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Asymptotics and Time-Scaling of Stop-and-Go Waves in Car-Following Models

Guy Fayolle* Jean-Marc Lasgouttes*

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Abstract

It is known that waves, known as *stop-and-go waves* or *phantom jams*, can appear spontaneously in dense traffic. This causes a situation where drivers are confronted with consecutive phases of acceleration and braking. While waves are well understood in the setting of macroscopic models, there are no such results for the car-following model. This paper gives an asymptotic approximation of the speed and the shape of those waves. It relies on the well-known saddle-point method in order to describe the trajectory of a vehicle caught in such a wave.

Keywords: IDM (Intelligent driver model), Stop-and-go waves, Saddle-point.

1 Introduction

The phenomenon of *stop-and-go waves*, also known as *phantom jams*, has been documented early in the second part of the XXth century [6]. More recently, it has been demonstrated in the field [10, 11], that vehicles on a highway can suffer from stop-and-go traffic conditions without any external intervention. These waves usually travel backward with respect to traffic.

Shock waves, that typically happen when the number of lanes is reduced, are a well-studied property of macroscopic traffic models, like the popular Lighthill-Whitham model [8]. This led to an elaborate shock wave theory (see [6] for details).

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A better alternative is found in the form of car-following models (also known as follow-the-leader models), which describe the acceleration of a vehicle with a human driver as a function of the vehicle’s speed, its distance to the previous vehicle and optionally the relative speed between the two. They are also used today to describe the control loop of autonomous vehicles or advanced driver-assistance systems (ADAS), like adaptive cruise control. Although the history of these models is long [6, 12], two models stand out and are often referred to: the Optimal Velocity Model [1, 2], which only depends on the vehicle’s speed and the inter-vehicle distance, and the more elaborate Intelligent Driver Model [14] and its variants [9, 13].

We will be interested in this study in situations of traffic *string instability* conditions (see e.g. [3, 15]), in which deviations from the equilibrium will tend to be amplified, giving rise to stop-and-go traffic. This instability propagates in the traffic as a wave [6]. Unlike [5], which considered the effect of driver reaction time, our focus here is on the characterization of this wave at a sufficiently long distance (typically $N = 20$ vehicles). A parameter of interest is the velocity of the wave, which can be used to anticipate the arrival of a wave known by some means to have appeared upstream in the traffic. We propose to evaluate the wave asymptotically using the *saddle-point method*.

The organization of this paper is as follows. Section 2 describes the linearization of our car-following model, states the problem properly, and establishes some preliminary properties of interest. Section 3 proposes some first results on wave propagation in finite time, while Section 4 presents the main results when the time scales linearly with N , culminating with Theorem 4.5. Extensions of these results are discussed in Section 5. Finally, Appendix A gives a concise description of the saddle point method, and Appendix B gives hints on why only a bound is obtained in Section 3 instead of a real asymptotic approximation.

2 The Model

We consider a sequence of vehicles numbered $j = 0, 1, \dots, N$ in a single lane, where vehicle 0 is the first. Each vehicle $j > 0$ is characterized by its position $\hat{x}_j(t)$, its speed $\hat{v}_j(t) \stackrel{\text{def}}{=} \hat{x}'_j(t)$ and finally its acceleration $\hat{a}_j(t) \stackrel{\text{def}}{=} \hat{x}''_j(t)$. The latter is computed in car following models as a function of what happens directly in front of it, through three values:

- the inter-vehicle gap $\Delta \hat{x}_j \stackrel{\text{def}}{=} \hat{x}_j(t) - \hat{x}_{j-1}(t)$,

- the inter-vehicle relative speed $\Delta\hat{v}_j \stackrel{\text{def}}{=} \hat{v}_j(t) - \hat{v}_{j-1}(t)$,
- the vehicle's speed $\hat{v}_j(t)$.

The general form used in this paper is

$$\hat{a}_j(t) = f(\Delta\hat{x}_j(t), \Delta\hat{v}_j(t), \hat{v}_j(t)), \quad (1)$$

where f is a continuously differentiable function on \mathbb{R}^3 . Note that, in the case of the optimal velocity model, the inter-vehicle relative speed is not taken into account and f will be simpler. Such a very general non linear formulation does not lend itself to explicit solutions. A classical way to study them (as done in e.g. [3, 15]) is to linearize them around the equilibrium defined by the equation

$$0 = f(\Delta x^*, 0, v^*).$$

Here, Δx^* and v^* are respectively the stationary heading and the stationary speed. Dropping the hat on the variables to denote the solutions of the linearized model, the new equation of interest is

$$a_j(t) = \alpha(\Delta x_j(t) - \Delta x^*) + \beta\Delta v_j(t) + \gamma(v_j(t) - v^*), \quad (2)$$

where the parameters α , β and γ stand for the following partial derivatives at equilibrium

$$\alpha \stackrel{\text{def}}{=} \frac{\partial f}{\partial \Delta x}(\Delta x^*, 0, v^*), \quad \beta \stackrel{\text{def}}{=} \frac{\partial f}{\partial \Delta v}(\Delta x^*, 0, v^*), \quad \gamma \stackrel{\text{def}}{=} -\frac{\partial f}{\partial v}(\Delta x^*, 0, v^*).$$

In what follows, we will assume that these parameters are non-negative. This corresponds to a reduced acceleration when the danger grows.

