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# Trajectory Tracking in Linear Complementarity Systems with and without State Jumps: A Passivity Approach <sup>\*</sup>

Aya Younes, Félix Miranda-Villatoro, Bernard Brogliato

<sup>a</sup>Univ. Grenoble-Alpes, Inria, CNRS, LJK, Grenoble INP, Grenoble, 38000, France

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## Abstract

This work is largely concerned with trajectory tracking in linear complementarity systems (LCS). The main analytical tool for stability analysis and control design is passivity and the associated linear matrix inequalities (LMIs) for feedback passification. Cases with and without state-jumps, with and without parametric uncertainties, are analysed. Theoretical findings are illustrated with examples from circuits with set-valued, nonsmooth electronic components, networks with unilateral interactions, and mechanical system with unilateral spring.

*Keywords:* Linear complementarity systems, Passivity, Lyapunov stability, Trajectory tracking, State jumps, circuits, networks, mechanical systems.

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## 1. Introduction

Complementarity dynamical systems are an important class of nonsmooth nonlinear systems, that has received attention in the automatic control and applied mathematics literature because of their applications in many fields like electrical circuits with nonsmooth components, mechanical systems with unilateral contact, networks with unilateral interactions, economics with projected dynamical systems, genetics, traffic flow and neural networks, *etc.*, see [1, 2, 3]. Trajectory tracking for fully-actuated complementarity Lagrangian systems has been tackled in [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. The stabilization problem is studied in [18, 19, 20], the output regulation problem is analyzed in [21]. The Lyapunov stability of linear complementarity systems (LCS) has been tackled in [22, 18, 23, 24, 25]. The trajectory tracking problem for mechanical systems with unilateral constraints and/or impacts is given a solution in [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 26, 27] using various strategies. It is noteworthy that in most of these articles, the desired trajectory is suitably modified in the neighborhood of state jumps (impacts), and/or the tracking error dynamics 'stability is tailored to the problem. The tracking of cyclic trajectories which undergo constrained/unconstrained/impacting phases of motion (including multiple constraints and multiple impacts phenomena) is studied in [9, 10, 11, 12, 13] for  $n$ -degree-of-freedom (dof) systems. Basically, the controllers are extended passivity-based algorithms which switch between three sub-controllers (persistent contact with a constant number of activated constraints, mode obtained from the deactivation of one or several constraints, impacting transition phase between two persistent contact modes when the number of activated constraints increases). The authors introduce what they call weak stability, which extends Lyapunov stability by disregarding the Lyapunov function variation during the transition phases (nonmonotonic Lyapunov functions are used). Trajectory tracking for particles evolving in planar billiards is studied in [4, 5, 6, 7, 8]. The objective is to control a two-dof particle inside a closed compact domain when trajectories collide with the domain's boundary and never undergo persistent contact with it (*i.e.*, a kind of vibro-impact systems [28, section 1.3.2]). Desired trajectory spreading is introduced in [17, 16] to cope with the peaking (jump-mismatch) phenomenon which is a typical behavior of systems with state jumps [29].

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Email addresses: [aya.younes@inria.fr](mailto:aya.younes@inria.fr) (Aya Younes), [felix.miranda-villatoro@inria.fr](mailto:felix.miranda-villatoro@inria.fr) (Félix Miranda-Villatoro), [bernard.brogliato@inria.fr](mailto:bernard.brogliato@inria.fr) (Bernard Brogliato)

This article is dedicated to trajectory tracking in LCS, which make a class of complementarity systems different from Lagrangian complementarity systems studied in the above references. The main differences between Lagrangian systems with unilateral constraints and impacts, and LCS as in this paper, are that impacts in mechanical systems are state-dependent, see (60), and passivity does not hold when the complementarity variables (*i.e.*, the contact force multiplier and the gap function which usually is a nonlinear function of the position) are used as the input and output variables to define the supply rate (see Remark 8.1). In passive LCS, state jumps are essentially exogenous (*i.e.*, they are triggered by the signal  $Fu(t)$  in the variable  $w$  in (2) below), and passivity holds between the complementarity variables. Moreover the complementarity variables are not of the same form in mechanical systems and in LCS. In the latter they take the form of a linear function of the state, the multiplier and an exogenous signal (see Definition 1.3 below and  $w$  in (2)), while unilateral constraints in Lagrangian systems always have  $D = 0$  and  $F = 0$  (hence excluding the strong passivity of the closed-loop system), and usually are a nonlinear function of one part of the state only, *i.e.*, positions. However even linear unilaterally constrained Lagrangian systems do not fit within the class of LCS studied in this article, because of the different nature of state jumps as explained above. One major consequence is that state jumps can be triggered instantaneously by the exogenous signal, which is not possible in mechanical systems where exogenous signals (external forces and torques) do not act directly in the complementarity constraints. The way the desired trajectories are designed in the above works and in this article, are quite different as well. When the unilateral constraints are replaced by unilateral springs, things differ significantly as illustrated in section 8. Certainly, the frameworks that are the closest to what is presented in the sequel, are in [21] and [14] (though the class of studied systems is not the same, and the robustness analysis is not led in [14]). This is detailed below in Remark 3.7.

The main contributions of this article are to propose a solution for the trajectory tracking in LCS, in the case where all parameters are known, and when parameter uncertainties are taken into account. The case of absolutely continuous solutions is treated, as well as the case of solutions with state jumps (when parameters are known). Passivity and maximal monotonicity of suitable operators are the tools which allow to design stable closed-loop systems. Matrix inequalities stemming from passivity are central in the design process. The theoretical developments are illustrated by numerical examples on nonsmooth circuits, networks with unilateral interactions, and mechanics with unilateral springs.

This article is organised as follows: useful notations and definitions are given in the next paragraph; section 2 is dedicated to recall the main results about the well-posedness (existence and uniqueness of solutions) of LCS, as well as state jump mappings; section 3 presents in details the control design and the error dynamics analysis, in the known parameters case, with and without state jumps; section 4 is devoted to extend the results to the case when parametric uncertainties are considered; a toy scalar example is studied in section 5, a circuit example is analysed in section 6, a network with unilateral interactions is presented in section 7, and a simple mechanical system with a unilateral spring is presented in section 8; conclusions end the article in section 9, and several useful tools and results are provided in Appendix A through Appendix G.

*Notation and Definitions.* The elements of a matrix  $M \in \mathbf{R}^{n \times m}$  are denoted  $M_{ij}$ . A matrix  $M \in \mathbf{R}^{n \times n}$ , possibly nonsymmetric, is said to be positive definite,  $M > 0$  (resp. positive semi-definite,  $M \succcurlyeq 0$ ) if  $x^T M x > 0$  for all  $x \neq 0 \in \mathbf{R}^n$  (resp.  $x^T M x \geq 0$ ).  $M$  is a P-matrix if all its principal minors are positive; a positive definite matrix is a P-matrix. The minimum and the maximum eigenvalues of  $M$  are denoted  $\lambda_{\min}(M)$  and  $\lambda_{\max}(M)$ . The maximum singular value of  $M \in \mathbf{R}^{n \times m}$  is denoted  $\sigma_{\max}(M)$ ,  $\text{Im}(M)$  is its image,  $\text{Ker}(M)$  is its null space. The singular values are ordered as  $\sigma_{\max}(M) = \sigma_1(M) \geq \sigma_2(M) > \dots > \sigma_r(M) > 0$ , where  $r = \text{rank}(M)$ . Let  $M = M^T$ , its eigenvalues are real and ordered as  $\lambda_{\max}(M) = \lambda_1(M) \geq \lambda_2(M) \geq \dots \geq \lambda_n(M) = \lambda_{\min}(M)$ .  $M^\dagger$  denotes the Moore-Penrose pseudo-inverse of  $M$ . Let  $1 \leq p < +\infty$ , a Lebesgue integrable function  $f : I \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$  belongs to  $\mathcal{L}_{loc}^p(I; \mathbf{R})$  if  $(\int_I \|f(t)\|^p dt)^{\frac{1}{p}} < +\infty$  for any open set  $I$  (extended spaces). If  $I = \mathbf{R}^n$  then  $f \in \mathcal{L}^p(\mathbf{R}^n; \mathbf{R})$ . Also  $f \in \mathcal{L}^\infty(\mathbf{R}^n; \mathbf{R})$  if  $f(\cdot)$  is Lebesgue integrable and  $\|f(x)\| < +\infty$  almost everywhere on  $\mathbf{R}^n$ . We denote  $\|f\|_{L,p} \triangleq (\int_I \|f(t)\|^p dt)^{\frac{1}{p}}$  for  $1 \leq p < +\infty$ , and  $\|f\|_{L,\infty} \triangleq \sup_{t \in I} \|f(t)\|$  (when  $I$  is obvious from the context it may be avoided). The range of the function  $f(\cdot)$  is  $\text{Im}(f)$ , its domain is  $\text{dom}(f) = \{x \mid f(x) < +\infty\}$ . Right and left limits of the function  $f(\cdot)$  at  $t$  are denoted  $f(t^+)$  and  $f(t^-)$ . Let  $S$  be a nonempty set,  $\text{rint}(S)$  is its relative interior [30]. Let  $S$  and  $S'$  be closed sets,  $d_H(S, S')$  denotes the Hausdorff distance between  $S$  and  $S'$ . AC is for absolute continuity, BV is for bounded variation, LBV is locally BV, and RCLBV is for right-continuous LBV. Let  $S \subseteq \mathbf{R}^m$  be a closed convex set containing

0. Its polar cone  $S^\circ = \{x \in \mathbf{R}^m \mid x^\top v \leq 0, \forall v \in S\} = -S^*$  where  $S^*$  is the dual cone. Let us recall some useful definitions [1, 30, 31, 32, 33].

**Definition 1.1.** Let  $S \subset \mathbf{R}^n$  be a closed, non-empty and convex set. The normal cone to  $S$  at a point  $x \in S$  is given by

$$\mathcal{N}_S(x) = \left\{ g \in \mathbf{R}^n \mid g^\top(z - x) \leq 0, \forall z \in S \right\}. \quad (1)$$

For  $x \notin S$  one usually sets  $\mathcal{N}_S(x) = \emptyset$ . The tangent cone  $\mathcal{T}_S(\cdot)$  is the polar to the normal cone.

The domain of a set-valued mapping  $T : \mathbf{R}^n \rightrightarrows \mathbf{R}^n$  is  $\text{Dom}(T) = \{x \in \mathbf{R}^n \mid T(x) \neq \emptyset\}$ .

**Definition 1.2.** A multivalued mapping  $T : \text{Dom}(T) \subseteq \mathbf{R}^n \rightrightarrows \mathbf{R}^n$  is monotone if for all  $u_1, u_2 \in \text{Dom}(T)$ , for all  $v_1 \in T(u_1), v_2 \in T(u_2)$ , then

$$\langle u_1 - u_2, v_1 - v_2 \rangle \geq 0.$$

It is maximal if its graph cannot be enlarged without destroying monotonicity.

The normal cone to a convex closed nonempty set, defines a maximal monotone set-valued mapping. More generally, let  $\varphi : \mathbf{R}^n \rightarrow \mathbf{R} \cup \{+\infty\}$  be a proper, convex and lower semicontinuous function. Its subdifferential at  $x$  is defined as the set  $\partial\varphi(x) = \{g \in \mathbf{R}^n \mid \varphi(v) - \varphi(x) \geq g^\top(v - x), \forall v \in \mathbf{R}^n\}$ , and it defines a maximal monotone mapping. The normal cone in (1) is the subdifferential of the indicator function defined as  $\Psi_S(x) = 0$  if  $x \in S$ ,  $\Psi_S(x) = +\infty$  if  $x \notin S$ . Another function plays an important role: the support function of the set  $S$ , defined as  $\sigma_S(y) = \sup_{v \in S} v^\top y$ . It is related to the indicator function as:  $\eta \in \mathcal{N}_S(v) = \partial\Psi_S(v) \Leftrightarrow v \in \partial\sigma_S(\eta) = \mathcal{N}_S^{-1}(\eta)$  (the indicator and the support functions are conjugate functions, and their subdifferentials are inverse mappings).

**Definition 1.3.** A linear complementarity system (LCS) with inputs is a nonsmooth nonlinear dynamical system defined as:

$$\begin{cases} (a) & \dot{x}(t) = Ax(t) + B\lambda(t) + Eu(t) \quad (dt\text{-almost everywhere}) \\ (b) & 0 \leq \lambda(t) \perp w(t) = Cx(t) + D\lambda(t) + Fu(t) \geq 0 \end{cases} \quad (2)$$

where  $x \in \mathbf{R}^n$ ,  $\lambda \in \mathbf{R}^m$  and  $w \in \mathbf{R}^m$ ,  $A, B, C, D, E$  and  $F$  are constant matrices of appropriate dimensions,  $u : \mathbf{R}_+ \rightarrow \mathbf{R}^p$  is an exogenous signal or a control input. The constraint in (2) (b) is a linear complementarity problem (LCP) in  $\lambda$ , denoted  $\text{LCP}(D, Cx + Fu)$ .

A mode of the LCP (2) (b) corresponds to a pair  $(\lambda, w)$  with  $\lambda_i = 0$  and  $w_i > 0$  (noncontact indices),  $\lambda_j > 0$  and  $w_j = 0$  (contact indices),  $\lambda_k = w_k = 0$  (degenerate indices),  $i \in \mathcal{I} \subseteq \{1, \dots, m\}$ ,  $j \in \mathcal{J} \subseteq \{1, \dots, m\}$ ,  $k \in \mathcal{K} \subseteq \{1, \dots, m\}$ ,  $\mathcal{I} \cup \mathcal{J} \cup \mathcal{K} = \{1, \dots, m\}$ ,  $\mathcal{I} \cap \mathcal{J} \cap \mathcal{K} = \emptyset$ ,  $\mathcal{I} \cap \mathcal{J} = \emptyset$ ,  $\mathcal{I} \cap \mathcal{K} = \emptyset$ , and  $\mathcal{J} \cap \mathcal{K} = \emptyset$ . The switching times between modes  $i$  and  $j$  correspond to junction times, or detachment times. It is noteworthy that most of the results in this article apply to linear cone complementarity systems (LCCS), with (b')  $K \ni \lambda(t) \perp w(t) = Cx(t) + D\lambda(t) + Fu(t) \in K^*$ , where  $K \subseteq \mathbf{R}^m$  is a nonempty closed convex cone,  $K^* = \{z \in \mathbf{R}^m \mid z^\top w \geq 0, \forall w \in K\}$  is its dual cone. Using tools from convex analysis it is possible to rewrite equivalently (2) as the differential inclusion (DI) [1]:

$$\dot{x}(t) \in Ax(t) + Eu(t) - B(D + \mathcal{N}_{\mathbf{R}^m}^{-1})^{-1}(Cx(t) + Fu(t)) \triangleq -H(t, x(t)), \quad (3)$$

where  $\lambda(t) \in -(D + \mathcal{N}_{\mathbf{R}^m}^{-1})^{-1}(Cx(t) + Fu(t))$ . We shall see later that the DI may sometimes be written in a slightly different way when the input is given a specific form, see (17) below. A state  $x^*$  is an equilibrium point of (2) if and only if there exist  $\lambda^*$  and  $w^* \in \mathbf{R}^n$  such that the mixed LCP (MLCP)

$$\begin{cases} 0 = Ax^* + B\lambda^* \\ 0 \leq \lambda^* \perp w^* = Cx^* + D\lambda^* \geq 0 \end{cases}$$

holds. Let us define Moreau's first order sweeping process:

**Definition 1.4.** Moreau's first order sweeping process (FOSwP) is a differential inclusion of the form:

$$\dot{x}(t) \in -\mathcal{N}_{S(t)}(x(t)) + f(t, x)$$

where  $\mathcal{N}_{S(t)}(x(t))$  is the normal cone to the closed convex non-empty set  $S(t) \subseteq \mathbf{R}^n$  at  $x(t)$ , defined in (1.1), and  $f(t, x)$  is a single-valued map such that  $f : \mathbf{R} \times \mathbf{R}^n \rightarrow \mathbf{R}^n$ .

Passivity plays an important role in the sequel of this work.

**Definition 1.5.** A quadruple  $(A, B, C, D)$  is said to be passive, or dissipative with respect to the supply rate  $u^\top y$ , if there exists a non-negative function  $V : \mathbf{R}^n \rightarrow \mathbf{R}_+$ , called a storage function, such that for all  $t_0 \leq t_1$  and all time functions  $(u, x, y) \in \mathcal{L}_2([t_0, t_1]; \mathbf{R}^m \times \mathbf{R}^n \times \mathbf{R}^m)$  such that  $\dot{x}(t) = Ax(t) + Bu(t)$ ,  $y(t) = Cx(t) + Du(t)$ , the following inequality holds:

$$V(x(t_0)) + \int_{t_0}^{t_1} u^\top(t)y(t) dt \geq V(x(t_1)) \quad (4)$$

The inequality (4) is called the dissipation inequality. Equivalently, the following LMI

$$\begin{pmatrix} A^\top P + PA & PB - C^\top \\ B^\top P - C & -D - D^\top \end{pmatrix} \preceq 0 \quad (5)$$

has a solution  $P = P^\top \succcurlyeq 0$ . Then  $V(x) = \frac{1}{2}x^\top Px$ . The quadruple  $(A, B, C, D)$  is said strictly state passive if the LMI (5) holds with  $P = P^\top \succ 0$  and  $A^\top P + PA + \epsilon P \preceq 0$  for some  $\epsilon > 0$ . It is said strongly passive if the LMI (5) holds with strict inequality and  $P = P^\top \succ 0$ .

