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Temporal Duration-Based Probabilistic Argumentation Frameworks

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Abstract. In recent years, Dung-style Argumentation Frameworks have been studied with a focus on time, for instance, through intervals of availability being added to arguments and relations, resulting in varying outputs of Dung semantics over time. In this paper, consider the probability distribution of arguments over time intervals. Based on this temporal probabilistic model, we study how these frameworks can be transformed into a probabilistic argumentation according to the constellation approach and how they can be interpreted within the epistemic approach. The epistemic approach, in particular, relies on the notion of defeat to select significant conflicts based on probability distributions. We also introduce a temporal acceptability of arguments based on the concept of defence, allowing for finer results in time. Finally, we extend our models (constellation and epistemic) to account for events that have a duration, i.e. that can occur for several consecutive instants of time.

Keywords: Temporal Argumentation · Probabilistic Argumentation · Epistemic Approach · Constellation Approach · Extended Dung's Semantics.

1 Introduction

Argumentation Theory studies how conclusions can be drawn starting from a given set of facts or premises, and, in the field of Artificial Intelligence, it provides tools for modelling human-fashioned logical reasoning where the available information may be discordant. A simple yet powerful representation of conflicting information is provided by the Abstract Argumentation Frameworks [27], or AFs in short. AFs are directed graphs where the nodes are called arguments, and the edges represent conflict relations, called attacks, between two arguments. Several “semantics” have been introduced to analyse the acceptability of the arguments in an AF. A semantics defines a criterion for evaluating the arguments, allowing one to distinguish those that are acceptable (i.e. those that would prevail in a debate) from those that should be rejected. Following the first characterisation given in [27], several additions have been proposed to enhance the expressive power of AFs. To give some examples, an AF can be endowed with a support relation [16, 20, 21, 23], a similarity relation [2, 3, 5, 4, 6], probability [31], and also a notion of time [18, 19, 22, 38, 39].

Continuing along the same line of research, we propose a characterisation of AF that encapsulates *both* notions of *time* and *probability*, two aspects that have never been

studied together in the existent literature. In classes of AFs that also consider the temporal aspect, arguments and attacks are only available within specific time intervals [22]. Therefore, studying the acceptability of arguments in this case involves considering their availability. As for probability, which can be introduced in AFs as a value to associate with attacks and arguments, the literature suggests two main approaches: the constellation [33] and the epistemic [37] ones. The former approach involves examining all the possible worlds (graphs) generated by the different combinations of probabilistic arguments. Each argument with a probability less than 1 may or may not exist in a given world; thus, the topology of the AF changes accordingly. In the latter approach, on the other hand, probability denotes a degree of belief assigned to arguments based on a criterion of justifiability. This degree can be used, for example, to label arguments as *in*, *out*, or *undec* (if the probability is greater/less/equal to 0.5, respectively), to compute the probability of an extension by summing the probability values together, or to check properties (i.e., constraints) on the AF.

This paper combines the temporal and the probabilistic aspects. In detail, we introduce a framework in which the arguments are comparable to events occurring within a specified time interval with a given probability distribution. We assume that each argument occurred for sure within the specified time interval, but there is uncertainty about the exact time instant. Therefore, we use the notion of argument availability over time, similar to [22], while not explicitly handling probability as a constellation or strictly adhering to the epistemic approach. To illustrate the practical relevance of the proposed system, let us present the following example.

Example 1. We want to solve a murder case for which the police have gathered some evidence about a suspect. We are aware of five arguments reported by witnesses and medical experts, which describe events that occurred in connection with the victim's death. Each event is placed in a precise time interval, which, for convenience, we represent as a set of dots. The arguments are as follows:

- a) witness A recalls seeing the suspect in a bar between 1 pm and 8 pm (i.e. in the interval $\{1, \dots, 8\}$);
- b) the bartender B is sure that the suspect was not in the bar at 6 pm (i.e. $\{6\}$);
- c) witness C reports that the suspect was in a fight with the victim between 4 pm and 7 pm (i.e. $\{4, \dots, 7\}$);
- d) according to the doctor, the victim died between 5 pm and 9 pm (i.e. $\{5, \dots, 9\}$);
- e) whenever he goes to the bar, the suspect reportedly gets drunk between 3 pm and 7 pm, according to some acquaintances (i.e. $\{3, \dots, 7\}$).

The attacks between arguments *a*, *b*, *c*, *d* and *e* are given in Figure 1, which provides a static representation of the events without taking time into account to detect conflicts. Therefore, arguments *a* and *b* attack each other since the suspect may or may not be in the bar; *b* also attacks *e* since the suspect must have been at the bar to get drunk; arguments *a* and *d*, then, attack each other because the suspect could not have killed the victim while at the bar; finally, there is a symmetrical attack between *c* and *d* since the victim could not have fought while already dead. Instead, arguments *a* and *c* do not contradict each other since the fight could have occurred in the bar.

The arguments' probability distribution over time is then represented in Figure 2. In this example, we use a uniform distribution for arguments a , c and e , while arguments d and e follow a normal distribution. Finally, argument b holds with probability 1 at 6 pm. Note that one can choose different probability distributions to represent various types of uncertainty.



Fig. 1. The AF G_0 describing the events of Example 1.

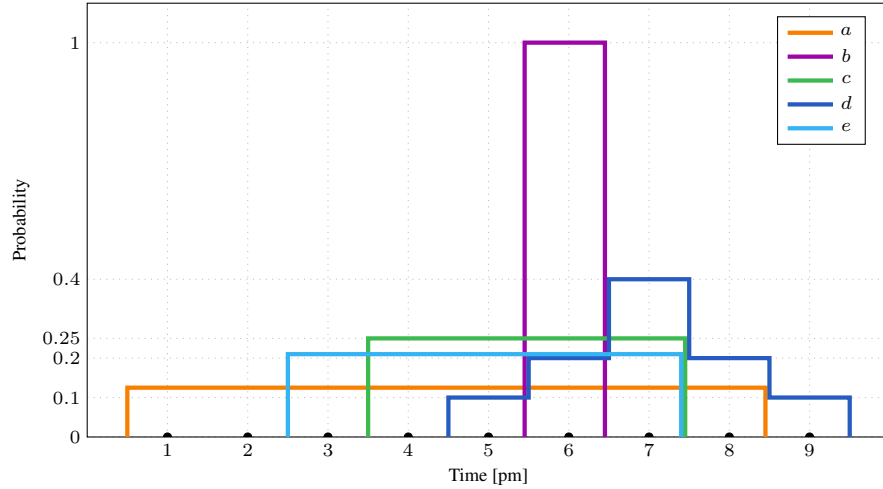


Fig. 2. Probability distribution over time for the arguments in G_0 .

Since events can be uncertain over time, the notion of conflict between arguments needs to be revised. For example, two contradictory arguments, such as a) “the suspect was at the bar” and b) “the suspect was not at the bar”, may not be in conflict if they were held at different times. Additionally, by analysing the probability distribution of each event over time, we can quantify the conflict between the two arguments.

Systems already present in the literature, like [22, 31], that separately exploit the temporal and probabilistic information of arguments and attacks cannot model situations of the type given in the example above, where deriving an acceptable conclusion must take both factors (time and probability) into account simultaneously. The study conducted in this paper is a follow-up to works we initiated in [12, 13] that introduced a representation for Temporal Probabilistic Argumentation Frameworks (TPAFs), an

extension to classical AFs for handling the temporal and probabilistic aspects of argumentation. In addition to the framework, we also presented a method for deriving conflict between arguments and evaluating their acceptability. In this extended paper, we include the following additional content.

- We introduce the notion of minimal Temporal AFs according to criteria based on either i) a set of arguments or ii) an interval of time, in order to restrict the analysis of the framework to the sufficient subset of arguments needed.
- We show how to map TPAFs into well-known constellation Probabilistic Argumentation Frameworks (PrAF).
- We present a new approach for computing arguments' acceptability by exploiting time availability and probability distributions (epistemic approach).
- We extend TPAFs into Temporal Duration-based Probabilistic Argumentation Frameworks (TDPAFs) to accommodate arguments with duration over time.

The remainder of this paper is organised as reported below. Section 2 summarises the basic notions concerning AFs and extension-based semantics; in Section 3, we show how to restrict a TAF to obtain a minimal representation with respect to arguments or time; in Section 4, we formalise TPAFs, then i) we propose a transformation from TPAFs to PrAFs and ii) we provide new defeat functions to extend the Dung's semantics and the consistent temporal acceptability; Section 5 extends TPAFs into TDPAFs to consider arguments with duration over time; finally, Section 6 concludes the paper with final remarks and possible future work.

2 Preliminaries

In this section, we recall the formal definition of an Abstract Argumentation Framework and the related extension-based semantic [27].

Definition 1 (AF). *An Abstract Argumentation Framework (AF) is a pair $\langle \mathcal{A}, \mathcal{R} \rangle$ where \mathcal{A} is a set of arguments, and \mathcal{R} is a binary relation on \mathcal{A} .*

Consider two arguments a, b belonging to an AF $F = \langle \mathcal{A}, \mathcal{R} \rangle$. We denote with $(a, b) \in \mathcal{R}$ an attack from a to b ; we can also say that b is *defeated* by a . The notion of attack can also be extended to sets of arguments: we will say that an argument a attacks a set of arguments $S \subseteq \mathcal{A}$ if $\exists b \in S$ such that $(a, b) \in \mathcal{R}$. Moreover, we use the notation $\mathcal{R} \downarrow_S = \{(a, b) \in \mathcal{R} \mid a, b \in S\}$ for restricting the attack relation \mathcal{R} to the only arguments in S . Among the arguments of an AF, we can select those not attacked by any other argument using the definition given in [7].

Definition 2 (Unattacked arguments). *Given an AF $F = \langle \mathcal{A}, \mathcal{R} \rangle$, a set of arguments $S \subseteq \mathcal{A}$ is unattacked if and only if $\nexists a \in (\mathcal{A} \setminus S)$ such that a attacks S .*

For b to be *acceptable*, we require that every argument that defeats b is defeated in turn by some other argument of the AF.

Definition 3 (Acceptable argument). Given an AF $\langle \mathcal{A}, \mathcal{R} \rangle$, an argument $a \in \mathcal{A}$ is acceptable to $D \subseteq \mathcal{A}$ if and only if $\forall b \in \mathcal{A}$ such that b is attacking a , $\exists d \in D$ such that d is attacking b . We say that a is defended by D .

One can further refine the set of selected arguments by using the notion of defence as a criterion for distinguishing acceptable arguments in the framework.

Definition 4 (Extension-based semantics). Let $\langle \mathcal{A}, \mathcal{R} \rangle$ be an AF. A set $E \subseteq \mathcal{A}$ is conflict-free if and only if $\nexists a, b \in E$ such that $(a, b) \in \mathcal{R}$. A conflict-free subset E is then **admissible**, if each $a \in E$ is defended by E ; **complete**, if it is admissible and $\forall a \in \mathcal{A}$ defended by E , $a \in E$; **stable**, if it is admissible and attacks every argument in $\mathcal{A} \setminus E$; **preferred**, if it is complete and \subseteq -maximal; **grounded**, if it is complete and \subseteq -minimal.

We also need the notion of time intervals for reasoning with temporal aspects of arguments. For example, in Timed AFs [22], each argument is associated with temporal intervals that express the time in which the argument is available.

Definition 5 (Temporal interval). Let \mathbf{T} be the discrete universe of time points. A temporal interval is a subset $I = \{t_1, t_2, \dots, t_n\}$ of \mathbf{T} where t_1 and t_n are respectively the minimum and maximum bounds of I . Moreover, $I = \{t_i\}$ denotes the instant t_i and $I = \emptyset$ is not allowed.

Definition 6 (TAF). A Timed Abstract Argumentation Framework (TAF) is a tuple $\langle \mathcal{A}, \mathcal{R}, Av \rangle$ where \mathcal{A} is a set of arguments, \mathcal{R} is a binary relation on \mathcal{A} , and $Av : \mathcal{A} \rightarrow \wp(\mathbf{T})$ is the availability function for arguments.³

Example 1 (Continued). Consider the AF $\mathbf{G}_0 = \langle \mathcal{A}, \mathcal{R} \rangle$ given in Figure 1 and the availability function illustrated in Figure 3. The tuple $\mathbf{G}_1 = \langle \mathcal{A}, \mathcal{R}, Av \rangle$ is a Timed Abstract Argumentation Framework (TAF).

