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Abstract

We propose a new algorithm to compute the topology of a real algebraic space curve. The novelties of this algorithm are a new technique to achieve the lifting step which recovers points of the space curve in each plane fiber from several projections and a weaker notion of generic position. As distinct to previous work, our *sweep generic position* does not require that x-critical points have different x-coordinates. The complexity of achieving this sweep generic position property is thus no longer a bottleneck in term of complexity. The bit complexity of our algorithm is $\widetilde{\mathcal{O}}(d^{18} + d^{17}\tau)$ where d and τ bound the degree and the bitsize of the integer coefficients, respectively, of the defining polynomials of the curve and polylogarithmic factors are ignored. To the best of our knowledge, this improves upon the best currently known results at least by a factor of d^2.

Keywords: Real algebraic space curve, Topology, Bit complexity.

1. Introduction

Algebraic curves are widely used in computer aided geometric design and geometric modeling. For example, implicit surface-surface intersection computation is strongly related to algebraic space curves. Algebraic curves are also basic research objects in classical algebraic geometry. One basic task in the study of an algebraic curve is to compute its topology. Computing the topologies of an algebraic plane curve and an algebraic space curve are fundamental steps to compute the topology of an algebraic surface [10, 16]. Furthermore, some control problems implicitly need the topology information of algebraic (space) curves [29].

Previous work. There have been many papers which studied the computation of the topology of an algebraic plane curve, see for instance [1, 3, 5, 8, 9, 13, 18, 19, 22, 23, 27, 33], but only a few studying the topology of an algebraic space curve [2, 12, 14, 20, 21, 25]. Almost all the existing work [2, 12, 14, 20, 25] computing the topology of an algebraic space curve requires the curve to be in a so-called generic position. Although the definitions of generic positions vary in the literature, they all include the condition that the x-critical points have different x-coordinates. Checking whether an algebraic space curve is in generic position is not a trivial task, and finding a shearing of the

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coordinate system so that the sheared curve is in generic position is a complexity bottleneck [14, 24]. Note that shearing a curve may reduce sparsity if present in its defining polynomials. Even though this is unlikely to increase the asymptotic complexity, it may affect the running times of some implementations. The CAD method can compute the topology of an algebraic space curve, regardless it is in generic position or not. On the other hand, the CAD method requires many resultant computations in the lifting step, which is a complexity bottleneck of the method, as explained by Lazard [28].

The complexity for computing the topology of an algebraic plane curve is well studied, the record bound is $\tilde{\mathcal{O}}(d^6 + d^5\tau)$ for an input polynomial of degree d and bitsize τ [15, 27]. Only few results are known for the complexity of computing the topology of an algebraic space curve given by two polynomials of degree d and bitsize τ. Diatta et. al [16] present a bit complexity of $\tilde{\mathcal{O}}(d^{21}\tau)$ under the assumption that the input space curve is in generic position. Cheng et. al [12] propose a method without a generic position hypothesis with complexity $\tilde{\mathcal{O}}(d^{37}\tau)$. Jin and Cheng [24] give a bit complexity of $\tilde{\mathcal{O}}(d^{20} + d^{19}\tau)$ via the computation of a strong generic position [11]. Katsamaki et al. [26] address the case where the curve in given by a parameterizetion instead of implicitly. Their algorithm works for any dimension of the embedding space and does not need any generic position assumption since it mostly computes in the parameter domain. Nonetheless, for the three dimensional case, they do not report an isotopic approximation since they do not handle knots for space curves.

Contributions. Let the space curve $\mathcal{C}$ be defined by $\{(x, y, z) \in \mathbb{R}^3, f(x, y, z) = g(x, y, z) = 0\}$ with f and g integer polynomials of total degree at most d. Our algorithm follows a classical projection/lifting method followed by a sweep plane connection. We denote the square-free part of the resultant of f and g with respect to z by $h_0(x, y)$ and the resultant of $h_0(x, y)$ and $\frac{\partial h_0(x,y)}{\partial y}$ with respect to y by $r(x)$. The curve $\{h_0(x, y) = 0\}$ is thus the projection of $\mathcal{C}$ on the xy-plane and $r(x)$ encodes its x-critical points.

For the lifting step, we compute the $\mathcal{O}(d^2)$ space points of $\mathcal{C}$ on the fibers $\{x = \alpha\}$, where α is a real root of $r(x)$. Our new lifting algorithm uses the following subroutine (Section 3.3.1): given n points with two coordinates but providing only the first coordinates and n linear combinations of the coordinates of these points, one can recover the second coordinates of these n points. To apply this subroutine in a fiber $x = \alpha$, we consider the linear coordinate transformations $\phi_s : (\alpha, y, z) \to (\alpha, y + sz, z)$ for d^2 different values of s. The d^2 linear combinations of the y and z coordinates, $y - sz$, are then given by solving the triangular systems $\{r(x), h_s(x, y)\}$, where h_s is the resultant of the polynomials $f \circ \phi_s$ and $g \circ \phi_s$ with respect to the z-variable (Section 3.3.2).

To ease our sweep-plane connection algorithm in the x-direction, we require that the curve $\mathcal{C}$ has (a) no asymptote in the z-direction, (b) a finite number of points on any x-fiber plane $\{x = \alpha\}$ for $\alpha \in \mathbb{R}$. We show how to shear the coordinate system to achieve this *sweep generic position* (Section 3.1). It is worth noting that we do not require that the x-fibers contain only one x-critical point as in classical generic positions. The complexity of achieving this sweep generic position is thus no longer a bottleneck in term of complexity. The connections between two fibers are recovered using two different plane projections of $\mathcal{C}$ to solve possible ambiguities when two space components project to the same plane component.

Our method does not need the curve to be reduced, that is, the ideal generated by f and g needs not to be radical, our only assumption is that f and g are coprime. This is important for computing the topology of a surface

since its polar curve is, in general, not reduced [16].

Based on the state-of-the-art complexity result in [15] for isolating bivariate triangular systems and computing the topology of algebraic plane curves, we analyze the bit complexity of our algorithm for computing the topology of an algebraic space curve. The bit complexity of our algorithm is $\tilde{\mathcal{O}}(d^{18} + d^{17}\tau)$, where d and τ are the degree bound and the bit size bound of the coefficients of the defining polynomials of the algebraic space curve. To the best of our knowledge, this improves upon the former results by at least a factor of d^2.

2. Notation and preliminaries

Let $\mathbb{R}$ and $\mathbb{C}$ be the fields of real and complex numbers, and let $\mathbb{Z}$ be the ring of integers. For a polynomial $P(x)$ in $R[x]$ with R a ring, the leading coefficient of P with respect to x is denoted $\mathrm{Lc}_x(P)$. Let $h(x, y) \in \mathbb{Z}[x, y]$, we denote the algebraic plane curve defined by $\{h = 0\}$ as $\mathcal{C}_h$. Let p be a point on $\mathcal{C}_h$, we call p an **x-critical point** if $h(p) = \partial_y h(p) = 0$, and a **singular point** if $h(p) = \partial_x h(p) = \partial_y h(p) = 0$, where $\partial_x h$ and $\partial_y h$ are the partial derivatives with respect to x and y.

We always use $\mathcal{C}$ to denote an algebraic space curve defined by two coprime polynomials $f(x, y, z)$ and $g(x, y, z)$ in $\mathbb{Z}[x, y, z]$, that is $\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 | f(x, y, z) = g(x, y, z) = 0\}$ with $\gcd(f, g) = 1$. We denote by d the maximum of the total degrees of f and g. We call **x-fiber** a plane of equation $x = \alpha$ for $\alpha \in \mathbb{R}$.

The curve $\mathcal{C}$ is called **reduced** when the ideal generated by f and g is radical. Our algorithm does not require that $\mathcal{C}$ is reduced and does not compute the radical. The singularities of a variety are geometric features so they are naturally defined from the ideal of the variety. Let the ideal of the curve $\mathcal{C}$ be generated by the polynomials $(f_i)_{i=1,\ldots,n}$ and let J be the Jacobian matrix of the f_i, that is, its rows are the gradients of the f_i. A point P of $\mathcal{C}$ is called **regular** if J has rank 2 at P. A point of $\mathcal{C}$ which is not regular is called a **singular point or singularity**. A point P of $\mathcal{C}$ is called **x-critical** if it is either singular or it is regular and the tangent line of $\mathcal{C}$ at P is in an x-fiber. Note that the tangent line is the kernel of the Jacobian matrix. The x-coordinates of the x-critical points give all the x-fiber planes one may have to consider for a sweep plane algorithm. On the other hand, when projecting the space curve $\mathcal{C}$ to the plane curve $\pi(\mathcal{C})$, an x-critical point of $\mathcal{C}$ may no longer be an x-critical point of $\pi(\mathcal{C})$, where the projection map is

$$\begin{array}{rccc} \pi: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ & (x, y, z) & \longrightarrow & (x, y). \end{array}$$

To better understand this fact, the geometric characterization of a singularity in terms of intersection of local branches is useful. Geometrically, a regular point of a space curve is a point where the curve has a well defined tangent line given by the kernel of the Jacobian matrix J. For a singular point, there are at least 2 (maybe complex) curve branches passing through the point. An x-critical point of $\mathcal{C}$ is called **cylindrical** if its projection under π is not an x-critical point of the plane curve $\mathcal{C}_{h_0}$, where $h_0 = \mathrm{Squarefree}(\mathrm{Res}_z(f, g))$. Note that $\pi(\mathcal{C}) \subset \mathcal{C}_{h_0}$ and the equality holds when the leading coefficients of f and g with respect to z do not vanish simultaneously and one considers the complex points projecting to real points. A cylindrical singular point P of $\mathcal{C}$ occurs when all branches that intersect at P have the same projection and there is no other branch of $\mathcal{C}$ projecting to another branch of $\mathcal{C}_{h_0}$ passing through $\pi(P)$, see Figure 1. A cylindrical regular x-critical point of $\mathcal{C}$ occurs when the branch of $\mathcal{C}$ passing through P has a