When LCS are considered, then the multiplier  $\lambda$  is chosen as the input  $u$  while  $w$  is chosen as the output  $y$  for the passivity property. Passivity is then to be seen as a structural property related to internal variables of the LCS, and motivated by the fact that it corresponds to a physical property in some applications like circuits.

**Definition 1.6.** [33, Definitions 2.34, 2.58, 2.78] The transfer function  $H(s) = C(sI - A)^{-1}B + D \in \mathbb{C}^{m \times m}$  is positive real (PR) if:  $H(s)$  has no pole in  $\mathbf{Re}[s] > 0$ ,  $H(s)$  is real for all positive real  $s$ ,  $H(s) + H^*(s) \succcurlyeq 0$  for all  $\mathbf{Re}[s] > 0$ . The transfer function  $H(s) \in \mathbb{C}^{m \times m}$  is strictly positive real (SPR) if  $H(s - \epsilon)$  is PR for some  $\epsilon > 0$  [33, Definition 2.58] and it is strong SPR (SSPR) if  $H(s)$  is analytic in  $\mathbf{Re}[s] \geq 0$  and  $\mathbf{Re}[H(j\omega)] \geq \delta > 0$ , for all  $\omega \in [-\infty, \infty]$  and some  $\delta \in \mathbf{R}$  [33, Definition 2.78].

Strict state passivity is related to SPR transfer functions, while strong passivity is related to SSPR transfer functions [33, 34]. Using the Schur complement theorem (see, e.g., [33, Theorem A.65]), strong passivity implies that  $D \succ 0$  and  $-A^\top P - PA \succ 0$ .

## 2. Well-posedness and State-jumps of Time-Varying LCS

### 2.1. LCS with External Inputs

LCS as in (2) are nonlinear nonsmooth dynamical systems with inputs, whose well-posedness (existence, uniqueness, and continuous dependence of solutions) is tackled in [18, 22, 35, 36, 37, 38, 23, 21]. The main well-posedness studies for LCS are summarized and commented in Appendix B. Most of these results use the passivity of  $(A, B, C, D)$ , or conditions implied by passivity. This allows to use the maximal monotonicity of the operator  $H(t, \cdot)$  in (3) for each  $t$  using a result established in [23, 39, 40].

#### 2.1.1. Multipliers boundedness (AC solutions)

Well-posedness analyses usually focus solely on the state  $x(\cdot)$ . However, the boundedness of the multiplier is an important feature, since it may be used for feedback in this article. The condition in item 2 (b) ii) guarantees the existence of a solution to the LCP in (2). If the conditions in item 3 are verified, then explicit upper-bounds  $\|\dot{x}(t)\| \leq \dot{v}(t) + \alpha(t)\|x(t)\|$  can be obtained for some integrable  $\alpha(\cdot)$  and AC  $v(\cdot)$  [41, Theorem 4.3] [42, section 5.2, Theorem 2.1]. In our case, these upper-bounds depend on the properties of the set-valued map  $t \mapsto S(t)$  (which in turn depend on  $Fu(\cdot)$  being AC, or Lipschitz continuous, or LBV [37, Proposition 3.2]). Following the developments in [21] (see item 2. (c) in Appendix B),  $\lambda = \lambda^{\text{im}}(t, x) + \lambda^{\text{ker}}$ , where  $x \mapsto \lambda^{\text{im}}(t, x) \in \text{Im}(D + D^\top)$  is Lipschitz continuous, while  $\lambda^{\text{ker}} \in \text{Ker}(D + D^\top)$ . Moreover,  $\lambda^{\text{im}}$  is the least norm element in the (possibly) set-valued right-hand side of the DI, see [21, Equ. (15), Lemma 3].

It is inferred that in case the existence of AC solutions is proved, then  $B\lambda(\cdot)$  is a bounded selection of the (possibly set-valued) right-hand side in (2). The part of  $\lambda$  inside  $\text{Ker}(B)$  does not play any role in  $x(\cdot)$ , but we may need the whole multiplier vector for feedback purpose. It is therefore reasonable to assume that the minimum norm element

is available and that it satisfies a linear growth condition in  $\|x\|$ . Finally, it is worth recalling that even if  $x(\cdot)$  is AC, the multiplier  $\lambda(\cdot)$  may have discontinuities at junction times when  $w(\cdot)$  reaches the boundary (*i.e.*,  $w = 0$ ), when  $D$  is not a P-matrix. In the case of item 1, the multipliers are AC when  $Fu(t)$  is AC since they are a Lipschitz continuous function of AC functions.

From a more practical point of view, multipliers in LCS can be calculated by constructing the contact LCP [1, section 2.4.1]. Thus they appear to be a nonsmooth function of both state and input. The uniqueness of  $\lambda$  can be deduced from the conditions in item 2 (a) and (b), where passivity is the main property, plus some constraints qualification.

### 2.1.2. State jumps

Let the solution  $x(\cdot)$  of (2) be RCLBV. Then  $x(\cdot)$  may undergo discontinuities and the LCS has to be interpreted in the Measure Differential Inclusion (MDI) formalism [42, 43, 1]. In other words, the DI (3) has to be embedded into an MDI:  $dx \in (Ax(t) + Eu(t))dt - B(D + \mathcal{N}_{\mathbf{R}_+^m}^{-1})^{-1}(Cx(t^+) + Fu(t))d\nu$ , where  $dx$  is the differential measure associated with the RCLBV function  $x(\cdot)$  [42],  $dt$  is the Lebesgue measure, and  $d\nu$  is a specific Radon measure, see [21, Definition 4] for a rigorous introduction. The dissipation inequality extends to MDI, see [28, section 5.4.4.5] [33, section 7.2.4.1]. Let the set of time instants at which the state undergoes a jump, be denoted as  $\mathcal{J}_x$ , and the set of time instants at which  $Fu(\cdot)$  is discontinuous be denoted as  $\mathcal{J}_{Fu}$ . Then if conditions of item 2 (a) hold,  $\mathcal{J}_x \subseteq \{0\} \cup \mathcal{J}_{Fu}$  [18, 37]. In other words, the state can jump only initially, or at times of discontinuity in  $Fu(\cdot)$  (or in the set  $S(t)$  in the FOSwP formalism). *This means that state jumps can be chosen and dwell times can be imposed (contrarily to complementarity mechanical systems), by suitably choosing  $Fu(t)$ .*

In the case of BV solutions, and under the conditions stated above, the term  $B\lambda$  is a Dirac measure at times of state jumps, and we may denote  $\lambda = \lambda^{imp} + \lambda^{reg}$ . The input in (11) below, may be defined with the complete  $\lambda$  ("impulsive" input), or just with its function part  $\lambda^{reg}$ . In both cases, the stability analysis has to incorporate the jumps. At an instant of jump  $t$ , using (3) and [21, section 3.4], we have:

$$x(t^+) - x(t^-) \in -B(D + \partial\sigma_{S(t^+)})^{-1}(Cx(t^+)), \quad (6)$$

with  $S(t^+) = \mathbf{R}_+^m - Fu(t^+)$ . State jump rules formulations are given in [1, Lemma 2.3], compiling results in [18, 37, 35, 44]. They are based on a maximum dissipation principle (a kind of plastic impact, inspired from [45]). Let us introduce such rules briefly. If  $(A, B, C, D)$  is passive with  $P = P^\top > 0$  and if  $0 \in \mathcal{K} \triangleq \{z \in \mathbf{R}^n \mid Cz + Fu(t^+) \in Q_D^*\}$  (notice that the set  $\mathcal{K} \neq \emptyset$  by the condition in item 2 (b) ii)),  $Q_D = \{z \in \mathbf{R}^m \mid 0 \leq z \perp Dz \geq 0\}$ , then the jumps dissipate "energy", *i.e.*,  $V(x(t^+)) \leq V(x(t^-))$  [44, Lemma 3], and the state jump rule is given by:

$$x(t^+) = \operatorname{argmin}_{x \in \mathcal{K}} \frac{1}{2}(x - x(t^-))^\top P(x - x(t^-)). \quad (7)$$

Notice that  $Q_D$  is a polyhedral set, and so is its dual cone [31, Lemma 6.45, Theorem 3.52]. If  $D > 0$  then  $Q_D = \{0\}$ ,  $Q_D^* = \mathbf{R}^m$ ,  $\mathcal{K} = \mathbf{R}^n$  and using (7),  $x(t^+) = x(t^-)$ . If  $D = 0$  then  $Q_D = \mathbf{R}_+^m = Q_D^*$ ,  $\mathcal{K} = \{z \in \mathbf{R}^n \mid Cz + Fu(t^+) \geq 0\}$ , hence  $0 \in \mathcal{K}$  if and only if  $Fu(t) \geq 0$ . Thus strongly passive systems have continuous-time solutions, but strictly state passive systems may have state jumps. Notice that:  $x(t^+) - x(t^-) = B\sigma$ , with  $\lambda^{imp} = \sigma\delta_t$ , and  $0 \leq \sigma \perp D\sigma \geq 0$ . As shown in [18, 1, 44] the state jump does not depend on a particular choice of the storage function matrix  $P$ . The meaning of the complementarity conditions at discontinuity times is explained in [18, Theorems 6.1, 9.1], see also [1, Lemma 2.3].

### 2.1.3. Zeno behaviour

LCS undergo two classes of events, *i.e.*, switching between modes of the LCP (2) (b) (for AC solutions), and state discontinuities (for BV solutions). Zenoness with AC solutions is tackled in [46, 22, 47] when  $Fu(\cdot) = 0$ . It states that if  $B \operatorname{SOL}(D, Cx)$  is a singleton, then the LCS (2) is Zeno-free. This holds if  $D$  is a P-matrix. Thus junction and detachment times are separated (dwell time). The conditions stated in [35, Theorem 7] also prevent Zeno behaviour in LCS with external inputs and state jumps, by imposing condition ii) in item 2 (b) between jumps.

## 2.2. Time-Varying LCS (TVLCS)

We may also encounter TVLCS of the form:

$$\begin{cases} (a) & \dot{x}(t) = A(t)x(t) + B(t)\lambda(t) + E(t)u(t) \quad (\text{almost everywhere}) \\ (b) & 0 \leq \lambda(t) \perp w(t) = C(t)x(t) + D(t)\lambda(t) + F(t)u(t) \geq 0 \end{cases}$$

when we deal with uncertainties which may be time-varying. The TVLCS is rewritten equivalently as:

$$\begin{aligned}\dot{x}(t) \in & A(t)x(t) + E(t)u(t) - B(t)(D(t) + \mathcal{N}_{\mathbb{R}_+^m}^{-1})^{-1}(C(t)x(t) + F(t)u(t)) \\ & = A(t)x(t) + E(t)u(t) - B(t)(D(t) + \mathcal{N}_{\mathbb{R}_+^m - F(t)u(t)}^{-1})^{-1}(C(t)x(t))\end{aligned}\quad (8)$$

Clearly, some of the results described above can be used to analyse (8), in particular cases [37, 36]. One such case is as follows. Assume that  $A = A(t)$ ,  $E = E(t)$ ,  $F = F(t)$ ,  $D = 0$ ,  $C$  and  $B$  are constant. Assume that there exists  $P = P^\top > 0$  such that  $PB = C^\top$ . Then (8) is rewritten equivalently as:

$$\dot{x}(t) \in A(t)x(t) + E(t)u(t) - BN_{\mathbb{R}_+^m - F(t)u(t)}(Cx(t))\quad (9)$$

After the classical state change  $z = Rx$ ,  $R^2 = P$ ,  $R = R^\top > 0$  [1], (9) is rewritten equivalently as:

$$\dot{z}(t) \in RA(t)R^{-1}z(t) + RE(t)u(t) - \mathcal{N}_{\mathbb{R}_+^m - F(t)u(t)}(z(t)),$$

which is a FOSwP (Definition 1.4) with affine perturbation. Then the material in item 3 in Appendix B can be used to analyse the well-posedness with both AC and RCLBV solutions. The general problem of well-posedness of (8) is not tackled in this work.

### 3. Trajectory Tracking in the Nominal Case

In this section, it is assumed that the plant model and parameters have no uncertainties. That is, the matrices  $A$ ,  $B$ ,  $C$ ,  $D$ ,  $E$ ,  $F$  in (2) are exactly known.

#### 3.1. Controller Design

Let us consider the LCS with external input in (2). Let us define the system which generates the desired trajectory to be tracked, as follows:

$$\begin{cases} \dot{x}_d(t) = Ax_d(t) + B\lambda_d(t) + Eu_d(t) \\ 0 \leq \lambda_d(t) \perp w_d(t) = Cx_d(t) + D\lambda_d(t) + Fu_d(t) \geq 0 \end{cases}\quad (10)$$

where  $u_d(\cdot)$  is an input. Following the LCS of the desired system in (10), the designer has to perform a preliminary analysis of the desired dynamics to determine a suitable trajectory for tracking. This analysis may involve numerical methods or the approaches presented in [48, 49]. The aim is to design a feedback controller such that the error dynamics with state vector  $e \stackrel{\Delta}{=} x - x_d$  possesses some stability property, to be defined later. Let us state the following that will be used later:

**Assumption 3.1.** *The solution  $x_d : \mathbb{R}_+ \rightarrow \mathbb{R}^n$  of the LCS (10) is AC and uniformly bounded, and the multiplier vector  $\lambda_d(u_d, x_d)$  is a bounded function of time.*

Conditions such that this holds can be obtained from the results stated in Appendix B, applied to (10).

**Assumption 3.2.** *The state  $x(\cdot)$  and the multiplier  $\lambda(\cdot)$  are available for measurement.*

In practice, the multiplier may be a physical quantity (like voltages and currents in circuits, contact force in mechanics), and Assumption 3.2 is reasonable (and studying state observers in the loop is outside the scope of this article). Thus the feedback controller in the plant (2) is chosen generically as:

$$u(x, \lambda, t) = K[x - x_d(t)] + G[\lambda - \lambda_d(t)] + u_d(t).\quad (11)$$

Inserting (11) into (2) for some feedback gains  $K \in \mathbb{R}^{p \times n}$  and  $G \in \mathbb{R}^{p \times m}$  gives rise to the closed-loop LCS:

$$\begin{cases} \dot{x}(t) = (A + EK)x(t) + (B + EG)\lambda(t) - EKx_d(t) - EG\lambda_d(t) + Eu_d(t) \quad (\text{a.e.}) \\ 0 \leq \lambda(t) \perp w(t) = (C + FK)x(t) + (D + FG)\lambda(t) - FKx_d(t) - FG\lambda_d(t) + Fu_d(t) \geq 0 \end{cases}\quad (12)$$

It is important to note that in general both  $\lambda_d(\cdot)$  and  $u_d(\cdot)$  in (10) may be discontinuous time-functions, hence (12) is in general an LCS as (2) with a potentially discontinuous term  $-FKx_d(t) - FG\lambda_d(t) + Fu_d(t)$  inside the complementarity

constraints. As a consequence jumps in  $x(\cdot)$  may occur unless conditions hold (see the end of section 3.2). We shall come back on state jumps in section 3.3. Rewriting (10) equivalently as

$$\begin{cases} \dot{x}_d(t) = (A + EK)x_d(t) + (B + EG)\lambda_d(t) + Eu_d(t) - EKx_d(t) - EG\lambda_d(t) \\ 0 \leq \lambda_d(t) \perp w_d(t) = (C + FK)x_d(t) + D\lambda_d(t) + Fu_d(t) - FKx_d(t) \geq 0 \end{cases} \quad (13)$$

gives rise to the error dynamics:

$$\begin{aligned} \dot{e}(t) &= (A + EK)e(t) + (B + EG)(\lambda(t) - \lambda_d(t)) \\ 0 \leq \begin{pmatrix} w(t) \\ w_d(t) \end{pmatrix} &= \begin{pmatrix} C + FK \\ -C - FK \end{pmatrix} e(t) + \begin{pmatrix} D + FG & -FG \\ 0 & D \end{pmatrix} \begin{pmatrix} \lambda(t) \\ \lambda_d(t) \end{pmatrix} + \begin{pmatrix} 0 & C \\ C + FK & -FK \end{pmatrix} \begin{pmatrix} x(t) \\ x_d(t) \end{pmatrix} \\ &+ \begin{pmatrix} Fu_d(t) \\ Fu_d(t) \end{pmatrix} \perp \begin{pmatrix} \lambda(t) \\ \lambda_d(t) \end{pmatrix} \geq 0 \end{aligned} \quad (14)$$

Clearly, the LCS in (14) is well-posed if both (10) and (12) are. However, the LCS in (14) cannot be used for well-posedness directly because  $x(\cdot)$  acts as an exogenous signal in complementarity constraints. The closed-loop's well-posedness has to be tackled with (12) and (13) (or (10)). In view of the structure of the closed-loop LCS in (12), let us state the following assumptions, which will be used in the sequel for both well-posedness and stability purposes.