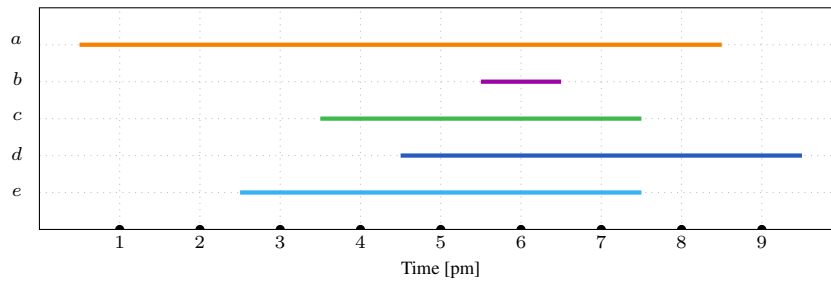


Fig. 3. Example of availability function for \mathbf{G}_1 .

³ We use $\wp(\mathbf{T})$ to indicate the powerset of \mathbf{T} .

Besides time, AFs can also express uncertainty in the form of probability over arguments and attacks. The literature considers two main approaches towards probabilistic AFs: the *constellation* approach and *epistemic* one. In the constellation approach [11, 25, 28, 29], the uncertainty resides in the topology of the considered AF, where the probability values quantify the potential existence of arguments or attacks. Consequently, the constellation approach generates a set of AFs representing all the possible worlds where each argument/attack may exist or not and defines a probabilistic distribution over their extensions. Below, we report the case in which the probability is expressed for attacks while arguments are certain.

Definition 7 (PrAF). *A Probabilistic Abstract Argumentation Framework (PAF) is a tuple $\langle \mathcal{A}, \mathcal{R}, P_R \rangle$ where \mathcal{A} is a set of arguments, \mathcal{R} is a binary relation on \mathcal{A} , and P_R is a joint probability distribution over \mathcal{R} .*

In the epistemic approach [32, 37] instead, the topology of a graph is fixed, and the probability assignments quantify the existing uncertainty of arguments: the more likely an agent is to believe in an argument, the less likely it is to believe in an argument attacking it.

3 Minimal TAFs according to arguments or time

A TAF modelled on a real situation may contain information on various events in different time intervals, some of which are unnecessary to analyse a specific case. For example, a lawyer may want to focus on a subset of arguments to prepare the defence. At the same time, an investigator might only be interested in examining a specific time slot to solve a certain case. When removing information from an argumentative system (including AFs and their extensions), one must be careful not to delete elements that could alter the justification status of the remaining arguments. In other words, only arguments, attacks, and TAFs, time intervals, which are irrelevant to the acceptability assessment, can be ignored. In the literature, several authors have already studied how modifying an AF can also produce changes at the semantic level [9, 24, 26]. The opposite problem, i.e. to preserve arguments' acceptability while changing the underlying framework, has also been investigated, leading to notions such as robustness [15] and equivalence [10] for abstract argumentation.

In this work, we want to tackle the problem from a different, more practical perspective, allowing non-expert users to select a portion of information to consider. Definitions 8 and 9 show how to restrict a TAF to a precise subset of arguments and a time interval, respectively. In both cases, after considering what information should be included in the final framework, we must reintroduce the arguments and attacks necessary to preserve acceptability. This step is necessary because the justification status of an argument may depend on arguments outside the set in which the user is interested, and these arguments must, therefore, be reconsidered in the final evaluation. As a criterion to determine which arguments must necessarily be reintroduced into the TAF after the user has decided which information to keep, we use directionality [7, 8], which can be summed up as “the justification status of an argument should only depend on the justification status of its attackers”. Many of the most commonly used semantics for AFs (like complete, grounded and preferred) satisfy directionality.

Definition 8 (Minimal TAF according to arguments). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, Av \rangle$ be a TAF and $S \subseteq \mathcal{A}$ a set of arguments. The TAF obtained from \mathbf{G} by focusing on S is $\mathbf{G}_{\downarrow S} = \langle \mathcal{A}', \mathcal{R}_{\downarrow \mathcal{A}'}, Av' \rangle$, where $\mathcal{A}' = S \cup \{b \in \mathcal{A} \text{ such that there exists a path from } b \text{ to some } a \in S\}$ and $Av' : \mathcal{A}' \rightarrow \wp(\mathbf{T})$ is such that $Av'(a) = Av(a)$ for all $a \in \mathcal{A}'$.

Example 1 (Continued). Consider the TAF \mathbf{G}_1 and suppose the detective in charge of the case wants to analyse the situation by focusing on a restricted subset of arguments, i.e. a , b and c . The TAF obtained from \mathbf{G}_1 by focusing on $\{a, b, c\}$ is $\mathbf{G}_{1\downarrow S} = \langle \mathcal{A}', \mathcal{R}_{\downarrow \mathcal{A}'}, Av' \rangle$ where $\mathcal{A}' = \{a, b, c, d\}$, $\mathcal{R}_{\downarrow \mathcal{A}'} = \{(a, b), (a, d), (b, a), (c, d), (d, a), (d, c)\}$ and Av' is as given in Figure 4. Argument d , not included in the set specified by the detective, is reintroduced in $\mathbf{G}_{1\downarrow S}$ since it is linked to (attacks) arguments a and c and is, therefore, essential for assessing their acceptability.

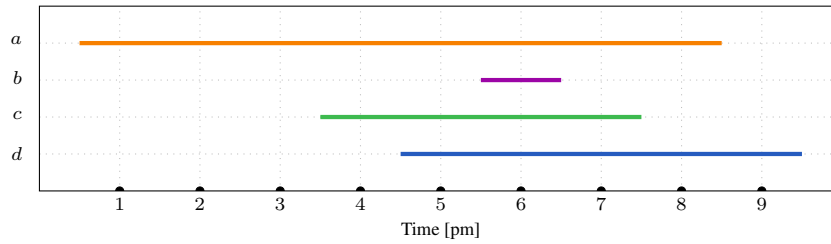


Fig. 4. Availability function for $\mathbf{G}_{1\downarrow S}$.

Given a TAF \mathbf{G} and a subset of its arguments S on which we want to focus, via Definition 8 we obtain a new TAF containing all and only the information necessary to analyse the acceptability of the arguments of S (for the notions of acceptability that satisfy directionality). In particular, $\mathbf{G}_{\downarrow S}$ will include all the arguments of S plus the arguments transitively related to them by the attack relation. Consequently, we have that the arguments in \mathcal{A}' are not attacked.

Proposition 1 (Unattacked minimal TAF). Let $\mathbf{G}_{\downarrow S} = \langle \mathcal{A}', \mathcal{R}_{\downarrow \mathcal{A}'}, Av' \rangle$ be the TAF obtained from $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, Av \rangle$ by focusing on a set of arguments $S \subseteq \mathcal{A}$. Then \mathcal{A}' is unattacked (in the sense of Definition 2).

Proof (Proposition 1). By Definition 8, we know that \mathcal{A}' contains all arguments from which it is possible to reach an element of S . Therefore, $\nexists a \in (\mathcal{A} \setminus \mathcal{A}')$ such that a attacks \mathcal{A}' . □

The availability function Av' given in Definition 8 identifies two new time extremes t_0 and t_n that coincide, respectively, with the smallest and largest t_i in which an argument of \mathcal{A}' is available. If we want to study the acceptability of arguments available within a given time interval, we resort instead to Definition 9.

Definition 9 (Minimal TAF according to time). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, Av \rangle$ be a TAF and $I = \{t_1, \dots, t_n\}$ an interval of time. The TAF obtained from \mathbf{G} by focusing on I is $\mathbf{G} \downarrow^I = \mathbf{G} \downarrow_S$, where $S = \{a \in \mathcal{A} \text{ for which } \exists t_i \in Av(a) \mid t_1 \leq t_i \leq t_n\}$.

In other words, focusing on a time interval amounts to first identifying the arguments available within the interval and then focusing on those arguments.

Example 1 (Continued). Starting again from the TAF \mathbf{G}_1 , our detective now wants to get an idea of what happened in the final moments of the incident, from 8 pm to 9 pm ($I = \{8, \dots, 9\}$). According to Definition 9, $\mathbf{G}_1 \downarrow^I = \mathbf{G}_1 \downarrow_S$ where $S = \{a, d\}$ is the set of arguments available within the interval I . Then, by Definition 8, $\mathbf{G}_1 \downarrow_S = \langle \mathcal{A}', \mathcal{R} \downarrow_{\mathcal{A}'}, Av' \rangle$ where $\mathcal{A}' = \{a, b, c, d\}$, $\mathcal{R} \downarrow_{\mathcal{A}'} = \{(a, b), (a, d), (b, a), (c, d), (d, a), (d, c)\}$ and Av' is as depicted in Figure 4.

4 Temporal Point-based Probabilistic Argumentation Frameworks

To reason about uncertain events in time using argumentation-based tools, we must first be able to represent arguments' probabilistic and temporal aspects in a single framework. To this end, we instantiate the generic framework proposed in [17], in which arguments are evaluated over time, and we associate each argument with the probability of its occurrence at a given time. We obtain in this way a Temporal Probabilistic Argumentation Framework (TPAF).

Definition 10 (TPAF). A Temporal Probabilistic Argumentation Framework (TPAF) is a tuple $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ such that: \mathcal{A} is a finite set of arguments; $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ is the attack relation; $\mathcal{P}^I : \mathcal{A} \rightarrow [0, 1]$ is the probability distribution of an argument over a time interval I .

Example 1 (Continued). Consider the TPAF \mathbf{G}_2 obtained by combining the AF \mathbf{G}_0 of Figure 1 with the probability distribution of Figure 2. The time points in the considered interval $I = \{1, \dots, 9\}$ represent the hours of the day. For each argument, we know the probability of its occurrence at a certain instant, e.g. $\mathcal{P}^{\{7\}}(d) = 0.4$. Furthermore, we have that $\mathcal{P}^{\{1, \dots, 9\}}(x) = \sum_{t \in \{1, \dots, 10\}} \mathcal{P}^{\{t\}}(x) = 1$ for all arguments x in \mathbf{G}_2 , i.e. the probability distribution of any argument is equal to 1 over the whole interval.

When an argument has a probability of occurring equal to zero, it should not be considered in the reasoning process. Therefore, we extract, for each argument, the instants in which its probability is positive, i.e. when the argument can occur.

Definition 11 (Positive probability over time). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF, and $a \in \mathcal{A}$ an argument. We define the set of non-null probability of a by $\mathcal{T}(a) = \{t \in \mathbf{T} \mid \mathcal{P}^{\{t\}}(a) > 0\}$.

Example 1 (Continued). For the arguments in \mathbf{G}_2 we have that $\mathcal{T}(a) = \{1, \dots, 8\}$, $\mathcal{T}(b) = \{6\}$, $\mathcal{T}(c) = \{4, \dots, 7\}$, $\mathcal{T}(d) = \{5, \dots, 9\}$ and $\mathcal{T}(e) = \{3, \dots, 7\}$.

Depending on the user's needs, it is useful to be able to focus the study of a TPAF on a specific subset of arguments or time intervals. To this extent, we can adapt Definitions 8 and 9 to also work with TPAFs.

Definition 12 (Minimal TPAF according to arguments). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF and $S \subseteq \mathcal{A}$ a set of arguments. The TPAF obtained from \mathbf{G} by focusing on S is $\mathbf{G}_{\downarrow S} = \langle \mathcal{A}', \mathcal{R}_{\downarrow \mathcal{A}'}, \mathcal{P}' \rangle$, where $\mathcal{A}' = S \cup \{b \in \mathcal{A} \text{ such that there exists a path from } b \text{ to some } a \in S\}$ and $\mathcal{P}' : \mathcal{A}' \rightarrow [0, 1]$ is such that $\mathcal{P}'(a) = \mathcal{P}(a)$ for all $a \in \mathcal{A}'$.

Definition 13 (Minimal TPAF according to time). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF and $I = \{t_1, \dots, t_n\}$ an interval of time. The TPAF obtained from \mathbf{G} by focusing on I is $\mathbf{G}_{\downarrow I} = \mathbf{G}_{\downarrow S}$, where $S = \{a \in \mathcal{A} \text{ for which } \exists t_i \in \mathcal{T}(a) \mid t_1 \leq t_i \leq t_n\}$.