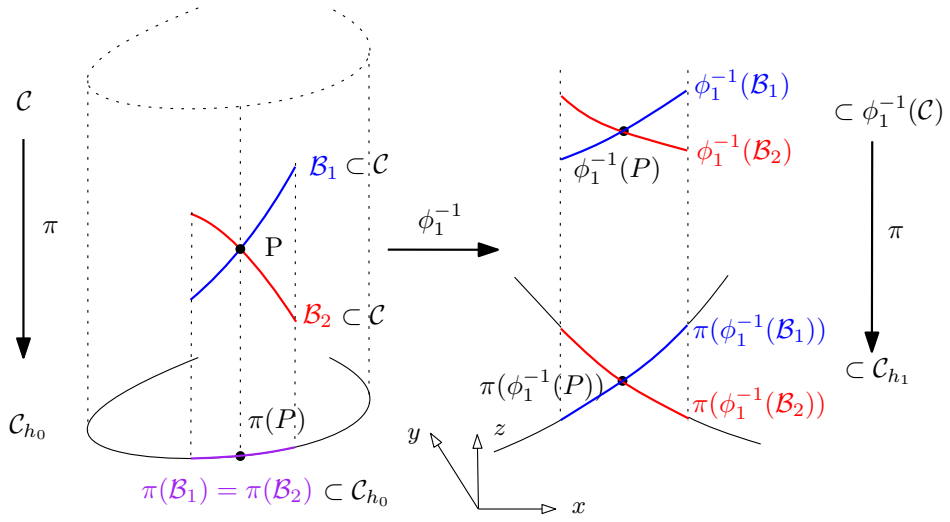

Figure 1: P is a cylindrical singular point of $\mathcal{C}$, its image, $\phi_1^{-1}(P)$, on the sheared curve projects to $\pi(\phi_1^{-1}(P))$ which is a singular point of $\mathcal{C}_{h_1}$.

tangent line in the z-direction, and there is no other branch of $\mathcal{C}$ projecting to another branch of $\mathcal{C}_{h_0}$ passing through $\pi(P)$, see Figure 2. Lemma 3.3 shows that all x-critical points of $\mathcal{C}$ are witnessed either as x-critical points of the projection $\mathcal{C}_{h_0}$ or as x-critical points of $\mathcal{C}_{h_1}$, which is the projection of a shearing of $\mathcal{C}$. Our algorithm uses these two plane curves to identify all the x-critical points of $\mathcal{C}$. We define the shearing function:

$$\begin{array}{rccc} \phi_s : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & (x,y,z) & \longrightarrow & (x, y+sz, z) \end{array} \tag{1}$$

The sheared curve of $\mathcal{C}$ by the shear ϕ_s is defined by $\phi_s^{-1}(\mathcal{C}) = \{(x,y,z) \,|\, f \circ \phi_s(x,y,z) = g \circ \phi_s(x,y,z) = 0\} = \{(x, y-sz, z) \,|\, (x,y,z) \in \mathcal{C}\} = \phi_{-s}(\mathcal{C})$.

3. Algorithm

In this section, we detail our deterministic algorithm for computing the topology of the space curve $\mathcal{C}$ given by the coprime polynomials f and g in $\mathbb{Z}[x,y,z]$. Our algorithm follows a classical projection/lifting method followed by a sweep plane connection:

1. Shear the coordinate system such that the space curve is in a sweep generic position.
2. Projection: Project the space curve $\mathcal{C}$ onto the xy-plane and compute the topology of the plane curve $\mathcal{C}_{h_0}$, where $h_0(x,\, y) = \text{Squarefree}(\text{Res}_z(f,g))$.
3. Lifting: Lift some plane points of $\mathcal{C}_{h_0}$ to obtain the corresponding space points of $\mathcal{C}$.
4. Connection: Connect the space points by line segments to recover the topology of $\mathcal{C}$.

The topology of the curve $\mathcal{C}$ is then implicitly encoded by an embedded graph isotopic to $\mathcal{C}$ in $\mathbb{R}^3$ whose vertices are the computed space points in the x-fibers, and edges are straight line segments connecting the vertices. We explain each step in detail in the following sections.

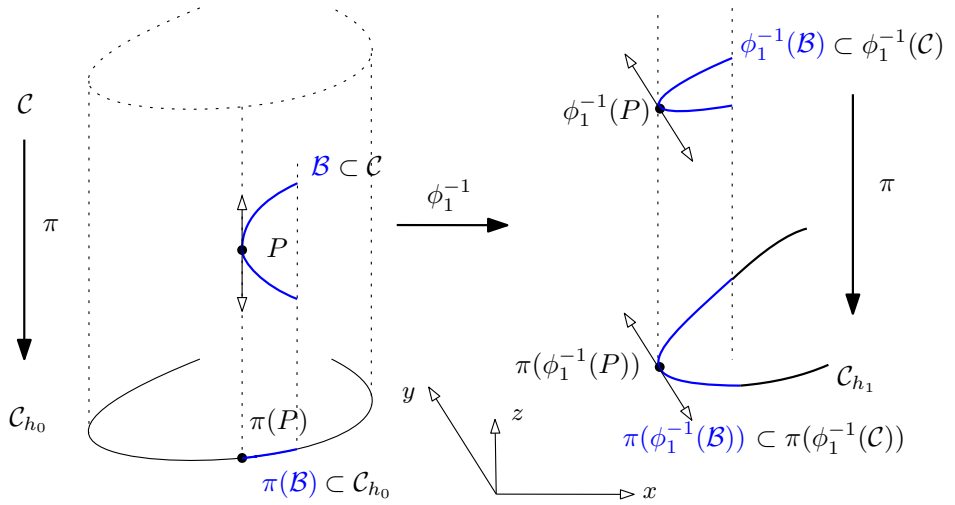

Figure 2: P is a regular cylindrical x-critical point of $\mathcal{C}$, its image, $\phi_1^{-1}(P)$, on the sheared curve projects to $\pi(\phi_1^{-1}(P))$ which is a regular x-critical point of $\mathcal{C}_{h_1}$.

3.1. Sweep generic position

To ease our sweep-plane algorithm in the x-direction, we require that the curve $\mathcal{C}$ has (a) no asymptote in the z-direction, (b) a finite number of points on any x-fiber plane $\{x = \alpha\}$ for $\alpha \in \mathbb{R}$. We show how to achieve these requirements by applying shearings of the coordinate system as detailed in Algorithm 1. Since such shearings do not change the topology of the curve, we then compute the topology of the sheared curve. Note that shearings preserve the fact that the polynomials remain coprime. The proof of correctness of Algorithm 1 follows from the following lemmas.

Lemma 3.1. *The polynomials $\tilde{f}$ and $\tilde{g}$ output by Algorithm 1 are such that (a) the leading coefficient of $\tilde{f}$ with respect to z is a non-zero constant and (b) the resultant of $\tilde{f}$ and $\tilde{g}$ with respect to z has no factor containing only the variable x.*

Proof. Writing $f = \sum_{0\leq i+j+k\leq d} c_{i,j,k}x^iy^jz^k = \sum_{0\leq i+j\leq d} c_{i,j,d-i-j}x^iy^jz^{d-i-j} + \sum_{0\leq i+j+k<d} c_{i,j,k}x^iy^jz^k$ yields that the leading coefficient of $f(x+uz, y+vz, z)$ with respect to z is $L(u,v) = \sum_{0\leq i+j\leq d} c_{i,j,d-i-j}u^iv^j$, which is not the null polynomial since there exists one $c_{i,j,d-i-j} \neq 0$. If for all $\alpha \in \{0,\dots,d\}$, the univariate polynomial $L(\alpha, v)$ is the null polynomial, then the polynomial $\Pi_{\alpha=0}^{\alpha=d}(u-\alpha)$, which is of degree $d+1$, divides $L(u,v)$. This is in contradiction with the fact that L has degree at most d in u. Hence there exists $\alpha \in \{0,\dots,d\}$ such that the univariate polynomial $L(\alpha, v)$ is not the null polynomial, it is of degree at most d in v, so that there exists $\beta \in \{0,\dots,d\}$ such that it does not vanish it. Hence, the choice of (α, β) ensures that $\mathrm{Lc}_z(\hat{f})$ is a constant and since the second shear of Line 8 does not modify this leading coefficient, $\mathrm{Lc}_z(\tilde{f})$ is also a constant.

For point (b), the condition of Line 4 shows that $h(x,y)$ has no factor containing only the variable x. We need not to transform the coordinate system. Thus $\gamma = 0$, which implies $\tilde{f} = \hat{f}$ and $\tilde{g} = \hat{g}$. Then h is the resultant of $\tilde{f}$ and $\tilde{g}$ and it has no factor containing only the variable x according to the condition of Line 4.

If $h(x,y)$ has a factor containing only the variable x, we need to transform the coordinate system to find a γ such that $h(x+\gamma y, y)$ has no such factor. The leading coefficient of $h(x+\gamma y, y)$ with respect to y is the polynomial

Algorithm 1: SGP: Sweep generic position

Input : $f, g \in \mathbb{Z}[x, y, z]$ coprime defining a curve $\mathcal{C}$.

Output: $\tilde{f}, \tilde{g} \in \mathbb{Z}[x, y, z]$ coprime defining a curve isotopic to $\mathcal{C}$ that has (a) no asymptote in the z-direction, (b) a finite number of points on any x-fiber plane $\{x = \alpha\}$ for $\alpha \in \mathbb{R}$.

1 Let $f = \sum_{0 \leq i+j+k \leq d} c_{i,j,k} x^i y^j z^k$, evaluate the bivariate polynomial $\sum_{0 \leq i+j \leq d} c_{i,j,d-i-j} u^i v^j$ on the integer grid $\{0, \ldots, d\} \times \{0, \ldots, d\}$, choose (α, β) such that it does not vanish it.

2 $\hat{f}(x, y, z) = f(x + \alpha z, y + \beta z, z)$, $\hat{g}(x, y, z) = g(x + \alpha z, y + \beta z, z)$.

