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Flux Reconstruction Type Auxiliary Solvers for a Trefftz Domain Decomposition Method Dealing with Time-Harmonic Electromagnetism

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6th June 2024



Context of the work

- Increasing need of 3D simulations of time-harmonic electromagnetic waves: **high-frequency**, **heterogeneous** environments, **wide** domains (in terms of wavelengths)...

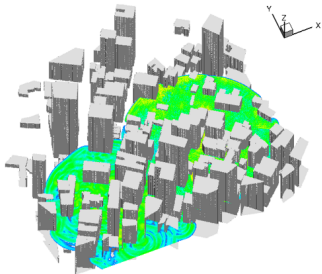


Figure: FDTD simulation on a Manhattan mesh of $700 \times 600 \times 300\text{m}$, Thibault Volpert (DEMR ONERA)

Example of computation challenge:
field radiated by two antennas in Manhattan

- **Large simulation domain.**
- **Wide range of frequencies** (1-18 GHz).

⇒ Need of **efficient methods** to solve such ambitious cases!

Plan of the talk

1. 3D Maxwell: limitations and Trefftz method
2. FR for wave propagation problems
3. Quasi-Trefftz approach

The 3D Maxwell problem: current limitations and Trefftz method interests

Current issues: memory and iterative resolution

Time-harmonic Maxwell equations for the electromagnetic field

$\mathbf{Y} := (\mathbf{E}, \mathbf{H}) \in [\mathbf{H}(\text{curl}, \Omega)]^6$:

$$i\kappa\mathbf{M}\mathbf{Y} + \sum_{j=1}^3 \frac{\partial \mathbf{F}^j \mathbf{Y}}{\partial x_j} = \mathbf{0} \text{ in } \Omega, \quad (1)$$

with **impedance Boundary Conditions** (BCs)

$$(\mathbf{n}_{\partial\Omega} \times \mathbf{E}) \times \mathbf{n}_{\partial\Omega} + Z_{\partial\Omega} \mathbf{n}_{\partial\Omega} \times \mathbf{H} = \mathbf{g} \text{ on } \partial\Omega. \quad (2)$$

Many classic methods as Finite Differences [Yee 1966], Finite Elements [Nédélec 1980], Discontinuous Galerkin [Fezoui et al. 2005], but limitations for current applications:

- Wide domains lead to **very large linear systems**.
- Traditional direct solvers induce a **prohibitive memory consumption**.
- Usual numerical methods are **not well adapted to iterative resolution**.

Considered numerical method: the Trefftz approach

1. Belongs to the Discontinuous Galerkin methods, with the Galerkin space $\mathbb{X} := \prod_{T \in \mathcal{T}_h} \mathbb{X}_T$ made up of **local solutions** (i.e. in each mesh cell $T \in \mathcal{T}_h$) of the Maxwell equations.

2. Use of the **reciprocity formula** verified in each cell

$$\forall T \in \mathcal{T}_h, \int_{\partial T} \gamma_{\times}^T \mathbf{H}^T \cdot \overline{\gamma_t \mathbf{E}'^T} + \gamma_t \mathbf{E}^T \cdot \overline{\gamma_{\times}^T \mathbf{H}'^T} = 0, \quad (3)$$

in addition to the introduction of **numerical traces** [Sirdey 2022]:

Find $\mathbf{Y} = (\mathbf{E}, \mathbf{H}) \in \mathbb{X}$ such that $\forall \mathbf{Y}' = (\mathbf{E}', \mathbf{H}') \in \mathbb{X}$,

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \widehat{\gamma_{\times}^T \mathbf{H}^T} \cdot \overline{\gamma_t \mathbf{E}'^T} + \widehat{\gamma_t \mathbf{E}^T} \cdot \overline{\gamma_{\times}^T \mathbf{H}'^T} = \ell(\mathbf{E}'),$$

where $\widehat{\cdot}$ stands for *upwind* (or Riemann) **numerical traces** and with

$$\gamma_t \mathbf{u}^T = \mathbf{u}^T - (\mathbf{u}^T \cdot \mathbf{n}_T) \mathbf{n}_T \text{ and } \gamma_{\times}^T \mathbf{u}^T = \mathbf{n}_T \times \mathbf{u}^T.$$

The Trefftz approach: properties and classic limits

1. The formulation is posed on the **mesh skeleton** and adaptable to an **iterative resolution (contraction property)** [Cessenat, Després 1998].
2. A relevant choice of the numerical traces naturally implies formulation **coercivity**.
3. Basis functions have a **physical meaning**: leads to a **reduced numerical pollution** [Ihlenburg, Babuška 1995].