Example 1 (Continued). Focusing on the set of arguments $S = \{a, b, c\} \subseteq \mathcal{A}$, we obtain the TPAF $\mathbf{G}'_2 = \mathbf{G}_{\downarrow S} = \langle \mathcal{A}', \mathcal{R}_{\downarrow \mathcal{A}'}, \mathcal{P}' \rangle$ with $\mathcal{A}' = \{a, b, c, d\}$, $\mathcal{R}_{\downarrow \mathcal{A}'} = \{(a, b), (a, d), (b, a), (c, d), (d, a), (d, c)\}$ and \mathcal{P}' as given in Figure 5.

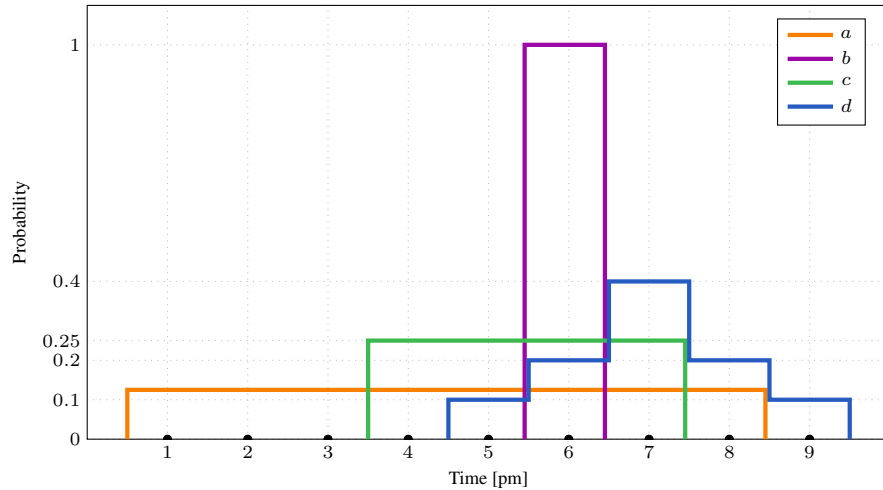


Fig. 5. Probability distribution over time for the arguments in \mathbf{G}'_2 .

The extension of Proposition 1 to TPAFs is straightforward. To reason about these argumentation graphs where we have a probability distribution on the occurrence of arguments, we have two options:

1. compute the probability of conflicts of each argument to obtain a probabilistic argumentation graph (PrAF, see Section 4.1), allowing us then to apply the constellation approach; or
2. follow the epistemic approach of probabilistic argumentation graphs where the objective is to determine and keep only significant attacks (see Section 4.2).

4.1 Constellation Approach

In the *constellation* paradigm [34], probability values determine the likelihood of both arguments and attacks to be part of an AF, thus generating different frameworks (i.e. worlds) with a distinct existence probability.

Definition 14 (Probability of a world). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, P_R \rangle$ be a PrAF and $\omega = \langle \mathcal{A}', \mathcal{R}' \rangle$ be a subgraph denoted by $\omega \sqsubseteq \mathbf{G}$ ⁴. The probability of subgraph ω , denoted $p(\omega)$, is

$$\left(\prod_{att \in \mathcal{R}'} P_R(att) \right) \times \left(\prod_{att \in \mathcal{R} \setminus \mathcal{R}'} (1 - P_R(att)) \right).$$

Since the uncertainty lies in the topology of the PrAF, to evaluate the acceptability of an argument a , we need first to compute its acceptance probability in all the AFs induced from the given PrAF and then to aggregate all obtained values. Let us see an example of a PrAF and its constellation.

Example 2. Let us consider the PrAF $\mathbf{F} = \langle \{a, b, c\}, \{(c, b), (c, a), (b, a)\}, P_R \rangle$ such that $P_R((c, b)) = 0.3$, $P_R((c, a)) = 0.2$, and $P_R((b, a)) = 0.1$ where its constellation is represented in Figure 6.

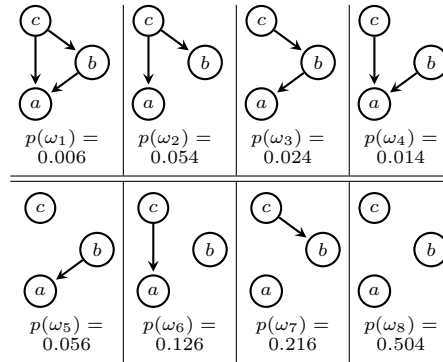


Fig. 6. Constellation of a PrAF \mathbf{F} .

An AF can be endowed with probability on both arguments and attacks. In [35], the authors proposed a way to transform any AF having probability on arguments and attacks to a PrAF with probability only on attacks (or only on arguments) thanks to a probabilistic-attack normal form (or probabilistic-argument normal form). They showed that all these forms are equivalent, i.e. share the same probabilistic distribution on their extensions. As illustrated in Figure 7, the transformation consists of adding a new argument, “n”, which attacks all the arguments such that the probability of the attack is

⁴ The notation $\omega \sqsubseteq \mathbf{G}$ stands for: $\mathcal{A}' \subseteq \mathcal{A}$ and $\mathcal{R}' = \{(a, b) \in \mathcal{R} \mid a \in \mathcal{A}' \text{ and } b \in \mathcal{A}'\}$: ω is a subgraph of an \mathbf{G} .

equal to 1 minus the probability of the argument. In the following, we consider PrAFs with only probability values on attacks, which allows us to focus on the uncertainty of conflicts according to the possible occurrence of arguments over time.

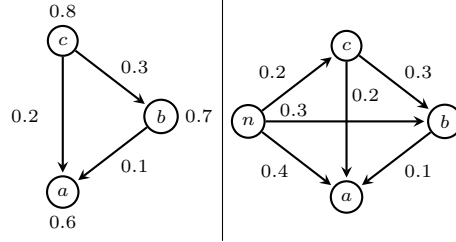


Fig. 7. A PrAF (left) and its probabilistic-attack normal form (right).

We propose instantiating TPAF in PrAF using the probability distributions of the arguments. When two arguments have joint occurrence probabilities (intersection), we combine them to obtain a probability of intersection (that the two events co-occur). This probability is computed between two arguments in conflict and can thus represent the probability of conflict (because an attack is valid if the arguments co-occur).

Definition 15 (Probability of intersection). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF, $a, b \in \mathcal{A}$ are arguments, and $I \subseteq \mathbf{T}$ is the intersected positive time interval of a and b , i.e. $I = \mathcal{T}(a) \cap \mathcal{T}(b)$. We define the probability (of intersection) that two events occur at the same instant by $PI(a, b) = \sum_{t \in I} \mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(b)$.

If the probability distributions are uniform, then the probability of the intersection of two arguments corresponds to the number of instants in which they can co-occur divided by the product of the size of the arguments' positive probability.

Proposition 2 (Special case of uniform distribution). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF, and $a, b \in \mathcal{A}$ are arguments. If the probability distribution of the two arguments a and b are uniform, then we can calculate the probability of intersection as follows: $PI(a, b) = \frac{|\mathcal{T}(a) \cap \mathcal{T}(b)|}{|\mathcal{T}(a)| \times |\mathcal{T}(b)|}$.

Proof (Proposition 2). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF, $a, b \in \mathcal{A}$ are arguments, and $I \subseteq \mathbf{T}$ is the intersected positive time interval of a and b , i.e. $I = \mathcal{T}(a) \cap \mathcal{T}(b)$. Let a and b having a uniform probability distribution, i.e. $\forall t \in \mathcal{T}(a), \mathcal{P}^{\{t\}}(a) = \frac{1}{|\mathcal{T}(a)|}$ and $\forall t \in \mathcal{T}(b), \mathcal{P}^{\{t\}}(b) = \frac{1}{|\mathcal{T}(b)|}$.

The probability (of intersection) that two events occur at the same instant is computed by $PI(a, b) = \sum_{t \in I} \mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(b)$. Therefore $PI(a, b) = \sum_{t \in \mathcal{T}(a) \cap \mathcal{T}(b)} \mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(b) = \sum_{t \in \mathcal{T}(a) \cap \mathcal{T}(b)} \frac{1}{|\mathcal{T}(a)|} \times \frac{1}{|\mathcal{T}(b)|} = \frac{|\mathcal{T}(a) \cap \mathcal{T}(b)|}{|\mathcal{T}(a)| \times |\mathcal{T}(b)|}$. \square

The transformation of a TPAF into a PrAF can be formalised as follows.

Definition 16 (Transformation of a TPAF into a PrAF). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF. We define $\mathbf{G}' = \langle \mathcal{A}, \mathcal{R}, P_R \rangle$ be the PrAF transformation of \mathbf{G} , such that for all $(a, b) \in \mathcal{R}$, $P_R(a, b) = PI(a, b)$.

Example 1 (Continued). Referring to the TPAF \mathbf{G}'_2 , we can obtain the corresponding transformed PrAF by computing:

- $P_R(a, b) = P_R(b, a) = PI(a, b) = PI(b, a) = 0.125$,
- $P_R(a, d) = P_R(d, a) = PI(a, d) = PI(d, a) = 0.125 \times 0.1 + (0.125 \times 0.2) \times 2 + 0.125 \times 0.4 = 0.1125$, and
- $P_R(c, d) = P_R(d, c) = PI(c, d) = PI(d, c) = 0.25 \times 0.1 + 0.25 \times 0.2 + 0.25 \times 0.4 = 0.175$.

After transforming the TPAF into a PrAF, we can use any semantics designed for this latter paradigm (see [29] for examples of applicable semantics).

4.2 Epistemic Approach

The epistemic paradigm [30] uses probability theory to represent degrees of belief in arguments and attacks. Regarding the semantics of the epistemic framework, probabilities on arguments and relations are used to validate or ignore attacks. We adapt the epistemic concept to TPAFs, using the probability distributions between conflicting arguments as a threshold to determine whether to consider an attack. In the following definition, we propose three variants of *defeat* (where the first two are the weak and strong versions of the same idea) to determine whether an attack is valid.

Definition 17 (α -Defeats). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF and consider two arguments $a, b \in \mathcal{A}$ such that $(a, b) \in \mathcal{R}$, and the interval $I \subseteq \mathbf{T}$ such that $I = \mathcal{T}(a) \cap \mathcal{T}(b)$, i.e. I is the intersected positive time interval of a and b . Given a threshold $\alpha \in]0, 1]$, we say that

- a **weakly sum defeats** b according to α if $\sum_{t \in I} \mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(b) \geq \alpha$ and $\exists t \in I$ such that $\mathcal{P}^{\{t\}}(a) \geq \mathcal{P}^{\{t\}}(b)$ (in this case we write $\Delta_w^\alpha(a, b) = \top$, otherwise $\Delta_w^\alpha(a, b) = \perp$);
- a **strongly sum defeats** b according to α if $\sum_{t \in I} \mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(b) \geq \alpha$ and $\forall t \in I$, $\mathcal{P}^{\{t\}}(a) \geq \mathcal{P}^{\{t\}}(b)$ (in this case we write $\Delta_s^\alpha(a, b) = \top$, otherwise $\Delta_s^\alpha(a, b) = \perp$);
- a **one point defeats** b according to α if $\exists t \in I$ such that $\mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(b) \geq \alpha$ and $\mathcal{P}^{\{t\}}(a) \geq \mathcal{P}^{\{t\}}(b)$ (in this case we write $\Delta_o^\alpha(a, b) = \top$, otherwise $\Delta_o^\alpha(a, b) = \perp$).

Example 1 (Continued). Consider the TPAF \mathbf{G}'_2 and let set $\alpha = 0.1$.

For the arguments $a, b \in \mathcal{A}$, we have $\mathcal{T}(a) \cap \mathcal{T}(b) = \{6\}$ and:

- $\sum_{t \in \{6\}} \mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(b) = \mathcal{P}^{\{6\}}(a) \times \mathcal{P}^{\{6\}}(b) = 0.125 \times 1 = 0.125 \geq \alpha$.
- $\mathcal{P}^{\{6\}}(a) = 0.125$ and $\mathcal{P}^{\{6\}}(b) = 1$, then $\mathcal{P}^{\{6\}}(a) < \mathcal{P}^{\{6\}}(b)$.

- Therefore, $\Delta_o^\alpha(a, b) = \Delta_w^\alpha(a, b) = \Delta_s^\alpha(a, b) = \perp$ and $\Delta_s^\alpha(b, a) = \Delta_w^\alpha(b, a) = \Delta_o^\alpha(b, a) = \top$.