**Assumption 3.3.** *There exist matrices  $K$  and  $G$  such that the plant's closed-loop quadruple  $(A + EK, B + EG, C + FK, D + FG)$  is strictly state passive.*

From Assumption 3.3, there exist gain matrices  $K$  and  $G$  such that the following bilinear matrix inequality (BMI) with unknowns  $P$ ,  $K$  and  $G$ :

$$M \triangleq \begin{pmatrix} (A + EK)^\top P + P(A + EK) & P(B + EG) - (C + FK)^\top \\ (B + EG)^\top P - (C + FK) & -(D + FG) - (D + FG)^\top \end{pmatrix} \preceq \begin{pmatrix} -\epsilon P & 0 \\ 0 & 0 \end{pmatrix}, \quad (15)$$

has a solution  $P = P^\top > 0$ . This may be replaced by the more stringent assumptions:

**Assumption 3.4.** *There exist matrices  $K$  and  $G$  such that the plant's closed-loop quadruple  $(A + EK, B + EG, C + FK, D + FG)$  is strongly passive.*

From Assumption 3.4, there exist gain matrices  $K$  and  $G$  such that the bilinear matrix inequality:

$$M < 0 \quad (16)$$

has a solution  $P = P^\top > 0$ . For numerical purposes, it is important to note that the BMIs in (15) and (16) are transformed into LMIs, as detailed in Appendix C. Using convex analysis, the closed-loop system (12) is rewritten equivalently as the DI:

$$\dot{x}(t) \in (A + EK)x(t) - (B + EG)((D + FG) + \partial\sigma_{S(t)})^{-1}((C + FK)x(t) - EKx_d(t) - EG\lambda_d(t) + Eu_d(t)). \quad (17)$$

where  $S(t) = \{v \in \mathbf{R}_+^m \mid v + FKx_d(t) + FG\lambda_d(t) - Fu_d(t) \geq 0\}$  is closed nonempty convex for each  $t$ , and  $\sigma_{S(t)}(\cdot)$  is the support function of  $S(t)$ . The DI in (17) has the form

$$\dot{x}(t) \in -\mathcal{M}(t, x(t)) + f(t), \quad (18)$$

with  $f(t) = -EKx_d(t) - EG\lambda_d(t) + Eu_d(t)$  and  $\mathcal{M}(\cdot, \cdot)$  is a set-valued operator. In view of (10) and Assumption 3.1, depending on the system's parameters, the multiplier  $\lambda_d(\cdot)$  may be discontinuous at some instants, hence the set  $S(t)$  may also be discontinuous at those instants ( $D > 0$  prevents such jumps). The LCS in (10) is also equivalently rewritten as the DI:

$$\dot{x}_d(t) \in Ax_d(t) - B(D + \partial\sigma_{S_d(t)})^{-1}(Cx_d(t)), \quad (19)$$

with  $S_d(t) = \{v \in \mathbf{R}^m \mid v - Fu_d(t) \geq 0\}$ , and the set  $S_d(t)$  is continuous as long as  $u_d(t)$  is. Therefore the DI (19) also fits with (18). Let us now examine (14). Using that  $x(t) = e(t) + x_d(t)$ , the error dynamics may be rewritten as:

$$\dot{e}(t) \in \tilde{A}e(t) - \tilde{B}(\tilde{D} + \mathcal{N}_{\tilde{S}(t,e(t))}^{-1})^{-1}(\tilde{C}e(t)) \quad (20)$$



with  $\tilde{S}(t, e) = \{z \in \mathbf{R}^{2m} \mid z + \tilde{E}e + \tilde{F}(t) \geq 0\}$ ,  $\tilde{E} = \begin{pmatrix} 0 \\ C + FK \end{pmatrix}$ ,  $\tilde{F}(t) = \begin{pmatrix} Cx_d(t) + Fu_d(t) \\ Cx_d(t) + Fu_d(t) \end{pmatrix}$ ,  $\tilde{D} = \begin{pmatrix} D + FG & -FG \\ 0 & D \end{pmatrix}$ ,  $\tilde{C} = \begin{pmatrix} C + FK \\ -C - FK \end{pmatrix}$ ,  $\tilde{B} = (B + EG \quad -B - EG)$ ,  $\tilde{A} = A + EK$ . The DI in (20) has a state and time-dependent polyhedral set  $\tilde{S}(\cdot, \cdot)$  which renders its study more complex. For each  $t$  and  $e$  the normal cone  $\mathcal{N}_{\tilde{S}(t, e)}$  defines a maximal monotone mapping, which may allow to recast (20) in the framework of item 2 (f) in Appendix B, and deduce conditions such that (20) has a Lipschitz solution. A second equivalent way to write the LCS (14) is:

$$\dot{e}(t) \in \tilde{A}e(t) - \tilde{B}(\tilde{D} + \mathcal{N}_{\tilde{S}(t)}^{-1})^{-1}(\tilde{C}e(t)) \Leftrightarrow \begin{cases} \dot{e}(t) = \tilde{A}e(t) + \tilde{B}\lambda(t) \\ 0 \leq \lambda(t) \perp \tilde{C}e(t) + \tilde{D}\lambda(t) + \tilde{F}(t) \geq 0 \end{cases}, \quad (21)$$

with  $\tilde{A} = \tilde{A}$ ,  $\tilde{B} = \tilde{B}$ ,  $\tilde{C} = \begin{pmatrix} C + FK \\ 0 \end{pmatrix}$ ,  $\tilde{D} = \tilde{D}$ ,  $\tilde{S}(t) = \{z \in \mathbf{R}^{2m} \mid z + \tilde{F}(t) \in \mathbf{R}_+^{2m}\}$ ,  $\tilde{F}(t) = \begin{pmatrix} Cx_d(t) + Fu_d(t) \\ Cx_d(t) + Fu_d(t) \end{pmatrix}$ . It is noteworthy that some conditions have to be imposed so that the construction of the DIs in (20) and (21) is possible, see section 3.3 for more details. Now we have at our disposal several (equivalent) formalisms for the closed-loop plant dynamics (12) and (17), the desired trajectory generator (10) and (19), and the error dynamics (14) and (20) and (21). This is useful for the well-posedness analyses relying on the various results in [18, 35, 36, 37, 21]. A difficulty is to determine under which conditions the passivity in Assumptions 3.3 or 3.4, implies the passivity of the quadruples  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ , or  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ , see (26).

**Remark 3.5.** *[Closed-loop system's well-posedness] In the framework of this article, the well-posedness of the plant dynamics is not a fundamental issue, since it is only the closed-loop system which is used in the analysis. In a similar way as feedback can be used to make an unstable system, stable in closed-loop, it may be used to render an ill-posed plant's model, well-posed in closed loop. The well-posedness of the closed-loop plant LCS (12) (equivalently the DI in (17)) can be inferred from [37, 36, 21, 50, 23], assuming that the desired signals  $x_d(\cdot)$ ,  $\lambda_d(\cdot)$  and  $u_d(\cdot)$  are AC or  $\mathcal{L}_{loc}^2(\subseteq \mathcal{L}_{loc}^1)$ , and that some basic constraint qualification conditions hold. Clearly in this setting Assumption 3.1 is important. When  $D + FG > 0$  (this is the case if Assumption 3.4 holds) then the results reported in item 1 in Appendix B apply. If  $D + FG = 0$  (which may be the case if Assumption 3.3 holds) then the results in item 3 in Appendix B apply. When  $D + FG \succcurlyeq 0$  but not null, then the various results in item 2 in Appendix B can be used. This requires to check the constraint qualifications listed in item 2 (a) through (f). This is not tackled in this article whose primary goal is tracking control.*

**Remark 3.6.** *The trajectory tracking problem as tackled in this work, can be interpreted as a synchronization problem between the master system (10), and the plant (2).*

**Remark 3.7.** *The output regulation problem tackled in [21] differs from the problem tackled in this work. First, the desired systems are different: The desired system to be tracked is defined by the dynamics of the quadruple  $(A_r, G_r, H_r, J_r)$  as mentioned in [21] which is different from the real system defined. It is noticeable that the controller  $u$  in [21] is introduced only in the ordinary differential equation (ODE) of the real system, but not in the variational inequality (VI). Besides, there is no presence of a desired controller  $u_d$  in the desired system, which is autonomous. Second, the objectives are different: In the presence of uncertainties, the real and the desired systems have different dynamics. Hence, in this case, the aim of the regulation problem in [21] is to design a controller using internal model principle that achieves a zero steady-state regulation error:  $e \triangleq x - \Pi x_r$  for some matrix  $\Pi$ . But, the aim of the tracking problem tackled in this work, in the presence of uncertainties, is to ultimately bound the tracking error:  $e \triangleq x - x_d$ , where  $x_d(\cdot)$  is generated by desired dynamics. Third, the controller in [21] is state feedback, while state and multiplier feedback are considered in this article.*

*The authors in [14] study tracking for Measure Differential Inclusions (MDIs), which differ from LCS investigated in this paper. They give sufficient conditions for the uniform convergence of MDIs with maximal monotonicity properties. In the tracking control of MDIs, the authors considered a desired trajectory  $x_d$  of locally bounded variation, and control inputs with an impulsive part (which we avoid, see section 3.3 and condition (C1)). In our approach, the desired trajectory  $x_d$  is designed independently by a LCS as shown in (10). Another difference with [14] is that the*

authors are interested in designing a controller such that the MDI of the closed-loop system is uniformly convergent with zero tracking error, while this paper is primarily concerned with studying the stability of the error dynamics without explicitly considering the convergence of the closed-loop system.

### 3.2. Error Dynamics Stability Analysis (no state jumps)

This section is dedicated to the stability analysis of the error dynamics (14), and it is assumed that all trajectories are AC. The arguments used for the proof of Proposition 3.8 are similar to those employed in [51, 52, 23], and are given for completeness.

**Proposition 3.8.** *Suppose that Assumptions 3.1, 3.2 and 3.3 hold, and that the solution  $x(\cdot)$  of the closed-loop LCS (12) is AC on  $\mathbb{R}_+$ . Then, the error dynamics in (14) has a globally exponentially stable equilibrium point  $e^* = 0$ .*

*Proof.* Let  $\Delta\lambda(t) \triangleq \lambda(t) - \lambda_d(t)$  and  $\Delta w(t) \triangleq w(t) - w_d(t) = (C + FK)e(t) + (D + FG)\Delta\lambda(t)$ . Consider the Lyapunov function candidate  $V(e) = e^\top P e$ . From the assumptions  $e(\cdot)$  is AC and thus it has a derivative almost everywhere. Along the error dynamics trajectories (14) it holds that:

$$\dot{V}(t) = e^\top [(A + EK)^\top P + P(A + EK)]e + 2e^\top P(B + EG)\Delta\lambda(t)$$

From the complementarity conditions in (12) and (13) we obtain equivalently:

$$\lambda(t) \in -\mathcal{N}_{S(t)}((C + FK)x(t) + (D + FG)\lambda(t))$$

and

$$\lambda_d(t) \in -\mathcal{N}_{S(t)}((C + FK)x_d(t) + (D + FG)\lambda_d(t))$$

with  $S(t)$  defined after (17). It follows from the monotonicity of the normal cone mapping that:

$$[w(t) - w_d(t)]^\top [\lambda(t) - \lambda_d(t)] = \Delta w(t)^\top \Delta\lambda(t) \leq 0. \quad (22)$$

In matrix form (the time argument is dropped in the right-hand side):

$$\begin{aligned} \dot{V}(t) &= \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top M \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} + \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top \begin{pmatrix} 0 & (C + FK)^\top \\ C + FK & D + FG + (D + FG)^\top \end{pmatrix} \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} \\ &= \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top M \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} + 2\Delta\lambda^\top \Delta w \leq \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top M \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} \leq 0, \end{aligned} \quad (23)$$

where  $(C + FK)e = \Delta w - (D + FG)\Delta\lambda$ . So, for all  $e$  and  $\Delta\lambda$  and using the strict state passivity, it is inferred:

$$\dot{V}(t) \leq -\epsilon e(t)^\top P e(t) = -\epsilon V(t).$$

Classical arguments yield  $V(t) \leq V(0) \exp(-\epsilon t)$ . Using the inequality:  $\lambda_{\min}(P)\|e\|^2 \leq e^\top P e$ , the following is obtained

$$\|e(t)\|^2 \leq \frac{V(t)}{\lambda_{\min}(P)} \leq \frac{V(0)}{\lambda_{\min}(P)} \exp(-\epsilon t)$$

Therefore, the equilibrium point of the error dynamics is globally exponentially stable.  $\square$

Notice that using (14) it follows that  $\lambda(t) - \lambda_d(t)$  converges exponentially fast to  $\text{Ker}(B - EG)$ .

*Comments on passivity.* Following on from the proof of Proposition 3.8, the supply rate in (22) indicates that the passivity, in this case, is studied between the input  $\lambda - \lambda_d$  and the output  $w - w_d$ . The variation of the storage function in (23) is written as  $\dot{V}(t) \leq 2\Delta\lambda^\top \Delta w$ . This shows the incremental passivity [53, 33] of the LCS defined in (12).

*Continuity of the solutions.* A sufficient condition for no state-jump is that both matrices  $D$  and  $D+FG$  are P-matrices, and  $u_d(\cdot)$  is continuous. Then (10) has continuously differentiable solutions  $x_d(\cdot)$  with uniqueness for any initial data, and  $\lambda_d(\cdot)$  is time-continuous (being a Lipschitz continuous function of  $x_d$  and  $u_d$ ). Thus the trajectories of the closed-loop system (12) (or (17)) are also state-jump free. Let now  $G = 0$  in (11). Then  $\lambda_d(\cdot)$  does not appear in (12). Thus provided both  $x_d(\cdot)$  and  $u_d(\cdot)$  are continuous, no state jump occurs, except possibly at the initial time. These results follow by applying, *e.g.*, items 1 or 2 (a) in Appendix B. When  $D = 0$  and/or  $D + FG = 0$  (which cannot be excluded by Assumption 3.3), item 3 in Appendix B can be used. We may also rely on item 2 (f) to analyse (20) and guarantee it has Lipschitz continuous solutions for the admissible initial state. Let us check conditions i) of item 2 (f) in Appendix B under Assumption 3.3 ( $\Rightarrow D + FG \succcurlyeq 0$ ) and  $D \succcurlyeq 0$  (most of the conditions assuring Assumption 3.1 imply it):

(a)  $\tilde{D} \succcurlyeq 0 \Leftrightarrow \tilde{D} + \tilde{D}^\top \succcurlyeq 0$ . Using Lemma E.1, equivalently we have:

1.  $\text{Im}(G^\top F^\top) \subseteq \text{Im}(D + D^\top)$ ,

2.  $D + FG + (D + FG)^\top \succcurlyeq FG(D + D^\top)^\dagger G^\top F^\top$ . A sufficient condition for this to hold, is  $D + FG > 0$ , and  $\sigma_{\max}(FG(D + D^\top)^\dagger G^\top F^\top) < \sigma_{\max}(D + FG + (D + FG)^\top) = \lambda_{\max}(D + FG + (D + FG)^\top)$ . From [54, Corollary 9.6.5] we have:  $\sigma_{\max}(FG(D + D^\top)^\dagger G^\top F^\top) \leq \sigma_{\max}^2(FG)\sigma_{\max}((D + D^\top)^\dagger)$ . If  $D + D^\top = 0$  the matrix inequality is satisfied. If  $D + D^\top \neq 0$ ,  $\text{rank}(D + D^\top) = d$ , then  $\sigma_{\max}((D + D^\top)^\dagger) = \sigma_d^{-1}(D + D^\top)$  [54, Fact 6.3.28], and  $\sigma_d(D + D^\top) = \lambda_d(D + D^\top)$  since  $D + D^\top \succcurlyeq 0$  is symmetric. Thus a sufficient condition is  $\sigma_{\max}^2(FG) < \lambda_d(D + D^\top)\lambda_{\max}(D + FG + (D + FG)^\top)$ . Another sufficient condition, following similar steps, is  $D + D^\top > 0$ ,  $\sigma_{\max}(FG) < \lambda_d(D + FG + (D + FG)^\top)\lambda_{\max}(D + D^\top)$ . We see that in both cases we allow for  $D + D^\top$  of  $D + FG + (D + FG)^\top$  to be low rank, which hampers  $\tilde{D} > 0$ . Finally, we may use Lemmas E.1, E.2, as well as the matrix decompositions described after Lemma E.2, to get necessary and sufficient conditions.

(b)  $\text{Im}(\tilde{D}) \subseteq \text{Im}(\tilde{C}) \Leftrightarrow \text{Im}(D + FG) + \text{Im}(FG) \subseteq \text{Im}(C + FK)$  and  $\text{Im}(D) \subseteq \text{Im}(C + FK)$ .

(c)  $\text{Ker}(\tilde{D} + \tilde{D}^\top) \subseteq \text{Ker}(P\tilde{B} - \tilde{C}^\top)$  for some  $R^{n \times n} \ni P = P^\top > 0 \Leftrightarrow \{(x, y) \in R^{m \times m} \mid ((D + FG) + (D + FG)^\top)x - FGy = 0, -G^\top F^\top x + (D + D^\top)y = 0\} \subseteq \{(x, y) \in R^{m \times m} \mid x - y \in \text{Ker}(P(B + EG) - (C + FK)^\top)\}$ . This is verified if  $FG = 0$  and Assumption 3.3 holds, since in this case  $\text{Ker}(D + D^\top) \subseteq \text{Ker}(P(B + EG) - (C + FK)^\top)$  [35] [33, section 3.8]. If  $D$  and  $D + FG$  are P-matrices, and if  $FG(D + D^\top)^{-1}G^\top F^\top \neq D + FG + (D + FG)^\top$ , then  $\text{Ker}(\tilde{D} + \tilde{D}^\top) = \{0\}$ , hence the inclusion holds. If  $FG(D + D^\top)^{-1}G^\top F^\top = D + FG + (D + FG)^\top$ , then  $\text{Ker}(\tilde{D} + \tilde{D}^\top) = R^m \times R^m$ , hence the inclusion holds only if  $P(B + EG) = (C + FK)^\top$ .