Assume that the vehicles are organized at $t = 0$ such that $x_j(0) = -j\Delta x^*$. If the system were strictly at equilibrium, the trajectory of vehicle j would be

$$x_j^*(t) \stackrel{\text{def}}{=} v^*t - j\Delta x^*.$$

In what follows, we assume that a local perturbation of the trajectory $x_0(t)$ of the first vehicle occurs, for example a change of speed during a few seconds at $t = 0$. We want to understand how this disturbance propagates to the N -th vehicle, when N is large.

To evaluate the variation around this equilibrium value, let $X_j(z)$ denote the Laplace transform of $x_j(t) - x_j^*(t)$:

$$X_j(z) \stackrel{\text{def}}{=} \int_0^\infty (x_j(t) - x_j^*(t))e^{-zt} dt,$$

Then, setting $\delta \stackrel{\text{def}}{=} \beta + \gamma$, the transfer function associated to (2) is equal to [15]

$$Q(z) \stackrel{\text{def}}{=} \frac{X_{j+1}(z)}{X_j(z)} = \frac{\beta z + \alpha}{D(z)}, \quad \text{where } D(z) = z^2 + \delta z + \alpha, \quad (3)$$

and the inverse Laplace transform formula yields the relation, for any integer N ,

$$x_N(t) = x_N^*(t) + \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} e^{tz} X_0(z) Q^N(z) dz. \quad (4)$$

In this study, we are interested in the behavior of $x_N(t)$ when N and maybe also t are large.

The core problem: Estimate, as $N \rightarrow \infty$, the following integral, where ϕ is a Laplace transform:

$$I(t; N) = \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} e^{tz} \phi(z) Q^N(z) dz. \quad (5)$$

where $\phi(z)$ is sufficiently smooth and bounded at infinity on the contours of interest.

It is well-known that the dynamical system associated to Q is always *stable*, since the poles of Q (the two zeros of D) are located in the left half complex plane, and there will be denoted by d_0, d_1 , with

$$\Re(d_1) \leq \Re(d_0) \leq 0. \quad (6)$$

In the following, we will assume however that the system is *string unstable*, which is tantamount to

$$2\alpha > \mu \stackrel{\text{def}}{=} \delta^2 - \beta^2. \quad (7)$$

We begin by establishing some preliminary important inequalities, the substance of which will be commented on below.

Lemma 2.1. *Assume (7) holds, i.e. $\mu \leq 2\alpha$. Then*

$$\alpha > \beta\gamma. \quad (8)$$

If, in addition,

$$\delta^2 \geq 4\alpha, \quad (9)$$

then the poles $d_1 < d_0 < 0$ of Q are real and

$$\gamma \leq \beta(\sqrt{2} - 1). \quad (10)$$

Proof. From definition (7), we have

$$\mu - 2\alpha = \gamma(\gamma + 2\beta) - 2\alpha = \gamma^2 + 2(\beta\gamma - \alpha),$$

and (8) follows immediately. On the other hand, when (9) and (7) simultaneously holds, we can write using (9)

$$\delta^2 - \beta^2 < 2\alpha \leq \frac{\delta^2}{2},$$

which implies $\delta^2 = (\beta + \gamma)^2 < 2\beta^2$ and (10), concluding the proof of the lemma. \square

3 An estimate of $|I(t; N)|$ for t finite, as $N \rightarrow \infty$

As shown in Appendix B, the calculation of a precise estimate for $I(t; N)$ defined by (5) would require intricate technicalities. In fact, the presence of the poles d_0, d_1 in the left half-plane, see (6), renders the use of the saddle-point method practically out of question here. Therefore, we only exhibit a bound on the modulus $|I(t; N)|$, as $N \rightarrow \infty$.

To this end, we will work in the right half-plane. By Cauchy's theorem, we can write

$$I(t; N) = \frac{1}{2i\pi} \int_{u-i\infty}^{u+i\infty} e^{tz} \phi(z) Q^N(z) dz, \quad (11)$$

where u is a *finite positive* real number.

Setting

$$J(t, N) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\phi(z)| e^{tu} e^{N \log |Q(u+iy)|} dy, \quad (12)$$

the main substance of this section is the following lemma.

Proposition 3.1. *There exists $u_0 > 0$ such that*

$$|I(t, N)| \leq J(t, N) \sim \frac{e^{tu_0}}{4\pi} \Gamma\left(\frac{1}{4}\right) \phi(u_0) Q^N(u_0) (cN)^{-\frac{1}{4}}, \quad \text{as } N \rightarrow \infty, \quad (13)$$

with

$$Q(u_0) < 1, \quad \text{and} \quad c = \frac{1}{2} \frac{(\alpha + \beta u_0)^2}{D^4(u_0)}, \quad (14)$$

which means in particular that $\lim_{N \rightarrow \infty} I(t, N) = 0$ exponentially fast.

Proof. It suffices to show the existence of $u > 0$, such that, on the vertical line of equation $z = u + iy$, we have $|Q(z)| < 1$, uniformly for all real y .