For arguments $a, d \in \mathcal{A}$, we have $\mathcal{T}(a) \cap \mathcal{T}(d) = \{5, \dots, 8\}$ and:

- $\nexists t \in \{5, \dots, 8\}$ such that $\mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(d) \geq \alpha$, so $\Delta_o^\alpha(a, d) = \Delta_o^\alpha(d, a) = \perp$.
- $\sum_{t \in \{5, \dots, 8\}} \mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(d) = 0.125 \times 0.1 + (0.125 \times 0.2) \times 2 + 0.125 \times 0.4 = 0.1125 > \alpha$.
- $\mathcal{P}^{\{5\}}(a) > \mathcal{P}^{\{5\}}(d)$, therefore $\Delta_w^\alpha(a, d) = \top$.
- $\mathcal{P}^{\{6\}}(d) > \mathcal{P}^{\{6\}}(a)$, so $\Delta_w^\alpha(d, a) = \top$.
- a and d are more likely at different times, consequently $\Delta_s^\alpha(a, d) = \Delta_s^\alpha(d, a) = \perp$.

Finally, for arguments $c, d \in \mathcal{A}$, we have $\mathcal{T}(c) \cap \mathcal{T}(d) = \{5, \dots, 7\}$ and:

- $\mathcal{P}^{\{7\}}(c) \times \mathcal{P}^{\{7\}}(d) = 0.1 \geq \alpha$ and $\mathcal{P}^{\{t\}}(c) \times \mathcal{P}^{\{t\}}(d) < \alpha$ for any other $t \neq 7$.
- $\sum_{t \in \{5, \dots, 7\}} \mathcal{P}^{\{t\}}(c) \times \mathcal{P}^{\{t\}}(d) = 0.25 \times 0.1 + 0.25 \times 0.2 + 0.25 \times 0.4 = 0.175 \geq \alpha$.
- $\mathcal{P}^{\{5\}}(c) > \mathcal{P}^{\{5\}}(d)$ and $\mathcal{P}^{\{7\}}(d) > \mathcal{P}^{\{7\}}(c)$.
- Hence, $\Delta_w^\alpha(c, d) = \Delta_o^\alpha(d, c) = \Delta_w^\alpha(d, c) = \top$, and $\Delta_o^\alpha(c, d) = \Delta_s^\alpha(c, d) = \Delta_s^\alpha(d, c) = \perp$.

The previous example shows that the weak defeat condition does not imply the strong one. On the other hand, the strong defeat implies the weak one.

Proposition 3 (Link between α -defeats). *Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF, $a, b \in \mathcal{A}$ are arguments and $\alpha \in]0, 1]$. If $\Delta_o^\alpha(a, b) = \top$ then $\Delta_w^\alpha(a, b) = \top$ and $\Delta_s^\alpha(a, b) = \top$ then $\Delta_w^\alpha(a, b) = \top$.*

Proof (Proposition 3). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF, $a, b \in \mathcal{A}$ are arguments, the interval $I = \mathcal{T}(a) \cap \mathcal{T}(b)$ and $\alpha \in]0, 1]$.

- $\Delta_o^\alpha(a, b) = \top$ implies $\Delta_w^\alpha(a, b) = \top$:
 - If $\exists t \in I$ such that $\mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(b) \geq \alpha$ (first condition of Δ_o^α) then $\sum_{t \in I} \mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(b) \geq \alpha$ (first condition of Δ_w^α) because any probability is non-negative.
 - The second condition of Δ_o^α and Δ_w^α ($\exists t \in I$ such that $\mathcal{P}^{\{t\}}(a) \geq \mathcal{P}^{\{t\}}(b)$) are the same, hence they implicate each other.
- $\Delta_s^\alpha(a, b) = \top$ implies $\Delta_w^\alpha(a, b) = \top$:
 - The first condition of Δ_s^α and Δ_w^α ($\sum_{t \in I} \mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(b) \geq \alpha$) are the same, hence they implicate each other.
 - If $\forall t \in I, \mathcal{P}^{\{t\}}(a) \geq \mathcal{P}^{\{t\}}(b)$ (second condition of Δ_s^α) then $\exists t \in I$ such that $\mathcal{P}^{\{t\}}(a) \geq \mathcal{P}^{\{t\}}(b)$ (second condition of Δ_w^α) because the universal quantifier, i.e. \forall , implies the existential one, i.e. \exists .

□

Note that there is no implication between Δ_o^α and Δ_s^α , since the first condition of Δ_o^α ($\exists t \in I$ such that $\mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(b) \geq \alpha$) implies that of Δ_s^α ($\sum_{t \in I} \mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(b) \geq \alpha$), whereas the second condition of Δ_s^α ($\forall t \in I,$

$\mathcal{P}^{\{t\}}(a) \geq \mathcal{P}^{\{t\}}(b)$ implies that of $\Delta_w^\alpha (\exists t \in I \text{ such that } \mathcal{P}^{\{t\}}(a) \geq \mathcal{P}^{\{t\}}(b))$.

The α -defeat can identify significant attacks within a TPAF and discard those not as meaningful according to the specified threshold α . This results in a TPAF that only presents conflicts deemed important enough to be considered. Hereafter, we will use Δ to refer to a generic defeat function.

Definition 18 (Restricted TPAF). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF, and Δ a temporal probabilistic defeat function. We denote the restricted graph by $\Delta\text{-}\mathbf{G} = \langle \mathcal{A}, \mathcal{R}', \mathcal{P} \rangle$, where: the attacks are restricted according to Δ such that $\mathcal{R}' \subseteq \mathcal{R}$ and $\forall (a, b) \in \mathcal{R}$, $(a, b) \in \mathcal{R}'$ if and only if $\Delta(a, b) = \top$.

Example 1 (Continued). Recall that $PI(x, y) = \sum_{t \in \mathcal{T}(x) \cap \mathcal{T}(y)} \mathcal{P}^{\{t\}}(x) \times \mathcal{P}^{\{t\}}(y)$, then:

- $PI(a, b) = PI(b, a) = 0.125$,
- $PI(a, d) = PI(d, a) = 0.0125 + 0.025 + 0.025 + 0.05 = 0.1125$, and
- $PI(c, d) = PI(d, c) = 0.025 + 0.05 + 0.1 = 0.175$.

With $\alpha = 0.1$, $\Delta_w^\alpha(a, d) = \Delta_w^\alpha(b, a) = \Delta_w^\alpha(c, d) = \Delta_w^\alpha(d, a) = \Delta_w^\alpha(d, c) = \top$, while $\Delta_w^\alpha(a, b) = \perp$, and thus the attacks in the resulting TPAF $\Delta_w^\alpha\text{-}\mathbf{G}'_2$ are as shown in Figure 8.



Fig. 8. Restricted TPAF $\Delta_w^\alpha\text{-}\mathbf{G}'_2$ obtained for $\alpha = 0.1$.

Concerning the strong sum defeat, we expect to have a restricted TPAF with fewer attacks than those in Figure 8. In detail, $\Delta_s^\alpha(b, a) = \top$, while $\Delta_s^\alpha(a, b) = \Delta_s^\alpha(a, d) = \Delta_s^\alpha(c, d) = \Delta_s^\alpha(d, a) = \Delta_s^\alpha(d, c) = \perp$. A representation of the resulting graph is given in Figure 9.



Fig. 9. Restricted TPAF $\Delta_s^\alpha\text{-}\mathbf{G}'_2 = \Delta_o^\alpha\text{-}\mathbf{G}'_2$ obtained for $\alpha = 0.1$.

For the one point defeat, then, we obtain the restricted TPAF shown in Figure 10, where $\Delta_o^\alpha(b, a) = \Delta_o^\alpha(d, c) = \top$, while $\Delta_o^\alpha(a, b) = \Delta_o^\alpha(a, d) = \Delta_o^\alpha(c, d) = \Delta_o^\alpha(d, a) = \perp$.

We now extend the notion of conflict-freeness to TPAFs by considering a defeat function Δ .



Fig. 10. Restricted TPAF $\Delta_o^\alpha\text{-G}'_2$ obtained for $\alpha = 0.1$.

Definition 19 (Δ -conflict-free). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF, $S \subseteq \mathcal{A}$ a set of arguments and Δ a defeat function. We say that S is Δ -conflict-free, and we write $S \in \Delta\text{-cf}(\mathbf{G})$, if and only if $\nexists a, b \in S$ such that $\Delta(a, b) = \top$.

Example 1 (Continued). Referring to the previous example, we can verify that the set $\{a, c, d\}$ is Δ_s^α -conflict-free in \mathbf{G}'_2 . However, the same set is not Δ_w^α -conflict-free since, for instance, $\Delta_w^\alpha(c, d) = \top$. The sets of Δ -conflict-free arguments are:

- $\Delta_s^\alpha\text{-cf}(\mathbf{G}'_2) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$;
- $\Delta_o^\alpha\text{-cf}(\mathbf{G}'_2) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}\}$;
- $\Delta_w^\alpha\text{-cf}(\mathbf{G}'_2) = \{\emptyset, \{c\}, \{b\}, \{b, c\}, \{a\}, \{a, c\}, \{d\}, \{b, d\}\}$.

We use the Δ -conflict-free notion to study when an argument a is defended from the attack of another single argument b . In particular, we want to know at which instants of time argument b attacks a and, simultaneously, is defeated by other arguments.

Definition 20 (Δ -SingleDefence of a from b by S). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF, Δ a defeat function, $a, b \in \mathcal{A}$ two arguments such that $\Delta(b, a) = \top$, and $S \subseteq \mathcal{A}$ such that $a \cup S$ is a Δ -conflict-free set of arguments. We define the Δ -SingleDefence of a from b with respect to S as $\Delta\text{-1def}(a, b, S) = \emptyset$ if $\nexists c \in S$ such that $\Delta(c, b) = \top$, otherwise

$$\Delta\text{-1def}(a, b, S) = \mathcal{T}(a) \cap \bigcup_{c \in \{x \mid x \in S, \Delta(x, b) = \top\}} \mathcal{T}(b) \cap \mathcal{T}(c)$$

The previous definition determines when an argument is Δ -defended by a set of arguments in a TPAF against a single attacker. In the following, we generalise this notion of defence against a set of attackers.

Definition 21 (Δ -Defence of a with respect to S). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF, $a \in \mathcal{A}$ and $S \subseteq \mathcal{A}$ such that $a \cup S$ is a Δ -conflict-free set of arguments. The Δ -defence for a with respect to S , is defined as $\Delta\text{-def}(a, S) = \mathcal{T}(a)$ if $\nexists b \in \mathcal{A}$ such that $\Delta(b, a) = \top$, otherwise

$$\Delta\text{-def}(a, S) = \bigcap_{b \in \{x \mid x \in \mathcal{A}, \Delta(x, a) = \top\}} (\mathcal{T}(a) \setminus \mathcal{T}(b)) \cup \Delta\text{-1def}(a, b, S)$$

According to Definition 21, an argument is defended in all time instants where either it is not attacked or an argument defends it.

Example 1 (Continued). Consider the TPAF $\Delta_w^\alpha\text{-}\mathbf{G}'_2$ of Figure 8. For $BD = \{b, d\}$, we have that $\Delta_w^\alpha\text{-1def}(d, a, BD) = \{5, \dots, 9\} \cap ((\{6\} \cap \{1, \dots, 8\}) \cup (\{5, \dots, 9\} \cap \{1, \dots, 8\})) = \{5, \dots, 8\}$. We also compute $\Delta_w^\alpha\text{-1def}(d, c, BD) = \{5, \dots, 9\} \cap (\{5, \dots, 9\} \cap \{4, \dots, 7\}) = \{5, \dots, 7\}$. Then, since $\Delta_w^\alpha(a, d) = \Delta_w^\alpha(c, d) = \top$, we have that $\Delta_w^\alpha\text{-def}(d, BD) = ((\{5, \dots, 9\} \setminus \{1, \dots, 8\}) \cup \{5, \dots, 8\}) \cap ((\{5, \dots, 9\} \setminus \{4, \dots, 7\}) \cup \{5, \dots, 7\}) = \{5, \dots, 9\}$. Moreover, $\Delta_w^\alpha\text{-def}(b, BD) = \mathcal{T}(b) = \{6\}$ since $\nexists x \in \mathcal{A}$ such that $\Delta_w^\alpha(x, b) = \top$.