Limits of the classic Plane Waves (PWs) choice:

- Numerical dependence phenomena: **ill-conditioned basis** [Congreve et al. 2019].
- **No adaptability of the basis** to the local properties: expected singularities, complex interference phenomena...

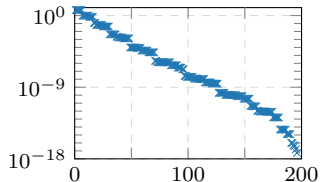


Figure: Eigenvalues of the mass matrix for 196 PWs.

⇒ Possible algebraic adaptations [Barucq et al. 2021].

⇒ Introduction of **well-conditioned basis** by parameterising the space of local solutions by **polynomial boundary conditions**.

Quasi-Trefftz approach

Definition of the local problem

A local solution \mathbf{Y}^T in T is parameterised by a tangential field $\mathbf{g}^T \in \mathbb{V}$ such that

$$\gamma_t \mathbf{E}^T + Z_{\partial T} \gamma_{\times}^T \mathbf{H}^T = \mathbf{g}^T \text{ on } \partial T. \quad (4)$$

\implies Consider a finite-dimensional subspace $\mathbb{V}_h \subset \mathbb{V}$ and the associated Maxwell solutions [Fure et al. 2020].

Construction of \mathbb{V}_h :

- Consider a cell $T \in \mathcal{T}_h$.
- Define a mesh of $\mathcal{T}(\partial T)$.
- Consider a piecewise polynomial \mathbf{g}^T of degree k_{QT} on $\mathcal{T}(\partial T)$.

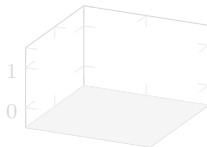


Figure: 2D example of the mesh of ∂T for $k_{QT} = 1$.

\implies **Associated Maxwell solutions are unknown:** need of an **auxiliary solver** to compute approximations!

\implies **Other Quasi-Trefftz approaches:** generalised PWs [Imbert-Gérard, Després 2013], polynomial Quasi-Trefftz functions [Imbert-Gérard, Sylvand 2023], Embedded Trefftz DG [Lehrenfeld, Stocker 2023], BEM local solver [Barucq et al. 2017]...

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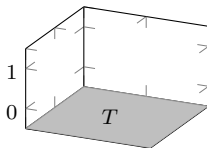


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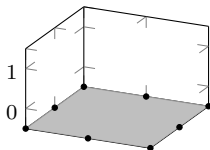


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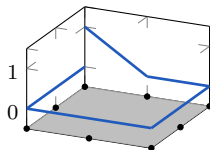


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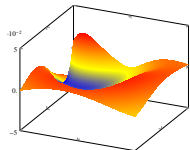


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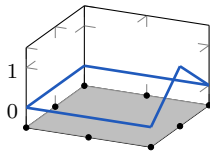


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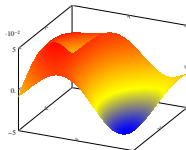


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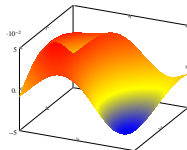


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The Flux Reconstruction method for wave propagation problems

FR principle

Consider the homogeneous **1D model system** on $\Omega = [0, L]$:
Find $\mathbf{y} = (u, v) : \Omega \rightarrow \mathbb{C}^2$ such that

$$\begin{cases} i\kappa\mathbf{y} + \frac{d\phi}{dx} = \mathbf{0}, \\ \phi = \mathbf{F}\mathbf{y}, \end{cases} \quad \text{in } \Omega \text{ with } \mathbf{F} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad (5)$$

and the BCs

$$\begin{cases} u(0) - Z_1 v(0) = g_1, \\ u(L) + Z_2 v(L) = g_2. \end{cases} \quad (6)$$



Presentation for a **linear** equation!