3 Compute $h(x, y) = \mathrm{Res}_z(\hat{f}(x, y, z), \hat{g}(x, y, z)) = \sum_{i=0,\ldots,2d^2} c_i(x) y^i$.

4 **if** $\gcd(c_0(x), \ldots, c_{2d^2}(x))$ *is a constant* **then**

5 $\quad \gamma = 0$

6 **else**

7 $\quad$ Let $h(x, y) = \sum_{0 \leq i+j \leq d'} c_{i,j} x^i y^j$ with d' the total degree of h. Evaluate the univariate polynomial $\sum_{0 \leq i \leq d'} c_{i,d'-i} s^i$ at integer values starting from 0 until an integer $\gamma \leq d'$ is such that it does not vanish it.

8 **return** $\tilde{f}(x, y, z) = \hat{f}(x + \gamma y, y, z)$, $\tilde{g}(x, y, z) = \hat{g}(x + \gamma y, y, z)$.

$\sum_{0 \leq i \leq d'} c_{i,d'-i} s^i$ as defined in Line 7. A choice of γ such that it does not vanish ensures that this leading is a nonzero constant. Thus $h(x + \gamma y, y)$ has no factor containing only the variable x. By the specialization property of the resultant [4, Prop. 8.48], $h(x + \gamma y, y) = \mathrm{Res}_z(\hat{f}(x, y, z), \hat{g}(x, y, z))(x + \gamma y, y) = \mathrm{Res}_z(\hat{f}(x + \gamma y, y, z), \hat{g}(x + \gamma y, y, z))(x, y) = \mathrm{Res}_z(\tilde{f}(x, y, z), \tilde{g}(x, y, z))(x, y)$, thus $\mathrm{Res}_z(\tilde{f}(x, y, z), \tilde{g}(x, y, z))(x, y)$ has no factor containing only the variable x, which is condition (b).

□

The next lemma shows that the properties of the output of Algorithm 1 established in Lemma 3.1 are sufficient conditions for the correctness of the algorithm.

Lemma 3.2. *If the leading coefficient of f with respect to z is a non-zero constant and the resultant of f and g with respect to z has no factor containing only the variable x, then $\mathcal{C}$ has (a) no asymptote in the z direction and (b) a finite number of points on any x-fiber.*

Proof. An asymptote in the z-direction is a solution to the bivariate system $\{\mathrm{Lc}_z(f), \mathrm{Lc}_z(g)\}$ where $\mathrm{Lc}_z(T)$ is the leading coefficient of $T \in \mathbb{Z}[x, y, z]$ with respect to z. The assumption that $\mathrm{Lc}_z(f)$ a non-zero scalar is thus a sufficient condition to avoid such asymptotes.

For point (b), by contradiction, assume that there exists a fiber $x = \alpha$ which is not finite, that is the system $\{f(\alpha, y, z), g(\alpha, y, z)\}$ is not 0-dimensional. In other words $f(\alpha, y, z)$ and $g(\alpha, y, z)$ have a non-constant gcd $G(y, z)$ in $\mathbb{C}[y, z]$. Let f_1 and g_1 in $\mathbb{C}[y, z]$ be defined such that $f(\alpha, y, z) = G(y, z) f_1(y, z)$ and $g(\alpha, y, z) = G(y, z) g_1(y, z)$.

If the degree of G with respect to z is 0, then its degree with respect to y is non-zero. Let β be a root of this $G(y)$, then $f(\alpha, \beta, z) = G(\beta) f_1(\beta, z) = 0$ so that (α, β) is a common solution to all the coefficients of f with respect

to z and in particular its leading term. This is in contradiction with the hypothesis that this leading coefficient is a constant. One thus has that the degree of G with respect to z is at least 1. The polynomials $f(\alpha, y, z)$ and $g(\alpha, y, z)$ thus have a common factor depending on z which implies that their resultant with respect to z vanishes. On the other hand, since the leading coefficient of f with respect to z is a constant, it does not vanish at α and the specialization property of the resultant yields that for a constant c:

$$\mathrm{Res}_z(f(x,y,z), g(x,y,z))(\alpha, y) = c\,\mathrm{Res}_z(f(\alpha,y,z), g(\alpha,y,z))(y) = 0.$$

The bivariate polynomial $\mathrm{Res}_z(f(x,y,z), g(x,y,z)) \in \mathbb{Z}[x,y]$ identically vanishes at α, thus, $m(x)$, the minimal polynomial of α divides all its coefficients with respect to y. The polynomial $m(x)$ is thus a factor of the resultant $\mathrm{Res}_z(f(x,y,z), g(x,y,z))$. This contradicts the hypothesis and concludes the proof. □

3.2. Bivariate triangular system solving and plane curve topology

As subroutines of our algorithm, we solve bivariate triangular systems and compute the topology of plane curves. There are many algorithms for computing the topology of plane curves, however, most of them shear the coordinate system which is not desirable for our algorithm. To avoid such a change of coordinates, we use the recent results of [15] that achieve the best complexity for computing the topology of a plane curve in its original coordinate system. Also in [15], an algorithm to solve triangular bivariate systems is given with a detailed complexity analysis taking into account the degrees and bitsizes of both polynomials of the system. This latter complexity is critical for the use of amortization in the analysis of our algorithm.

3.3. Lifting: computing the space key points on $\mathcal{C}$

In this step, we compute space points of $\mathcal{C}$ on x-fibers $\{x = \alpha\}$, from the plane projections of $\mathcal{C}$ and shearings of $\mathcal{C}$, where α is a real roots of $r^*(x)$ defined in (3). We call these space points of $\mathcal{C}$ the **space key points of $\mathcal{C}$**. We will use d^2 plane curves $\mathcal{C}_{h_m}$ defined, for the integers $(s_m)_{0\le m\le d^2}$, by the polynomials

$$h_m(x,y) = \mathrm{Squarefree}(\mathrm{Res}_z(f\circ\phi_{s_m}, g\circ\phi_{s_m})) = \mathrm{Squarefree}(\mathrm{Res}_z(f(x, y+s_m z, z), g(x, y+s_m z, z))). \tag{2}$$

In the following, we assume $s_m = m$ for $0 \le m \le d^2$ but keep the variable s_m to emphasize that it is a shear parameter. With lcm standing for the least common multiple, we define

$$r_t(x) = \mathrm{Res}_y(h_t, \partial_y h_t), \text{ for } t = 0 \text{ or } 1, \quad r^*(x) = \mathrm{lcm}(r_0(x), r_1(x)). \tag{3}$$

Lemma 3.3 shows that all x-critical points of $\mathcal{C}$ are witnessed either as x-critical points of the projection $\mathcal{C}_{h_0}$ or as x-critical points of $\mathcal{C}_{h_1}$.

Lemma 3.3. *Let P be an x-critical point of $\mathcal{C}$, then $\pi(P)$ is an x-critical point of $\mathcal{C}_{h_0}$ or $\pi(\phi_1^{-1}(P))$ is an x-critical point of $\mathcal{C}_{h_1}$, where $h_1 = \mathrm{Squarefree}(\mathrm{Res}_z(f\circ\phi_1, g\circ\phi_1))$.*

Proof. If $\pi(P)$ is not an x-critical point of $\mathcal{C}_{h_0}$, then the point P is thus a cylindrical x-critical point of $\mathcal{C}$. Hence, there are two possibilities: P is a cylindrical singular point of $\mathcal{C}$ or P is a cylindrical regular x-critical point.

In the first case, any two distinct branches passing through P are mapped on the sheared curve $\phi_1^{-1}(\mathcal{C})$ on two different branches, whose projections by π will intersect at $\pi(\phi_1^{-1}(P))$, see Figure 1. More explicitly, let $\gamma_1(t) = (x(t), y(t), z_1(t))$ and $\gamma_2(t) = (x(t), y(t), z_2(t))$ be to different local branches of $\mathcal{C}$ at P, defined in a neighborhood of 0 such that $\gamma_1(0) = \gamma_2(0) = P$. One thus has $z_1(t) \neq z_2(t)$ for any $t \neq 0$. The images of these local branches by ϕ_1^{-1} are local branches of $\phi_1^{-1}(\mathcal{C})$ at $\phi_1^{-1}(P)$ defined by $(x(t), y(t) - z_1(t), z_1(t))$ and $(x(t), y(t) - z_2(t), z_2(t))$. The projection by π thus gives the local branches of $\mathcal{C}_{h_1}$ at $\pi(\phi_1^{-1}(P))$ defined by $(x(t), y(t) - z_1(t))$ and $(x(t), y(t) - z_2(t))$ which are crossing at $\pi(\phi_1^{-1}(P))$ since $z_1(t) \neq z_2(t)$ for any $t \neq 0$ and $z_1(0) = z_2(0)$. This implies that $\pi(\phi_1^{-1}(P))$ is a singular point of $\mathcal{C}_{h_1}$. Hence, $\pi(\phi_1^{-1}(P))$ is an x-critical point of $\mathcal{C}_{h_1}$.

In the second case, the tangent line equation to $\mathcal{C}$ at P is $\{x = P_x, y = P_y\}$, where (P_x, P_y, P_z) are the coordinates of P, so that the tangent line to the sheared curve $\phi_1^{-1}(\mathcal{C})$ at $\phi_1^{-1}(P)$ has equation $\{x = P_x, y + z = P_y\}$ that projects by π to the line $\{x = P_x\}$ on the plane, see Figure 2. This line is thus a tangent line to $\mathcal{C}_{h_1}$ at $\pi(\phi_1^{-1}(P))$. If there is no other branch of $\phi_1^{-1}(\mathcal{C})$ that projects to a branch of $\mathcal{C}_{h_1}$ passing through $\pi(\phi_1^{-1}(P))$, this point is a regular x-critical point of $\mathcal{C}_{h_1}$. If there are other branches of $\phi_1^{-1}(\mathcal{C})$ that project to branches of $\mathcal{C}_{h_1}$ passing through $\pi(\phi_1^{-1}(P))$, this point is a singular x-critical point of $\mathcal{C}_{h_1}$.

In both cases, $\pi(\phi_1^{-1}(P))$ is an x-critical point of $\mathcal{C}_{h_1}$, which concludes the proof. □

This lemma thus yields the following corollary.