Focusing on the argument a and the set $AC = \{a, c\}$, we have $\Delta_w^\alpha\text{-1def}(a, b, AC) = \{1, \dots, 8\} \cap \emptyset = \emptyset$, since b is not attacked by any other argument, and $\Delta_w^\alpha\text{-1def}(a, d, AC) = \{1, \dots, 8\} \cap ((\{5, \dots, 9\} \cap \{1, \dots, 8\}) \cup (\{5, \dots, 9\} \cap \{4, \dots, 7\})) = \{5, \dots, 8\}$. Therefore $\Delta_w^\alpha\text{-def}(a, AC) = ((\{1, \dots, 8\} \setminus \{6\}) \cup \emptyset) \cap ((\{1, \dots, 8\} \setminus \{5, \dots, 9\}) \cup \{5, \dots, 8\}) = \{1, 2, 3, 4, 5, 7, 8\}$. We observe that a is not defended at $t = 6$ when its undefeated attacker b is available with probability 1.

We now define Δ -admissible, Δ -complete, Δ -preferred, Δ -stable and Δ -grounded semantics for TPAFs.

Definition 22 (Δ -Semantics). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF, Δ a defeat function and consider a set of arguments $E \subseteq \mathcal{A}$. We say that:

- E is a Δ -admissible extension of \mathbf{G} , denoted by $E \in \Delta\text{-ad}(\mathbf{G})$ if and only if for all $a \in E$ it holds that $\mathcal{T}(a) = \Delta\text{-def}(a, E)$;
- E is a Δ -complete extension of \mathbf{G} , denoted by $E \in \Delta\text{-co}(\mathbf{G})$ if and only if E is a Δ -admissible extension of \mathbf{G} and E contains all the arguments a such that $\mathcal{T}(a) = \Delta\text{-def}(a, E)$;
- E is a Δ -preferred extension of \mathbf{G} , denoted by $E \in \Delta\text{-pr}(\mathbf{G})$ if and only if E is a \subseteq -maximal Δ -complete extension;
- E is a Δ -stable extension of \mathbf{G} , denoted by $E \in \Delta\text{-st}(\mathbf{G})$ if and only if E is Δ -admissible and for all $b \in \mathcal{A} \setminus E$, there exists $a \in E$ such that a defeats b , i.e., $\Delta(a, b) = \top$;
- E is the Δ -grounded extension of \mathbf{G} , denoted by $E \in \Delta\text{-gr}(\mathbf{G})$ if and only if E is the \subseteq -minimal Δ -complete extension.

Example 1 (Continued). Consider the set of arguments $BD = \{b, d\}$. We know from the previous example that $\Delta_w^\alpha\text{-def}(b, BD) = \{6\} = \mathcal{T}(b)$ and $\Delta_w^\alpha\text{-def}(d, BD) = \{5, \dots, 9\} = \mathcal{T}(d)$. Hence, $BD \in \Delta_w^\alpha\text{-ad}(\mathbf{G}'_2)$. BD is also a Δ_w^α -complete extension of \mathbf{G}'_2 since we obtain $\Delta_w^\alpha\text{-def}(x, BD) = \mathcal{T}(x)$ only for $x \in \{b, d\}$. On the contrary, the singleton $\{c\}$, which is a Δ_w^α -admissible extension, is not Δ_w^α -complete since $\Delta_w^\alpha\text{-def}(d, \{c\}) = \mathcal{T}(d)$. The other Δ_w^α -complete extensions besides $\{b, d\}$ are indeed $\{b\}$ and $\{b, c\}$. We can now identify the grounded extension $\{b\}$ which is the \subseteq -minimal Δ_w^α -complete.

The two following propositions show that the Δ -semantics satisfy classical properties in non-temporal frameworks.

Proposition 4 (Unicity of the Δ -grounded extension). For any $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF, there always exists one and only one Δ -grounded extension.

Proof (Proposition 4). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF and $E \subseteq \mathcal{A}$ a subset of arguments. From Definition 22, $E \in \Delta\text{-gr}(\mathbf{G})$ if and only if E is the \subseteq -minimal Δ -complete extension. First, by definition, there cannot exist two \subseteq -minimal sets of a set of sets. Hence if there exists a grounded extension, it is unique. Second, from Definition 22:

- E is a Δ -admissible extension of \mathbf{G} , denoted by $E \in \Delta\text{-ad}(\mathbf{G})$ if and only if for all $a \in E$ it holds that $\mathcal{T}(a) = \Delta\text{-def}(a, E)$;
- E is a Δ -complete extension of \mathbf{G} , denoted by $E \in \Delta\text{-co}(\mathbf{G})$ if and only if E is a Δ -admissible extension of \mathbf{G} and E contains all the arguments a such that $\mathcal{T}(a) = \Delta\text{-def}(a, E)$.

We have therefore two cases, either it exists arguments undefeated or not. In the case of the existence of undefeated arguments, these arguments are in some Δ -admissible extensions, and they will be in each Δ -complete extension. Moreover, the \subseteq -minimal Δ -complete extension contains all these undefeated arguments (with their defended arguments) which is the grounded extension. In the case where there are no undefeated arguments, the Δ -complete semantics has always the empty set as an extension. Therefore, the \subseteq -minimal Δ -complete extension is the empty set, which is consequently also the grounded extension. □

The relations between Δ -admissible, Δ -preferred, Δ -stable, and Δ -complete extensions are given below.

Proposition 5 (Relation between semantics). *Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF. Then:*

1. *Let $E \subseteq \mathcal{A}$. Then, E is \subseteq -maximal Δ -admissible if and only if E is a \subseteq -maximal Δ -complete extension;*
2. *A Δ -preferred extension is also a Δ -complete extension;*
3. *A Δ -stable extension is also a Δ -preferred extension;*
4. *The Δ -grounded extension is a subset of all Δ -preferred and Δ -stable extensions.*

Proof (Proposition 5). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF. Then:

1. From Definition 22, any Δ -complete extension E is a Δ -admissible one such that it contains its defended arguments. Then, E is \subseteq -maximal Δ -admissible if and only if E is a \subseteq -maximal Δ -complete extension;
2. From Definition 22, a Δ -preferred extension is a \subseteq -maximal Δ -complete extension. Therefore, a Δ -preferred extension is also a Δ -complete extension;
3. Let E a \subseteq -maximal Δ -complete extension. From the Definition 22, if an argument x is conflict-free with E it must be in E otherwise if x is in defeat with an argument $a \in E$, E must defend a , i.e. $\exists b \in E$ such that b defeat x . Therefore, a Δ -stable extension is a \subseteq -maximal Δ -complete extension and also a Δ -preferred extension;
4. Given that the Δ -grounded extension is the \subseteq -minimal Δ -complete extension and given that a Δ -stable extension and a Δ -preferred extension are Δ -complete extensions, the Δ -grounded extension is a subset of all Δ -preferred and Δ -stable extensions.

□

Let us now define the notion of skeptical acceptability according to a Δ semantics.

Definition 23 (Δ -Skeptical acceptability). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF, and let $\{E_1, \dots, E_n\}$ be the set of Δ -extensions of \mathbf{G} , with respect to a semantics between admissible (ad), complete (co), preferred (pr), stable (st) and grounded (gr). An argument $a \in \mathcal{A}$ is Δ -skeptical acceptable under Δ -s where $s \in \{ad, co, pr, st, gr\}$, denoted by $a \in \Delta$ -sk-s(\mathbf{G}), if and only if $\forall E \in \{E_1, \dots, E_n\}, a \in E$.

Example 1 (Continued). Knowing that the sets $\{b\}$, $\{b, c\}$, and $\{b, d\}$ represents all and only Δ_w^α -complete extensions of \mathbf{G}'_2 , we conclude that $\{b\} \in \Delta_w^\alpha$ -sk-co(\mathbf{G}'_2), while $\{c\} \notin \Delta_w^\alpha$ -sk-co(\mathbf{G}'_2). Recalling that $\{c\} \in \Delta_w^\alpha$ -ad(\mathbf{G}'_2), we can also verify that $\{d\} \notin \Delta_w^\alpha$ -sk-ad(\mathbf{G}'_2).

We show that the Δ -semantics (Definition 22) and the classical semantics (Definition 4) produce the same extensions for AFs derived from restricted TPAFs, using a defeat function Δ .

Theorem 1 (Link between the Dung's semantics and the Δ -semantics). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF, Δ a defeat function and Δ - \mathbf{G} the restricted graph of \mathbf{G} according to Δ . The following holds.

- If E is a Δ -admissible extension of \mathbf{G} , then E is an admissible extension of Δ - \mathbf{G} .
- If E is a Δ -complete extension of \mathbf{G} , then E is a complete extension of Δ - \mathbf{G} .
- If E is a Δ -preferred extension of \mathbf{G} , then E is a preferred extension of Δ - \mathbf{G} .
- If E is a Δ -stable extension of \mathbf{G} , then E is a stable extension of Δ - \mathbf{G} .
- If E is a Δ -grounded extension of \mathbf{G} , then E is a grounded extension of Δ - \mathbf{G} .

Proof (Theorem 1). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF and Δ a defeat function. Let $E \subseteq \mathcal{A}$ such that E is Δ -conflict-free.

- Recall the two definitions of Δ -admissible and classic admissible extensions:
 - From Definition 22, $E \in \Delta$ -ad(\mathbf{G}) if and only if for all $a \in E$ it holds that $\mathcal{T}(a) = \Delta$ -def(a, E).
 - From Definition 4, a conflict-free subset E is admissible if each $a \in E$ is defended by E .
- 1. First, let show that if E is Δ -conflict-free in \mathbf{G} then E is conflict-free in Δ - \mathbf{G} . From Definition 19, E is Δ -conflict-free if and only if $\nexists a, b \in E$ such that $\Delta(a, b) = \top$. Therefore, given that there do not exist significant attacks between arguments in E , there do not exist attacks between arguments of E in Δ - \mathbf{G} , i.e. E is conflict-free in the restricted graph Δ - \mathbf{G} .
- 2. Secondly let us show that if in \mathbf{G} , $\forall a \in E$ it holds that $\mathcal{T}(a) = \Delta$ -def(a, E) then in Δ - \mathbf{G} , each $a \in E$ is defended by E . From Definition 21, an argument a is Δ -defended by E if there exists some argument in E which defends a (typologically). Please note, additionally, that the fact of being defended in the reduced graph Δ - \mathbf{G} does not imply that it is totally defended ($\mathcal{T}(a) = \Delta$ -def(a, E)) in \mathbf{G} .

From these two previous points, we can deduce that if E is a Δ -admissible extension of \mathbf{G} , then E is an admissible extension of $\Delta\text{-}\mathbf{G}$.

- Recall the two definitions of Δ -complete and classic complete extensions:
 - From Definition 22, E is a Δ -complete extension of \mathbf{G} , denoted by $E \in \Delta\text{-co}(\mathbf{G})$ if and only if E is a Δ -admissible extension of \mathbf{G} and E contains all the arguments a such that $\mathcal{T}(a) = \Delta\text{-def}(a, E)$;
 - From Definition 4, E is complete, if it is admissible and $\forall a \in \mathcal{A}$ defended by E , $a \in E$.

Since the definition of the complete semantics depends on the admissible semantics, we refer to the previous proof for the link between the Δ -admissible and admissible extensions.
Hence, if E is a Δ -complete extension of \mathbf{G} , then E is a complete extension of $\Delta\text{-}\mathbf{G}$.
- Recall the two definitions of Δ -preferred and classic preferred extensions:
 - E is a Δ^I -preferred extension of \mathbf{G} , denoted by $E \in \Delta\text{-pr}(\mathbf{G})$ if and only if E is a \subseteq -maximal Δ -admissible extension;
 - E is preferred if it is complete and \subseteq -maximal.

Since the definition of the preferred semantics depends on the complete semantics, we refer to the previous proof for the link between the Δ -complete and complete extensions.
Hence, if E is a Δ -preferred extension of \mathbf{G} , then E is a preferred extension of $\Delta\text{-}\mathbf{G}$.
- Recall the two definitions of Δ -grounded and classic grounded extensions:
 - E is the Δ -grounded extension of \mathbf{G} , denoted by $E \in \Delta\text{-gr}(\mathbf{G})$ if and only if E is the \subseteq -minimal Δ -complete extension.
 - E is grounded if it is complete and \subseteq -minimal.