The various conditions stated in Appendix B are sufficient only, so it may be that some of them are unnecessary in some cases (*e.g.*, condition **b**) above is needed for continuity arguments using the Vladimirov's pseudo-distance defined in [55] as done in [21, 50, 56], but it is not necessary at all [1, section 5.3] [36]). Condition **b**) is not necessary for the well-posedness of (14) when both  $D$  and  $D + FG$  are P-matrices. The point is also that the error system (14) or (20), is a specific interconnection of both subsystems, which does not necessarily inherits good properties of the subsystems (*e.g.*,  $D \succcurlyeq 0$  and  $D + FG \succcurlyeq 0$  may not imply  $\tilde{D} \succcurlyeq 0$ ). Conditions in **a**) and **c**) are necessary for the passivity of  $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$  [35] [33, section 3.8].

The stability proof shows that under the proposition's assumptions (like the continuity of the solutions) the generalized equation  $0 \in \tilde{A}e^* - \tilde{B}(\tilde{D} + \mathcal{N}_{\tilde{S}(t)}^{-1})^{-1}(\tilde{C}e^*)$  has the unique solution  $e^* = 0$  for all  $t \geq 0$ .

### 3.3. Error Dynamics Stability Analysis including State Jumps

An important question is whether or not the tracking control framework developed above is suitable for trajectories with discontinuities. Let us study how to relax Assumption 3.1 and the continuity of the closed-loop state  $x(\cdot)$ . State jumps are known to add difficulty to the trajectory tracking problem, especially when the jump times are unknown. The so-called peaking phenomenon (or jump-mismatch), due to non-synchronized jumps in the plant and the desired trajectories, has long been known to be one of the obstacles. The jumps in  $x(\cdot)$  and  $x_d(\cdot)$  may arise from different reasons:

1. Let the conditions in section 2.1 item 2 (a) or item 3 hold for  $(A, B, C, D)$ . Apart from possible initial discontinuity at  $t = t_0$ , jumps at  $t$  in  $x_d(\cdot)$  occur only if  $Fu_d$  is discontinuous at  $t$  and  $\text{rank}(D) < m$ .
2. Let the conditions in section 2.1 item 2 (a) or item 3 hold for  $(A + EK, B + EG, C + FK, D + FG)$ . Jumps at  $t$  in the closed-loop state occur only if  $\text{rank}(D + FG) < m$  and  $-FKx_d - FG\lambda_d + Fu_d$  is discontinuous at  $t$ .

The following is deduced, where  $\mathcal{T}_f$  denotes the set of discontinuity times of the function  $f(\cdot)$ :

**Lemma 3.9.** Let  $FG\lambda_d(\cdot)$  and  $u_d(\cdot)$  be bounded functions of time. (a)  $\mathcal{T}_{x_d} \subseteq \{t_0\} \cup \mathcal{T}_{Fu_d}$  and (b)  $\mathcal{T}_x \subseteq \{t_0\} \cup \mathcal{T}_{Fu_d} \cup \mathcal{T}_{FG\lambda_d}$ .

*Proof.* (a) is obvious. (b):  $\mathcal{T}_x \subseteq \{t_0\} \cup \mathcal{T}_{-FKx_d - FG\lambda_d + Fu_d} \subseteq \{t_0\} \cup \mathcal{T}_{FKx_d} \cup \mathcal{T}_{FG\lambda_d} \cup \mathcal{T}_{Fu_d} \subseteq \{t_0\} \cup \mathcal{T}_{FG\lambda_d} \cup \mathcal{T}_{Fu_d}$  since  $\mathcal{T}_{FKx_d} \subseteq \mathcal{T}_{Fu_d}$ .  $\square$

Lemma 3.9 indicates that when  $FG\lambda_d$  is continuous, then either  $x_d(\cdot)$  jumps while  $x(\cdot)$  is continuous, or the inverse, or they jump simultaneously. This will be illustrated on examples. Notice that when  $x_d(\cdot)$  jumps, then  $\lambda_d$  is a Dirac measure, and the meaning of the feedback in (11) has to be carefully studied, as well as the mere meaning of the complementarity constraints in (12) which contain the term  $FG\lambda_d$ . Also depending on  $FG$ , the sum of two Dirac measures, one stemming from  $\lambda_d$  (due to a jump in  $u_d(t)$ ) and one stemming from  $\lambda$  (due to a jump in  $-FKx_d - FG\lambda_d + Fu_d$ ) could occur in the differential part of the closed-loop dynamics (see section 5): we may call this a forbidden situation (this is not to be confused with the case of Measure Differential Inclusions, or with passive LCS with state jumps [1, 37, 18]). It is inferred that:

**(C1)** If  $x_d(\cdot)$  jumps, we impose  $FG = 0$ . A sufficient condition to prevent forbidden situations, is that  $FG = 0$  if  $u_d(\cdot)$  is discontinuous. In other words, only continuous  $u_d(\cdot)$  is allowed if  $FG \neq 0$ .

In view of condition (C1), we can refine item 2 just above: Jumps in the closed-loop state can occur at  $t$  if:

1.  $FG = 0$  and  $-FKx_d + Fu_d$  is discontinuous at  $t$ ,
2. or  $FG \neq 0$ ,  $x_d$  is continuous at  $t$  and  $\lambda_d$  jumps at  $t$  (which can occur if  $u_d$  is continuous at  $t$  at a junction time with the constraint boundary  $(Cx_d + D\lambda_d + Fu_d)_i = 0$  for some  $1 \leq i \leq m$ ).

Conditions in item 2 are not straightforward, because we want that  $FG \neq 0$ , but  $FG$  also multiplies  $\lambda$ , see (12), and passivity implies  $D + FG \succcurlyeq 0$ . The scalar case  $m = 1$  is analyzed in section 5.

The first step is to characterize the jumps in (14), then to study the variation  $\Delta V(e(t)) = V(e(t^+)) - V(e(t^-)) = (e(t^+) - e(t^-))^T P(e(t^+) - e(t^-))$ .

### 3.3.1. Case $G = 0$ and $D = 0$

In this case  $\lambda_d$  does not enter (12), and  $\tilde{D} = 0$ . Let Assumptions 3.1 and 3.3 hold, and let us study the case when  $x(\cdot)$  jumps while  $x_d(\cdot)$  is continuous. Then  $(A + EK, B, C + FK, 0)$  is strictly state passive, and  $PB = (C + FK)^T$  for some  $P = P^T > 0$ . The closed-loop system in (17) becomes:

$$\dot{x}(t) \in (A + EK)x(t) - BN_{S(t)}((C + FK)x(t)) - EKx_d(t) + Eu_d(t). \quad (24)$$

where  $S(t) = \{v \in \mathbf{R}_+^m \mid v - FKx_d(t) + Fu_d(t) \geq 0\}$ . This DI can be rewritten equivalently as a FOSwP (Definition 1.4, see item 3 in Appendix B and [1, 37, 40]), as:

$$\dot{\zeta}(t) \in R(A + EK)R^{-1}\zeta(t) + RE(-Kx_d(t) + u_d(t)) - N_{\Phi(t)}(\zeta(t))$$

where  $R^2 = P$ ,  $R = R^T > 0$ ,  $\zeta = Rx$ ,  $\Phi(t) = \{Rx \mid (C + FK)x - FKx_d(t) + Fu_d(t) \in S(t)\}$ . We see that  $\Phi(t) \neq \emptyset$  for each  $t$  if and only if there exists  $x$  such that  $(C + FK)x - FKx_d(t) + Fu_d(t) \geq 0$ . This is guaranteed if a condition as in Appendix B item 3 iii) holds, i.e.,  $\text{Im}(C + FK) - \mathbf{R}_+^m = \mathbf{R}^m$  (which is a constraint qualification). Assume in addition that  $u_d(\cdot)$  is continuous on  $\mathbf{R}_+$ . Suppose that  $Cx_d(0) + Fu_d(0) \geq 0$ , hence  $x_d(\cdot)$  is continuous at  $t = 0$ . Then a jump in  $x(\cdot)$  (thus in  $e(\cdot)$ ) can occur only at the initial time and  $V(0^+) - V(0^-) = e(0^+)^T Pe(0^+) - e(0^-)^T Pe(0^-) = (e(0^+) + e(0^-))^T P(e(0^+) - e(0^-)) = (x(0^+) + x(0^-) - 2x_d(0))^T P(x(0^+) - x(0^-)) = x(0^+)^T Px(0^+) - x(0^-)^T Px(0^-) - 2x_d(0)^T P(x(0^+) - x(0^-))$ . Now using the passivity of  $(A + EK, B, C + FK, 0)$ , we have (see the paragraph State Jumps at the end of 2):  $x(0^+)^T Px(0^+) - x(0^-)^T Px(0^-) \leq 0$ , provided that  $0 \in \{z \in \mathbf{R}^n \mid (C + FK)z - FKx_d(0) + Fu_d(0) \geq 0\}$ , equivalently  $0 \in \Phi(0)$ , equivalently  $-FKx_d(0) + Fu_d(0) \geq 0$ . In this case,  $V(0^+) - V(0^-) \leq -2x_d(0)^T P(x(0^+) - x(0^-))$ . We have  $P(x(0^+) - x(0^-)) \in -\mathcal{N}_{\mathcal{K}}(x(0^+))$  [1, Lemma 2.3]. Thus it is necessary and sufficient that  $x_d(0) \in (\mathcal{N}_{\mathcal{K}}(x(0^+)))^\circ = \mathcal{T}_{\mathcal{K}}(x(0^+))$  to guarantee that the right-hand side is nonpositive. Therefore, the following has been proved:

**Lemma 3.10.** Assume that  $u_d(\cdot)$  is continuous,  $G = 0$ ,  $D = 0$ ,  $-FKx_d(0) + Fu_d(0) \geq 0$ , and  $x_d(0) \in \mathcal{T}_{\mathcal{K}}(\text{proj}_P[\mathcal{K}; x(t_0^-)])$ . Then at an initial state jump, we have  $V(0^+) - V(0^-) \leq 0$ , where  $P = P^T > 0$  is a solution of the passivity LMI associated with the triple  $(A + EK, B, C + FK)$  and  $V(t) = V(e(t))$ .

The condition on  $x_d(0)$  is certainly not easy to check in general (but  $x_d(0) = 0$  is always suitable). The interest of Lemma 3.10 is that it allows for a jump in  $x(0)$  while  $x_d(0)$  does not jump. But applying it at any time of state jump  $t_k > 0$  implies to impose a suitable desired state, which may not be possible in our framework where  $x_d$  is generated by (10). Under the same conditions, assume that  $Fu_d$  has a discontinuity at time  $t_c$ . From (10) and (12), both  $x(\cdot)$  and  $x_d(\cdot)$  may jump at  $t_c$ , so  $\lambda$  and  $\lambda_d$  are Dirac measures at  $t_c$ . From (12) the discontinuity may act in both terms  $FKx_d$  and  $Fu_d$ . Using (17) and (19), this implies that the jumps' magnitudes in the sets  $S(t)$  and  $S_d(t)$ , may not be equal. The post-jump states are computed using (7), where  $\mathcal{K} = \{z \in \mathbb{R}^n \mid (C + FK)z - FKx_d(t) + Fu_d(t) \geq 0\}$  for the closed-loop plant, and  $\mathcal{K}_d = \{z \in \mathbb{R}^n \mid Cz + Fu_d(t) \geq 0\}$  for the desired system.

### 3.3.2. Extension for further jumps at $t > 0$

Let us now place ourselves in another perspective. In the foregoing subsection it has been assumed that  $x_d(\cdot)$  is time-continuous (since  $u_d(\cdot)$  is assumed to be time-continuous). Notice that condition (C1) can be relaxed, if it is admitted that the state  $x_d(\cdot)$  of the dynamics in (10) can be arbitrarily reset to some value at arbitrary times, without considering  $\lambda_d$  as a Dirac measure. In such a case, the desired dynamics is no longer autonomous (the desired state has to be modified online, a common feature in trajectory tracking for systems undergoing state jumps [12, 13, 17]). Though this resetting strategy is not tackled in this article, it leads us to consider the following. Consider (12), together with Assumption 3.3.

**Proposition 3.11.** *Let  $V(e) = e^\top P e$ , with  $P$  a solution of the closed-loop passivity LMI, and let  $\{0\} \in \mathcal{K} = \{z \in \mathbb{R}^n \mid (C + FK)z - FKx_d - FG\lambda_d(t^+) + Fu_d(t) \in \mathcal{Q}_{D+FG}^*\}$ . Assume that  $x_d(t^+) \in \mathcal{T}_{\mathcal{K}}(\text{proj}_P[\mathcal{K}; x(t^-)])$  and  $x_d(t^+)^\top P x(t^-) \geq x_d(t^-)^\top P x(t^-)$ , then  $\Delta V(e(t)) \leq (x_d(t^+) - x_d(t^-))^\top P (x_d(t^+) + x_d(t^-))$ .*

*Proof.* At a state jump time:

$$\begin{aligned}
\Delta V(e(t)) &= (x(t^+) - x(t^-))^\top P (x(t^+) + x(t^-)) - (x_d(t^+) - x_d(t^-))^\top P (x_d(t^+) + x_d(t^-)) \\
&\quad - (x_d(t^+) - x_d(t^-))^\top P (x(t^+) + x(t^-)) + (x_d(t^+) - x_d(t^-))^\top P (x_d(t^+) + x_d(t^-)) \\
&\leq -(x(t^+) - x(t^-))^\top P (x_d(t^+) + x_d(t^-)) - (x_d(t^+) - x_d(t^-))^\top P (x(t^+) + x(t^-)) \\
&\quad + (x_d(t^+) - x_d(t^-))^\top P (x_d(t^+) + x_d(t^-)) \\
&= -(x_d(t^+) + x_d(t^-))^\top P (x(t^+) - x(t^-)) - (x_d(t^+) - x_d(t^-))^\top P (x(t^+) - x(t^-)) \\
&\quad - 2(x_d(t^+) - x_d(t^-))^\top P x(t^-) + (x_d(t^+) - x_d(t^-))^\top P (x_d(t^+) + x_d(t^-)) \\
&= -2x_d(t^+)^\top P (x(t^+) - x(t^-)) - 2(x_d(t^+) - x_d(t^-))^\top P x(t^-) \\
&\quad + (x_d(t^+) - x_d(t^-))^\top P (x_d(t^+) + x_d(t^-))
\end{aligned} \tag{25}$$

The first term in the first equality in (25) is nonpositive from the passivity. It is noteworthy that we cannot infer the same conclusions about the last term in (25) from (13) because these dynamics are equivalent to that in (10). Let us consider the last equality in (25). We know that  $P(x(t^+) - x(t^-)) \in -\mathcal{N}_{\mathcal{K}}(x(t^+))$  [1, Lemma 2.3]. Thus the nonpositivity of the first term is equivalent to  $x_d(t^+) \in (\mathcal{N}_{\mathcal{K}}(x(t^+)))^\circ = \mathcal{T}_{\mathcal{K}}(x(t^+))$ . Using (7) the first condition follows. The second condition is obvious.  $\square$

Thus under the conditions of Proposition 3.11,  $\Delta V(e(t)) \leq 0$  if  $x_d(t^+)^\top P x_d(t^+) \leq x_d(t^-)^\top P x_d(t^-)$ , which means that the desired state jump is dissipative with respect to the closed-loop storage function (in general there is no reason that it should satisfy this property). This makes a set of constraints that the desired state  $x_d(t^+)$  has to satisfy. Note that the conditions of Proposition 3.11 are sufficient only. Examples show that they may not be satisfied, while  $\Delta V(e(t)) \leq 0$ , see section 5.4. Also, we note that a reset mechanism is not needed when  $G = 0$  (then condition C1) holds true, while the characterization of  $\Delta V(e(t))$  as in (25) remains valid.