Corollary 3.4. *The x-coordinates of all the x-critical points of $\mathcal{C}$ are zeros of r^*.*

On the other hand, some zeros of r^* do not witness x-critical points of $\mathcal{C}$ but only x-critical points of $\mathcal{C}_{h_0}$ or $\mathcal{C}_{h_1}$. For instance, when two distinct branches of $\mathcal{C}$ intersect in projection and thus generate a singular point in the projection, this point is sometimes called an apparent singularity [14]. All the x-fibers given by zeros of r^* together with intermediate fibers are enough to recover the topology of $\mathcal{C}$ from the topologies of the two plane curves $\mathcal{C}_{h_0}$ and $\mathcal{C}_{h_1}$ as explained in Section 3.4.

3.3.1. Recovering points from multiple projections

Our lifting step is based on the following combinatorial observation and its corollary. We use in this section the function

$$\begin{array}{rccc} \psi_s : & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \\ & (x, y) & \longrightarrow & x + sy \end{array}$$

Lemma 3.5. *Let $P = \{(\alpha_1, \beta_{1,1}), \ldots, (\alpha_1, \beta_{1,n_1}), (\alpha_2, \beta_{2,1}), \ldots, (\alpha_2, \beta_{2,n_2}), \ldots, (\alpha_l, \beta_{l,1}), \ldots, (\alpha_l, \beta_{l,n_l})\} \subset \mathbb{R}^2$ with cardinality $\sum_{i=1}^{l} n_i = n$. For $s \in \mathbb{R} \setminus 0$, define $\psi_s(P) = \{p_x + sp_y \,|\, (p_x, p_y) \in P\}$ and $T_{s,j} = \{\frac{a - \alpha_j}{s} \,|\, a \in \psi_s(P)\}$. For any $m > n - \min_i\{n_i\}$ nonzero distinct $s_1, \ldots, s_m \in \mathbb{R}$, one has for all $j \in \{1, \ldots, l\}$*

$$\bigcap_{i=1}^{m} T_{s_i,j} = \{\beta_{j,1}, \ldots, \beta_{j,n_j}\}.$$

Proof. It is enough to prove the case $j = 1$, that is, $\bigcap_{i=1}^{m} T_{s_i,1} = \{\beta_{1,1}, \ldots, \beta_{1,n_1}\}$, since the other cases can be proved in a similar way.

Denote $Y = \{\beta_{1,1}, \cdots, \beta_{1,n_1}\}$. One has $\forall t \in \{1, \ldots, n_1\}, \forall i \in \{1, \ldots, m\}, \alpha_1 + s_i\beta_{1,t} \in \psi_{s_i}(P)$, and $\frac{(\alpha_1 + s_i\beta_{1,t}) - \alpha_1}{s_i} = \beta_{1,t} \in T_{s_i,1}$, then $\beta_{1,t} \in \bigcap_{i=1}^m T_{s_i,1}$, thus $Y \subset \bigcap_{i=1}^m T_{s_i,1}$.

Next, we prove $\bigcap_{i=1}^m T_{s_i,1} \subset Y$. Assume that $\bigcap_{i=1}^m T_{s_i,1} \nsubseteq Y$, then there exists $\beta^* \in \bigcap_{i=1}^m T_{s_i,1}$ and $\beta^* \notin Y$. Therefore we have $\forall i \in \{1, \ldots, m\}, \exists t_i \in \{1, \ldots, l\}, k_i \in \{1, \ldots, n_i\}$ s.t. $\frac{(\alpha_{t_i} + s_i\beta_{t_i,k_i}) - \alpha_1}{s_i} = \beta^*$, i.e.,

$$\alpha_{t_i} + s_i\beta_{t_i,k_i} = \alpha_1 + s_i\beta^*. \tag{4}$$

We claim that $\forall t_i \in \{1, \ldots, m\}, t_i \neq 1$. Otherwise, if $t_i = 1$, we can get $\alpha_1 + s_i\beta_{1,k_1} = \alpha_1 + s_i\beta^*$ from (4), i.e., $\beta^* = \beta_{1,k_1} \in Y$. It is a contradiction with $\beta^* \notin Y$. Hence, we have that $s_i, i \in \{1, \ldots, m\}$ has m choices while (t_i, k_i) $(t_i \neq 1)$ has only $n - n_1$ choices. By assumption $m > n - n_1$, thus, there exist two different values $s_{i_1} \neq s_{i_2}$ and the same values t_{i^*}, k_{i^*}, where $1 \leq i_1 \neq i_2 \leq m$, $1 < t_{i^*} \leq m, 1 \leq k_{i^*} \leq n_{t_{i^*}}$ s.t.

$$\alpha_{t_{i^*}} + s_{i_1}\beta_{t_{i^*},k_{i^*}} = \alpha_1 + s_{i_1}\beta^*. \tag{5}$$

$$\alpha_{t_{i^*}} + s_{i_2}\beta_{t_{i^*},k_{i^*}} = \alpha_1 + s_{i_2}\beta^*. \tag{6}$$

Subtracting Equation (6) from Equation (5), we have $(s_{i_1} - s_{i_2})\beta_{t_{i^*},k_{i^*}} = (s_{i_1} - s_{i_2})\beta^*$, i.e., $\beta^* = \beta_{t_{i^*},k_{i^*}}$. Substituting $\beta^* = \beta_{t_{i^*},k_{i^*}}$ into Equation (5), we get $\alpha_{t_{i^*}} = \alpha_1$. This results contradicts the condition $1 < t_{i^*}$. Therefore, we have $\bigcap_{i=1}^m T_{s_i,1} \subset Y$. □

Corollary 3.6. *Assume that the point set P has at most n distinct points in $\mathbb{R}^2$ and that only their first coordinates are known. If for $m\,(\geq n)$ given distinct nonzero $s_1, \cdots, s_m \in \mathbb{R}$, one knows $\psi_{s_i}(P) = \{p_x + s_i p_y |\, (p_x, p_y) \in P\}$ for $i = 1 \ldots m$, then one can recover the second coordinates of all the points in P.*

Proof. Using the notation of Lemma 3.5 for the point set P, it is enough to note that $m \geq n > n - 1 \geq n - \min_i n_i$. The second coordinates are recovered via the construction of the sets $T_{s_i,j}$. □

We give an example to illustrate the theorem and its corollary.

Example 3.7. *Assume that $P = \{(1,1), (1,2), (2,2)\}$ but we know only their first coordinates $\{1, 2\}$. We notice that the total number of points in P is 3, so we set $s_1 = 1, s_2 = 2, s_3 = 3$ and compute*

$$\psi_{s_1}(P) = \{p_x + s_1 p_y \,|\, (p_x, p_y) \in P\} = \{2, 3, 4\},$$
$$\psi_{s_2}(P) = \{p_x + s_2 p_y \,|\, (p_x, p_y) \in P\} = \{3, 5, 6\},$$
$$\psi_{s_3}(P) = \{p_x + s_3 p_y \,|\, (p_x, p_y) \in P\} = \{4, 7, 8\}.$$

Next we begin to recover the second coordinates. We compute

$$T_{s_1,1} = \{1, 2, 3\}, T_{s_2,1} = \{1, 2, 2.5\}, T_{s_3,1} = \{1, 2, 7/3\} \text{ and } T_{s_1,1} \cap T_{s_2,1} \cap T_{s_3,1} = \{1, 2\}.$$

Hence, we get the two points $(1,1), (1,2) \in P$ with first coordinate 1. Similarly, we compute

$$T_{s_1,2} = \{0, 1, 2\}, T_{s_2,2} = \{0.5, 1.5, 2\}, T_{s_3,2} = \{2/3, 5/3, 2\} \text{ and } T_{s_1,2} \cap T_{s_2,2} \cap T_{s_3,2} = \{2\}.$$

The point with first coordinate 2 of P is thus $(2,2) \in P$. Hence, we recover the second coordinates of the points in P.

Algorithm 2: Space key points

Input : The plane key points on the fiber $x = x_i$ of $\mathcal{C}_{h_m}$: $(x_i, y_{i,j}^m), 1 \le j \le l_i^m$, $0 \le m \le d^2$.

Output: The space key points on the fiber $x = x_i$ of $\mathcal{C}$: $P_{i,j,k}(x_i, y_{i,j}, z_{i,j,k})$, where $y_{i,j} = y_{i,j}^0$, $1 \le j \le l_i^0 = l_i$, $1 \le k \le l_{i,j}$.

1 Compute $t_{j,m} = y_{i,j}^0 / s_m$ for $1 \le j \le l_i^0, 1 \le m \le d^2$.

2 Compute $R_m = \{y_{i,j}^m / s_m, 1 \le j \le l_i^m\}$ for $1 \le m \le d^2$.

3 Compute $T_{s_m,j} := \{t_{j,m} - R_m\} = \{t_{j,m} - a \,|\, \forall a \in R_m\}$ for $1 \le j \le l_i^0, 1 \le m \le d^2$.

4 Compute $\bigcap_{1 \le m \le d^2} T_{s_m,j} = [z_{i,j,1}, \ldots, z_{i,j,k}, \ldots, z_{i,j,l_{i,j}}]$ for $1 \le j \le l_i^0$.

5 **return** *The point set* $\{P_{i,j,k}(x_i, y_{i,j}, z_{i,j,k}), 1 \le j \le l_i^0, 1 \le k \le l_{i,j}\}$.