Since the definition of the grounded semantics depends on the complete semantics, we refer to its previous proof for the link between the Δ -complete and complete extensions.
Hence, if E is a Δ -grounded extension of \mathbf{G} , then E is a grounded extension of $\Delta\text{-}\mathbf{G}$.
- Recall the two definitions of Δ -stable and classic stable extensions:
 - E is a Δ -stable extension of \mathbf{G} , denoted by $E \in \Delta\text{-st}(\mathbf{G})$ if and only if E is Δ -admissible and for all $b \in \mathcal{A} \setminus E$, there exists $a \in E$ such that a defeats b , i.e., $\Delta(a, b) = \top$;
 - E is stable if it is admissible and attacks every argument in $\mathcal{A} \setminus E$;

Since the definition of stable semantics depends on the admissible semantics, we refer to its previous proof for the link between the Δ -admissible and admissible extensions.
Moreover, given that the attacks in the restricted graph $\Delta\text{-}\mathbf{G}$ depends from the defeat attacks in \mathbf{G} , then if in \mathbf{G} , for all $b \in \mathcal{A} \setminus E$, there exists $a \in E$ such that a defeats b , i.e., $\Delta(a, b) = \top$; then in $\Delta\text{-}\mathbf{G}$, every argument in $\mathcal{A} \setminus E$ is attacked.
Hence, if E is a Δ -stable extension of \mathbf{G} , then E is a stable extension of $\Delta\text{-}\mathbf{G}$.

□

The previous result does not offer a complete characterization (if and only if), as it only considers the graph topology as in classical AFs. This means that it would be possible to have a defence between arguments, but the duration of that defence may only cover part of the availability of an argument, depending on its temporality in TPAF. Therefore, constructing a restricted graph and applying classical semantics can be seen as a relaxed version of Δ -semantics. Indeed, the Δ -semantics is required to accept only the completely defended arguments. In contrast, if we apply the Dung semantics on a restricted graph, it will be enough that an argument has a defended instant to be accepted.

Example 3. Consider as a counterexample the TPAF $\mathbf{H} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ where $\mathcal{A} = \{a, b, c\}$, $\mathcal{R} = \{(a, b), (b, c)\}$ and \mathcal{P} is given in Figure 11. For $\alpha = 0.2$, we have that $\mathbf{H} \equiv \Delta_w^\alpha\text{-}\mathbf{H}$, since all three attacks are preserved in the restricted TPAF. It is easy to verify that the set $\{a, c\}$ is an admissible extension of $\Delta_w^\alpha\text{-}\mathbf{H}$. However, $\{a, c\}$ is not a Δ_w^α -admissible extension of \mathbf{H} . In fact, $\Delta_w^\alpha\text{-}1\text{def}(c, b, \{a, c\}) = \{1, \dots, 4\} \cap (\{1, \dots, 2\} \cap \{1\}) = \{1\}$, and so $\Delta_w^\alpha\text{-def}(c, \{a, c\}) = (\{1, \dots, 4\} \setminus \{1, \dots, 2\}) \cup \{1\} = \{1, 3, 4\} \neq \mathcal{T}(c) = \{1, \dots, 4\}$.

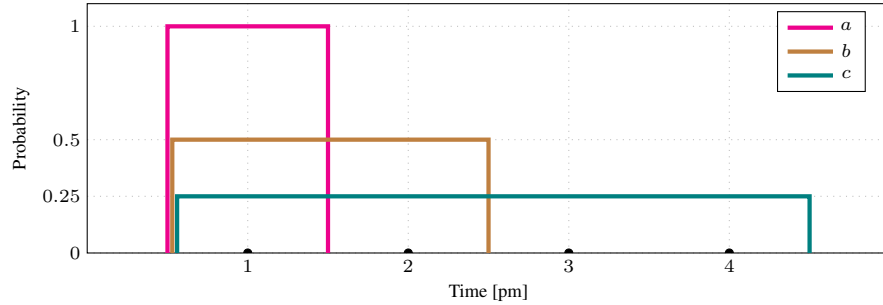


Fig. 11. Probability distribution \mathcal{P} for the arguments in $\mathbf{H} = \langle \{a, b, c\}, \{(a, b), (b, c)\}, \mathcal{P} \rangle$.

4.3 Consistent Temporal Δ -Acceptability

As we saw in the previous section, the notion of a skeptically acceptable argument only identifies arguments that are acceptable in each instant of the studied interval. However, knowing if a given argument is acceptable in some time instants and defeated in others is also interesting. For this purpose, we introduce a finer-grained notion of acceptability over time, which extracts the instants in which an argument is defended. We first define the notion of minimal temporal Δ -acceptability, which extracts all the instants in which an argument is defended according to all maximal consistent sets of arguments containing the evaluated argument.

Definition 24 (Minimal temporal Δ -acceptability). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF and $a \in \mathcal{A}$. We define the minimal temporal Δ -acceptability of a according to a defeat function Δ by $\text{min-}\Delta\text{-acc}(a) = \bigcap_{S \in \text{maxFree-}\Delta(a)} \Delta\text{-def}(a, S)$, where $\text{maxFree-}\Delta(a)$ is the set of all \subseteq -maximal Δ -conflict-free set of arguments containing a .

Example 1 (Continued). Again, consider the TPAF \mathbf{G}'_2 . We want to compute the minimal temporal Δ_w^α -acceptability of a . Firstly, we identify the set $\text{maxFree-}\Delta_w^\alpha(a) = \{\{a, c\}\}$. Thus we obtain $\text{min-}\Delta_w^\alpha\text{-acc}(a) = \Delta_w^\alpha\text{-def}(a, \{a, c\}) = \{1, 2, 3, 4, 5, 7, 8\}$. Similarly, we compute $\text{min-}\Delta_w^\alpha\text{-acc}(b) = \Delta_w^\alpha\text{-def}(b, \{b, c\}) \cap \Delta\text{-def}(b, \{b, d\}) = \{6\}$, $\text{min-}\Delta_w^\alpha\text{-acc}(c) = \Delta_w^\alpha\text{-def}(c, \{a, c\}) \cap \Delta_w^\alpha\text{-def}(c, \{b, c\}) = \{4, \dots, 7\}$, and $\text{min-}\Delta_w^\alpha\text{-acc}(d) = \Delta_w^\alpha\text{-def}(d, \{b, d\}) = \{5, \dots, 9\}$. Figure 12 shows the minimal temporal Δ_w^α -acceptability for all the arguments in \mathbf{G}'_2 .

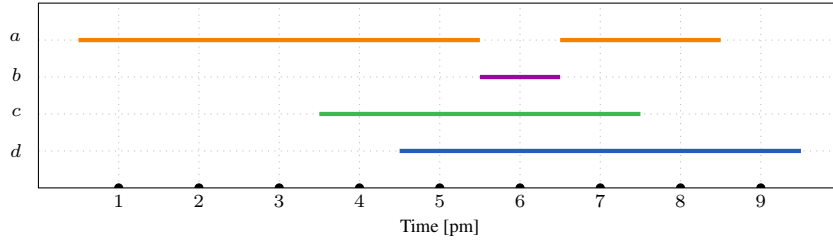


Fig. 12. Minimal temporal Δ_w^α -acceptability for the arguments in \mathbf{G}'_2 .

The above example shows that some arguments share instants of minimal temporal Δ_w^α -acceptability, even though they defeat each other. This happens for arguments a and d at $t = 5, 7, 8$, and for c and d at $t = 5, 6, 7$. To prevent mutually defeating arguments from being acceptable at the same time, we refine the notion of minimal temporal Δ -acceptability by imposing that an argument cannot be acceptable at the same time as its attackers. We introduce the following definition.

Definition 25 (Consistent temporal Δ -acceptability). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF and $a \in \mathcal{A}$. We define the consistent temporal Δ -acceptability of a according to Δ by $\text{con-}\Delta\text{-acc}(a) = \text{min-}\Delta\text{-acc}(a) \setminus \bigcup_{b \in \mathcal{A} \text{ such that } \Delta(b,a)=\top} \text{min-}\Delta\text{-acc}(b)$.

Example 1 (Continued). We compute the consistent temporal Δ_w^α -acceptability of all arguments in \mathbf{G}'_2 (see Fig. 13 for a visual reference). Starting with a , we have $\text{con-}\Delta_w^\alpha\text{-acc}(a) = \text{min-}\Delta_w^\alpha\text{-acc}(a) \setminus (\text{min-}\Delta_w^\alpha\text{-acc}(b) \cup \text{min-}\Delta_w^\alpha\text{-acc}(d)) = \{1, 2, 3, 4, 5, 7, 8\} \setminus (\{6\} \cup \{5, \dots, 9\}) = \{1, 2, 3, 4\}$. For b we compute $\text{con-}\Delta_w^\alpha\text{-acc}(b) = \text{min-}\Delta_w^\alpha\text{-acc}(b) \setminus \emptyset = \{6\}$. For c , $\text{con-}\Delta_w^\alpha\text{-acc}(c) = \text{min-}\Delta_w^\alpha\text{-acc}(c) \setminus \text{min-}\Delta_w^\alpha\text{-acc}(d) = \{4, 5, 6, 7\} \setminus \{5, \dots, 9\} = \{4\}$. Finally, for d we obtain $\text{con-}\Delta_w^\alpha\text{-acc}(d) = \text{min-}\Delta_w^\alpha\text{-acc}(d) \setminus (\text{min-}\Delta_w^\alpha\text{-acc}(a) \cup \text{min-}\Delta_w^\alpha\text{-acc}(c)) = \{5, \dots, 9\} \setminus (\{1, 2, 3, 4, 5, 7, 8\} \cup \{4, \dots, 7\}) = \{9\}$.

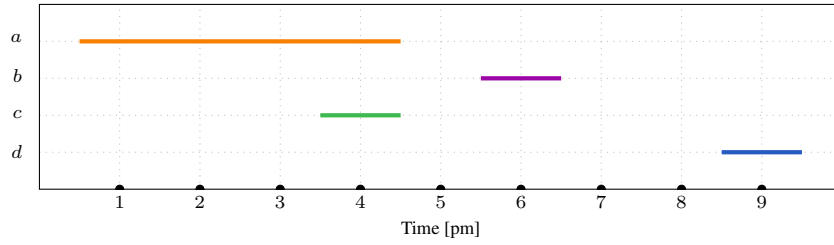


Fig. 13. Consistent temporal Δ_w^α -acceptability for the arguments in \mathbf{G}'_2 .

In Figure 13, only the arguments a and c are considered acceptable simultaneously (for $t = 4$), based on the notion of Δ_w^α -acceptability. It is also noteworthy that no arguments are considered acceptable for $t = 5, 7, 8$. This is because, during those instants, no argument prevails over the other in terms of weakly sum defeat, i.e., no part of the dialogue presented in Example 1 is more convincing than the other.

By further analysing Figure 13, we can comprehend the sequence of events preceding the victim's death. A plausible reconstruction could be as follows. The suspect was in the bar between 1 pm and 4 pm, and he fought with the victim at 4 pm. So the fight happened inside the bar. For sure, by 6 pm, the suspect was not in the bar. At 9, the victim died. This re-enactment gives us a clearer idea of the situation and helps us unravel the events that took place during the considered time interval based on the information in our possession.

In the case of a symmetric attack, keeping the consistency of the initial attack relation (coming from the relation \mathcal{R}) is sufficient to keep one defeat between the arguments. Consequently, when no defeat is detected between these arguments (initially in conflict), they may have consistent instants in common.

Proposition 6 (Consistency with the attack relation). *Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF, $a, b \in \mathcal{A}$ two arguments such that $(a, b), (b, a) \in \mathcal{R}$. For any defeat function Δ , if $\Delta(a, b) = \top$ or $\Delta(b, a) = \top$ then $\text{con-}\Delta\text{-acc}(a) \cap \text{con-}\Delta\text{-acc}(b) = \emptyset$.*

Proof (Proposition 6). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P} \rangle$ be a TPAF, $a, b \in \mathcal{A}$ two arguments such that $(a, b), (b, a) \in \mathcal{R}$, a defeat function Δ and $\Delta(a, b) = \top$ or $\Delta(b, a) = \top$.

From Definition 25, $\text{con-}\Delta\text{-acc}(a) = \text{min-}\Delta\text{-acc}(a) \setminus \bigcup_{b \in \mathcal{A} \text{ such that } \Delta(b, a) = \top} \text{min-}\Delta\text{-acc}(b)$. Hence:

- $\text{con-}\Delta\text{-acc}(a) \subseteq \text{min-}\Delta\text{-acc}(a)$, then
- if $\Delta(b, a) = \top$ then
 - $\text{con-}\Delta\text{-acc}(a) = \text{con-}\Delta\text{-acc}(a) \setminus \text{min-}\Delta\text{-acc}(b)$, then
 - $\text{con-}\Delta\text{-acc}(a) = \text{con-}\Delta\text{-acc}(a) \setminus \text{con-}\Delta\text{-acc}(b)$, then
 - $\text{con-}\Delta\text{-acc}(a) \cap \text{con-}\Delta\text{-acc}(b) = \emptyset$.