Setting $Y = y^2$, we obtain after a direct algebra

$$F(Y, u) \stackrel{\text{def}}{=} |Q(z)|^2 = \frac{A(u) + \beta^2 Y}{(D(u) - Y)^2 + B(u)Y}, \quad (15)$$

together with

$$\frac{\partial F(Y, u)}{\partial Y} = \frac{-\beta^2 Y^2 - 2A(u)Y + A(u)[2D(u) - B(u)] + \beta^2 D(u)^2}{[(D(u) - Y)^2 + B(u)Y]^2}, \quad (16)$$

where

$$A(u) = (\alpha + \beta u)^2, \quad B(u) = (\delta + 2u)^2.$$

To prove (13), it suffices to choose some $u > 0$ such that $|F(Y, u)| < 1, \forall Y \geq 0$. From (15), we have the equivalence

$$|F(Y, u)| < 1 \iff Y^2 + (B(u) - 2D(u) - \beta^2)Y + D^2(u) - A(u) > 0, \quad (17)$$

keeping in mind that $Y = y^2$ is a positive variable.

We note that $D^2(u) - A(u) = u(u + \gamma)(D(u) + \beta u + \alpha) > 0, \forall u > 0$. On the other hand, the coefficient of Y in (17), which is equal to

$$2u^2 + 2\delta u + \mu - 2\alpha,$$

remains positive for some u_m properly chosen, thus ensuring the validity of the inequality (17) on the real axis $u \geq u_m$.

The first step is to find $u_0 \geq u_m$, such that $F(Y, u_0) \leq F(0, u_0), \forall Y \geq 0$. For this purpose, by using equation (16), it suffices to take u_0 as being the first positive root of the 4-th degree polynomial.

$$P(u) \stackrel{\text{def}}{=} A(u)[2D(u) - B(u)] + \beta^2 D(u)^2,$$

since in this case the numerator of (16) is strictly negative, so that $F(Y, u_0)$ is an increasing function of $Y \geq 0$. It is also worth noting that, by inequality (7),

$$P(0) = \alpha^2(2\alpha - \mu) > 0,$$

which implies that $\frac{\partial F(Y, u)}{\partial Y}$ is not always positive for $0 \leq u \leq u_0$.

The change of variable $y^2 = Y$ in the integral (12) allows to rewrite $J(t, N)$ as

$$J(t, N) = \frac{e^{tu_0}}{2\pi} \int_0^\infty \frac{1}{\sqrt{Y}} e^{N \frac{\log F(Y, u_0)}{2}} dY, \quad (18)$$

and we are in a position to estimate (18) by Laplace's method as follows.

Since $P(u_0) = 0$, we get from (16)

$$\frac{\partial^2 F(Y, u_0)}{\partial Y^2} \Big|_{Y=0} = -\frac{2A(u_0)}{D^4(u_0)},$$

whence

$$\log F(Y, u_0) = \log \frac{A(u_0)}{D^2(u_0)} - \frac{A(u_0)}{D^4(u_0)} Y^2 + \mathcal{O}(Y^3).$$

By using the saddle-point method, presented in Appendix A, we obtain the first term of the asymptotic expansion of $J(t, N)$

$$J(t, N) \sim \frac{e^{tu_0}}{2} \Gamma\left(\frac{1}{4}\right) e^{aN} (cN)^{-\frac{1}{4}}, \quad \text{as } N \rightarrow \infty,$$

where c is given by (14) and

$$a = \frac{1}{2} \log \frac{A(u_0)}{D^2(u_0)} = \log Q(u_0), \quad \text{whence } e^{aN} = Q^N(u_0),$$

remarking that (17) is satisfied, which shows that a is negative. The proof of the proposition is concluded. \square

4 Linear time scaling and estimate of $I(\tau N; N)$, as $N \rightarrow \infty$

In this section, using the saddle-point method, we analyze the behavior of the system on a time scale proportional to its size. Therefore, for any $\tau > 0$, the integral of interest is:

$$I(\tau N; N) = \frac{1}{2i\pi} \int_{-i\infty}^{+i\infty} \phi(z) [e^{\tau z} Q(z)]^N dz.$$

A candidate saddle-point for this integral satisfies $\frac{d}{dz} [e^{\tau z} Q(z)] = 0$, that is

$$\tau = T(z) \stackrel{\text{def}}{=} -\frac{Q'(z)}{Q(z)} = \frac{\beta z^2 + 2\alpha z + \alpha\gamma}{(\beta z + \alpha)D(z)}. \quad (19)$$

While a saddle-point z is obviously a function of τ , it will be in fact more convenient in the sequel to reverse the dependence, and to consider τ as a function of z .

As we shall see later, the variations of T do matter, especially its derivative given by

$$T'(z) = \frac{P_{T'}(z)}{(z^2 + \delta z + \alpha)^2(\beta z + \alpha)^2},$$

with $P_{T'}(z) \stackrel{\text{def}}{=} -\beta^2 z^4 - 4\alpha\beta z^3 - 2\alpha(2\beta\gamma + \alpha)z^2$

$$+ 2\alpha(\alpha(\beta - \gamma) - \beta\gamma(\beta + \gamma))z + \alpha^2(2\alpha - \mu). \quad (20)$$

Note that when z is real, $T'(\cdot)$ has the same sign as its numerator $P_{T'}(\cdot)$.