3.3.2. Space key points: Algorithm 2

Based on the above results, we show how to recover the space key points from plane key points on the curves $\mathcal{C}_{h_m}$. The first step is to solve the triangular systems $\{r^*(x), h_m(x, y)\}$ for $0 \le m \le d^2$. Let $\{x_1, \ldots, x_l\}$ denote the real roots of $r^*(x)$ such that $x_i < x_j$ for $1 \le i < j \le l$. We call the solutions of the above triangular systems in the fibre $x = x_i$ the **plane key points** for this fibre, we denote them by $(x_i, y_{i,j}^m), 1 \le j \le l_i^m$, $0 \le m \le d^2$. In addition we simplify the notation $y_{i,j}^0$ as $y_{i,j}$. Let $\{P_{i,j,k}(x_i, y_{i,j}, z_{i,j,k}), 1 \le j \le l_i^0, 1 \le k \le l_{i,j}\}$ be the space key points of $\mathcal{C}$. By the definitions of h_m in Equation (2) and ϕ_{s_m} in Equation (1), one has $\mathcal{C}_{h_m} = \pi \circ \phi_{s_m}(\mathcal{C})$, so the values $y_{i,j}^m = y_{i,j}^0 - s_m z_{i,j,k}$, thus Corollary 3.6 enables us to recover the values $z_{i,j,k}$ as detailed in Algorithm 2.

Since Algorithm 2 will be applied on interval input, we introduce the basics of interval arithmetic. An interval $\mathbf{x} = [a, b]$, $a \le b$ is a set of real numbers defined by $[a, b] = \{x \big| a \le x \le b\}$. If $\mathbf{x}$ and $\mathbf{y}$ are intervals and $\odot$ denotes one of the arithmetic operators $+$, $-$, $\times$ and $/$, then $\mathbf{x} \odot \mathbf{y}$ is defined by

$$\mathbf{x} \odot \mathbf{y} = \{x \odot y | x \in \mathbf{x}, y \in \mathbf{y}\}$$

Any real number a is considered to be the interval $a = [a, a]$, which means that the expressions such as $a\mathbf{x}, a + \mathbf{x}, \mathbf{x}/a$, and $(-1)\mathbf{x} = -\mathbf{x}$ are well defined. Note that when $\odot$ represents $/$, one assumes $0 \notin \mathbf{y}$. The **width** of an interval $[a, b]$ is $w([a, b]) = b - a$. We refer to [32] for more details. For example, $[1, 2] + [3, 4] = [4, 6]$, $[1, 2] - [3, 4] = [-3, -1]$, and $[1, 2] \times [3, 4] = [3, 8]$.

When Algorithm 3 calls Algorithm 2, all coordinates of points in Algorithm 2 are algebraic numbers represented by intervals derived from the isolation of polynomials. In Step 4 of Algorithm 2, several interval representations of the same exact real algebraic number need to be identified to this same number and need not be mistakenly identified to another nearby distinct algebraic number. Of course, two interval representations of the same number overlap since they both contain this number. On the other hand, when two interval representations overlap they may represent two different algebraic numbers in which case refining the interval representations will eventually make them disjoint. We detail our solution to this issue using separating bounds for polynomials.

Let i, j be fixed. A sufficient condition for Algorithm 2 to be correct when working with intervals is to compute intervals for the algebraic number in each $T_{s_m,j}$ such that (c1): two intervals do not intersect if and only if they isolate different algebraic number in $\cup_{1 \le m \le d^2} T_{s_m,j}$. Let δ be a lower bound on the minimum distance between

distinct elements of $\cup_{1\leq m\leq d^2} T_{s_m,j}$. If the widths of the isolating intervals of the algebraic numbers in the $T_{s_m,j}$ are smaller than $\delta/2$ then condition (c1) is satisfied. Such a lower bound δ can be computed as follows. Let $\mathrm{Sep}(h_m)$ be the separation bound of the solutions of $h_m(x_i, y)$. For a set of real numbers E, define the separation of E, denoted $\mathrm{Sep}(E)$, as the minimum distance between distinct elements of E. The elements of $T_{s_m,j}$ are the solutions of $h_m(x_i, y)$ shifted by the value $y^0_{i,j}$ and divided by s_m, thus $\mathrm{Sep}(T_{s_m,j}) = \mathrm{Sep}(h_m)/s_m \geq \mathrm{Sep}(h_m)/d^2$. The separation of the union $\cup_{1\leq m\leq d^2} T_{s_m,j}$ is thus larger than $1/d^2$ times the minimum distance between two distinct values, one being a root of h_{m_1} and the other a root of h_{m_2} (m_1 may be equal to m_2). On the other hand, the minimum distance between two distinct algebraic numbers, one being a root of h_{m_1} and the other a root of h_{m_2} is at least $\mathrm{Sep}(h_{m_1}h_{m_2})$. One can thus define $\delta = 1/d^2 \min_{1\leq m_1\leq m_2\leq d^2} \mathrm{Sep}(h_{m_1}h_{m_2})$ as a lower bound on $\mathrm{Sep}(\cup_{1\leq m\leq d^2} T_{s_m,j})$.

It remains to express the condition that the isolating intervals of the algebraic numbers in the $T_{s_m,j}$ are smaller than $\delta/2$ in terms of widths of the intervals of the input of the algorithm which are isolating the solutions $y^m_{i,j'}$ of $h_m(x_i, y)$ for $0 \leq m \leq d^2$. Assume that all the solutions $y^m_{i,j'}$ are given by intervals $[y^m_{i,j'}]$ of widths smaller than $\delta/4$. By construction $T_{s_m,j} = \{(y^0_{i,j} - y^m_{i,j'})/s_m, 1 \leq j' \leq l^m_i\}$, so that the computed intervals are of widths $w(([y^0_{i,j}] - [y^m_{i,j'}])/s_m]) \leq w([y^0_{i,j}]) + w([y^m_{i,j'}]) \leq \delta/2$.

According to the analysis above, we thus obtain the following result for the interval version of Algorithm 2.

Lemma 3.8. *Let $\delta = 1/d^2 \min_{1\leq m_1\leq m_2\leq d^2} \mathrm{Sep}(h_{m_1}h_{m_2})$, note that δ has the same asymptotic bound as $\mathrm{Sep}(h_m)$ and [15, Proposition 25] yields $\mathrm{Sep}(h_m) = 2^{-\tilde{\mathcal{O}}(d^8+d^7\tau)}$, also a non-asymptotic bound may be computed following the proof of [15, Proposition 25]. Assume the input isolating intervals of the algebraic numbers $y^m_{i,j}$ have widths smaller than $\delta/4$, then Algorithm 2 is correct, that is it returns disjoint intervals for the different algebraic numbers $z_{i,j,k}$, for $1 \leq k \leq l_{i,j}$.*

3.4. Connections between space points of $\mathcal{C}$

In this subsection, we compute the connections between the space key points of $\mathcal{C}$ on the fibers $x = x_i$ and $x = x_{i+1}$ for $1 \leq i \leq l-1$, where x_i are the roots of r^* defined in (3). By the definition of r^* and Corollary 3.4, between these fibers, there are no x-critical points of $\mathcal{C}$ nor $\mathcal{C}_{h_0}$ nor $\mathcal{C}_{h_1}$. The connections between two x-fibers can thus be done by straight line segments. These connections can be recovered from the ones in the topology of $\mathcal{C}_{h_0}$ except when several space points project to the same plane point. To separate branches of $\mathcal{C}$ that project to the same branch of $\mathcal{C}_{h_0}$, we add intermediate fibers defined by the $x^+_i \in \mathbb{Q}$:

$$x^+_0 < x_1 < x^+_1 < \cdots < x_i < x^+_i < x_{i+1} < \cdots < x^+_{l-1} < x_l < x^+_l \tag{7}$$

In practice, x^+_i is chosen as the mid-point between the isolating bounds of the intervals isolating the x_i. So connecting the points between the fibers $x = x_i$ and $x = x_{i+1}$ is decomposed into connecting the points between the fibers $x = x_i$ and $x = x^+_i$, and connecting the points between the fibers $x = x^+_i$ and $x = x_{i+1}$. By solving the bivariate systems $\{f(x^+_i, y, z), g(x^+_i, y, z)\}$, we obtain the space points of $\mathcal{C}$ on the intermediate fibers. Together with the space key points already computed, one has all space points in all fibers.

Our method is very similar to the one of Alcazar and Sendra [2]: they use the topology of the plane projections of the space curve along the z-axis and the y-axis. We also use the projection along the z-axis, that is $\mathcal{C}_{h_0}$, but our

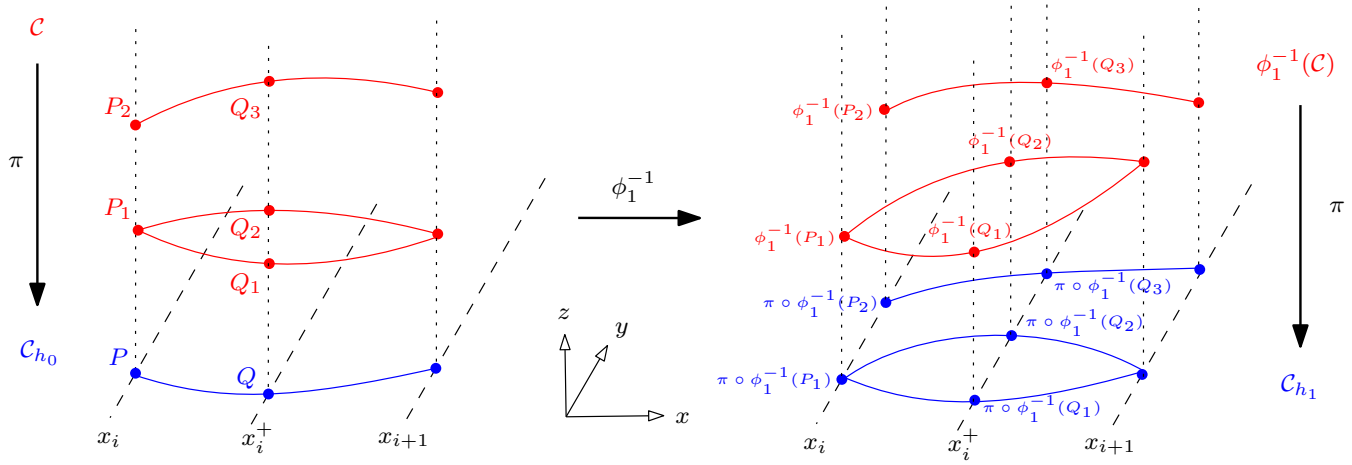

Figure 3: Connection between the x-critical fibers $x = x_i$ and $x = x_{i+1}$ via the intermediate fiber $x = x_i^+$.

second curve is the sheared projection $\mathcal{C}_{h_1}$ instead of the projection along the y-axis. Still this does not make any difference with respect to the main property of these two plane curves and projections: points of $\mathcal{C}$ that are mapped to the same point of $\mathcal{C}_{h_0}$ by π are mapped to distinct points of $\mathcal{C}_{h_1}$ by $\pi \circ \phi_1^{-1}$. This property implies that connections between points of $\mathcal{C}$ that remain ambiguous using only the information of the topology of $\mathcal{C}_{h_0}$ are resolved by using the information of the topology of $\mathcal{C}_{h_1}$. Alcazar and Sendra [2, §4.3.1] detail the connection method for the only non-trivial case of a point of $\mathcal{C}_{h_0}$ that is the projection of at least two points $\mathcal{C}$. Due to our weaker sweep generic assumption, we have in addition to handle the "cylindrical" case, that is when several branches of $\mathcal{C}$ project to the same branch of $\mathcal{C}_{h_0}$. In fact, it appears that this additional case is dealt with the same method. We thus present in Lemma 3.9 a unified method that applies to any cases. In addition, we illustrate the "cylindrical" case on Figure 3.