□

5 Temporal Duration-based Probabilistic Argumentation Frameworks

Until now, we considered all the events expressed by the arguments to only last for a single moment (duration = 1). In the following, we will generalise TPAF so that arguments can represent events lasting (continuously) over time (duration = n).

The probability distribution function will now represent the probability of the event's beginning. For each argument, we use a new function $\mathcal{D}(a)$ (for duration(a)), which returns the event's time duration for an argument a .

Definition 26 (TDPAF). A *Temporal Duration-based Probabilistic Argumentation Framework (TDPAF)* is a tuple $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P}, \mathcal{D} \rangle$ such that: \mathcal{A} is a finite set of arguments; $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ is the attack relation; $\mathcal{P}^I : \mathcal{A} \rightarrow [0, 1]$ is the probability distribution of an argument over a time interval I ; $\mathcal{D} : \mathcal{A} \rightarrow \mathbb{N} \setminus \{0\}$ is the duration of an argument.

Under the assumption that two arguments conflict if i) there is an attack between them and ii) there is at least one instant where the events intersect, we can generalise the results obtained in the previous section using the notion of cumulative probability, i.e. the probability that the event (which lasts over a period of time) will occur at a specific moment.

Definition 27 (Cumulative probability). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P}, \mathcal{D} \rangle$ be a TDPAF, $a \in \mathcal{A}$ an argument, and $\mathcal{D}(a) = n$. We define the cumulative probability of a at an instant $t \in \mathbf{T}$, denoted $\mathcal{CP}^{\{t\}}(a) = \sum_{i \in \{t-n-1, \dots, t\}} \mathcal{P}^{\{i\}}(a)$.

Given that the sum of the probability distribution of the event's beginning is equal to 1 and that the duration of the event is equal to n , we show that the sum of the cumulative probability is equal to n .

Proposition 7 (Sum of the cumulative probability). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P}, \mathcal{D} \rangle$ be a TDPAF, $a \in \mathcal{A}$ an argument, and $\mathcal{D}(a) = n$. Then $\sum_{t \in \mathbf{T}} \mathcal{CP}^{\{t\}}(a) = n$.

Proof (Proposition 7). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P}, \mathcal{D} \rangle$ be a TDPAF, $a \in \mathcal{A}$ an argument, and $\mathcal{D}(a) = n$. From Definition 27, the cumulative probability of a at an instant $t \in \mathbf{T}$, is computed by $\mathcal{CP}^{\{t\}}(a) = \sum_{i \in \{t-n-1, \dots, t\}} \mathcal{P}^{\{i\}}(a)$, i.e. each cumulative probability

of an instant t is calculated over a time window of size n ending on t . As a result, all probabilities (which in this case are possible starting points for a duration of n) will be considered in the next $n - 1$ instants (because each instant looks at its $n - 1$ predecessor instant). In other words, for all instant $t \in \mathbf{T}$, the probability $\mathcal{P}^{\{t\}}(a)$ will be sum n times from the instant t to the instant $t + n - 1$ in $\sum_{t \in \mathbf{T}} \mathcal{CP}^{\{t\}}(a)$. Hence,

$\sum_{t \in \mathbf{T}} \mathcal{CP}^{\{t\}}(a) = \sum_{t \in \mathbf{T}} \mathcal{P}^{\{t\}}(a) \times n = n \times \sum_{t \in \mathbf{T}} \mathcal{P}^{\{t\}}(a)$. Finally, given that the sum of probability is equal to 1, i.e. $\sum_{t \in \mathbf{T}} \mathcal{P}^{\{t\}}(a) = 1$, we obtain $\sum_{t \in \mathbf{T}} \mathcal{CP}^{\{t\}}(a) = n$.

□

We also guarantee that the cumulative probability value is less than or equal to 1 for any instant of time.

Proposition 8 (Maximal value for an instant of the cumulative probability). *Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P}, \mathcal{D} \rangle$ be a TDPAF, $a \in \mathcal{A}$ an argument, and $\mathcal{D}(a) = n$. Then $\forall t \in \mathbf{T}$, $\mathcal{CP}^{\{t\}}(a) \leq 1$.*

Proof (Proposition 8). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P}, \mathcal{D} \rangle$ be a TDPAF, $a \in \mathcal{A}$ an argument, and $\mathcal{D}(a) = n$. From Definition 27, the cumulative probability of a at an instant $t \in \mathbf{T}$, is computed by $\mathcal{CP}^{\{t\}}(a) = \sum_{i \in \{t-n-1, \dots, t\}} \mathcal{P}^{\{i\}}(a)$. In the worst case, instant t considers all the starting points in its calculation, i.e. $\mathcal{T}(a) \subseteq \{t-n-1, \dots, t\}$. However, given that $\sum_{t \in \mathbf{T}} \mathcal{P}^{\{t\}}(a) = 1$, we always have $\forall t \in \mathbf{T}$, $\mathcal{CP}^{\{t\}}(a) \leq 1$. □

Depending on the user's needs, again, it is useful to be able to focus the study of a TDPAF on a specific subset of arguments or time intervals. To this extent, we can adapt Definitions 8 and 12, and Definitions 9 and 13 to work with TDPAFs.

Definition 28 (Minimal TDPAF according to arguments). *Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P}, \mathcal{D} \rangle$ be a TDPAF and $S \subseteq \mathcal{A}$ a set of arguments. The TDPAF obtained from \mathbf{G} by focusing on S is $\mathbf{G}_{\downarrow S} = \langle \mathcal{A}', \mathcal{R}_{\downarrow \mathcal{A}'}, \mathcal{P}', \mathcal{D}' \rangle$, where $\mathcal{A}' = S \cup \{b \in \mathcal{A} \text{ such that there exists a path from } b \text{ to some } a \in S\}$, $\mathcal{P}' : \mathcal{A}' \rightarrow [0, 1]$ is such that $\mathcal{P}'(a) = \mathcal{P}(a)$ for all $a \in \mathcal{A}'$ and $\mathcal{D}' : \mathcal{A}' \rightarrow \mathbb{N} \setminus \{0\}$ is such that $\mathcal{D}'(a) = \mathcal{D}(a)$ for all $a \in \mathcal{A}'$.*

Definition 29 (Minimal TDPAF according to time). *Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P}, \mathcal{D} \rangle$ be a TDPAF and $I = \{t_1, \dots, t_n\}$ an interval of time. The TDPAF obtained from \mathbf{G} by focusing on I is $\mathbf{G}_{\downarrow I} = \mathbf{G}_{\downarrow S}$, where $S = \{a \in \mathcal{A} \text{ for which } \exists t_i \in \mathcal{T}(a) \mid t_1 \leq t_i \leq t_n\}$.*

Example 4. Using the concept of \mathcal{D} , we can extend Example 1 (the murder case) to a more realistic setting where events can occur over several instants, named \mathbf{G}'_3 . Note that the range of information is restricted on the arguments from a to d , as the argument e does not belong to the minimal TAF (or TPAF or TDPAF).

- a) witness A recalls seeing the suspect in a bar between 1 pm and 8 pm (i.e. in the interval $\{1, \dots, 8\}$): with a duration $\mathcal{D}(a) = 3$;
- b) the bartender B is sure that the suspect was not in the bar at 6 pm (i.e. $\{6\}$): with a duration $\mathcal{D}(b) = 2$;
- c) witness C reports that the suspect was in a fight with the victim between 4 pm and 7 pm (i.e. $\{4, \dots, 7\}$): with a duration $\mathcal{D}(c) = 1$;
- d) according to the doctor, the victim died between 5 pm and 9 pm (i.e. $\{5, \dots, 9\}$): with a duration $\mathcal{D}(d) = 1$;

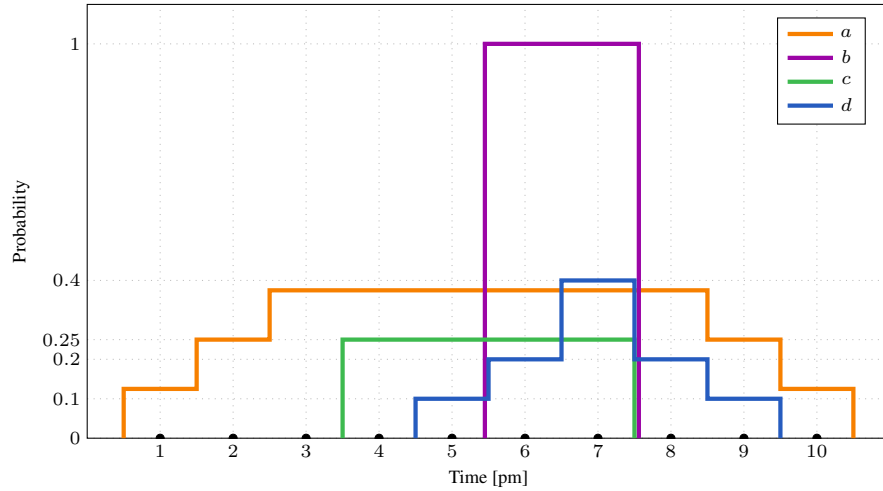


Fig. 14. Cumulative Probability distribution over time for the arguments in the TDPAF \mathbf{G}'_3 .

5.1 Constellation Approach

In the case of arguments occurring over a set of instants (greater than 1), the probability of intersection can be extended by the cumulative probability instead of the classical probability distribution. However, the probability of intersection must be between $[0, 1]$ (as it is a probability), but since it is possible to have several instants in conflict (probability equal to 1, see argument b in Fig. 14), the sum of the probabilities of conflict of all instants (i.e. the probability of intersection) can be greater than 1. Therefore, in the following definition, extending the probability of intersection for the cumulative probability, the maximum value is limited to 1.

Definition 30 (General probability of intersection). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P}, \mathcal{D} \rangle$ be a TDPAF, $a, b \in \mathcal{A}$ are arguments, and $I \subseteq \mathbf{T}$ is the intersected positive time interval of a and b , i.e. $I = \mathcal{T}(a) \cap \mathcal{T}(b)$. We define the general probability of intersection that two events occur at the same instant by $\overline{PI}(a, b) = \min \left(1, \sum_{t \in I} \mathcal{CP}^{\{t\}}(a) \times \mathcal{CP}^{\{t\}}(b) \right)$.

We show in the following proposition that Definition 30 is a generalisation of Definition 15, given the equivalence of the result for the case where an argument has a duration of 1.

Proposition 9 (Generalisation of PI). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P}, \mathcal{D} \rangle$ be a TDPAF, $a, b \in \mathcal{A}$ are arguments with a \mathcal{D} equal to 1, and $I \subseteq \mathbf{T}$ is the intersected positive time interval of a and b , i.e. $I = \mathcal{T}(a) \cap \mathcal{T}(b)$. Therefore $PI(a, b) = \overline{PI}(a, b)$.

Proof (Proposition 9). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P}, \mathcal{D} \rangle$ be a TDPAF, $a, b \in \mathcal{A}$ are arguments with a \mathcal{D} equal to 1, and $I \subseteq \mathbf{T}$ such that $I = \mathcal{T}(a) \cap \mathcal{T}(b)$. From Definition 27, the cumulative probability of a at an instant $t \in \mathbf{T}$, is

computed by $\mathcal{CP}^{\{t\}}(a) = \sum_{i \in \{t-n-1, \dots, t\}} \mathcal{P}^{\{i\}}(a)$, then when $n = 1$, we have $\mathcal{CP}^{\{t\}}(a) = \sum_{i \in \{t-1, \dots, t\}} \mathcal{P}^{\{i\}}(a) = \mathcal{P}^{\{t\}}(a)$. Hence, in this case, $\overline{PI}(a, b) = \min\left(1, \sum_{t \in I} \mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(b)\right)$. Note that $\sum_{t \in I} \mathcal{P}^{\{t\}}(a) \leq 1$ and $\sum_{t \in I} \mathcal{P}^{\{t\}}(b) \leq 1$ (because $I \subseteq \mathbf{T}$) then for any probability distribution, the sum of the product of the elements (which are less than or equal to 1) are always less than or equal to 1, i.e. $\sum_{t \in I} \mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(b) \leq 1$. Therefore $\overline{PI}(a, b) = \sum_{t \in I} \mathcal{P}^{\{t\}}(a) \times \mathcal{P}^{\{t\}}(b) = PI(a, b)$. \square

About Proposition 2, which is specific to the uniform distribution, we cannot generalise it in the same way as we did in Definition 30 because, using durations, each instant (from a uniform distribution) does not have the same cumulative probability (see argument a in Fig 14). The transformation of a TDPAF into a PrAF can be generalised as follows.

