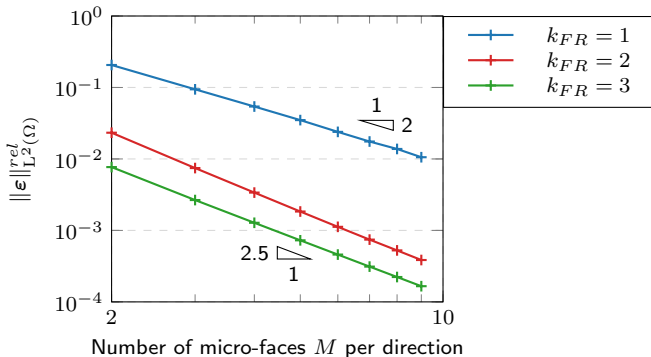


- Need of  $k_{FR} \geq k_{QT} + 2$  for quasi-optimal orders, but **no interest in over-resolving!**

## Numerical experiments: convergence in the number of basis functions

Is using quasi-solutions enough?  
How to adapt the FR order to the Trefftz one?

⇒ **Local basis enrichment** through refinement of the micro-mesh  $\mathcal{T}_h(\partial T)$ .



- Need of  $k_{FR} \geq k_{QT}$  to be sufficiently 'quasi-solution'.
- Improvement for  $k_{FR} \geq k_{QT} + 1$ , but **limited interest in over-resolving!**







## Conclusion and perspectives

- Classic numerical schemes are **not well adapted to wide domain simulations**, contrary to the **Trefftz method**.
- The classic choice of **Plane Waves** leads to limitations: we introduce a **Quasi-Trefftz approach**, in which a FR solver computes approximate Maxwell solutions associated to polynomial BCs in each cell.
- Good numerical **independence properties** of the basis functions.
- Numerically,  $k_{FR} \geq k_{QT} + 2$  for **quasi-optimality**, but **no need to over-resolve!**

### Perspectives:

- **Theoretical well-posedness conditions and a priori error estimates** in progress.
- Adaptable for **heterogeneous** configurations.
- Possible **preconditioning** techniques and algebraic adaptations.
- Possible **hybrid approach** by mixing with PW basis.

# Bibliography

-  [Sirdey 2022] M. Sirdey. 'Méthode itérative de Trefftz pour la simulation d'ondes électromagnétiques en trois dimensions', PHD, Université de Pau et des Pays de l'Adour, 2022.
-  [Yee 1966] Kane Yee. Numerical solution of initial boundary value problems involving maxwell's equations in isotropic media. IEEE Transactions on Antennas and Propagation, 14(3):302–307, 1966.
-  [Nédélec 1980] J. C. Nedelec, 'Mixed finite elements in 3', Numer. Math., vol. 35, no. 3, pp. 315–341, 1980.
-  [Fezoui et al. 2005] L. Fezoui, S. Lanteri, S. Lohrengel, and S. Piperno, 'Convergence and stability of a discontinuous Galerkin time-domain method for the 3D heterogeneous Maxwell equations on unstructured meshes', ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique, vol. 39, no. 6, pp. 1149–1176, 2005.
-  [Cessenat, Després 1998] O. Cessenat, B. Després. 'Application of an Ultra Weak Variational Formulation of Elliptic PDEs to the Two-Dimensional Helmholtz Problem', SIAM Journal on Numerical Analysis, vol. 35, pp. 255-299, 1998.
-  [Ihlenburg, Babuška 1995] F. Ihlenburg, I. Babuška. 'Finite element solution of the Helmholtz equation with high wave number Part I : The h-version of the FEM', Computers Mathematics with Applications, vol. 30, pp. 9-37, 1995.



[Congreve et al. 2019] S. Congreve, J. Gedicke, I. Perugia. 'Numerical Investigation of the Conditioning for Plane Wave Discontinuous Galerkin Methods', in Numerical Mathematics and Advanced Applications ENUMATH 2017, pp. 493-500, 2019, Springer, Cham.



[Barucq et al. 2021] H. Barucq, A. Bendali, J. Diaz, and S. Tordeux, 'Local strategies for improving the conditioning of the plane-wave Ultra-Weak Variational Formulation', Journal of Computational Physics, vol. 441, p. 110449, 2021.



[Fure et al. 2020] H. Fure, S. Pernet, M. Sirdey, S. Tordeux, 'A discontinuous Galerkin Trefftz type method for solving the two dimensional Maxwell equations', SN Partial Differential Equations and Applications, vol. 1, pp. 1-25, 2020.



[Imbert-Gérard, Després 2013] L.-M. Imbert-Gérard, B. Després. 'A generalized plane wave numerical method for smooth non constant coefficients'. IMA Journal of Numerical Analysis 34, 2024.



[Imbert-Gérard, Sylvand 2023] L.-M. Imbert-Gérard, G. Sylvand. 'Three types of quasi-Trefftz functions for the 3D convected Helmholtz equation: construction and approximation properties'. arXiv preprint, 2023.



[Lehrenfeld, Stocker 2023] C. Lehrenfeld and P. Stocker, 'Embedded Trefftz discontinuous Galerkin methods', International Journal for Numerical Methods in Engineering, vol. 124, no. 17, pp. 3637-3661, 2023.





[Barucq et al. 2017] H. Barucq, A. Bendali, M. Fares, V. Mattesi and S. Tordeux, 'A symmetric Trefftz-DG formulation based on a local boundary element method for the solution of the Helmholtz equation'. *Journal of Computational Physics*, Vol. 330, pp. 1069-92, 2017.



[Huynh 2007] H.T. Huynh, 'A Flux Reconstruction Approach to High-Order Schemes Including Discontinuous Galerkin Methods'. *AIAA Paper 2007-4079*, pp. 1-42, 2007.



[Rivet et al. 2023] M. Rivet, S. Pernet, S. Tordeux, 'Flux reconstruction method for time-harmonic linear propagation problems: 1D a priori error analysis'. HAL preprint, 2023.



[Rivet et al. 2024] M. Rivet, S. Pernet, S. Tordeux, 'Optimised Correction Polynomial Functions for the Flux Reconstruction Method in Time-Harmonic Electromagnetism'. HAL preprint, 2024.