4.1 Asymptotic expansions

It appears that any characteristic value of interest can be parameterized by the real part u of the relevant saddle-point z , which satisfies $T(z) = \tau$. The question is to know whether we can construct a meaningful bijection between the τ 's and the u 's, keeping in mind that τ must be a positive real number, which implies in particular $\Im(T(z)) = 0$ and $\Re(T(z)) \geq 0$.

It turns out that it is difficult to consider all positive values of τ from the outset, and, consequently, we will first discuss in terms of u .

In fact, it is possible to classify saddle-points by looking at the sign of $T'(u)$, as explained in the following lemma.

Lemma 4.1. *Assume $u \in \mathbb{R}$.*

1. *When $T'(u) < 0$, the function $y \mapsto |Q(u + iy)|$, $y \in \mathbb{R}$, has a global maximum at $y = 0$.*
2. *When $T'(u) > 0$, then y_0 , defined below, is real and $T(u + iy_0) \in \mathbb{R}$.*

$$y_0 = \frac{1}{\beta} \sqrt{-(\beta u + \alpha)^2 + \sqrt{\Delta'}}, \quad (21)$$

$$\Delta' = \alpha(\alpha - \beta\gamma) \left[4\beta^2 u^2 + 2\beta(\beta\delta + 2\alpha)u + \alpha\beta\delta + \alpha^2 \right]. \quad (22)$$

Furthermore, y_0 is also the maximum of $y \mapsto |Q(u + iy)|$ for $y > 0$.

Proof. Before using the explicit form of Q , it is useful to establish a few generic relationships. If we choose an arbitrary complex point $z = u + iy$,

then the variation of the modulus $|Q|$ along a vertical line is given by

$$\begin{aligned}\frac{\partial}{\partial y}|Q(u+iy)|^2 &= \frac{\partial}{\partial y}Q(u+iy)Q(u-iy) = iQ'(z)Q(\bar{z}) - iQ(z)Q'(\bar{z}) \\ &= 2|Q(z)|^2\Im(T(z)).\end{aligned}$$

Thus, the extrema of $y \mapsto |Q(u+iy)|$ are reached when $\Im(T(z)) = 0$, and therefore correspond to possibly admissible real saddle-points.

Setting now $Y = y^2$, we have, with the notation of (15) and (16),

$$\begin{aligned}F(Y, u) &= |Q(u+iy)|^2 = \frac{A(u) + \beta^2 Y}{(D(u) - Y)^2 + B(u)Y}, \\ \text{where } A(u) &= (\alpha + \beta u)^2, \quad B(u) = (\delta + 2u)^2.\end{aligned}$$

To analyze the behavior of $F(Y, u)$ viewed as a function of Y , we study its derivative, already computed in (16),

$$\frac{\partial F(Y, u)}{\partial Y} = \frac{P_{F'}(Y, u)}{[(D(u) - Y)^2 + B(u)Y]^2}, \quad (23)$$

where

$$P_{F'}(Y, u) \stackrel{\text{def}}{=} -\beta^2 Y^2 - 2A(u)Y + A(u)(2D(u) - B(u)) + \beta^2 D^2(u).$$

Although this can be easily verified from the expression above, it is interesting to note that, for any Q ,

$$\frac{\partial F(Y, 0)}{\partial Y} = \lim_{y \rightarrow 0} \frac{1}{2y} \frac{\partial}{\partial y} |Q(u+iy)|^2 = Q(u)^2 T'(u).$$

Therefore, when $T'(u) < 0$, $P_{F'}(Y, u)$ is always negative and $F(Y, u)$ decreases for all $Y \geq 0$, so that $|Q(u+iy)|$ is maximal for $y = 0$, which proves part 1. of the lemma.

Assume now $T'(u) > 0$. Then $P_{F'}(Y, u)$, seen as a quadratic polynomial in Y , has two roots of opposite signs, the positive one being $Y_0 = y_0^2$ as defined in (21)-(22). Note that some algebraic simplification takes place, which keeps the degree in u of Δ' reasonably low. \square

When $T'(u) < 0$, then u is a saddle-point associated to $\tau = T(u)$, which allows us to compute the principal part of the integral $I(\tau N; N)$, as we did in Section 3 for $|I(t, N)|$, using the *saddle-point method* (also known as method of steepest descent).

Lemma 4.2. *When $u > \Re(d_0)$ (see (6)), and $T'(u) < 0$, the following asymptotic formula holds, as $N \rightarrow \infty$:*

$$I(T(u)N; N) \approx \frac{\phi(u)[e^{uT(u)}Q(u)]^N}{\sqrt{2\pi N|T'(u)|}}.$$

Proof. The proof is a direct application of the theory presented in [4, IX.1.] or in [7, V,78]. Since $Q(z)$ has no pole in the right half-plane (see Section 2), by Cauchy's theorem, $I(T(u)N; N)$ can be rewritten as

$$I(T(u)N; N) = \frac{1}{2i\pi} \int_{u-i\infty}^{u+i\infty} \phi(z)[e^{T(u)z}Q(z)]^N dz, \quad (24)$$

where the integral is taken along the vertical line going through the real saddle-point of abscissa u .