### 3.3.3. Using the DI (21)

The formalism in (20) is not convenient for the state jumps analysis, because it involves a state-dependent set. Let us recall the error dynamics in (21) to analyze the state discontinuities.

$$\dot{e}(t) \in \bar{A}e(t) - \bar{B}(\bar{D} + \mathcal{N}_{S(t)}^{-1})^{-1}(\bar{C}e(t)) \Leftrightarrow \begin{cases} \dot{e}(t) = \bar{A}e(t) + \bar{B}\lambda(t) \\ 0 \leq \lambda(t) \perp \bar{C}e(t) + \bar{D}\lambda(t) + \bar{F}(t) \geq 0 \end{cases} ,$$

with  $\bar{A} = \tilde{A}$ ,  $\bar{B} = \tilde{B}$ ,  $\bar{C} = \begin{pmatrix} C + FK \\ 0 \end{pmatrix}$ ,  $\bar{D} = \tilde{D}$ ,  $\bar{S}(t) = \{z \in \mathbb{R}^{2m} \mid z + \bar{F}(t) \in \mathbb{R}_+^{2m}\}$ ,  $\bar{F}(t) = \begin{pmatrix} Cx_d(t) + Fu_d(t) \\ Cx_d(t) + Fu_d(t) \end{pmatrix}$ . At a state jump time  $t$ , the DI in (21) may be written as:

$$e(t^+) - e(t^-) \in -\bar{B}(\bar{D} + \mathcal{N}_{\bar{S}(t^+)}^{-1})^{-1}(\bar{C}e(t^+)).$$

which has a solution if the quadruple  $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$  is passive with positive definite storage function. Then (7) holds. Thus it is inferred that  $\Delta V(e(t)) \leq 0$  provided that  $0 \in \mathcal{K} = \{e \in \mathbb{R}^n \mid \bar{C}e + \bar{F}(t) \in \mathcal{Q}_{\bar{D}}^*\}$ , and:

$$\begin{pmatrix} -P(A + EK) - (A + EK)^\top P & -P(B + EG) + (C + FK)^\top & P(B + EG) \\ -(B + EG)^\top P + C + FK & D + FG + (D + FG)^\top & FG \\ (B + EG)^\top P & (FG)^\top & D + D^\top \end{pmatrix} \succcurlyeq 0, \quad (26)$$

where  $P = P^\top > 0$  is a solution to  $-M \succcurlyeq 0$  with  $M$  in (15). Notice that (26) holds only if  $M \leq 0$ , hence only if  $P$  is also a solution of the plant's closed-loop LMI. Thus, the solution  $P$  of (26) must be the solution of BMI in (16). From Assumption 3.3 we have  $-M \succcurlyeq 0$ . Using Lemma E.1, we infer that (26) holds if and only if:

1.  $\bar{D} + \bar{D}^\top \succcurlyeq 0$  (see item a) at the end of section 3.2 for sufficient conditions),
2.  $\text{Im} \begin{pmatrix} -(B + EG)^\top P + C + FK \\ (B + EG)^\top P \end{pmatrix} \subseteq \text{Im}(\bar{D} + \bar{D}^\top)$ ,
- 3.

$$-M \succcurlyeq \begin{pmatrix} -P(B + EG) + (C + FK)^\top & P(B + EG) \end{pmatrix} (\bar{D} + \bar{D}^\top)^\dagger \begin{pmatrix} -(B + EG)^\top P + C + FK \\ (B + EG)^\top P \end{pmatrix} \quad (27)$$

Some comments arise:

- A sufficient condition for (27) to hold is  $\bar{D} = -\bar{D}^\top \iff D + FG = -(D + FG)^\top$ ,  $D = -D^\top$ ,  $FG = 0$ . This is allowed by strict state passivity in Assumption 3.3. However item 2 then implies that  $-(B + EG)^\top P + C + FK = 0$  and  $(B + EG)^\top P = 0$ , hence  $C + FK = 0$  and  $B + EG = 0$  since  $P$  is full-rank. Thus in (12) the ODE part and the complementarity part are decoupled.
- Assume that  $D = \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix}$ ,  $D_1 > 0$ . Item 1 implies that  $FG = \begin{pmatrix} (FG)_1 & 0 \\ 0 & 0 \end{pmatrix}$ . Then  $\bar{D} + \bar{D}^\top = \begin{pmatrix} \bar{D}_1 + \bar{D}_1^\top & 0 \\ 0 & 0 \end{pmatrix}$ , and  $(\bar{D} + \bar{D}^\top)^\dagger = \begin{pmatrix} (\bar{D}_1 + \bar{D}_1^\top)^\dagger & 0 \\ 0 & 0 \end{pmatrix}$ , with  $\bar{D}_1 + \bar{D}_1^\top = \begin{pmatrix} D_1 + (FG)_1 + D_1 + (FG)_1^\top & 0 & -(FG)_1 \\ 0 & 0 & 0 \\ -(FG)_1^\top & 0 & D_1 + D_1^\top \end{pmatrix}$ . Similar calculations can be done for  $M$  in (16), and also for the ranges inclusion in item 2. Pursuing the calculations and matrices partitions allows to simplify (27).
- Assume that  $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$  is passive. A jump in  $e(\cdot)$  occurs at  $t$  only if  $\bar{F}(\cdot)$  jumps at  $t$ . In turn  $\bar{F}(\cdot)$  jumps at  $t$  if and only if  $Cx_d + Fu_d$  jumps at  $t$ , while  $x_d(\cdot)$  jumps only if  $u_d(\cdot)$  does. Hence  $e(\cdot)$  jumps at  $t$  only if  $u_d(\cdot)$  does. How is this related to the jumps in  $x(\cdot)$  in (12) and in  $x_d(\cdot)$  in (10)? As we saw above, if  $x_d(\cdot)$  jumps at  $t_c$  then  $\lambda_d$  is a Dirac measure at  $t_c$  and the complementarity constraints in (12) are meaningless at  $t_c$ . This means that applying an impulsive feedback control to an LCS (2) with  $FGu \neq 0$ , has to be avoided as pointed out in condition (C1). In fact, the DIs in (20), (21), (17), and (18) are constructed with the underlying assumption that the complementarity conditions can be rewritten equivalently as inclusion into a normal cone (see, e.g., [1, Equ. (B.1)] [30, Corollary 23.5.4]). For instance,  $f(t)$  in (18) is a Dirac measure if  $EG\lambda_d$  is. This might let one think that in this case,  $x(\cdot)$  jumps at  $t_c$ . We set that this is true if  $F = 0$  only, in which case  $x_d(\cdot)$  can jump only initially, see (10). Obviously, if  $\lambda_d$  is a Dirac measure, both formalisms (complementarity and inclusion) require further analysis for their understanding if  $FG \neq 0$ .
- The conditions in this paragraph are different from those of the foregoing paragraph, because now we consider both the desired system and the closed-loop plant simultaneously, instead of looking at (24) only. However, jumps in both  $x(\cdot)$  and  $x_d(\cdot)$  are permitted under some conditions as stated above. A quick examination of the LMI in (26) shows that the occurrence of state jumps is quite restricted in this context. Indeed, if  $D + D^\top = 0$ ,

then (26) implies that  $B + EG = 0$  and  $FG = 0$ . In fact, the LMI in (26) shows that studying state jumps from the passivity of the error dynamics in (21) is possible only in rare cases. This may be explained by the fact that the LCS in (12) is incrementally passive (see comment after the proof of Proposition 3.8) but the error dynamics in (21) is not passive as an operator between the two slack variables.

- The controller may be impulsive if  $G \neq 0$  and a jump occurs in  $x(\cdot)$ . Condition (C1) still applies.

### 3.3.4. Analysis of jump sets $\mathcal{K}$ and $\mathcal{K}_d$

These sets are crucial in the state jump characterization and computation, see (6). Let us consider:

$$\mathcal{K} = \{z \in \mathbf{R}^n \mid Cz + Fu_d(t^+) + FK(z - x_d(t^+)) \in \mathcal{Q}_{D+FG}^*\} \quad (28)$$

and

$$\mathcal{K}_d = \{z \in \mathbf{R}^n \mid Cz + Fu_d(t^+) \in \mathcal{Q}_D^*\} \quad (29)$$

In general, both sets are different. However if  $FG = 0$  then they differ only by the term  $FK(z - x_d(t^+))$ . Since  $x(t^+)$  belongs to  $\mathcal{K}$ , both sets are almost equal if the tracking error  $x(t^-) - x_d(t^-)$  and  $FK$  are very small. So if in addition both storage functions matrices  $P$  (calculated from the closed-loop system LMI) and  $P_d$  (computed from the desired dynamics LMI when this dynamics is passive) are close one to each other, both state jumps are almost the same as well. On the other hand, the state jump as computed in (7) is independent of  $P$  (or of  $P_d$ , respectively), provided it is a solution of the passivity LMI. Thus, if  $EG = 0$ , which implies that the "input" matrix of the closed-loop system is equal to the input matrix of the desired system, the forms of  $\mathcal{K}$  and  $\mathcal{K}_d$  in (28) and (29) will imply that both jump in  $x$  and  $x_d$  are close one to each other.

### 3.3.5. The jump-mismatch (peaking) phenomenon

This is a well-known phenomenon in trajectory tracking when state jumps are present, as recalled in the introduction. The problem that is faced in this study is twofold: 1) is the discontinuity mismatch issue present? If it is, how can it be coped with? In the scalar case treated in section 5, it will be shown that if both states jump then they jump at the same time. However, it is also possible that  $x$  jumps while  $x_d$  does not, and *vice versa*. Lemma 3.9 shows that when  $FG\lambda_d$  is continuous, then if both states jump they jump simultaneously.

## 4. Robustness Analysis: Tracking with Parametric Uncertainties

It is of interest to analyse the tracking problem when the plant's dynamics have uncertainties. In this case, the desired dynamics' matrices in (10), and the plant's model matrices, differ. The desired dynamics (10) has to be designed using a nominal plant model.

### 4.1. Controller Design

The controller  $u(t) = K_0[x(t) - x_d(t)] + G_0[\lambda(t) - \lambda_d(t)] + u_d(t)$  is designed from the plant's nominal quadruple  $(A_0, B_0, C_0, D_0)$ , along the same procedure as in the foregoing section. Therefore the desired system (10) is represented by the following LCS:

$$\begin{cases} \dot{x}_d(t) = A_0x_d(t) + B_0\lambda_d(t) + E_0u_d(t) \\ 0 \leq \lambda_d(t) \perp w_d(t) = C_0x_d(t) + D_0\lambda_d(t) + F_0u_d(t) \geq 0. \end{cases} \quad (30)$$

If strong passivity is used (Assumption 3.4) instead of strict state passivity as in the foregoing section (Assumption 3.3), the controller gains are computed assuming there exist matrices  $K_0$  and  $G_0$  such that the inequality (16) is satisfied for the nominal plant, *i.e.*,

$$M_0 \triangleq \begin{pmatrix} (A_0 + E_0K_0)^\top P_0 + P_0(A_0 + E_0K_0) & P_0(B_0 + E_0G_0) - (C_0 + F_0K_0)^\top \\ (B_0 + E_0G_0)^\top P_0 - (C_0 + F_0K_0) & -D_0 - F_0G_0 - (D_0 + F_0G_0)^\top \end{pmatrix} < 0 \quad (31)$$

has a solution  $P_0 = P_0^\top > 0$ . The plant dynamics is represented as follows

$$\begin{cases} \dot{x}(t) = (A_0 + \Delta A)x(t) + (B_0 + \Delta B)\lambda(t) + (E_0 + \Delta E)u(t) \\ 0 \leq \lambda(t) \perp w(t) = (C_0 + \Delta C)x(t) + (D_0 + \Delta D)\lambda(t) + (F_0 + \Delta F)u(t) \geq 0 \end{cases}$$

where  $\Delta A$ ,  $\Delta B$ ,  $\Delta C$ ,  $\Delta D$ ,  $\Delta E$  and  $\Delta F$  represent additive uncertainties. Sufficient conditions on the uncertainties upperbounds are calculated in the next section, so that some stability is guaranteed. The closed-loop system is given by:

$$\begin{cases} \dot{x}(t) = (A_0 + \Delta A + (E_0 + \Delta E)K_0)x(t) + (B_0 + \Delta B + (E_0 + \Delta E)G_0)\lambda(t) \\ \quad - (E_0 + \Delta E)(K_0x_d(t) + G_0\lambda_d(t) - u_d(t)) \\ 0 \leq \lambda(t) \perp w(t) = (C_0 + \Delta C + (F_0 + \Delta F)K_0)x(t) + (D_0 + \Delta D + (F_0 + \Delta F)G_0)\lambda(t) \\ \quad - (F_0 + \Delta F)(K_0x_d(t) + G_0\lambda_d(t) - u_d(t)) \geq 0 \end{cases} \quad (32)$$

It is noteworthy that the well-posedness of (32) may not be guaranteed for any uncertainties. Some of the results in Appendix B can be used. The following assumption is supposed to hold in this section:

**Assumption 4.1.** *The closed-loop system (32) is well-posed, i.e., it has unique AC solutions for any initial condition  $x(t_0) = x_0$  satisfying  $w(t_0, x_0) \geq 0$ , equivalently:*

$$x_0 \in \text{Dom}\left((D_0 + \Delta D + (F_0 + \Delta F)G_0 + \partial\sigma_{S(t)})^{-1}\right) = \text{Im}\left((D_0 + \Delta D + (F_0 + \Delta F)G_0 + \partial\sigma_{S(t)})\right),$$

with  $S(t) = \{v \in \mathbb{R}^m \mid v + (F_0 + \Delta F)(K_0x_d + G_0\lambda_d - u_d) \geq 0\}$ .

**Remark 4.2.** *It is clear that time-varying uncertainties give rise to a time-varying LCS in (32). As pointed out in section 2.2, it is only in few particular cases that the well-posedness of time-varying LCS has been studied. In this section stability relies on the fact that the closed-loop nominal system is strongly passive, and that the closed-loop plant feedthrough matrix  $(D_0 + \Delta D + (F_0 + \Delta F)G_0) > 0$ . Thus item 1 in Appendix B applies if this matrix is constant.*

#### 4.2. Error Dynamics Stability Analysis

The error dynamics is given by:

$$\begin{cases} \dot{e}(t) = (A_0 + E_0K_0)e(t) + (B_0 + E_0G_0)\Delta\lambda(t) + \Delta Ax(t) + \Delta B\lambda(t) + \Delta EK_0e(t) + \Delta EG_0\Delta\lambda(t) \\ \quad + \Delta Eu_d(t) \\ \Delta w(t) = w(t) - w_d(t) = (C_0 + F_0K_0)e(t) + (D_0 + F_0G_0)\Delta\lambda(t) + \Delta Cx(t) + \Delta D\lambda(t) + \Delta FK_0e(t) \\ \quad + \Delta FG_0\Delta\lambda(t) + \Delta Fu_d(t) \\ 0 \leq w(t) \perp \lambda(t) \geq 0 \quad \text{and} \quad 0 \leq w_d(t) \perp \lambda_d(t) \geq 0. \end{cases} \quad (33)$$

with  $\Delta\lambda(t) = \lambda(t) - \lambda_d(t)$ . Let:

$$p(x, t, \lambda) \triangleq \Delta Ax(t) + \Delta B\lambda(t) + \Delta EK_0e(t) + \Delta EG_0\Delta\lambda(t) + \Delta Eu_d(t)$$

and

$$q(x, t, \lambda) \triangleq \Delta Cx(t) + \Delta D\lambda(t) + \Delta FK_0e(t) + \Delta FG_0\Delta\lambda(t) + \Delta Fu_d(t).$$

Let us now state the stability result.

**Proposition 4.3.** *Let Assumptions 3.1, 3.2, 3.4 hold for the nominal system, and Assumption 4.1 hold for the closed-loop system. Let the inequalities*

$$\begin{aligned} \Delta A^\top \Lambda_A^{-1} \Delta A &\preceq I_n, \quad \Delta B^\top \Lambda_B^{-1} \Delta B \preceq I_m, \quad \Delta C^\top \Lambda_C^{-1} \Delta C \preceq I_n \\ \Delta D^\top \Lambda_D^{-1} \Delta D &\preceq I_m, \quad \Delta E^\top \Lambda_E^{-1} \Delta E \preceq I_p, \quad \Delta F \Lambda_F^{-1} \Delta F^\top \preceq I_m \end{aligned} \quad (34)$$

hold for any  $\Lambda_k = \Lambda_k^\top > 0$ ,  $k \in \{A, B, C, D, E, F\}$ , and assume there exist  $P_0$ ,  $K_0$  and  $G_0$  such that the matrix inequality

$$\begin{pmatrix} -(M_0)_{11} & -(M_0)_{12} & P_0 & P_0 & K_0^\top & I_n & 0 & 0 & 0 \\ -(M_0)_{12} & -(M_0)_{22} & 0 & 0 & 0 & 0 & G_0^\top & I_m & I_m \\ P_0 & 0 & \Lambda_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ P_0 & 0 & 0 & \tilde{\Lambda}_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ K_0 & 0 & 0 & 0 & \tilde{\Lambda}_F^{-1} & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & \frac{1}{2}I_n & 0 & 0 & 0 \\ 0 & G_0 & 0 & 0 & 0 & 0 & \tilde{\Lambda}_F^{-1} & 0 & 0 \\ 0 & I_m & 0 & 0 & 0 & 0 & 0 & \Lambda_2^{-1} & 0 \\ 0 & I_m & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\Lambda}_2^{-1} \end{pmatrix} > 0 \quad (35)$$



and  $P_0 = P_0^\top > 0$ , with  $\tilde{\Lambda}_F \triangleq I_p + \Lambda_F$ ,  $\tilde{\Lambda}_1 \triangleq \Lambda_A + 2\Lambda_E + \Lambda_B$ ,  $\tilde{\Lambda}_2 \triangleq \Lambda_C + \Lambda_D + 4I_m$ , holds. Then, the solution of the error dynamics in (33) is globally uniformly ultimately bounded. Note that  $(M_0)_{ij}$  denotes the element of the matrix  $M_0$  in (31) located at the  $i$ th row and  $j$ th column where  $i, j \in \{1, 2\}$ .

