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## Time-harmonic inverse wave problem in linear viscoelasticity

Florian Faucher<sup>1,\*</sup>, Ha Pham<sup>1</sup>, Otmar Scherzer<sup>2</sup>, H el ene Barucq<sup>1</sup>

<sup>1</sup>Team Makutu, Inria, University of Pau and Pays de l'Adour, TotalEnergies, CNRS, France.

<sup>2</sup>Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria.

\*Email: florian.faucher@inria.fr

### Abstract

We consider the quantitative inverse problem in which the physical properties of a viscoelastic medium are reconstructed from wave measurements. The inversion and direct modeling are carried out in the frequency domain to facilitate working with different attenuation models, which are frequency-dependent. The hybridizable discontinuous Galerkin (HDG) method is employed in forward modeling to discretize the time-harmonic linear viscoelastic equations, as well as in inversion. The inversion follows a minimization algorithm in which the elastic moduli are iteratively updated. We carry out reconstructions with attenuation model uncertainty and illustrate the performance of the method in ultrasound imaging experiment to reconstruct synthetic breast samples.

**Keywords:** Inverse problem; Viscoelastic models; Time-harmonic wave propagation; Full waveform inversion; nonlinear inversion.

### 1 Introduction

We consider the quantitative reconstruction of properties of a viscoelastic medium from measurements of waves. Wave propagation in such media displays loss of energy, called attenuation, and is mathematically modeled with linear viscoelastic equations, [1]. Compared to purely elastic material, viscoelastic ones are represented by additional parameters, increasing the complexity of inversion. Furthermore, various viscoelastic models exist to describe different attenuation behavior in material, [1]. In inversion, the precise viscoelastic model of the material is typically unknown, leading to *attenuation model uncertainty*, that is, inversion is carried out with a different attenuation model than the one to which observables or synthetic data are associated, [4]. In this work, we investigate different models of linear viscoelasticity which, in the frequency domain, can be united under the same form of constitutive law with complex-valued coefficients. Then, we carry out numerical exper-

iments of elastic reconstruction following an iterative minimization procedure, allowing for attenuation model uncertainty.

### 2 Time-harmonic viscoelastic modeling

At angular frequency  $\omega$  and for an interior source  $\mathbf{f}$ , the equation of motion of time-harmonic waves in viscoelastic medium is given in terms of the displacement  $\mathbf{u}$  and stress tensor  $\boldsymbol{\sigma}$  solutions to,

$$-\omega^2 \rho(\mathbf{x}) \mathbf{u}(\mathbf{x}, \omega) - \nabla \cdot \boldsymbol{\sigma}(\mathbf{x}, \omega) = \mathbf{f}(\mathbf{x}, \omega), \quad (1)$$

with  $\rho$  the density. The equation is coupled with absorbing or rigid boundary conditions.

**Constitutive law** The relation between the stress tensor  $\boldsymbol{\sigma}$  and the strain tensor  $\boldsymbol{\epsilon} = \nabla \mathbf{u} + (\nabla \mathbf{u})^t$  is given by the material constitutive law, which takes the following form in the frequency domain:

$$\boldsymbol{\sigma}(\mathbf{x}, \omega) = \mathbf{M}(\mathbf{x}, \omega) \boldsymbol{\epsilon}(\mathbf{x}, \omega), \quad (2)$$

where  $\mathbf{M}$  is the viscoelastic tensor which is *complex valued* and *frequency dependent* and takes different definitions according to the attenuation models. We give below the definition of viscoelastic models from [1], with auxiliary fourth-order tensors  $\mathbf{C}_\bullet$  and  $\boldsymbol{\eta}_\bullet$  that are real-valued and frequency independent.

- No attenuation:  $\mathbf{M}_0 := \mathbf{C}_0$ ;
- Kelvin–Voigt model:  $\mathbf{M}_{\text{kv}} := (\mathbf{C}_{\text{kv}} - i\omega \boldsymbol{\eta}_{\text{kv}})$ ;
- Maxwell model:  $\mathbf{M}_{\text{ma}} := -i\omega (\boldsymbol{\eta}_{\text{ma}}^{-1} - i\omega \mathbf{C}_{\text{ma}}^{-1})^{-1}$ ;
- Zener model:  $\mathbf{M}_{\text{ze}} := \left( \frac{\mathbf{C}_{\text{ze}} - i\omega \tau_\epsilon \boldsymbol{\eta}_{\text{ze}}}{1 - i\omega \tau_\sigma} \right)$ .