Our connection method first refines the topologies of $\mathcal{C}_{h_0}$ and $\mathcal{C}_{h_1}$ by adding all the fibers x_i and x_i^+. Figure 3 illustrates the connections between the points of $\mathcal{C}$ on the fibers $x = x_i$ and $x = x_i^+$ (the connection between the points on the fibers $x = x_i^+$ and $x = x_{i+1}$ is similar). For a plane curve segment $\widetilde{PQ}$ of $\mathcal{C}_{h_0}$ between two fibers, let P be its endpoint on $x = x_i$ and Q be its endpoint on $x = x_i^+$. We define the point sets $\pi^{-1}(P) \cap \mathcal{C} = \{P_1, \ldots, P_u\}$ and $\pi^{-1}(Q) \cap \mathcal{C} = \{Q_1, \ldots, Q_v\}$. Since the space curve $\mathcal{C}$ has no x-critical points in the fiber $x = x_i^+$, the number of space curve segments of $\mathcal{C}$ over $\widetilde{PQ}$ is exactly the number of points in $\pi^{-1}(Q) \cap \mathcal{C}$ and all these segments end at different Q_j. Our aim is to determine which Q_j's connects to each P_i. Since P_i may be an x-critical point of $\mathcal{C}$ it may connect 0, 1 or more Q_j's. This information is recovered using the topology of $\mathcal{C}_{h_0}$ and $\mathcal{C}_{h_1}$ as follows.

Lemma 3.9. *For a plane curve segment $\widetilde{PQ}$ of $\mathcal{C}_{h_0}$ between two fibers, let P be its endpoint on the critical fiber $x = x_i$ and Q be its endpoint on the intermediate fiber $x = x_i^+$. Let P_i be in $\pi^{-1}(P) \cap \mathcal{C}$ and Q_j be in $\pi^{-1}(Q) \cap \mathcal{C}$.*

The space points P_i and Q_j are connected in $\mathcal{C}$ if and only if the plane points $P = \pi(P_i)$ and $Q = \pi(Q_j)$ are connected in $\mathcal{C}_{h_0}$ and $\pi(\phi_1^{-1}(P_i))$ and $\pi(\phi_1^{-1}(Q_j))$ are connected in $\mathcal{C}_{h_1}$.

Proof. "⇒" If P_i connects to Q_j on $\mathcal{C}$, then, this connection is preserved by a shearing transformation and by projection. So, on the one hand $P = \pi(P_i)$ connects to $Q = \pi(Q_j)$ on $\mathcal{C}_{h_0}$ and on the other hand, $\phi_1^{-1}(P_i)$ connects to $\phi_1^{-1}(Q_j)$ on $\phi_1^{-1}(\mathcal{C})$ and $\pi(\phi_1^{-1}(P_i))$ connects to $\pi(\phi_1^{-1}(Q_j))$ on $\mathcal{C}_{h_1}$.

Algorithm 3: Space curve topology

Input: $f, g \in \mathbb{Z}[x, y, z]$ coprime defining a curve $\mathcal{C}$.

Output: The topology of $\mathcal{C}$.

0. $(f, g) = \textit{Algorithm 1}(f, g)$
1. Compute $h_m(x, y) = \text{Squarefree}(\text{Res}_z(f(x, y + s_m z, z), g(x, y + s_m z, z)))$, for $0 \leq s_m := m \leq d^2$.
   Compute $r_t(x) = \text{Res}_y(h_t, \partial_y h_t)$, $t \in \{0, 1\}$ and the topology of $\mathcal{C}_{h_0}$ and $\mathcal{C}_{h_1}$.
2. Compute $r^* = \text{lcm}(r_0(x), r_1(x))$.
   Add the points on the fibers corresponding to the real zeros of $\frac{r^*}{r_0}$ into the topology of $\mathcal{C}_{h_0}$.
   Add the points on the fibers corresponding to the real zeros of $\frac{r^*}{r_1}$ into the topology of $\mathcal{C}_{h_1}$.
3. Solve the triangular polynomial systems $\{r^*(x), h_m(x, y)\}$, for $1 \leq m \leq d^2$.
4. Compute the space key points of $\mathcal{C}$ on the fiber $x = x_i \in \mathbb{V}_{\mathbb{R}}(r^*)$ with Algorithm 2, for $1 \leq i \leq l$.
5. Compute x_i^+, $0 \leq i \leq l$, satisfying Equation (7).
6. Add the points on the fibers of $x = x_i^+$, $0 \leq i \leq l$, into the topology of $\mathcal{C}_{h_0}$ and $\mathcal{C}_{h_1}$.
7. Solve the bivariate systems $\{f(x_i^+, y, z), g(x_i^+, y, z)\}$, for $0 \leq i \leq l$.
8. Connect the points between the fibers (Section 3.4).

"$\Leftarrow$" Assume that P connects to Q and $\pi(\phi_1^{-1}(P_i))$ connects to $\pi(\phi_1^{-1}(Q_j))$. The first condition implies that there exists at least one space curve segment of $\mathcal{C}$ whose projection in $\mathcal{C}_{h_0}$ is $\widetilde{PQ}$. Since x_i^+ is not a zero of r^*, there is no x-critical point of $\mathcal{C}$ in the fiber $x = x_i^+$, thus each Q_j is not an x-critical point of $\mathcal{C}$ and Q_j connects to one and only one point in $\pi^{-1}(P)$. If P_i does not connect to Q_j, then there must exist one space point $P_{i'} \in \mathcal{C}$ on the fiber $x = x_i$ which connects to Q_j such that $\pi(P_{i'}) = P$ and $P_{i'} \neq P_i$. Since $P_{i'}$ connects to Q_j, $\pi(\phi_1^{-1}(P_{i'}))$ connects to $\pi(\phi_1^{-1}(Q_j))$ in $\mathcal{C}_{h_1}$. One then uses the important property of the two mappings π and $\pi \circ \phi_1^{-1}$: since $P_{i'}$ and P_i are distinct points in $\pi^{-1}(P)$, their projections on $\mathcal{C}_{h_1}$ are distinct points, $\pi(\phi_1^{-1}(P_{i'})) \neq \pi(\phi_1^{-1}(P_i))$. Meanwhile, $\pi(\phi_1^{-1}(P_i))$ also connects to $\pi(\phi_1^{-1}(Q_j))$. This implies there exist at least two different curve segments of $\mathcal{C}_{h_1}$ arriving from the left at $\pi(\phi_1^{-1}(Q_j))$. We conclude that $\pi(\phi_1^{-1}(Q_j))$ is an x-critical point of $\mathcal{C}_{h_1}$, which is a contradiction since the fiber $x = x_i^+$ does not contain any x-critical point of $\mathcal{C}_{h_1}$. We must then have that P_i connects to Q_j which concludes the proof. □

3.5. Main algorithm

We summarize in Algorithm 3 the different steps described in this section for the computation of the topology of the space curve $\mathcal{C}$.

4. Complexity

In this section, we analyze the bit complexity of our algorithms using the notation $\tilde{\mathcal{O}}(\cdot)$ to indicate that we omit poly-logarithmic factors. We first recall complexity results for computations with polynomials. In particular, our analysis uses the recent results of [15] for solving bivariate triangular systems and computing the topology of an

algebraic plane curve without shearings. Our complexity improvement with respect to the state of the art comes from (a) our weaker sweep generic position assumption that is no long a complexity bottleneck of the whole algorithm and, (b) our new lifting process exploiting the results of Diatta et al. [15].

4.1. Basic results for complexity

The bitsize of an integer n is the number of bits needed to represent it, that is $\lfloor \log n \rfloor + 1$ (log refers to the logarithm in base 2). For a rational q, its bitsize, $\mathcal{L}(q)$, is the maximum bitsize of its numerator and denominator. A multivariate polynomial is said to be of magnitude (d, τ) if its total degree is bounded by d, and the bitsize of its coefficients is bounded by τ.

Lemma 4.1 ([35]). *Let F and G be univariate polynomials of magnitudes (d, τ).*

- *Computing their gcd or the square-free part of F have bit complexity $\tilde{\mathcal{O}}(d^2\tau)$, and the gcd or the square-free part of F are of magnitude $(d, \mathcal{O}(d+\tau))$.*
- *Given a polynomial $H \in \mathbb{Z}[x]$ that divides F, computing $\frac{F}{H}$ has bit complexity $\tilde{\mathcal{O}}(d(d+\tau))$.*
- *Computing their lcm has bit complexity $\tilde{\mathcal{O}}(d^2\tau)$, and the lcm is of magnitude $(\mathcal{O}(d), \mathcal{O}(d+\tau))$.*

Lemma 4.2 ([4, 31, 36]). *Let F be a polynomial in $\mathbb{Z}[x]$ of magnitude (d, τ), the separation bound of F, that is the mininum distance between two complex roots, satisfies*

$$\mathrm{Sep}(F) \geq d^{-\frac{d+2}{2}}(d+1)^{\frac{1-d}{2}} 2^{\tau(1-d)} = 2^{-\tilde{\mathcal{O}}(d\tau)}.$$

Lemma 4.3 ([30, Theorem 5]). *For a polynomial F of magnitude (d, τ), one can compute isolating intervals for all its real roots of width less than 2^{-L} using $\tilde{\mathcal{O}}(d^3 + d^2\tau + dL)$ bit operations. Isolating intervals can be computed such that the sum of the bitsizes is $\tilde{\mathcal{O}}(d\tau)$.*

Lemma 4.4 ([35, Chap. 8]). *For two m-variate polynomials of magnitudes (d, τ), the magnitude of their product polynomial is $(\mathcal{O}(d), \mathcal{O}(\tau + \log d))$ and it is computed in $\mathcal{O}(d^m\tau)$ bit operations.*

A recursive Horner scheme yields the following results for multivariate polynomial evaluation.