*Proof.* The derivative of the Lyapunov function candidate  $V(t) = e^\top P_0 e$  along the closed-loop trajectories is calculated as follows:

$$\dot{V} = e^\top \left[ (A_0 + E_0 K_0)^\top P_0 + P_0 (A_0 + E_0 K_0) \right] e + 2e^\top P_0 (B_0 + E_0 G_0) \Delta \lambda + 2e^\top P_0 p(x, t, \lambda)$$

In matrix form

$$\begin{aligned} \dot{V}(t) = & \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix}^\top \begin{pmatrix} (A_0 + E_0 K_0)^\top P_0 + P_0 (A_0 + E_0 K_0) & P_0 (B_0 + E_0 G_0) \\ (B_0 + E_0 G_0)^\top P_0 & 0 \end{pmatrix} \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix} \\ & + 2e^\top P_0 p(x, t, \lambda) \\ & \pm \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix}^\top \begin{pmatrix} 0 & (C_0 + F_0 K_0)^\top \\ C_0 + F_0 K_0 & D_0 + F_0 G_0 + (D_0 + F_0 G_0)^\top \end{pmatrix} \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix} \end{aligned}$$

The purpose of the  $\pm$  term is to obtain the matrix in (31) and

$$\begin{aligned} \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix}^\top \begin{pmatrix} 0 & (C_0 + F_0 K_0)^\top \\ C_0 + F_0 K_0 & D_0 + F_0 G_0 + (D_0 + F_0 G_0)^\top \end{pmatrix} \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix} &= 2\Delta \lambda^\top [(C_0 + F_0 K_0)e + (D_0 + F_0 G_0)\Delta \lambda] \\ &= 2\Delta \lambda^\top (\Delta w - q(x, t, \lambda)) \end{aligned}$$

Then,

$$\dot{V}(t) = \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix}^\top M_0 \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix} + 2e^\top P_0 p(x, t, \lambda) + 2\Delta \lambda^\top (\Delta w - q(x, t, \lambda)).$$

Let us substitute the values of  $p$  and  $q$  and write explicitly the following

$$\begin{aligned} 2e^\top P_0 p(x, t, \lambda) - 2\Delta \lambda^\top q(x, t, \lambda) = & 2e^\top P_0 [\Delta A x + \Delta B \lambda + \Delta E K_0 e + \Delta E G_0 \Delta \lambda + \Delta E u_d \\ & \pm \Delta A x_d \pm \Delta B \lambda_d] \\ & - 2\Delta \lambda^\top [\Delta C x + \Delta D \lambda + \Delta F K_0 e + \Delta F G_0 \Delta \lambda \\ & + \Delta F u_d \pm \Delta C x_d \pm \Delta D \lambda_d]. \end{aligned}$$

Hence,

$$\begin{aligned} 2e^\top P_0 p(x, t, \lambda) - 2\Delta \lambda^\top q(x, t, \lambda) = & \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix}^\top \begin{pmatrix} (\Delta A + \Delta E K_0)^\top P_0 + P_0 (\Delta A + \Delta E K_0) & P_0 (\Delta B + \Delta E G_0) - (\Delta C + \Delta F K_0)^\top \\ (\Delta B + \Delta E G_0)^\top P_0 - (\Delta C + \Delta F K_0) & -\Delta D - \Delta F G_0 - (\Delta D + \Delta F G_0)^\top \end{pmatrix} \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix} \\ & + 2e^\top P_0 (\Delta A x_d + \Delta B \lambda_d + \Delta E u_d) - 2\Delta \lambda^\top (\Delta C x_d + \Delta D \lambda_d + \Delta F u_d) \end{aligned}$$

Let

$$\Delta M_0 \triangleq \begin{pmatrix} -(\Delta A + \Delta E K_0)^\top P_0 - P_0 (\Delta A + \Delta E K_0) & -P_0 (\Delta B + \Delta E G_0) + (\Delta C + \Delta F K_0)^\top \\ -(\Delta B + \Delta E G_0)^\top P_0 + (\Delta C + \Delta F K_0) & \Delta D + \Delta F G_0 + (\Delta D + \Delta F G_0)^\top \end{pmatrix}$$

Thus,

$$\dot{V} \leq - \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix}^\top (-M_0 + \Delta M_0) \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix} + 2e^\top P_0 (\Delta A x_d + \Delta B \lambda_d + \Delta E u_d) - 2\Delta \lambda^\top (\Delta C x_d + \Delta D \lambda_d + \Delta F u_d)$$

Let  $a(t) \triangleq \Delta A x_d + \Delta B \lambda_d + \Delta E u_d$  and  $b(t) \triangleq \Delta C x_d + \Delta D \lambda_d + \Delta F u_d$ . For any  $\Lambda_1^\top = \Lambda_1 > 0$  and  $\Lambda_2^\top = \Lambda_2 > 0$ , it holds that:

$$\begin{aligned} |2e^\top P_0 a(t)| &\leq e^\top P_0 \Lambda_1 P_0 e + a^\top(t) \Lambda_1^{-1} a(t) \\ |2\Delta \lambda^\top b(t)| &\leq \Delta \lambda^\top \Lambda_2 \Delta \lambda + b^\top(t) \Lambda_2^{-1} b(t) \end{aligned}$$

So,

$$\begin{aligned}\dot{V} &\leq -z^\top \left[ -M_0 + \Delta M_0 - \begin{pmatrix} P_0 \Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \right] z + a^\top(t) \Lambda_1^{-1} a(t) + b^\top(t) \Lambda_2^{-1} b(t) \\ &\leq -\lambda_{\min} \left[ -M_0 + \Delta M_0 - \begin{pmatrix} P_0 \Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \right] \|z\|^2 + \lambda_{\min}^{-1}(\Lambda_1) \|a(t)\|^2 + \lambda_{\min}^{-1}(\Lambda_2) \|b(t)\|^2\end{aligned}$$

where  $z(t) \triangleq \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix}$ . Let us prove that if the conditions, (34) and (35) hold, then the matrix  $-M_0 + \Delta M_0 - \begin{pmatrix} P_0 \Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$  is positive definite. For this purpose, the upper-bounding of  $\Delta M_0$  can be done term by term as:

$$\begin{pmatrix} e \\ \Delta \lambda \end{pmatrix}^\top \Delta M_0 \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix} = -2e^\top P_0 (\Delta A + \Delta E K_0) e - 2e^\top P_0 (\Delta B + \Delta E G_0) \Delta \lambda + 2e^\top (\Delta C + \Delta F K_0)^\top \Delta \lambda + 2\Delta \lambda^\top (\Delta D + \Delta F G_0) \Delta \lambda$$

Thus, for any  $\Lambda_k = \Lambda_k^\top > 0$ ,  $k \in \{A, B, C, D, E, F\}$

$$2e^\top P_0 (\Delta A + \Delta E K_0) e \geq -e^\top \left[ P_0 (\Lambda_A + \Lambda_E) P_0 + \Delta A^\top \Lambda_A^{-1} \Delta A + K_0^\top \Delta E^\top \Lambda_E^{-1} \Delta E K_0 \right] e$$

$$2\Delta \lambda^\top (\Delta B + \Delta E G_0)^\top P_0 e \geq -e^\top \left[ P_0 (\Lambda_B + \Lambda_E) P_0 \right] e - \Delta \lambda^\top \left[ \Delta B^\top \Lambda_B^{-1} \Delta B + G_0^\top \Delta E^\top \Lambda_E^{-1} \Delta E G_0 \right] \Delta \lambda$$

$$2\Delta \lambda^\top (\Delta C + \Delta F K_0) e \geq -\Delta \lambda^\top \left[ \Lambda_C + \Delta F \Lambda_F^{-1} \Delta F^\top \right] \Delta \lambda - e^\top \left[ \Delta C^\top \Lambda_C^{-1} \Delta C + K_0^\top \Lambda_F K_0 \right] e$$

$$2\Delta \lambda^\top (\Delta D + \Delta F G_0) \Delta \lambda \geq -\Delta \lambda^\top \left[ \Lambda_D + \Delta D^\top \Lambda_D^{-1} \Delta D + G_0^\top \Lambda_F G_0 + \Delta F \Lambda_F^{-1} \Delta F^\top \right] \Delta \lambda$$

Assume that the constraints on uncertainties in (34) stated in Proposition 4.3 are satisfied, then:

$$\begin{pmatrix} e \\ \Delta \lambda \end{pmatrix}^\top \Delta M_0 \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix} \geq - \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix}^\top \begin{pmatrix} P_0 \tilde{\Lambda}_1 P_0 + K_0^\top \tilde{\Lambda}_F K_0 + 2I_n & 0 \\ 0 & \tilde{\Lambda}_2 + G_0^\top \tilde{\Lambda}_F G_0 \end{pmatrix} \begin{pmatrix} e \\ \Delta \lambda \end{pmatrix}$$

Thus:

$$\begin{aligned}z^\top \left[ -M_0 + \Delta M_0 - \begin{pmatrix} P_0 \Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \right] z \geq \\ z^\top \begin{pmatrix} -(M_0)_{11} - P_0 \Lambda_1 P_0 - P_0 \tilde{\Lambda}_1 P_0 - K_0^\top \tilde{\Lambda}_F K_0 - 2I_n & -(M_0)_{12} \\ -(M_0)_{12} & -(M_0)_{22} - \tilde{\Lambda}_2 - \Lambda_2 - G_0^\top \tilde{\Lambda}_F G_0 \end{pmatrix} z\end{aligned}\tag{36}$$

Applying the Schur complement Theorem for positive definiteness [33, Theorem A.65], it is inferred that the matrix obtained in (36) is positive definite if and only if the matrix inequality (35) in Proposition 4.3 holds. The inequality obtained in (35) is a bilinear matrix inequality BMI due to the upper-left sub-matrix  $M_0$ . Then, in order to solve this inequality, it must be transformed into LMI (see Appendix C). It follows that

$$M_{0_{lm}} \triangleq - \begin{pmatrix} -Q_0 A_0^\top - Q_0 A_0 - N_0^\top E_0^\top - E_0 N_0 & -B_0 - E_0 G_0 + Q_0 C_0^\top + N_0^\top F_0^\top \\ -B_0^\top - G_0^\top E_0^\top + C_0 Q_0 + F_0 N_0 & D_0 + F_0 G_0 + (D_0 + F_0 G_0)^\top \end{pmatrix}\tag{37}$$

with  $Q_0 = P_0^{-1}$  and  $N_0 = K_0 Q_0$ . So, the BMI in (35) is written as:

$$\begin{pmatrix} -(M_{0_{lm}})_{11} & -(M_{0_{lm}})_{12} & I_n & I_n & N_0^\top & Q_0 & 0 & 0 & 0 \\ -(M_{0_{lm}})_{21} & -(M_{0_{lm}})_{22} & 0 & 0 & 0 & 0 & G_0^\top & I_m & I_m \\ I_n & 0 & \Lambda_1^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & \tilde{\Lambda}_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ N_0 & 0 & 0 & 0 & \tilde{\Lambda}_F^{-1} & 0 & 0 & 0 & 0 \\ Q_0 & 0 & 0 & 0 & 0 & \frac{1}{2} I_n & 0 & 0 & 0 \\ 0 & G_0 & 0 & 0 & 0 & 0 & \tilde{\Lambda}_F^{-1} & 0 & 0 \\ 0 & I_m & 0 & 0 & 0 & 0 & 0 & \Lambda_2^{-1} & 0 \\ 0 & I_m & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\Lambda}_2^{-1} \end{pmatrix} > 0\tag{38}$$

where  $(M_{0_{lin}})_{ij}$  denotes the element of the matrix  $M_{0_{lin}}$  in (37) located at the  $i$ th row and  $j$ th column with  $i, j \in \{1, 2\}$ . The LMI in (38) can be solved, under some conditions, in the new variables  $Q_0 = Q_0^\top > 0$ ,  $N_0$  and  $G_0$ . Thus, it is possible to say that the inequality:  $z^\top \left[ -M_0 + \Delta M_0 - \begin{pmatrix} P_0 \Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \right] z > 0$  holds. As a consequence of the result obtained about positive definiteness of the matrix obtained in (36) and due to symmetry, there exists  $\mu > 0$  such that  $-\lambda_{\min} \left[ M_0 + \Delta - M_0 - \begin{pmatrix} P_0 \Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \right] = -\mu < 0$ . According to Assumption 3.1 where  $u_d$ ,  $x_d$  and  $\lambda_d$  are bounded, the terms  $\|a(t)\|^2$  and  $\|b(t)\|^2$  are bounded such that  $\|a(t)\|^2 < \beta_1$  and  $\|b(t)\|^2 < \beta_2$  for some  $\beta_1, \beta_2 > 0$ . Then, it is inferred that:

$$\begin{aligned} \dot{V} &\leq -\mu \|z\|^2 + \lambda_{\min}^{-1}(\Lambda_1)\beta_1 + \lambda_{\min}^{-1}(\Lambda_2)\beta_2 \\ &\leq -\mu \|e\|^2 - \mu \|\Delta \lambda\|^2 + \lambda_{\min}^{-1}(\Lambda_1)\beta_1 + \lambda_{\min}^{-1}(\Lambda_2)\beta_2 \\ &\leq -\mu \|e\|^2 + \lambda_{\min}^{-1}(\Lambda_1)\beta_1 + \lambda_{\min}^{-1}(\Lambda_2)\beta_2. \end{aligned}$$

It shows that  $\dot{V} < 0$  outside the ball  $B_r(0) \subset \mathbf{R}^n$ , with  $r \triangleq \sqrt{\frac{\lambda_{\min}^{-1}(\Lambda_1)\beta_1 + \lambda_{\min}^{-1}(\Lambda_2)\beta_2}{\mu}}$ . Then, the solution of (33) is GUUB [57, 58]. According to Theorem D.1, the ultimate bound with  $\alpha_1(\|e\|) = \lambda_{\min}(P_0)\|e\|^2$  and  $\alpha_2(\|e\|) = \lambda_{\max}(P_0)\|e\|^2$  is given by

$$\|e\| \leq \alpha_1^{-1}(\alpha_2(r)) = \sqrt{\frac{\lambda_{\max}(P_0)r^2}{\lambda_{\min}(P_0)}} = \sqrt{\frac{\lambda_{\max}(P_0) \left( \lambda_{\min}^{-1}(\Lambda_1)\beta_1 + \lambda_{\min}^{-1}(\Lambda_2)\beta_2 \right)}{\lambda_{\min}(P_0)\mu}}$$

□

Notice that under the conditions of the proposition, then  $M_0 + \Delta M_0 > 0$ . This implies that the closed-loop system's quadruple in (32) is strongly passive (in other words the uncertainties do not destroy the strong passivity of the nominal closed-loop system). In particular this implies that  $D_0 + \Delta D + (F_0 + \Delta F)G_0 > 0$ . Consequently, Assumption 3.1 and item 1 in Appendix B guarantee that the solutions of (32) are AC as long as uncertainties are constant matrices. Time-varying uncertainties yield a closed-loop system which can be analyzed along the lines of section 2.2.

#### 4.3. Relaxing Strong Passivity to Strict State Passivity

It is of interest to relax the strong passivity condition (Assumption 3.4) of Proposition 4.3, with strict state passivity (Assumption 3.3). The strict state passivity BMI in (31) modified to  $M_0 \preceq \begin{pmatrix} -\epsilon P_0 & 0 \\ 0 & 0 \end{pmatrix}$ . This means that the nominal closed-loop system may have  $D_0 + F_0 G_0 \succcurlyeq 0$ , hence  $M_0 \preceq 0$ . Therefore we have to find conditions which guarantee:

$$-M_0 + \Delta M_0 - \begin{pmatrix} \epsilon' P_0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} P_0 \Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \succcurlyeq 0 \quad (39)$$

for some  $0 < \epsilon' < \epsilon$ , where  $M_0$  correspond to the strict state passivity LMI. To this end, we may rely on Lemma E.1 in Appendix E.

**Proposition 4.4.** *Let Assumptions 3.1, 3.2, 3.3 hold for the nominal system, and Assumption 4.1 holds for the closed-loop system. Then the matrix inequality in (39) holds if and only if:*

1.  $R \triangleq (D_0 + F_0 G_0) + (D_0 + F_0 G_0)^\top + (\Delta D + \Delta F G_0) + (\Delta D + \Delta F G_0)^\top - \Lambda_2 \succcurlyeq 0$ ,
2.  $Q \triangleq -(A_0 + E_0 K_0 + \frac{\epsilon'}{2} I_n)^\top P_0 - P_0 (A_0 + E_0 K_0 + \frac{\epsilon'}{2} I_n) - (\Delta A + \Delta E K_0)^\top P_0 - P_0 (\Delta A + \Delta E K_0) - P_0 \Lambda_1 P_0 \succcurlyeq 0$ , with  $-(A_0 + E_0 K_0)^\top P_0 - P_0 (A_0 + E_0 K_0) \succcurlyeq \epsilon P_0$ ,  $\epsilon > \epsilon' > 0$ ,
3.  $\text{Im}(S^\top) \subseteq \text{Im}(R)$ , with  $S \triangleq -P_0 (B_0 + E_0 G_0) + (C_0 + F_0 K_0)^\top - P_0 (\Delta B + \Delta E G_0) + (\Delta C + \Delta F K_0)^\top$ ,
4.  $Q \succcurlyeq S R^\top S^\top$ .

*Proof.* Notice that  $Q$  and  $R$  are symmetric by construction. The proof follows from Lemma E.1. □

The tracking error ultimate boundedness can be shown as follows.

**Proposition 4.5.** Assume that Proposition 4.4 holds, then the tracking error  $e$  is globally ultimately bounded as:

$$\|e\| \leq \sqrt{\frac{\lambda_{\max}(P_0)\eta_r}{\lambda_{\min}^2(P_0)\epsilon'}} \quad (40)$$

*Proof.* Under the proposition's assumptions, the rate of change of the Lyapunov function  $V(e) = e^\top P_0 e$  where  $P_0 = Q_0^{-1}$  is the solution of the strict state passivity LMI  $M_{0lin} \preceq \begin{pmatrix} -\epsilon Q_0 & 0 \\ 0 & 0 \end{pmatrix}$  where  $M_{0lin}$  is defined in (37) is:

$$\begin{aligned} \dot{V} &\leq -\begin{pmatrix} e \\ \Delta\lambda \end{pmatrix}^\top \left[ -M_0 + \Delta M_0 - \begin{pmatrix} P_0\Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \right] \begin{pmatrix} e \\ \Delta\lambda \end{pmatrix} + \underbrace{a^\top(t)\Lambda_1^{-1}a(t) + b^\top(t)\Lambda_2^{-1}b(t)}_{r(t)} \\ &\leq -\epsilon' e^\top P_0 e + r(t) \end{aligned}$$

where  $\epsilon' > 0$  and  $r(t)$  is a positive bounded term determined by the parametric uncertainties  $\Delta A$ ,  $\Delta B$ ,  $\Delta C$ ,  $\Delta D$ ,  $\Delta E$ , and  $\Delta F$  such that  $\|r(t)\| \leq \eta_r$ . So,

$$\dot{V} \leq -\epsilon' \lambda_{\min}(P_0) \|e\|^2 + \eta_r$$

Hence, the rate of change of the storage function  $\dot{V} < 0$  outside the ball  $B_r(0) \subset \mathbf{R}^n$ , with  $r \triangleq \sqrt{\frac{\eta_r}{\epsilon' \lambda_{\min}(P_0)}}$ . Therefore, the tracking error is GUUB [57, 58] and the expression of the ultimate bound is in (40).  $\square$

The difference between the case with strong passivity and the case with strict state passivity, is mainly that there is no  $-\|\Delta\lambda\|^2$  that helps to accelerate the convergence in the second case (see (4.2)). Moreover the strong passivity allows us to dispense with strict conditions on the structure of the uncertainties. In a sense, strict state passivity allows us to obtain a more fragile robustness.