In a small neighborhood of u , we have the following Taylor expansion

$$T(z)z + \log Q(z) = uT(u) + \log Q(u) - \frac{T'(u)}{2}(z-u)^2 + \mathcal{O}(z-u)^3,$$

where $T'(u) < 0$. Then the classical approach is to first show that the principal part of the integral (24) is the same as that of

$$I_r(T(u)N; N) = \frac{1}{2i\pi} \int_{u-ir}^{u+ir} [e^{T(u)z}Q(z)]^N dz, \quad (25)$$

where $r > 0$ is sufficiently small. In our case, this is true thanks to Lemma 4.1.

The second step is to find an appropriate *thalweg line* allowing us to use Laplace's method for real functions (see Appendix A). Here, the above Taylor expansion shows the vertical segment $[u-ir, u+ir]$ is perfectly suitable, since in that segment

$$-\frac{T'(u)(z-u)^2}{2} = -\frac{T'(u)(iy)^2}{2} = \frac{T'(u)y^2}{2} < 0.$$

Consequently, we can apply equation (27) from Appendix A, with $\alpha = 0$ and $\omega = \pi/2$. \square

Now, when $T'(u) > 0$, u cannot be a usable saddle-point. However, in this case, $z = u + iy$, with y defined by (21)-(22), is a saddle-point associated with $\tau = T(z) \in \mathbb{R}$. This yields the following result.

Lemma 4.3. *When $u > \Re(d_0)$ (see (6)) and $T'(u) > 0$, the following asymptotic formula holds, as $N \rightarrow \infty$:*

$$I(T(z)N; N) \approx \frac{2\Re[\phi(z)[e^{zT(z)}Q(z)]^N]}{\sqrt{2\pi N|T'(z)|}},$$

with $z = u + iy$ and y defined by (21)–(22).

Proof. Here we have to take into account the two complex conjugate saddle-points $u + iy_0$ and $u - iy_0$ located on the vertical line of abscissa u . Splitting the analogous of (24) into two parts

$$I(T(z)N; N) = \frac{1}{2i\pi} \int_{u-i\infty}^0 \phi(z)[e^{T(z)z}Q(z)]^N dz + \frac{1}{2i\pi} \int_0^{u+i\infty} \phi(z)[e^{T(z)z}Q(z)]^N dz,$$

we can, thanks to Lemma 4.1, estimate each part separately as done in Lemma 4.2. Then, by adding these two estimates (which are complex conjugate), we obtain the announced real principal part, which involves sinusoidal oscillations. \square

4.2 Main results

We show below that there exists a unique real pair (u_c, τ_c) such that, for all $0 < \tau < \tau_c$, Lemma 4.2 applies and $u = T^{-1}(\tau)$.

Lemma 4.4. *There exist two constants $\tau_c = \max_{u>0} T(u)$ and u_c with $\tau_c = T(u_c)$. In addition, $T(\cdot)$ is a decreasing bijection from $[u_c, \infty[$ to $[\tau_c, 0]$.*

Proof. A real extremum of $T(u)$ is given by a zero of $P_{T'}(u)$, whose second derivative is the quadratic polynomial

$$P_{T'}''(u) = -4(3\beta^2 u^2 + 6\alpha\beta u + 2\alpha\beta\gamma + \alpha^2),$$

with a reduced discriminant equal to $24\alpha\beta^2(\alpha - \beta\gamma)$, which is always positive due to (8). One sees at once that $P_{T'}''$ has two negative roots and $P_{T'}''(u) < 0$ for $u > 0$. Hence, $P_{T'}$ is concave on \mathbb{R}^+ . Since $P_{T'}(0) = \alpha^2(2\alpha - \mu) > 0$ and $\lim_{u \rightarrow \infty} P_{T'}(u) = -\infty$, there exists a unique $u_c > 0$ such that $T'(u_c) = 0$. \square

The range $[0, \tau_c]$ is the time interval that contains the start of the stop-and-go wave. Specifically, when $|e^{\tau u}Q(u)| < 1$, the wave is not yet present, and eventually there is a τ_w after which $|e^{\tau u}Q(u)| > 1$, signaling the wave front.

Theorem 4.5. *There exists a unique real $u_w > u_c$ satisfying*

$$T(u_w) = -\frac{\log Q(u_w)}{u_w} \stackrel{\text{def}}{=} \tau_w, \quad (26)$$

and the following asymptotic expansion holds, as $N \rightarrow \infty$:

$$x_N(\tau_w N) - x_N^*(\tau_w N) \approx \frac{X_0(u_w)}{\sqrt{2\pi|T'(u_w)|N}}.$$

When N is large, the N -th vehicle is affected time $t = \tau_w N$ by the wave generated at $t = 0$ at point $x_0(0) = 0$. The velocity of this wave is thus:

$$\frac{x_N(\tau_w N) - x_0(0)}{\tau_w N} = v^* - \frac{\Delta x^*}{\tau_w} + \mathcal{O}(N^{-3/2}).$$

Before presenting the proof of this theorem, it is worth noting that the description of the wave itself (when $\tau > \tau_w$) very likely does not match the original nonlinear model $\hat{x}_N(\tau N)$, due to the magnitude of deviation from equilibrium that it entails. Nonetheless, the description of the start of the wave should be faithful enough to the original nonlinear model (1). Nonetheless, our results for $\tau < \tau_c$ seem to be very valuable in practice.