For the numerical discretization of the viscoelastic wave equation, we use the HDG method, and refer to [5] for detailed description of its implementation for linear anisotropic elasticity. The implementation of the forward and inverse problems are carried out in the open-source software **Hawen**, [2].

### 3 Inverse problem and experiments

We consider viscoelastic isotropy with tensors  $\mathbf{C}$  and  $\boldsymbol{\eta}$  each having two degrees of freedom, respectively  $(\lambda, \mu)$  and  $(\hat{\lambda}, \hat{\mu})$ . Their actions on symmetric matrix  $\boldsymbol{\epsilon}$  are

$$\mathbf{C}_{\bullet} \boldsymbol{\epsilon} = \lambda_{\bullet} \text{Tr}(\boldsymbol{\epsilon}) + 2\mu_{\bullet} \boldsymbol{\epsilon}, \quad (3a)$$

$$\boldsymbol{\eta}_{\bullet} \boldsymbol{\epsilon} = \hat{\lambda}_{\bullet} \text{Tr}(\boldsymbol{\epsilon}) + 2\hat{\mu}_{\bullet} \boldsymbol{\epsilon}, \quad (3b)$$

with  $\text{Tr}$  denoting the trace. For all the above attenuation models, the constitutive law is written

$$\boldsymbol{\sigma} = \lambda_{\dagger} \text{Tr}(\boldsymbol{\epsilon}) \mathbf{I}_d + 2\mu_{\dagger} \boldsymbol{\epsilon}, \quad (4)$$

with  $\lambda_{\dagger}$  and  $\mu_{\dagger}$  complex-valued. Here we focus on the reconstruction of  $\lambda_{\bullet}$  and  $\mu_{\bullet}$ .

For inversion, we follow an iterative minimization of a misfit function that evaluates the discrepancy between the measurements and simulations starting from initial models. It is a Newton-type algorithm for nonlinear inversion, a methodology referred to as Full Waveform Inversion method in seismic applications, [6]. The gradient of the misfit function is computed with the adjoint-state method which needs to be adapted to the specificity of the HDG discretization, cf. [3] for acoustic waves.

We carry out a numerical experiment using a viscoelastic synthetic model of breast sample in ultrasound imaging in two dimensions. Data are generated for an acquisition setup with sources and receivers surrounding the sample of interest. The synthetic data are obtained for frequencies between 200 and 600kHz, using the Maxwell model of attenuation. For the reconstruction, we start from a constant background for initial models. The iterative minimization algorithm is carried out exclusively with the Kelvin–Voigt attenuation model, that is, using a different model from the one used to generate the data. To compare between models, we take  $\mathbf{M}_{\text{ma}} = \mathbf{M}_{\text{kv}}$  at 300kHz, defining ‘true’ models for Kelvin–Voigt attenuation. The reconstructed  $\mu_{\text{kv}}$  is shown in Figure 1 in which the different types of tissues are well distinguished, both in terms of structures and associated values of velocity.

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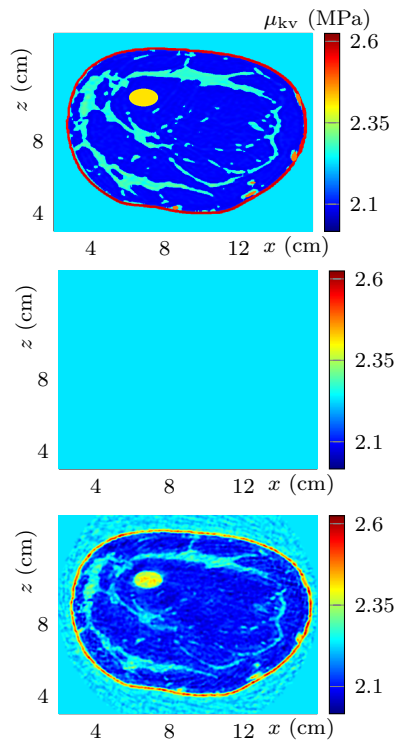


Figure 1: True model (top) and reconstruction (bottom). The initial model corresponds to a constant background (middle). The data used for inversion are synthetically generated using the Maxwell model of attenuation while the reconstruction is carried out with the Kelvin–Voigt model.

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