It is of interest to analyze under which conditions on the plant's nominal model and on the uncertainties, the conditions in items 1–4 in Proposition 4.4 hold. For item 1, assume that  $\Delta D = 0$ ,  $\Delta F = 0$ ,  $\Delta C = 0$  (the complementarity constraint has no uncertainty) then we may take  $\Lambda_2 = 0$  (since  $b(t) = 0$ ), so that  $R \succcurlyeq 0$  by closed-loop strict-state passivity. For item 2, using the fact that  $Q_0 > 0$ , Corollary E.4 can be used to derive sufficient condition such that  $Q > 0$ :  $\sigma_{\max}(Q_0) > \epsilon \lambda_{\max}(P_0) > 2\sigma_{\max}(P_0)(\sigma_{\max}(\Delta A) + \sigma_{\max}(\Delta EK_0)) + \sigma_{\max}^2(P_0)\sigma_{\max}(\Lambda_1)$ , where we used [54, Fact 9.14.15, Corollary 9.6.5]. Let us focus on items 3 and 4. We have  $-M_0 = \begin{pmatrix} Q_0 & S_0 \\ S_0^\top & R_0 \end{pmatrix}$ , and  $-M_0 + \Delta M_0 = \begin{pmatrix} P_0\Lambda_1 P_0 & 0 \\ 0 & \Lambda_2 \end{pmatrix} = \begin{pmatrix} Q_0 + \Delta Q & S_0 + \Delta S \\ S_0^\top + \Delta S^\top & R_0 + \Delta R \end{pmatrix}$ , where the matrices stem from (31) and items 1, 2 and 3 in Proposition 4.4. Using Lemma E.1 and E.2, the strict-state passivity LMI for the nominal system implies that  $Q_0 > Q_0 - \epsilon P_0 \succcurlyeq S_0 R_0^\dagger S_0^\top$ , and  $\text{Im}(S_0 R_0^\dagger S_0^\top) \subseteq \text{Im}(Q_0 - \epsilon P_0)$  (we also have  $\text{Im}(S_0 R_0^\dagger S_0^\top) \subseteq \text{Im}(Q_0) = \mathbf{R}^n$  since strict state passivity implies that  $Q_0 > 0$ ), and  $\text{Im}(S_0^\top) \subseteq \text{Im}(R_0) = \text{Im}(R_0^\dagger)$  [59, Proposition 8.1.7]. Assume that  $\text{Im}(\Delta S^\top) \subseteq \text{Im}(S_0^\top)$ . Then  $\text{Im}(S^\top) = \text{Im}(S_0^\top + \Delta S^\top) \subseteq \text{Im}(S_0^\top)$ . Thus item 3 is satisfied if  $\text{Im}(R) = \text{Im}(R_0)$  (which holds if  $\Delta D = 0$ ,  $\Delta F = 0$ ,  $\Delta C = 0$ , since in this case  $\Delta R = 0$ ). We can therefore state the following:

**Lemma 4.6.** Let Assumption 3.3 hold for the nominal system. Assume that  $\text{Im}(\Delta S^\top) \subseteq \text{Im}(S_0^\top)$ , and that  $\text{Im}(R) = \text{Im}(R_0)$ , then  $\text{Im}(S^\top) \subseteq \text{Im}(R)$ .

Notice that item 4 is  $Q_0 + \Delta Q \succcurlyeq (S_0 + \Delta S)(R_0 + \Delta R)^\dagger (S_0^\top + \Delta S^\top)$ . In the next proposition we derive conditions on the uncertainty matrices, that guarantee that the inequality in item 4 holds true when the nominal system is strictly state passive.

**Proposition 4.7.** Let Assumption 3.3 hold for the quadruple  $(A_0, B_0, C_0, D_0)$ . Assume that: (i)  $R \succcurlyeq 0$ , (ii)  $\text{Im}(\Delta R) \subseteq \text{Im}(R_0)$ , (iii)  $\text{rank}(R_0) = r$ ,  $R_0 \neq 0$ ,  $\sigma_{\max}(\Delta R) < \sigma_r(R_0)$ , (iv)  $\sigma_{\max}(OT) + \sigma_{\max}(O((R_0^\dagger \Delta R)^2)) + \sigma_{\max}(\Delta Q) < \epsilon \lambda_{\max}(P_0)$ , where  $OT$  is in (41). Then  $Q \succcurlyeq SR^\dagger S^\top$ .

*Proof.* Since  $\Delta R = \Delta R^\top$ , using [59, Fact 8.4.3] and [59, Fact 8.4.38] and (ii), we have  $(R_0 + \Delta R)^\dagger = (I_m + R_0^\dagger \Delta R)^\dagger (R_0^\dagger + R_0^\dagger \Delta R R_0^\dagger)^\dagger (I_m + \Delta R R_0^\dagger)^\dagger$ . Using Corollary E.4, [59, Fact 8.3.33], [59, Corollary 11.6.5], it follows from (iii) that  $I_m + R_0^\dagger \Delta R > 0$ . Thus  $(I_m + R_0^\dagger \Delta R)^\dagger = (I_m + R_0^\dagger \Delta R)^{-1} = \sum_{k=0}^{\infty} (-R_0^\dagger \Delta R)^k = I_m - R_0^\dagger \Delta R + \mathcal{O}((R_0^\dagger \Delta R)^2)$  [59, Proposition 11.3.10]. Therefore  $(R_0 + \Delta R)^\dagger = (I_m - R_0^\dagger \Delta R)(R_0^\dagger + R_0^\dagger \Delta R R_0^\dagger)^\dagger (I_m - \Delta R R_0^\dagger)^\dagger + \mathcal{O}((R_0^\dagger \Delta R)^2) = R_0^\dagger - R_0^\dagger \Delta R R_0^\dagger + \mathcal{O}((R_0^\dagger \Delta R)^2)$ . Consequently,  $(S_0 + \Delta S)(R_0 + \Delta R)^\dagger (S_0^\top + \Delta S^\top) = (S_0 + \Delta S)(R_0^\dagger - R_0^\dagger \Delta R R_0^\dagger + \mathcal{O}((R_0^\dagger \Delta R)^2))(S_0^\top + \Delta S^\top) = S_0 R_0^\dagger S_0^\top + OT + \mathcal{O}((R_0^\dagger \Delta R)^2)$ , where  $OT$  stands for other terms defined as:

$$OT = S_0 R_0^\dagger \Delta S^\top - S_0 R_0^\dagger \Delta R R_0^\dagger S_0^\top - S_0 R_0^\dagger \Delta R R_0^\dagger \Delta S^\top + \Delta S R_0^\dagger S_0^\top + \Delta S R_0^\dagger \Delta S^\top - \Delta S R_0^\dagger \Delta R R_0^\dagger S_0^\top - \Delta S R_0^\dagger \Delta R R_0^\dagger \Delta S^\top. \quad (41)$$

The result follows from  $Q_0 - S_0 R_0^\dagger S_0^\top \succcurlyeq \epsilon P_0$ , (iv), Corollary E.4 and [59, Fact 11.16.18], which guarantee that  $\epsilon P_0 > OT - \Delta Q + \mathcal{O}((R_0^\dagger \Delta R)^2)$ , so that  $Q_0 + \Delta Q > (S_0 + \Delta S)(R_0 + \Delta R)^\dagger (S_0^\top + \Delta S^\top)$ .  $\square$

Item (iv) in Proposition 4.7 means that a class of nonzero, sufficiently small uncertainties are allowed. The matrix  $\Lambda_1$  which appears in  $\Delta Q$  can be chosen small if  $\Delta A$ ,  $\Delta B$ ,  $\Delta E$  are small. As said above,  $\Delta R$  can be made small if the uncertainties inside the complementarity constraints are small.

## 5. Simple Scalar Example with State Jumps

This section and sections 6 through 7 are dedicated to analyze and to present numerical simulations of the above theoretical developments. The numerical simulations are done with the INRIA software package `siconos`<sup>1</sup>, and the LMIs are solved with `MOSEK 9.3.14` solver.

### 5.1. Dynamics and Closed-loop System

Consider the scalar LCS with  $d = 0$ :

$$\begin{cases} \dot{x}(t) = ax(t) + b\lambda(t) + u_1(t) \\ 0 \leq \lambda(t) \perp w(t) = cx(t) + u_2(t) \geq 0 \end{cases} \quad (42)$$

where  $a, b$ , and  $c \in \mathbf{R}$ ,  $u = (u_1, u_2)^\top$ ,  $u_1 = Eu$ ,  $u_2 = Fu$  with  $E = (1, 0)$  and  $F = (0, 1)$ . The desired system is represented by an LCS as follows:

$$\begin{cases} \dot{x}_d(t) = ax_d(t) + b\lambda_d(t) + u_{1d}(t) \\ 0 \leq \lambda_d(t) \perp w_d(t) = cx_d(t) + u_{2d}(t) \geq 0 \end{cases} \quad (43)$$

If  $a > 0$ , then the real system in (42) and the desired system (43) are unstable (*i.e.*, the quadruple  $(a, b, c, 0)$  has a real positive pole). But, in the context of trajectory tracking, the stability of the error dynamics is the main concern and not the stability of the plant and desired dynamics.

Let

$$\begin{aligned} u_1 &= k_1(x - x_d) + g_1(\lambda - \lambda_d) + u_{1d} \\ u_2 &= k_2(x - x_d) + g_2(\lambda - \lambda_d) + u_{2d} \end{aligned} \quad (44)$$

so that  $K = (k_1, k_2)^\top$  and  $G = (g_1, g_2)^\top$ . By substituting the expressions of  $u$  and  $v$  in (42), the closed-loop system is written in the form of (12) as follows:

$$\begin{cases} \dot{x}(t) = (a + k_1)x(t) + (b + g_1)\lambda(t) - k_1x_d(t) - g_1\lambda_d(t) + u_{1d}(t) \\ 0 \leq \lambda(t) \perp w(t) = (c + k_2)x(t) + g_2\lambda(t) - k_2x_d(t) - g_2\lambda_d(t) + u_{2d}(t) \geq 0. \end{cases} \quad (45)$$

**Remark 5.1.** Consider  $G = 0$ . Recall the conditions of well-posedness given in section 2.1, item 3 for the case when  $d = 0$ . Assume that there exists  $p > 0$  such that the quadruple  $(a, b, c, 0)$  is passive (*i.e.*, the LMI in (5) has a solution). Then the condition  $pb = c$  holds. In addition, if the closed-loop system in (45) is well-posed, then the condition  $pb = c + k_2$  holds. Therefore, knowing that  $p > 0$ , the variables  $b$ ,  $c$  and  $c + k_2$  should have the same sign. Notice also that the contact LCPs have the matrix  $cb$ , hence  $cb < 0$  implies no or several solutions for some  $u_{1d}(t)$  and  $u_{2d}(t)$ : passivity avoids this.

<sup>1</sup><https://nonsmooth.gricad-pages.univ-grenoble-alpes.fr/siconos/index.html>

## 5.2. State-jumps Analysis

State-jumps in  $x(\cdot)$  and  $x_d(\cdot)$  can occur for different reasons discussed in section 3.3. The purpose of this section is to analyze state jumps in the desired and closed-loop systems in different cases which illustrate the general developments.

### 5.2.1. If the controller gain $G = 0$

Notice that the dynamics (45) may not be well-posed due to the possible presence of Dirac measures that stem from both the complementarity conditions in (45) which create an impulsive multiplier  $\lambda$  and from the complementarity in (43) which implies an impulsive multiplier  $\lambda_d$ . Therefore we shall consider  $G = 0$  (which means that (C1) in section 3.3 is satisfied). In order to analyze state jumps, the following sets are calculated as follows (see section 2.1). For the desired system in (43):

$$\begin{aligned} Q_{d,D} &= \{\lambda_d \in \mathbf{R} \mid 0 \leq \lambda_d \perp d\lambda_d \geq 0\} = \{\lambda_d \in \mathbf{R} \mid 0 \leq \lambda_d \perp 0 \geq 0\} = \{\lambda_d \in \mathbf{R} \mid \lambda_d \in \mathbf{R}_+\} \\ Q_{d,D}^* &= \{w_d \in \mathbf{R} \mid \langle w_d, \lambda_d \rangle \geq 0\} = \{w_d \in \mathbf{R} \mid w_d \in \mathbf{R}_+\} \\ \mathcal{K}_d &= \{x_d \in \mathbf{R} \mid cx_d + u_{2d}(t^+) \in Q_{d,D}^*\} = \{x_d \in \mathbf{R} \mid cx_d \geq -u_{2d}(t^+)\} \end{aligned} \quad (46)$$

A jump can occur in the desired state  $x_d$  at time  $t = t_c$ ,  $t_c \in [0, +\infty)$ , if and only if  $cx_d(t_c^-) < -u_{2d}(t_c^+)$ . For the closed-loop system in (45):

$$\begin{aligned} Q_D &= \{\lambda \in \mathbf{R} \mid 0 \leq \lambda \perp d\lambda \geq 0\} = \{\lambda \in \mathbf{R} \mid \lambda \in \mathbf{R}_+\} \\ Q_D^* &= \{w \in \mathbf{R} \mid \langle w, \lambda \rangle \geq 0\} = \{w \in \mathbf{R} \mid w \in \mathbf{R}_+\} \\ \mathcal{K} &= \{x \in \mathbf{R} \mid (c + k_2)x - k_2x_d(t^+) + u_{2d}(t^+) \in Q_{d,D}^*\} = \{x \in \mathbf{R} \mid (c + k_2)x \geq k_2x_d(t^+) - u_{2d}(t^+)\} \end{aligned} \quad (47)$$

The state  $x$  of the closed-loop system in (45) jumps at any  $t = t_c$ ,  $t_c \in [0, +\infty)$ , if and only if  $cx(t_c^-) + k_2[x(t_c^-) - x_d(t_c^+)] < -u_{2d}(t_c^+)$ . Thus, when  $G = 0$ , the jumps at  $t > 0$  in  $x(\cdot)$  and  $x_d(\cdot)$  occur only if  $u_{2d}$  is discontinuous at  $t$ , as expected.

### 5.2.2. If the controller gain $G \neq 0$ and $x_d$ is continuous

In this case, a state jump in  $x$  (if any) is caused by the discontinuity in  $\lambda_d$ , which can arise when the complementarity problem switches mode (i.e.,  $w(t) = cx(t) + u_2(t)$  vanishes). The desired system's state is considered continuous to avoid a Dirac measure  $\lambda_d$  in complementarity conditions in (45). For the closed-loop system represented by  $(a + k_1, b + g_1, c + k_2, g_2)$  to be strictly passive, one must have  $g_2 > 0$  given that  $d = 0$ . But,  $g_2$  multiplies  $\lambda$  as shown in (45). The LCP of the closed-loop system (45) is

$$0 \leq \lambda(t) \perp w(t) = (c + k_2)x(t) + g_2\lambda(t) - k_2x_d(t) - g_2\lambda_d(t) + u_{2d}(t) \geq 0$$

has a unique piecewise continuous solution  $\lambda$ . Hence, the LCS in (45) is an ODE with an AC solution on  $\mathbf{R}_+$  provided that  $u_1(t)$  and  $u_2(t)$  are continuous (see section 2.1, item 1). Thus, the closed-loop system's state is jump-free in the scalar case with  $G \neq 0$  and the desired system's state  $x_d$  is continuous. Therefore, the error dynamics  $e(t)$  is continuous in this case. The explanation of this result can be further understood by analyzing the following sets of the closed-loop system:

$$\begin{aligned} Q_D &= \{\lambda \in \mathbf{R} \mid 0 \leq \lambda \perp (d + g_2)\lambda \geq 0\} = \{0\} \\ Q_D^* &= \{w \in \mathbf{R} \mid \langle w, \lambda \rangle \geq 0\} = \{\mathbf{R}\} \\ \mathcal{K} &= \{x \in \mathbf{R} \mid (c + k_2)x(t) + g_2\lambda(t) - k_2x_d(t) - g_2\lambda_d(t) + u_{2d}(t) \in Q_{d,D}^*\} = \{\mathbf{R}\} \end{aligned}$$

Using the optimization problem in (7),  $x(t^+) = x(t^-)$ . Thus, the closed-loop system's state  $x(t)$  is continuous even if  $\lambda_d$  is discontinuous. Therefore, the error dynamics  $e(t)$  is continuous.