Proof. Let us first define the function

$$f(u) \stackrel{\text{def}}{=} -\frac{\log Q(u)}{u}.$$

Equation (26) expresses the fact that the pair (τ_w, u_w) corresponds to a saddle-point in the sense of Lemma 4.4 and that $|e^{\tau_w u} Q(u)| = 1$. Let us prove its existence. In the vicinity of $u = 0+$, the following series approximations hold.

$$T(u) = \frac{\gamma}{\alpha} + \frac{2\alpha - \mu}{\alpha^2}u + \mathcal{O}(u^2), \quad f(u) = \frac{\gamma}{\alpha} + \frac{2\alpha - \mu}{2\alpha^2}u + \mathcal{O}(u^2),$$

so that, using (7), locally $T(u) > f(u)$. Conversely, as $u \rightarrow \infty$, $f(u) > T(u)$ since

$$T(u) = \frac{1}{u} + \mathcal{O}(u^{-2}) \quad f(u) = \frac{\log u}{u} + \mathcal{O}(u^{-2}).$$

Therefore, there is a crossing between the two functions, which proves the existence of u_w such that $T(u_w) = f(u_w)$.

The next easy thing to check is that any real u_w satisfying (26) is necessarily an extremum of f . Indeed, its derivative vanishes at u_w , since

$$f'(u_w) = -\frac{1}{u_w^2} \left[\frac{Q'(u_w)}{Q(u_w)} u - \log Q(u_w) \right] = \frac{1}{u} [T(u_w) - f(u_w)] = 0.$$

The sign of the second derivative of f at u_w is given by $T'(u_w)$:

$$f''(u_w) = \frac{1}{u_w^2} \left[(T'(u_w) - f'(u_w))u_w - T(u_w) + f(u_w) \right] = \frac{T'(u_w)}{u_w}.$$

Since $f(u)$ is increasing for small values of u , its first extremum cannot be a local minimum. Therefore, $T'(u_w) \leq 0$ and $u_w \geq u_c$. The case $u_w > u_c$ leads to a local maximum which is unique, because the next extremum of f would have to be a local minimum, which is not possible, since in this region $T'(u) < 0$. The case $T'(u_w) = 0$ is also possible, but the uniqueness of u_w still holds by continuity arguments.

The remainder of the proof is a simple application of Lemmas 4.2 and 4.4. \square

5 Conclusions and perspectives

In this paper, we have described the front of the wave (corresponding to $\tau < \tau_c$) created by a string unstable traffic in the context of a fairly general car-following model. Establishing the regime for $\tau > \tau_c$ is more difficult, but we can give some hints about what happens. In particular, we need to establish whether $T'(u)$ has a single zero for $u < 0$.

Our conjecture is that, for τ between τ_w and some $\tau_e > \tau_w$, the system experiences a stop-and-go wave corresponding to complex saddle-points, which lead to sinusoidal variations, as shown in Lemma 4.3. Then, for $\tau > \tau_e$, $T'(u)$ becomes negative again and the back of the wave appears.

The situation becomes even more complicated when $\delta^2 < 4\alpha$, since the poles of Q (the roots of D) are now complex conjugate.

Since the required additional results on the behavior of T' seem difficult to obtain at the moment, we plan to add to this work numerical experiments using an IDM model, in order to compare the asymptotic wave obtained from the linearized equation (2) to the result of the original system of N non-linear differential equations.

A Laplace's estimates and the saddle-point method

We take up the challenge of presenting the main ideas behind these methods on a single page!

Consider integrals of the form

$$\int_0^\infty g(x)e^{\lambda h(x)} dx$$

where $\lambda \rightarrow \infty$, $h(x)$ takes its maximum value (finite) at $x = 0$, with the following assumptions.

- The functions g and h are real, piece-wise continuous on the interval $]0, \infty[$.
- The integral $\int_0^\infty |g(x)|e^{\lambda h(x)}$ is convergent
- In the neighborhood of 0, we have

$$g(x) \sim Ax^p, \quad h(x) \sim a - cx^q + o(x^q),$$

with $p > -1$, $q > 0$, $c > 0$.

Then, the first term of the expansion with respect to λ , also called *the principal part*, is given by

$$I(\lambda) \sim \frac{A}{q} \Gamma\left(\frac{p+1}{q}\right) e^{a\lambda} (c\lambda)^{-\frac{p+1}{q}}.$$

For more details, the reader can consult [4].

Assume now that $h(z)$ and $g(z)$ take complex values and we want to analyze the integral

$$J(\lambda) = \int_{\mathcal{L}} g(z)e^{\lambda h(z)} dz,$$

where \mathcal{L} is contained in an open set of \mathcal{C} , in which $h(z)$ is analytic. The saddle-point method consists of returning to the real case by examining the lines of steepest descent of the surface $\mathcal{S} = \{\Re(h(z)) = \Re(h(z_0))\}$. After deforming properly the contour \mathcal{L} into a contour \mathcal{L}' , the *thalweg line* (going through the saddle-point z_0) on which $h'(z_0) = 0$, $h''(z_0) \neq 0$, $\{\Re(h(z)) \leq \Re(h(z_0))\}$ and $\Im(h(z)) = \text{Constant}$.