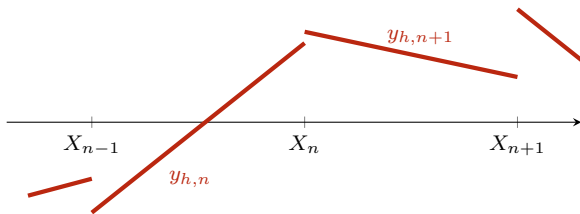
\implies Copy this equation at the discrete level on each cell of the mesh [Huynh 2007]:

$$\forall n \in \llbracket 1, N \rrbracket, i\kappa\mathbf{y}_h + \frac{d\tilde{\phi}_h}{dx} = \mathbf{0} \text{ in } [X_{n-1}, X_n]. \quad (7)$$

FR construction steps and properties

$$\forall n \in \llbracket 1, N \rrbracket, i\kappa \mathbf{y}_h + \frac{d\tilde{\phi}_h}{dx} = \mathbf{0} \text{ in } [X_{n-1}, X_n]. \quad (8)$$

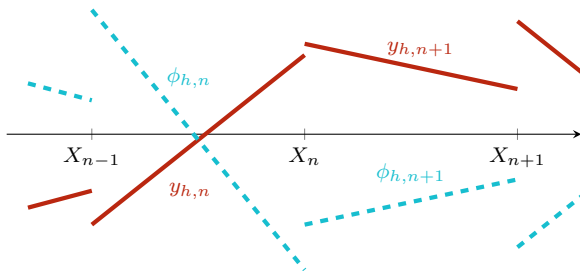
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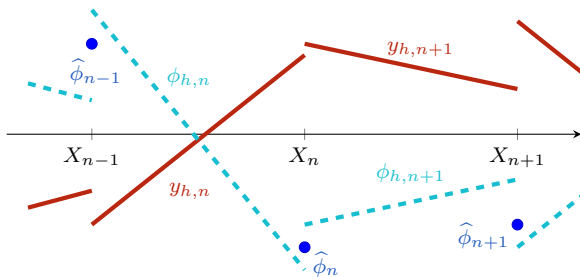
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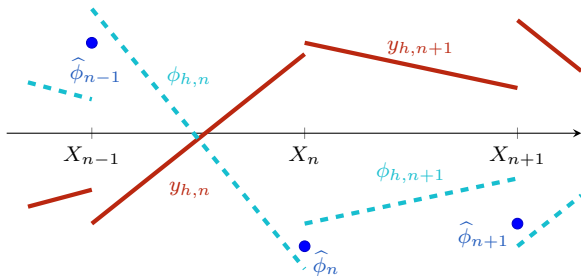
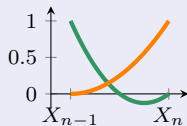
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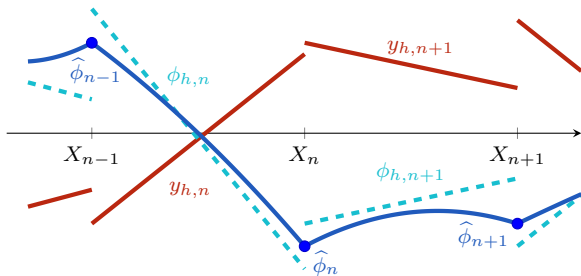
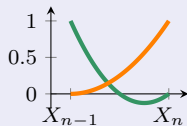
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- **Strong formulation:** no dependence on a quadrature rule.
- Choice of correction polynomials allows to **retrieve usual methods**: classic choice of Radau polynomials (nodal DG).
- Natural **generalisation to 3D** and adaptable to **unstructured meshes**.

Numerical results

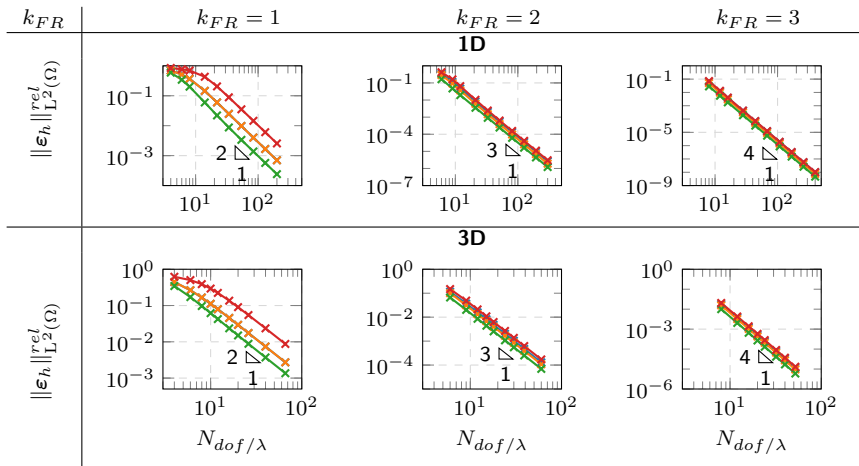


Table: h -convergence of the FR method in 1D and 3D for different correction polynomials (SD_CLo (— \times —), SD_IG (— \times —), FR_Rad (— \times —) and FR_G2 (— \times —)).