Lemma 4.5 ([7, Lemma 6]). *Let $F \in \mathbb{Z}[x_1, \ldots, x_m]$ be of magnitude (d, τ), and $q_1, \ldots, q_m$ be rational numbers each of bitsize σ, then evaluating $F(q_1, \ldots, q_m)$ has a bit complexity of $\tilde{\mathcal{O}}(d^m(\sigma+\tau))$, and the bitsize of $F(q_1, \ldots, q_m)$ is $\mathcal{O}(d\sigma + \tau)$.*

Corollary 4.6 ([6, Lemma 5]). *Let $F(x, y) \in \mathbb{Z}[x, y]$ be of magnitude (d, τ), the bit complexity of computing the square-free part of F is $\tilde{\mathcal{O}}(d^5 + d^4\tau)$.*

Lemma 4.7 ([17]). *Let F and G be polynomials in $\mathbb{Z}[y_1, \ldots, y_k][x]$ with $\deg_x(F) = p \geq q = \deg_x(G)$, $\deg_{y_i}(F) \leq p$, $\deg_{y_i}(G) \leq q$ and F has bitsize τ larger than σ, the bitsize of G. The resultant $\mathrm{Res}_x(F, G)$ can be computed in bit complexity $\tilde{\mathcal{O}}(q(p+q)^{k+1}p^k\tau)$, and it has magnitude $(2pq, \tilde{\mathcal{O}}(p\sigma + q\tau))$.*

Theorem 4.8 ([27],[6] Theorem 43). *Let $F(x, y)$, $G(x, y)$ in $\mathbb{Z}[x, y]$ be of magnitude (d, τ), solving the bivariate system $\{F(x, y) = G(x, y) = 0\}$ has bit complexity $\tilde{\mathcal{O}}(d^5(d+\tau))$.*

We use the following result for the complexity of computing the topology of a plane curve in its original coordinate system, that is identifying the x-fibers of its x-critical points without shearing.

Lemma 4.9 ([15]). *Let $F \in \mathbb{Z}[x,y]$ be a square-free polynomial of magnitude (d,τ). Computing the topology of the plane curve $\mathcal{C}(F) = \{(x,y) \in \mathbb{R}^2 \mid F(x,y) = 0\}$ without shearing has bit complexity $\tilde{\mathcal{O}}(d^6 + d^5\tau)$.*

The following result is used in the analysis of the lifting part of our algorithm.

Lemma 4.10 (Teissier [34]). *Let $F \in \mathbb{Z}[x,y]$. For an x-critical point $p = (\alpha, \beta)$ of the plane curve $\mathcal{C}_F$, it holds that*

$$\mathrm{mult}(\beta, F(\alpha, y)) = \mathrm{mult}(p, \{F, \partial_y F\}) - \mathrm{mult}(p, \{\partial_x F, \partial_y F\}) + 1,$$

where $\mathrm{mult}(\beta, F(\alpha,y))$ *denotes the multiplicity of β as a root of $F(\alpha, y) \in \mathbb{R}[y]$, ∂_x and ∂_y are the partial derivatives and* $\mathrm{mult}(p, \{G, H\})$ *is the multiplicity of p in the ideal generated by the bivariate polynomials G and H.*

4.2. Analysis of Algorithm 3

The proof of our main result, Theorem 4.11, is decomposed in several lemmas.

Theorem 4.11. *Let f and g be coprime polynomials in $\mathbb{Z}[x,y,z]$ of magnitude (d,τ), Algorithm 3 computes the topology of the algebraic space curve $\{(x,y,z) \in \mathbb{R}^3 \mid f(x,y,z) = g(x,y,z) = 0\}$ in $\tilde{\mathcal{O}}(d^{18} + d^{17}\tau)$ bit operations.*

Lemma 4.12. *The bit complexity of Step 0 of Algorithm 3, that is Algorithm 1, is $\tilde{\mathcal{O}}(d^7(d+\tau))$ and the output polynomials have bitsizes $\tilde{\mathcal{O}}(d+\tau)$.*

Proof. In Line 1 of Algorithm 1, a bivariate polynomial of magnitude (d,τ) is evaluated at $(d+1)^2$ points with a complexity of $\mathcal{O}(d^4(d+\tau))$ using Lemma 4.5. For Line 2, first consider the computation of $f(x+\alpha z, y, z) = \sum_{i=0,\ldots,d} c_i(y,z)(x+\alpha z)^i$. The computation of $(x+\alpha z)^i$ can be done with d bivariate multiplications of polynomials of magnitudes $(d, \log d)$, the output bitsize is $\tilde{\mathcal{O}}(d)$ and it is computed in $\tilde{\mathcal{O}}(d^4)$ according to Lemma 4.4. Then the d products of trivariate polynomials $c_i(y,z)(x+\alpha z)^i$ is computed in $\tilde{\mathcal{O}}(d^4(d+\tau))$. The second shearing has the same complexity and finally, the bitsize of $\hat{f}$ and $\hat{g}$ is $\tilde{\mathcal{O}}(d+\tau)$. In Line 3, computing the resultant of $\hat{f}$ and $\hat{g}$ with respect to z has complexity $\tilde{O}(d(d+d)^3 d^2(d+\tau)) = \tilde{O}(d^6(d+\tau))$ according to Lemma 4.7. The magnitude of the resultant $h(x,y)$ is $(\tilde{\mathcal{O}}(d^2), \tilde{\mathcal{O}}(d(d+\tau)))$. In Line 4, the computation of $\mathcal{O}(d^2)$ gcds of univariate polynomials of magnitudes $(\mathcal{O}(d^2), \tilde{\mathcal{O}}(d(d+\tau)))$ costs $\tilde{\mathcal{O}}(d^2(d^2)^2 d(d+\tau)) = \tilde{\mathcal{O}}(d^7(d+\tau))$. In Line 7, the $\tilde{\mathcal{O}}(d^2)$ evaluations of the univariate polynomial of magnitude $(\tilde{\mathcal{O}}(d^2), \mathcal{O}(d(d+\tau)))$ costs $\mathcal{O}(d^2 d^2 d(d+\tau)) = \mathcal{O}(d^5\tau)$. Finally, the shearing of Line 8 has the same complexity as the one in Line 2, that is $\tilde{\mathcal{O}}(d^4(d+\tau))$. □

Lemma 4.13. *The bit complexity of Steps 1 to 3 of Algorithm 3 is $\tilde{\mathcal{O}}(d^{13}(d+\tau))$.*

Proof. In Step 1, for $0 \le m \le d^2$, the bitsize of $s_m = m$ is $\mathcal{O}(\log d)$, thus the magnitudes of the sheared polynomials $f(x, y+s_m z, z)$ and $g(x, y+s_m z, z)$ are $(d, \mathcal{O}(d \log d + \tau)) = (d, \tilde{\mathcal{O}}(d+\tau))$ and they can be computed in $\tilde{\mathcal{O}}(d^4(d+\tau))$ as in Line 2 of Algorithm 1. By Lemma 4.7, the complexity of computing their resultant is:

$$\tilde{\mathcal{O}}(d(d+d)^3 \cdot d^2(d+\tau)) = \tilde{\mathcal{O}}(d^7 + d^6\tau),$$

and this resultant is of magnitude $(\mathcal{O}(d^2), \tilde{\mathcal{O}}(d^2+d\tau))$. According to Corollary 4.6, the computation of the square-free part for h_m has bit complexity

$$\tilde{\mathcal{O}}((d^2)^5 + (d^2)^4 \cdot (d^2 + d\tau)) = \tilde{\mathcal{O}}(d^{10} + d^9\tau).$$

Using again Lemma 4.7, the complexity of computing the resultant r_t is

$$\tilde{\mathcal{O}}(d^2(d^2 + d^2)^2 d^2(d^2 + d\tau)) = \tilde{\mathcal{O}}(d^{10} + d^9\tau).$$

Since m ranges from 0 to d^2 and t ranges from 0 to 1, computing all $h_m(x, y)$ and $r_t(x)$ has bit complexity

$$(d^2 + 1)(\tilde{\mathcal{O}}(d^7 + d^6\tau) + \tilde{\mathcal{O}}(d^{10} + d^9\tau)) + 2\,\tilde{\mathcal{O}}(d^{10} + d^9\tau) = \tilde{\mathcal{O}}(d^{12} + d^{11}\tau).$$

The last operation of Step 1 is the computation of the topology of $\mathcal{C}_{h_0}$ (and $\mathcal{C}_{h_1}$), according to Lemma 4.9, the bit complexity is $\tilde{\mathcal{O}}((d^2)^6 + (d^2)^5(d^2 + d\tau)) = \tilde{\mathcal{O}}(d^{12} + d^{11}\tau)$. Hence, the total complexity of Step 1 is $\tilde{\mathcal{O}}(d^{12} + d^{11}\tau)$.