In a neighborhood of z_0 , $h(z)$ and $g(z)$ are supposed to satisfy the conditions

$$h(z) \sim h(z_0) + d(z-z_0)^2 + \mathcal{O}(z-z_0)^3, \quad g(z) \sim g(z_0) + \mathcal{O}(z-z_0), \quad d = \rho e^{i\alpha}.$$

In this case, as $\lambda \rightarrow \infty$, we have (see e.g. formula (1.8.1) of [4, IX.1.]),

$$J(\lambda) \approx \sqrt{\frac{2\pi}{\lambda|h''(z_0)|}} e^{i\frac{\pi-\alpha}{2}} g(z_0) e^{\lambda h(z_0)}. \quad (27)$$

Although very powerful, this method may sometimes be difficult to implement, particularly with regard to the choice of \mathcal{L}' or because of the existence of several saddle-points.

B About the estimate of $I(t; N)$ for fixed t

Throughout this section, to avoid uninteresting technicalities, we take $\phi(z) = 1$ in the integral (5). Then, the integrand is a meromorphic function which, under the key assumption (7), has two poles of order N corresponding to the roots of $D(z)$. So, why not use Cauchy's residue theorem directly? Simply, because it is partly impossible to manipulate, as will be seen below!

Throughout this section, we suppose that (8) and (9) hold. Then the following properties hold.

- $D(z)$ has two negative real roots denoted by d_0, d_1 , with $d_1 \leq d_0 < 0$.
- The numerator of (19) has also two negative real roots, which are *saddle-points* denoted by z_0, z_1 , with $z_1 \leq z_0 < 0$.

Lemma B.1. *Under the above assumptions, we have*

$$d_1 < z_1 < d_0 < z_0 < 0. \quad (28)$$

Proof. We omit the details, which are of a purely algebraic nature. \square

By means of Cauchy's theorem, we can now rewrite the integral (5) in the form

$$I(t; N) = I_1(t; N) + I_2(t; N), \quad (29)$$

with

$$I_1(t; N) \stackrel{\text{def}}{=} \frac{1}{2i\pi} \int_{\mathcal{C}} e^{tz} Q^N(z) dz, \quad I_2(t; N) \stackrel{\text{def}}{=} \frac{1}{2i\pi} \int_{u-i\infty}^{u+i\infty} e^{tz} Q^N(z) dz,$$

where u is a negative real number satisfying $d_1 < u < z_1$ and the contour integral $I_1(t; N)$ is taken counter-clockwise. More precisely, the equality (29) follows from (28) and the fact that, in the negative half-plane,

$$e^{tz} Q^N(z) = \mathcal{O}(e^{t\Re(z)} |z|^{-N}), \quad \forall |z| \geq A \rightarrow \infty.$$

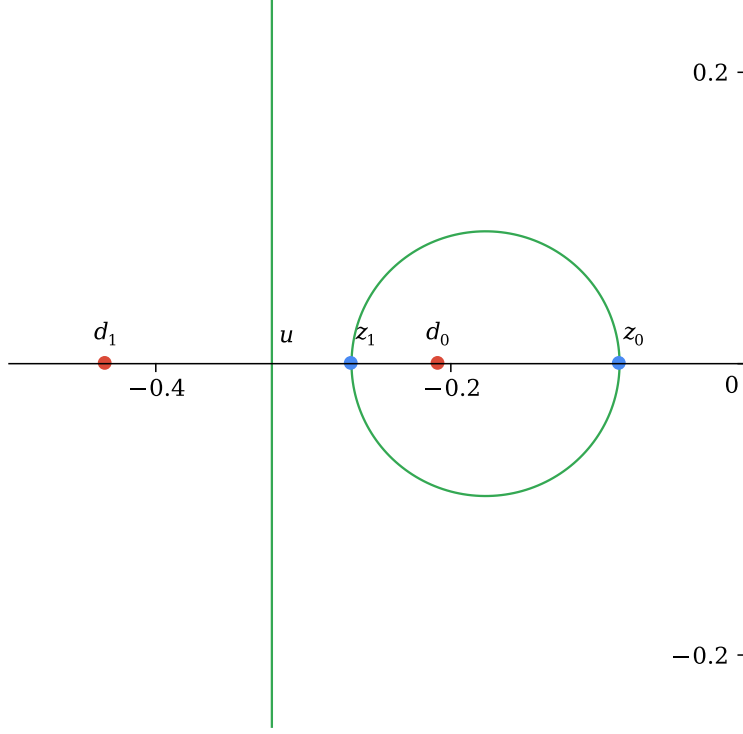


Figure 1: The integration contours.

Proposition B.2. *In the complex plane, the following properties hold.*

1. *If $z = x + iy$ belongs to the following family of cubic plane curves*

$$y[\beta(x^2 + y^2) + \alpha(2x + \gamma)] + KG(x, y) = 0, \quad (30)$$

where K is an arbitrary finite real constant, and

$$G(x, y) \stackrel{\text{def}}{=} (\beta x + \beta\delta - \alpha)y^2 + (\beta x + \alpha)D(x),$$

then $\Im(\log Q(z)) = K$.

2. *The curve obtained for $K = 0$ is decomposed into the real axis $y = 0$ and a circle C of radius R shown in Figure 1, given by the equation*

$$\left(x + \frac{\alpha}{\beta}\right)^2 + y^2 = R^2, \quad \text{where} \quad R^2 = \frac{\alpha(\alpha - \beta\gamma)}{\beta^2}, \quad (31)$$

admitting the real segment $[z_1, z_0]$ as its diameter, see Figure 1.