\Rightarrow **Dependence on the choice** of the correction polynomials: what is the **best choice**?

Error estimate and optimised correction polynomials

Developed an **explicit a priori error estimate** with incoming BCs, *i.e.* $Z_1 = Z_2 = 1$ [Rivet et al. 2023]:

$$\|\mathbf{y} - \mathbf{y}_h\|_{L^2(\Omega)}^2 \leq C(\kappa, h, L, P^{\rightarrow}) (|\mathbf{g}_1|^2 + |\mathbf{g}_2|^2) \quad (9)$$

\implies Choose the correction polynomial P^{\rightarrow} as [Rivet et al. 2024]

$$P^{\rightarrow} = \operatorname{argmin}_{\tilde{P} \in \mathbb{P}^{adm}} C(\kappa, h, L, \tilde{P}). \quad (10)$$

Error estimate and optimised correction polynomials

⇒ Direction-wise optimisation in the 3D case: **applicable even for general BCs!**

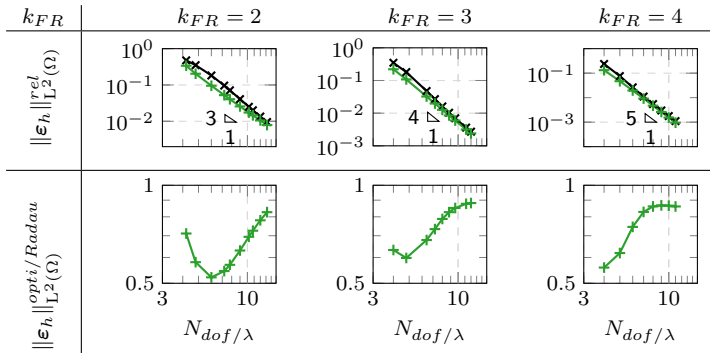


Table: General 3D simulation for Radau (—✖) and optimised (—+) polynomials.

⇒ Possible to find **better polynomials than the classic ones!**

A quasi-Trefftz approach based on a Flux Reconstruction auxiliary solver

Numerical experiments: basis quality

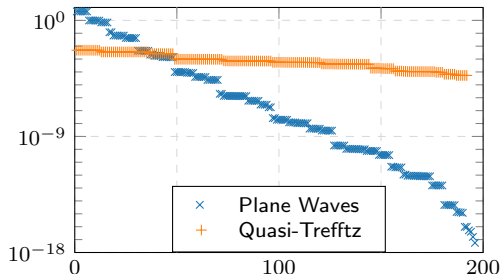


Figure: Eigenvalues of the mass matrix for 196 PWs and 192 Quasi-Trefftz basis functions.

Basis quality

- ⇒ **Adaptability** to the neighbouring dielectric parameters and BC refinement.
- ⇒ **Avoids spurious numerical modes** and **ill-conditioning** of the mass matrix!

Numerical experiments: choice of the FR order

Trefftz approach rests on the use of **exact solutions** of the Maxwell equations:

**Is using quasi-solutions enough?
How to adapt the FR order to the Trefftz one?**

- Uniform Cartesian mesh $\mathcal{T}_H(\Omega)$ of the domain Ω : N **macro-cells** per direction.
- Uniform Cartesian mesh $\mathcal{T}_h(\partial T)$ of ∂T : M **micro-faces** per direction.
- Piecewise polynomial BCs of degree $k_{QT} = 1$.

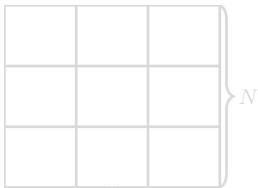


Figure: Mesh $\mathcal{T}_H(\Omega)$ for $N = 3$.



Figure: Mesh $\mathcal{T}_h(\partial T)$ for $M = 2$ and $k_{QT} = 1$.

\implies No theoretical *a priori* error estimates in $L^2(\Omega)$ -norm: **numerical convergence in mesh and number of basis functions** [Fure et al. 2020].