Definition 31 (Transformation of a TDPAF into a PrAF). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P}, \mathcal{D} \rangle$ be a TDPAF. We define $\mathbf{G}' = \langle \mathcal{A}, \mathcal{R}, P_R \rangle$ be the PrAF transformation of \mathbf{G} , such that for all $(a, b) \in \mathcal{R}$, $P_R(a, b) = \overline{PI}(a, b)$.

Example 1 (Continued). Referring to the TDPAF \mathbf{G}'_3 , we can obtain the corresponding transformed PrAF by computing:

- $P_R(a, b) = P_R(b, a) = \overline{PI}(a, b) = \overline{PI}(b, a) = 0.375 + 0.375 = 0.75$,
- $P_R(a, d) = P_R(d, a) = \overline{PI}(a, d) = \overline{PI}(d, a) = 0.375 \times 0.1 + 0.375 \times 0.2 + 0.375 \times 0.4 + 0.375 \times 0.2 + 0.25 \times 0.1 = 0.0375 + 0.075 + 0.15 + 0.075 + 0.025 = 0.3625$,
and
- $P_R(c, d) = P_R(d, c) = \overline{PI}(c, d) = \overline{PI}(d, c) = 0.25 \times 0.1 + 0.25 \times 0.2 + 0.25 \times 0.4 = 0.025 + 0.05 + 0.1 = 0.175$.

5.2 Epistemic Approach

For the epistemic approach, the only adjustment we need to make to generalize all the previous semantics to work on arguments occurring over a set of instants is to generalize the defeat function with \overline{PI} instead of PI .

Definition 32 (General α -defeats). Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P}, \mathcal{D} \rangle$ be a TDPAF and consider two arguments $a, b \in \mathcal{A}$ such that $(a, b) \in \mathcal{R}$, and the interval $I \subseteq \mathbf{T}$ such that $I = \mathcal{T}(a) \cap \mathcal{T}(b)$, i.e. I is the intersected positive time interval of a and b . Given a threshold $\alpha \in]0, 1]$, we say that

- a **weakly sum defeats** b according to α if $\sum_{t \in I} \mathcal{CP}^{\{t\}}(a) \times \mathcal{CP}^{\{t\}}(b) \geq \alpha$ and $\exists t \in I$ such that $\mathcal{CP}^{\{t\}}(a) \geq \mathcal{CP}^{\{t\}}(b)$ (in this case we write $\overline{\Delta}_w^\alpha(a, b) = \top$, otherwise $\overline{\Delta}_w^\alpha(a, b) = \perp$);
- a **strongly sum defeats** b according to α if $\sum_{t \in I} \mathcal{CP}^{\{t\}}(a) \times \mathcal{CP}^{\{t\}}(b) \geq \alpha$ and $\forall t \in I, \mathcal{CP}^{\{t\}}(a) \geq \mathcal{CP}^{\{t\}}(b)$ (in this case we write $\overline{\Delta}_s^\alpha(a, b) = \top$, otherwise $\overline{\Delta}_s^\alpha(a, b) = \perp$);

- a **one point defeats** b according to α if $\exists t \in I$ such that $\mathcal{CP}^{\{t\}}(a) \times \mathcal{CP}^{\{t\}}(b) \geq \alpha$ and $\mathcal{CP}^{\{t\}}(a) \geq \mathcal{CP}^{\{t\}}(b)$ (in this case we write $\overline{\Delta}_o^\alpha(a, b) = \top$, otherwise $\overline{\Delta}_o^\alpha(a, b) = \perp$).

It follows that Proposition 3 can be generalised for the generalised α -defeat functions given that the only modification from Definition 17 to Definition 32 is the replacement of $\mathcal{P}^{\{t\}}(a)$ and $\mathcal{P}^{\{t\}}(b)$ by $\mathcal{CP}^{\{t\}}(a)$ and $\mathcal{CP}^{\{t\}}(b)$. Having different constants does not change the approach to the proof.

Corollary 1 (Link between general α -defeats). *Let $\mathbf{G} = \langle \mathcal{A}, \mathcal{R}, \mathcal{P}, \mathcal{D} \rangle$ be a TDPAF, $a, b \in \mathcal{A}$ are arguments and $\alpha \in]0, 1]$. If $\overline{\Delta}_o^\alpha(a, b) = \top$ then $\overline{\Delta}_w^\alpha(a, b) = \top$ and $\overline{\Delta}_s^\alpha(a, b) = \top$ then $\Delta_w^\alpha(a, b) = \top$.*

Considering the other Definitions and Propositions in Section 4.2 and Section 4.3, they hold for TDPAF since the duration function is only used in the defeat function. Moreover, they also hold with these general α -defeat functions because these Definitions and Propositions are defined for any defeat function.

In the following, we illustrate the application of the previous definitions (of Sections 4.2, 4.3) by using the general defeat functions ($\overline{\Delta}^\alpha$) on the cumulative probability distribution (Fig 14). From the example for the semantics, using the $\overline{\Delta}_w^\alpha$ defeat the results are the same as before.

Example 1 (Continued). Consider the set of arguments $BD = \{b, d\}$. $\overline{\Delta}_w^\alpha$ - $\text{def}(b, BD) = \{6, 7\} = \mathcal{T}(b)$ and $\overline{\Delta}_w^\alpha$ - $\text{def}(d, BD) = \{5, \dots, 9\} = \mathcal{T}(d)$. Hence, $BD \in \overline{\Delta}_w^\alpha$ - $\text{ad}(\mathbf{G}'_3)$. BD is also a $\overline{\Delta}_w^\alpha$ -complete extension of \mathbf{G}'_3 since we obtain $\overline{\Delta}_w^\alpha$ - $\text{def}(x, BD) = \mathcal{T}(x)$ only for $x \in \{b, d\}$. On the contrary, the singleton $\{c\}$, which is a $\overline{\Delta}_w^\alpha$ -admissible extension, is not $\overline{\Delta}_w^\alpha$ -complete since $\overline{\Delta}_w^\alpha$ - $\text{def}(d, \{c\}) = \mathcal{T}(d)$. The other $\overline{\Delta}_w^\alpha$ -complete extensions besides $\{b, d\}$ are indeed $\{b\}$ and $\{b, c\}$. We can now identify the grounded extension $\{b\}$ which is the \subseteq -minimal $\overline{\Delta}_w^\alpha$ -complete.

Results for the $\overline{\Delta}_w^\alpha$ -skeptical acceptability are also equivalent to the case presented for the Δ -skeptical acceptability.

Example 1 (Continued). Knowing that the sets $\{b\}$, $\{b, c\}$, and $\{b, d\}$ represents all and only $\overline{\Delta}_w^\alpha$ -complete extensions of \mathbf{G}'_3 , we conclude that $\{b\} \in \overline{\Delta}_w^\alpha$ - $\text{sk-co}(\mathbf{G}'_3)$, while $\{c\} \notin \overline{\Delta}_w^\alpha$ - $\text{sk-co}(\mathbf{G}'_3)$. Recalling that $\{c\} \in \overline{\Delta}_w^\alpha$ - $\text{ad}(\mathbf{G}'_3)$, we can also verify that $\{d\} \notin \overline{\Delta}_w^\alpha$ - $\text{sk-ad}(\mathbf{G}'_3)$.

On the other hand, consistent temporal $\overline{\Delta}_w^\alpha$ -acceptability yields different outcomes than consistent temporal Δ -acceptability.

Example 1 (Continued). We compute the consistent temporal $\overline{\Delta}_w^\alpha$ -acceptability of all arguments in \mathbf{G}'_3 (see Fig. 15 for a visual reference). Starting with a , we have $\text{con-}\overline{\Delta}_w^\alpha$ - $\text{acc}(a) = \text{min-}\overline{\Delta}_w^\alpha$ - $\text{acc}(a) \setminus (\text{min-}\overline{\Delta}_w^\alpha$ - $\text{acc}(b) \cup \text{min-}\overline{\Delta}_w^\alpha$ - $\text{acc}(d)) = \{1, 2, 3, 4, 5, 7, 8, 9, 10\} \setminus (\{6, 7\} \cup \{5, \dots, 9\}) = \{1, 2, 3, 4, 10\}$. For b we compute $\text{con-}\overline{\Delta}_w^\alpha$ - $\text{acc}(b) = \text{min-}\overline{\Delta}_w^\alpha$ - $\text{acc}(b) \setminus \emptyset = \{6, 7\}$. For c , $\text{con-}\overline{\Delta}_w^\alpha$ - $\text{acc}(c) = \text{min-}\overline{\Delta}_w^\alpha$ - $\text{acc}(c) \setminus \text{min-}\overline{\Delta}_w^\alpha$ - $\text{acc}(d) = \{4, 5, 6, 7\} \setminus \{5, \dots, 9\} = \{4\}$. Finally, for d we

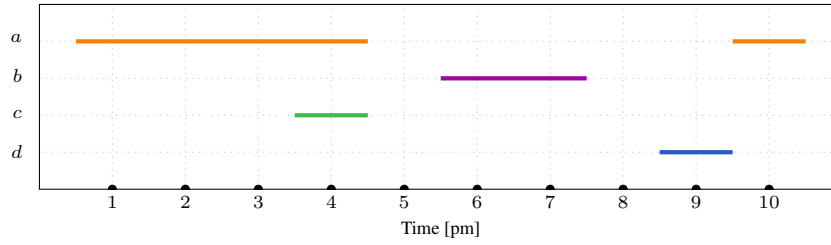


Fig. 15. Consistent temporal $\overline{\Delta}_w^\alpha$ -acceptability for the arguments in G'_3 .

obtain $\text{con-}\overline{\Delta}_w^\alpha\text{-acc}(d) = \text{min-}\overline{\Delta}_w^\alpha\text{-acc}(d) \setminus (\text{min-}\overline{\Delta}_w^\alpha\text{-acc}(a) \cup \text{min-}\overline{\Delta}_w^\alpha\text{-acc}(c)) = \{5, \dots, 9\} \setminus (\{1, 2, 3, 4, 5, 7, 8, 9, 10\} \cup \{4, \dots, 7\}) = \{9\}$.

Note that argument a has a duration of 3 (such that the instants of the event are consecutive), so there are only two possibilities according to $\text{con-}\overline{\Delta}_w^\alpha\text{-acc}(a)$, either a occurs over $\{1, 2, 3\}$ or during $\{2, 3, 4\}$.

6 Conclusion

We propose a system able to capture the temporal probabilistic nature of arguments and provide a tool for drawing conclusions starting from a set of conflicting facts/events for which the placement in time is uncertain. To evaluate the acceptability of arguments in TPAFs, we employ the probability of event occurrence by distributing uncertainty throughout timespans. This methodology can be applied in practical situations such as legal reasoning involving timed events (as in the example reported in the paper) or medical diagnosis based on the evolving symptoms of a patient.

The probability associated with timed arguments is subject to interpretations, which vary according to context and use case. For this reason, we propose different criteria for establishing if the conflict between arguments is significant enough to represent a defeat. Following the graph restriction process shown in Figures 8, 9, and 10, different semantics can be applied to evaluate the acceptability of arguments. For instance, we can use the Dung-style semantics (as described in Sections 4.1 and 4.2) or those more suitable for the notion of time, such as consistent temporal acceptability. The latter allows us to assess the acceptability of arguments over finer time scales (as discussed in Section 4.3).

In the future, we plan to carry on this work by examining other aspects of argumentation that relate uncertainty to the notion of time. In addition to uncertainty about when a given argument is valid, we may also consider probability associated with arguments and attacks (i.e. they are uncertain), as in the framework provided in [33]. Consequently, other criterion functions could be introduced for evaluating conflicts based on topological uncertainty. Moreover, by exploiting the uncertainty about arguments and attacks, along with the temporal duration of the events represented in TPAFs, we could evaluate the arguments through criteria that allow for greater differentiation between acceptability states, e.g. by extending ranking-based semantics [1, 14] to fit our framework. Instead of a probability score between 0 and 1, we could also consider quantitative

logic such as *Subjective Logic*, a type of probabilistic logic that explicitly considers epistemic uncertainty and source trust [36]. Finally, we could encompass a qualitative view of probability besides the quantitative one considered in this paper: in certain domains, it could be hard to condense uncertainty into a specific score and then to have fuzzy measures (e.g. a low/medium/high degree) or comparative ones (e.g. this event is more/less probable than this other event) could help the modelling of the considered problem.

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