3. For $K \neq 0$, all the curves go through the zeros of $D(z)$, including $G(x, y) = 0$, which is the curve corresponding to $K = \infty$.

Proof. From (3), we have

$$\log Q(z) = \log(\alpha + \beta z) - \log D(z),$$

as well as

$$\begin{cases} \arg(\alpha + \beta z) = \arctan\left(\frac{\beta y}{\alpha + \beta x}\right), \\ \arg D(z) = \arctan\left(\frac{y(\delta + 2x)}{x^2 - y^2 + \delta x + \alpha}\right). \end{cases}$$

By using the elementary trigonometric formula

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b},$$

and taking for the function \arg the principal determination of the argument, we infer, after a direct algebra, that the equality

$$\arg(\alpha + \beta z) = \arg D(z) + l,$$

where $0 \leq l \leq 2\pi$ is some constant, implies that z belongs to the cubic curve defined in (30) together with $\Im(Q(z)) = \tan(l) \stackrel{\text{def}}{=} K$.

The rest of the proof is straightforward and details are omitted. \square

B.1 Estimate of $I_1(t; N)$, for $t \geq 0$.

In polar coordinates, the equation of \mathcal{C} is given by

$$\begin{cases} x = R \cos \theta - \frac{\alpha}{\beta}, \\ y = R \sin \theta. \end{cases}$$

Then, setting $p = z + \frac{\alpha}{\beta}$, we can write, for $z \in \mathcal{C}$,

$$Q(z) = \frac{\beta p}{p^2 + p\left(\delta - \frac{2\alpha}{\beta}\right) + R^2} = \frac{\beta}{2R \cos \theta + \delta - \frac{2\alpha}{\beta}},$$

after using (31) and the identities $p\bar{p} = R^2$ and $p + \bar{p} = 2 \cos \theta$.

For $z \in \mathcal{C}$, it will be convenient to note $\tilde{Q}(\theta) \stackrel{\text{def}}{=} Q(z)$. Then, by (10), (9) and (31), we have

$$\beta\delta > \frac{\delta^2}{2} > 2\alpha,$$

together with

$$\left(\delta - \frac{2\alpha}{\beta}\right)^2 - 4R^2 = \delta^2 - 4\alpha > 0,$$

which shows that $\tilde{Q}(\theta)$ is always finite and positive. Consequently, for all $z \in \mathcal{C}$,

$$\log Q(z) = \log |Q(z)| = \log \tilde{Q}(\theta)$$

is a real quantity, and we are in a position to estimate $I_1(t; N)$ by means of the classical Laplace's method, which was our primary objective. It is immediate to check that $\tilde{Q}(\theta)$ has a unique maximum value

$$\tilde{Q}(\pi) = \max_{0 \leq \theta \leq 2\pi} \tilde{Q}(\theta) = \frac{\beta}{\delta - 2R - \frac{2\alpha}{\beta}}. \quad (32)$$

In addition, we have

$$\tilde{Q}(\pi) > \tilde{Q}(0) > 1,$$

since the inequality

$$\beta > \delta - 2R - \frac{2\alpha}{\beta}$$

holds as soon as $\alpha > \beta\gamma$. Hence, in the neighborhood of $\theta = \pi$, since $Q'(z_1) = 0$, we can write the following Taylor's expansion

$$\log \tilde{Q}(\theta) = \log \tilde{Q}(\pi) + m(\theta - \pi)^2 + \mathcal{O}((\pi - \theta)^3),$$

where

$$m = \frac{-R\tilde{Q}^2(\pi)}{\beta} < 0. \quad (33)$$

After an immediate change of variable, $I_1(t; N)$ takes the form

$$\begin{aligned} I_1(t; N) &= \frac{R}{2\pi} \int_0^{2\pi} e^{t(Re^{i\theta} - \alpha/\beta) + i\theta} e^{N \log \tilde{Q}(\theta)} d\theta \\ &= \frac{R}{2\pi} \int_0^{2\pi} e^{t(R \cos \theta - \alpha/\beta)} \cos(Rt \sin \theta + \theta) e^{N \log \tilde{Q}(\theta)} d\theta. \end{aligned} \quad (34)$$

On the circle \mathcal{C} , the function $\tilde{Q}(\theta)$ does reach its maximum. Hence, by using the classical expansion derived from Laplace's method (see e.g. [7]), after

having separated the real and imaginary parts in (34), we get, with the notation of equations (32), (33), and noting that $R + \alpha/\beta = -z_1$,

$$I_1(t; N) = Re^{tz_1} \frac{\tilde{Q}^N(\pi)}{\sqrt{2\pi N}} \left(\frac{-1}{\sqrt{-m}} + \mathcal{O}\left(\frac{1}{N}\right) \right), \quad (35)$$

which, by the way, is a negative quantity tending to $-\infty$, as $N \rightarrow \infty$.

Then, on the basis of the analysis carried out in Section 3, we have

$$I_2(t, N) = -I_1(t, N) + o(1),$$

noting, by Cauchy's theorem, that $I_2(t, N)$ is exactly equal to the residue of $I(t, N)$ around the pole d_1 in the left half-plane.

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