In Step 2, the polynomials $r_0(x)$ and $r_1(x)$ are of magnitudes $(d^4, \tilde{\mathcal{O}}(d^4 + d^3\tau))$. By Lemma 4.1, computing the lcm r^* of r_0 and r_1 has a bit complexity of $\tilde{\mathcal{O}}((d^4)^2 \cdot (d^4 + d^3\tau)) = \tilde{\mathcal{O}}(d^{11}(d + \tau))$, and the division of r^* by r_0 costs $\tilde{\mathcal{O}}((d^4) \cdot (d^4 + d^3\tau)) = \tilde{\mathcal{O}}(d^7(d + \tau)$. Adding the fibers of $\frac{r_0^*}{r_0}$ in the topology of $\mathcal{C}_{h_0}$ means solving the triangular system $\{\frac{r_0^*}{r_0}, h_0\}$. The polynomial $\frac{r_0^*}{r_0}$ is of magnitude $(\mathcal{O}(d^4), \tilde{\mathcal{O}}(d^4 + d^3\tau))$ and h_0 is of magnitude $(d^2, \tilde{\mathcal{O}}(d^2 + d\tau))$. According to [15, Proposition 27(a.1)], solving this triangular system is in $\tilde{\mathcal{O}}(d^{12} + d^{11}\tau)$. Adding the points on the fibers corresponding to the real zeros of $\frac{r^*}{r_1}$ into the topology of $\mathcal{C}_{h_1}$ has the same complexity.

Finally, in Step 3, each of the d^2 triangular systems solving $\{r^*(x), h_m(x, y)\}$ costs $\tilde{\mathcal{O}}(d^{12} + d^{11}\tau)$, thus they can all be solved in $\tilde{\mathcal{O}}(d^{14} + d^{13}\tau)$. □

Lemma 4.14. *The bit complexity of Step 4 of Algorithm 3, that is Algorithm 2, is* $\tilde{\mathcal{O}}(d^{17}(d + \tau))$.

Proof. First note that with fast arithmetic operations, addition, multiplication or division of rational numbers has a bit complexity that is softly linear in the bitsize of the input. The bit complexity of comparing two rationals is upper bounded by twice the smallest bitsize of the two rationals. The complexity of computing the intersection of two ordered sets is linear in the number of elements, the bit complexity is thus linear in the sum of the bitsizes of all elements in the sets. To apply Algorithm 2, one has to construct the $\mathcal{O}(d^4 \times d^2 \times d^2)$ sets $T^i_{s_m,j}$, each with $\mathcal{O}(d^2)$ elements. All operations performed by the algorithm, in particular the intersection computations, are softly linear in the sum of the bitsizes of all the elements of all the sets $T^i_{s_m,j}$. It is thus enough to compute this sum.

According to Lemma 3.8, for Algorithm 2 to be correct with intervals, it is sufficient to refine the input y coordinates up to width $1/4d^2 \min_{1\le m_1\le m_2\le d^2} \mathrm{Sep}(h_{m_1}h_{m_2}) = 2^{-\tilde{\mathcal{O}}(d^8+d^7\tau)}$. There are d^{10} elements in all the T sets, thus the total bit size is $\tilde{\mathcal{O}}(d^{18} + d^{17}\tau)$.

We now analyze the cost of refining all the y-intervals of one triangular system $\{r^*(x), h_m(x, y)\}$ to a bitsize $L = \tilde{\mathcal{O}}(d^8+d^7\tau)$. According to [15, Proposition 27(b)], the bit complexity of this refinement is bounded by $\tilde{\mathcal{O}}(L(N\mu+ n^2\sum\mu_x))$, with $N = \mathcal{O}(d^4)$ is the degree of r^*, $n = \mathcal{O}(d^2)$ is the degree of h_m, for x in the solution set $V(r^*)$ of r^*, $\mu_x = \max_{y|(x,y)\in V=V(r^*,h_m)} \mathrm{mult}(y, h_m(x, .))$ and $\mu = \max_{x\in V(r^*)} \mu_x \le n = \mathcal{O}(d^2)$. Teissier's Lemma 4.10 yields $\mathrm{mult}(y, h_m(x, .)) \le \mathrm{mult}((x, y), \{h_m, \partial_y h_m\}) + 1$. So that $\mu_x \le \max_{y|(x,y)\in V=V(r^*,h_m)}(\mathrm{mult}((x, y), \{h_m, \partial_y h_m\}) + 1)$

and

$$\begin{aligned}\sum_{x\in V(r^*)}\mu_x &\leq \#V(r^*/r_m)+\sum_{x\in V(r^*)\cap V(r_m)}\max_{y|(x,y)\in V}(\mathrm{mult}((x,y),\{h_m,\partial_y h_m\})+1)\\ &\leq N+\sum_{x\in V(r^*)\cap V(r_m)}\max_{y|(x,y)\in V}\mathrm{mult}((x,y),\{h_m,\partial_y h_m\})\end{aligned}$$

with $r_m(x)=\mathrm{Res}_y(h_m,\partial_y h_m)$. On the other hand, the Bézout bound for the system $\{h_m,\partial_y h_m\}$ yields

$$\sum_{x\in V(r_m)}\ \sum_{y|(x,y)\in V(h_m,\partial_y h_m)}\mathrm{mult}((x,y),\{h_m,\partial_y h_m\})\leq n^2=\mathcal{O}(d^4)$$

Thus $\sum_{x\in V(r^*)\cap V(r_m)}\max_{y|(x,y)\in V}\mathrm{mult}((x,y),\{h_m,\partial_y h_m\})$ is also bounded by $\mathcal{O}(d^4)$ and $\sum_{x\in V(r^*)}\mu_x=\mathcal{O}(d^4)$.

So, the bit complexity of refining all the y-intervals of one triangular system $\{r^*(x),h_m(x,y)\}$ is bounded by $\tilde{\mathcal{O}}(L(N\mu+n^2\sum\mu_x))=\tilde{\mathcal{O}}((d^8+d^7\tau)(d^4d^2+(d^2)^2d^4))=\tilde{\mathcal{O}}(d^{15}(d+\tau))$. Since we have d^2+1 curves h_m, the bit complexity of Step 4 is $\tilde{\mathcal{O}}(d^{17}(d+\tau))$. □

Lemma 4.15. *The bit complexity of Steps 5 to 8 of Algorithm 3 is $\tilde{\mathcal{O}}(d^{13}(d+\tau))$.*

Proof. In Step 5, the roots of $r^*(x)$ have already been isolated in Step 3. The rationals x_i^+ computed as the midpoints between the isolating intervals of the x_i have a total bitsize bounded by the total bitsize of the isolating intervals of the x_i, thus

$$\sum_{i=0}^{l}\mathcal{L}(x_i^+)=\tilde{\mathcal{O}}(d^8+d^7\tau)$$

according to Lemma 4.3.

In Step 6, $h_0(x,y)$ and $h_1(x,y)$ are of magnitudes $(\mathcal{O}(d^2),\tilde{\mathcal{O}}(d^2+d\tau))$ and one has to isolate the roots of $h_0(x_i^+,y)$ and $h_1(x_i^+,y)$. The univariate polynomials $h_0(x_i^+,y)$ and $h_1(x_i^+,y)$ are of magnitude $(\mathcal{O}(d^2),\tilde{\mathcal{O}}(d^2\mathcal{L}(x_i^+)+d\tau)$ and they are computed in $\tilde{\mathcal{O}}(d^4(\mathcal{L}(x_i^+)+d^2+d\tau))$ according to Lemma 4.5. According to Lemma 4.3, the bit complexity of the real root isolations for all $i=0,\ldots,l=\tilde{\mathcal{O}}(d^4)$, is

$$\tilde{\mathcal{O}}(\sum_{i=0}^{l}d^6+d^4(d^2\mathcal{L}(x_i^+)+d\tau))=\tilde{\mathcal{O}}(d^{10}+d^9\tau)+\mathcal{O}(d^6)\sum_{i=0}^{l}\mathcal{L}(x_i^+)=\mathcal{O}(d^6)\cdot\tilde{\mathcal{O}}(d^8+d^7\tau)=\tilde{\mathcal{O}}(d^{13}(d+\tau)).$$

The complexity of adding the fibers $x=x_i^+$ into the topology of $\mathcal{C}_{h_0}$ and $\mathcal{C}_{h_1}$ is thus $\tilde{\mathcal{O}}(d^{13}(d+\tau))$.

In Step 7, writing $f(x,y,z)=\sum_{j,k}c_{j,k}(x)y^jz^k$, the evaluation at x_i^+ is done via $\mathcal{O}(d^2)$ evaluations of univariate polynomials of degree at most d. According to Lemma 4.5, this costs $\mathcal{O}(d^3\mathcal{L}(x_i^+)+\tau)$ and $f(x_i^+,y,z)$ and $g(x_i^+,y,z)$ are of magnitudes $(d,d\mathcal{L}(x_i^+)+\tau)$. By Theorem 4.8, solving one of these systems has complexity $\tilde{\mathcal{O}}(d^5(d\mathcal{L}(x_i^+)+\tau))$, so that the total complexity of solving all these bivariate systems is:

$$\sum_{i=0}^{l}\tilde{\mathcal{O}}(d^5(d\mathcal{L}(x_i^+)+\tau))=\tilde{\mathcal{O}}(d^9\tau)+\tilde{\mathcal{O}}(d^6)\sum_{i=0}^{l}\mathcal{L}(x_i^+)=\tilde{\mathcal{O}}(d^{14}+d^{13}\tau).$$

Finally, in Step 8, the complexity of the connection, following Lemma 3.9, is bounded by the size of the graphs encoding the topologies of $\mathcal{C}_{h_0}$ and $\mathcal{C}_{h_1}$, it is thus in $\mathcal{O}(d^6)$. □

5. Conclusion

Using a new weaker sweep generic position assumption and a new lifting process, we improved the complexity of the problem of computing the topology of a space algebraic curve. On the other hand, we believe that our algorithm may not be efficient in practice since it uses many sheared projections and worst case separation bounds for polynomials. Our sweep generic position assumption is no longer a complexity bottleneck of the whole algorithm so that further improvements could be looked for in the lifting process. We also believe that this sweep generic position assumption could be generalized for space curves in higher dimensions.

Acknowledgement

The first author was partially supported by the National Key Research and Development Program of China grant 2022YFC3802102. The second author was supported by NSFC Grant 12001177.

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