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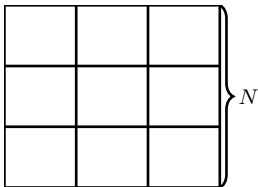


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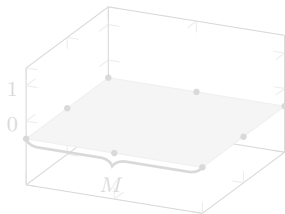


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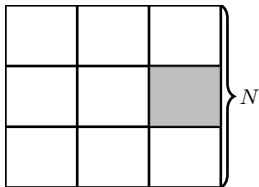


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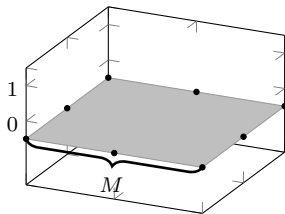


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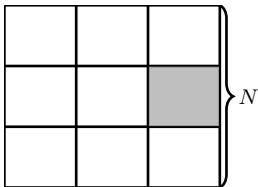


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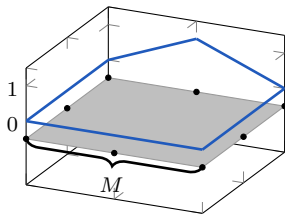


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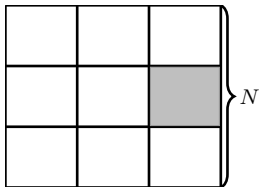


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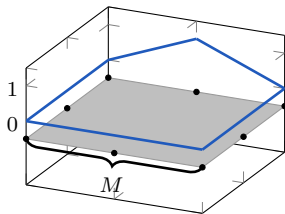


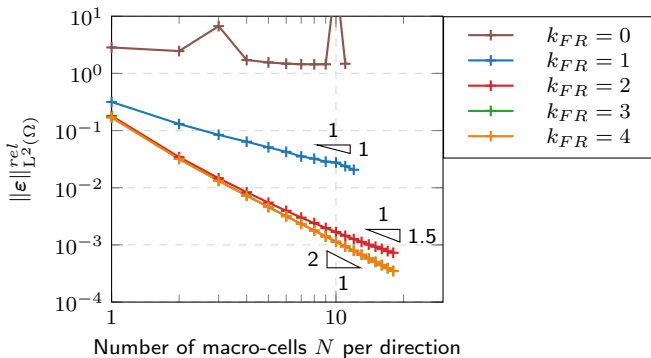
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Numerical experiments: mesh-convergence

Is using quasi-solutions enough?
How to adapt the FR order to the Trefftz one?

⇒ Mesh-convergence: refinement of the macro-mesh $\mathcal{T}_H(\Omega)$.

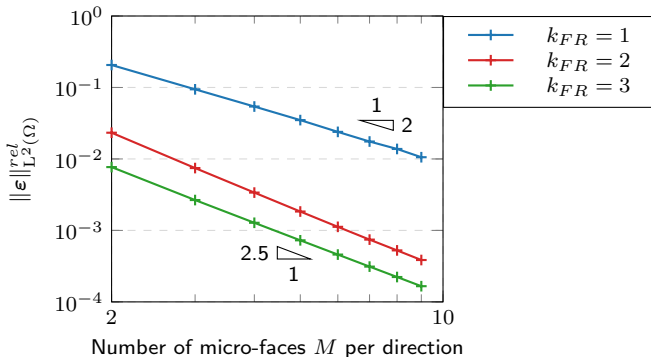


- Need of $k_{FR} \geq k_{QT} + 2$ for quasi-optimal orders, but **no interest in over-resolving!**

Numerical experiments: convergence in the number of basis functions

Is using quasi-solutions enough?
How to adapt the FR order to the Trefftz one?

⇒ **Local basis enrichment** through refinement of the micro-mesh $\mathcal{T}_h(\partial T)$.



- Need of $k_{FR} \geq k_{QT}$ to be **sufficiently 'quasi-solution'**.
- Improvement for $k_{FR} \geq k_{QT} + 1$, but **limited interest in over-resolving!**







Conclusion and perspectives

- Classic numerical schemes are **not well adapted to wide domain simulations**, contrary to the **Trefftz method**.
- The classic choice of **Plane Waves** leads to limitations: we introduce a **Quasi-Trefftz approach**, in which a FR solver computes approximate Maxwell solutions associated to polynomial BCs in each cell.
- Good numerical **independence properties** of the basis functions.
- Numerically, $k_{FR} \geq k_{QT} + 2$ for **quasi-optimality**, but **no need to over-resolve!**

Perspectives:

- **Theoretical well-posedness conditions and a priori error estimates** in progress.
- Adaptable for **heterogeneous** configurations.
- Possible **preconditioning** techniques and algebraic adaptations.
- Possible **hybrid approach** by mixing with PW basis.

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