### 5.3. Stability Analysis of Error Dynamics with State-jumps ( $G = 0$ )

This section is dedicated to the stability analysis of the error dynamics in (48) when state jumps occur. The error dynamics  $e(t) = x(t) - x_d(t)$  in the form of (14) is represented by:

$$\begin{aligned} \dot{e}(t) &= (a + k_1)e(t) + b(\lambda(t) - \lambda_d(t)) \\ 0 &\leq \begin{pmatrix} \lambda(t) \\ \lambda_d(t) \end{pmatrix} \perp \begin{pmatrix} w(t) \\ w_d(t) \end{pmatrix} = \begin{pmatrix} c + k_2 \\ -c - k_2 \end{pmatrix} e(t) + \begin{pmatrix} 0 & c \\ c + k_2 & -k_2 \end{pmatrix} \begin{pmatrix} x(t) \\ x_d(t) \end{pmatrix} + \begin{pmatrix} u_{2d}(t) \\ u_{2d}(t) \end{pmatrix} \geq 0 \end{aligned} \quad (48)$$

Let us now try the passivity criterion introduced in (26) to cope with state jumps directly from the error dynamics passivity. Let  $\bar{a} = a + k_1$ ,  $\bar{b} = \begin{pmatrix} b & -b \end{pmatrix}$  and  $\bar{c} = \begin{pmatrix} c + k_2 \\ 0 \end{pmatrix}$ . The passivity of the error dynamics represented by the quadruple  $(\bar{a}, \bar{b}, \bar{c}, 0)$  is determined by checking analytically if the BMI in (26) has a solution for the system (48). Consider

$$M_{ext} \triangleq \begin{pmatrix} -2p(a + k_1) & -pb + c + k_2 & pb \\ -bp + c + k_2 & 0 & 0 \\ bp & 0 & 0 \end{pmatrix} \geq 0.$$

From Lemma E.1 it follows that  $M_{ext} \succcurlyeq$  only if  $-pb + c + k_2 = pb = 0$ . Therefore it is inferred that this approach (imposing the passivity of the error dynamics with the LMI in (26)) is not fruitful in this case.

The following Lemma states the stability result.

**Lemma 5.2.** *Consider the dynamical systems in (45) and in (43). Suppose that Assumptions 3.2 and 3.3 hold. Assume that the states  $x(\cdot)$  and  $x_d(\cdot)$  have jumps for  $t > 0$  (which is possible since  $d = 0$  in the complementarity constraint). Then, the error dynamics in (48) has a globally asymptotically stable equilibrium point  $e^* = 0$ .*

*Proof.* see Appendix G.1 □

The next lemma states the existence of finite-time tracking.

**Corollary 5.3.** *Following on from Lemma 5.2, assume that  $u_{2d}(\cdot)$  is discontinuous at time instants  $t_k$ , with  $k = \{1, 2, \dots, n\}$  where  $n \in \mathbf{N}^*$ . For any  $t_k$ , if there exists  $u_{2d}(t_k^+) < \min\{-cx(t_k^-), -cx_d(t_k^-)\}$ , then both  $x$  and  $x_d$  jump to the value  $x_d(t_k^+) = x(t_k^+) = -\frac{1}{c}u_{2d}(t_k^+)$ . Therefore, there is perfect tracking for all  $t \geq t_k$  (i.e., finite-time tracking is achieved).*

*Proof.* see Appendix G.2 □

### 5.4. Numerical Applications with $G = 0$

Let us consider that the open-loop system with  $u_1 = u_2 = 0$  is not strictly passive but there exist  $k_1, k_2$ , and  $p$  such that the closed-loop quadruple  $(a + k_1, b, c + k_2, 0)$  is strictly state passive, equivalently the bilinear matrix inequality in (15) has a solution  $p > 0$  with  $\epsilon > 0$ . According to Appendix C, the following LMI is obtained

$$\begin{pmatrix} -2qa - 2N_1 - \epsilon q & -b + qc + N_2 \\ -b + cq + N_2 & 0 \end{pmatrix} \geq 0 \quad (49)$$

where  $q = p^{-1}$ ,  $N_1 = k_1q$  and  $N_2 = k_2q$ . The LMI in (49) has a solution such that:

$$k_1 = -1.198, \quad k_2 = 0.96 \quad \text{and} \quad p = 1.9607$$

for  $a = 0.5$ ,  $b = 1$  and  $c = 1$ . For the sake of numerical simulation, take  $a = 0.5$  (so that the solutions of systems in (43) and (42) diverge slowly).

**Example 1.** Let us take  $b = 1$ , and  $c = 1$ . The numerical simulation below for the closed-loop system and the desired system is implemented with a discontinuous controller  $u_{2d}(t)$  at  $t_k$  with  $k = \{1, 2, 3, 4\}$ . Take  $x(0) = 3$ ,  $x_d(0) = -3$ ,  $u_{1d} = 5 \sin 3t$  and  $u_{2d} = 4$  for  $t \leq 1s$  with the time step  $h = 0.01$ . Given that  $x_d(1^-) = -0.7$  and  $x(1^-) = 2.28$ , and following Corollary 5.3, let us choose  $u_{2d}(1^+)$  such that  $u_{2d}(1^+) = \min\{-x_d(1^-) - \delta, -x(1^-) - \delta\} < \min\{-x_d(1^-), -x(1^-)\}$  with  $\delta = 0.5$ . Based on the value of  $\delta$ ,  $u_{2d}(1^+) = \min\{0.2, -2.78\} = -2.78$ .

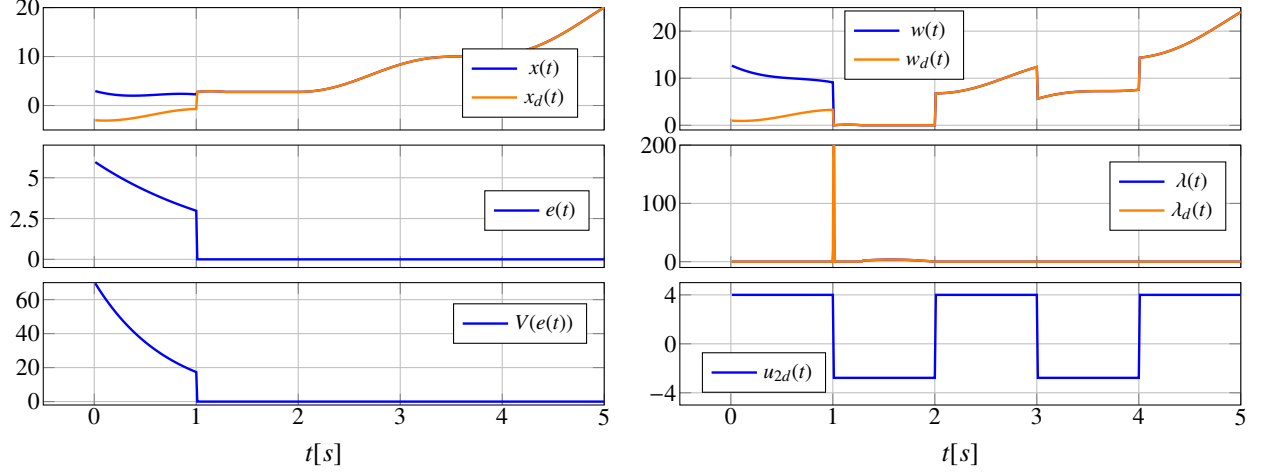


Figure 1: Numerical simulation of LCS where both  $x$  and  $x_d$  jump

Both  $x$  and  $x_d$  jumps at  $t = 1$  to the same value of  $-u_{2d}(t_1^+) = 2.87$ . Notice that the storage function's variation  $\Delta V$  is non-positive at the time of the state jump (i.e.,  $V(e(1^+)) - V(e(1^-)) \leq 0$ ). After the first jump of both states  $x$  and  $x_d$  at the same time  $t = 1$  and following Corollary 5.3, the controller  $u_{2d}$  can take any value at the time of discontinuities such that  $u_{2d}(t_{k+1}^+) \in \mathbf{R}$  and perfect tracking is preserved (i.e.,  $e = 0$ ).

**Example 2.** Let us illustrate the case when  $x_d$  jumps but  $x$  does not jump. Take  $x(0) = 3$ ,  $x_d(0) = -3$ . Knowing that  $x_d(1^-) = -0.7$  and  $x(1^-) = 2.28$  from the simulation in Figure 1, let us choose  $u_{2d}(1^+)$  such that  $x_d(1^-) \notin \mathcal{K}_d$  in (46) and  $x(1) \in \mathcal{K}$  in (47). Thus,

$$x_d(1^-) \notin \mathcal{K}_d \Leftrightarrow cx_d(1^-) + u_{2d}(1^+) < 0 \Rightarrow u_{2d}(1^+) < 0.7$$

and

$$x(1) \in \mathcal{K} \Leftrightarrow (c + k_2)x(1) - k_2x_d(1^+) + u_{2d}(1^+) \geq 0 \Rightarrow u_{2d}(1^+) \geq -2.28$$

with  $c = 1$  and  $k_2 = 0.96$  (the solution of LMI in (49)). Take  $u_{2d}(1^+) = -1$ .



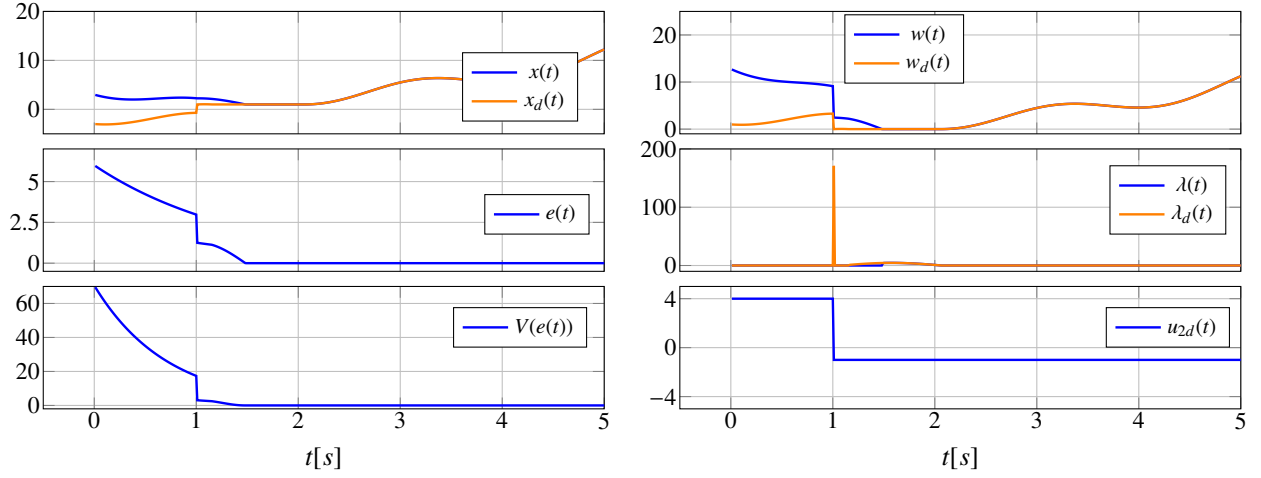


Figure 2: Numerical simulation of LCS when only  $x_d$  jumps

It is noticed from Figure 2 that the variation of the storage function  $V = pe^2$  is negative when  $x_d$  jumps at  $t = 1$ s even though the closed-loop system's state  $x$  does not jump at  $t = 1$ s. Let us create another jump in  $x_d$  in the transient regime of the error dynamics (i.e., before  $e = 0$  at  $t = 1.5$ s). Knowing that  $x_d(1.2^-) = 1$  and  $x(1.2^-) = 2.057$ , choose  $-2.057 \leq u_{2d}(1.2^+) < -1$  so that  $x_d(1.2^-) \notin \mathcal{K}_d$  and  $x(1.2) \in \mathcal{K}$ . Take  $u_{2d}(1.2^+) = -1.5$ .

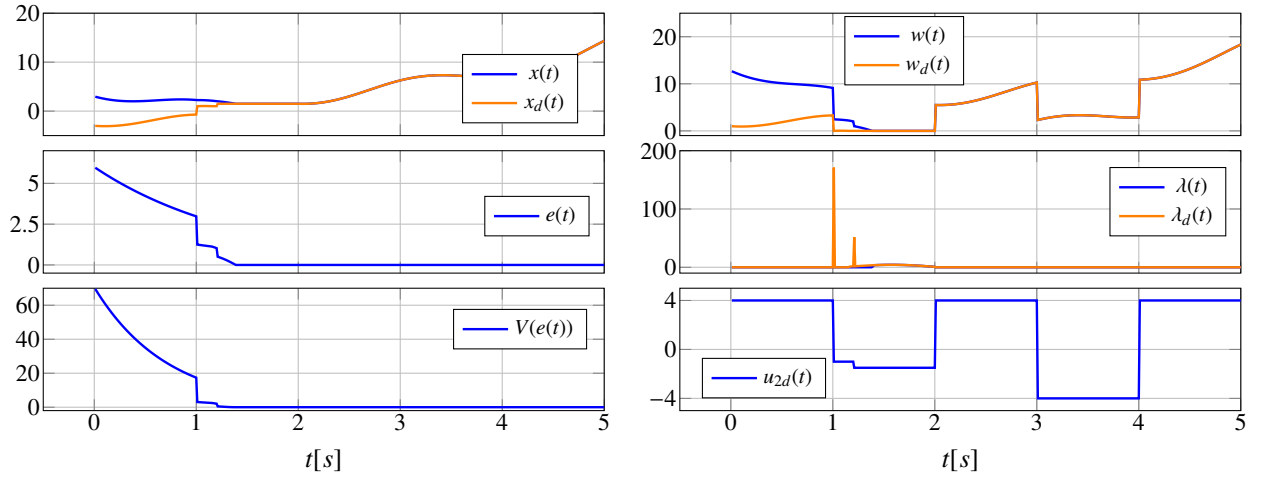


Figure 3: Numerical simulation of LCS where only  $x_d$  jumps

In Figure 3, the desired state  $x_d$  jumps at  $t = 1.2$ s such that  $x_d(1.2^+) = -u_{2d}(1.2^+) = 1.5$ . The value of the error  $e(1.2^+) = 0.557 < e(1^-) = 1.057$ , so the variation of the storage function  $V$ , computed in (G.1), is negative at the jump time  $t = 1.2$ . It is observed in Figure 3 that the error has an asymptotically stable equilibrium point  $e^* = 0$  which is consistent with the result in Lemma 5.2. It is important to state that, in the case of Figures 2 and 3, the necessary and sufficient conditions to have a jump in the closed-loop state  $x$  (i.e.,  $u_{2d}(1^+) < -2.28$ ) generate a jump as well in the desired state  $x_d$  at the same time, illustrating Corollary 5.3.

### 5.5. Numerical Examples such that Proposition 3.11's conditions are satisfied at state jump, with $G = 0$

The above conditions that guarantee a negative Lyapunov function jump, may not match with the sufficient conditions of Proposition 3.11. The goal in this section is to find examples such that the sufficient conditions of Proposition 3.11 are satisfied, as an illustration. Let  $a = 0.5$ ,  $b = 1$ ,  $c = 1$ ,  $p = 1.96$  and  $k_2 = 0.96$ .

**Example 3.** Let  $x(0) = -15$ ,  $x_d(0) = 1$ ,  $u_{1d}(t) = 5 \sin 3t$  and  $u_{2d}$  is a discontinuous function as shown in Figure 4. Let us create a jump at  $t = 1s$  such that the conditions in Proposition 3.11 are satisfied. At  $t = 1^-$ , the values of the states are  $x(1^-) = -2.06$  and  $x_d(1^-) = 5.89$ . Let us consider the case when  $x_d$  is continuous at  $t = 1s$  (i.e.,  $x_d(1^+) = x_d(1^-) = x_d(1)$ ) and  $x$  is discontinuous at  $t = 1$  (i.e.,  $x(1^+) = \frac{k_2 x_d^+ - u_{2d}^+}{c + k_2}$  from Appendix F). Assume that at  $t = 1s$ :

$$0 \in \mathcal{K} \Leftrightarrow -k_2 x_d(1^+) + u_{2d}(1^+) \geq 0 \Leftrightarrow u_{2d}(1^+) \geq 5.65$$

and

$$x_d(1^+) p x(1^-) \geq x_d(1^-) p x(1^-) \Leftrightarrow x_d(1) p x(1^-) = x_d(1) p x(1^-)$$

and

$$x_d(t^+) \in \mathcal{T}_{\mathcal{K}}(\text{proj}_p[\mathcal{K}; x(t^-)]) \Leftrightarrow x_d(t^+) \in \mathcal{T}_{\mathcal{K}}(x(t^+)) \Leftrightarrow x_d(t^+) \in (\mathcal{N}_{\mathcal{K}}(x(t^+)))^\circ.$$

Knowing that  $-p(x(1^+) - x(1^-)) \in \mathcal{N}_{\mathcal{K}}(x(1^+))$ , then the last condition is written as

$$\begin{aligned} x_d(1^+) \in (\mathcal{N}_{\mathcal{K}}(x(1^+)))^\circ &\Rightarrow -x_d(1^+) p(x(1^+) - x(1^-)) \leq 0 \\ &\Rightarrow x_d(1^+) p \left( \frac{k_2 x_d^+ - u_{2d}^+}{c + k_2} - x(1^-) \right) \leq 0 \\ &\Rightarrow u_{2d}(1^+) \geq 1.5 \end{aligned}$$

Let  $u_{2d}(1^+) \geq 5.56$  such that the above conditions (i.e., the conditions of Proposition 3.11) are satisfied and  $-5 \leq u_{2d}(1^+) < 9.6$  (i.e.,  $x_d(1) \in \mathcal{K}_d$  and  $x(1^-) \notin \mathcal{K}$  at  $t = 1s$ ), thus choose  $u_{2d}(1^+) = 6$ . The numerical simulation is shown in the Figure 4 with a time step  $h = 0.01s$ .

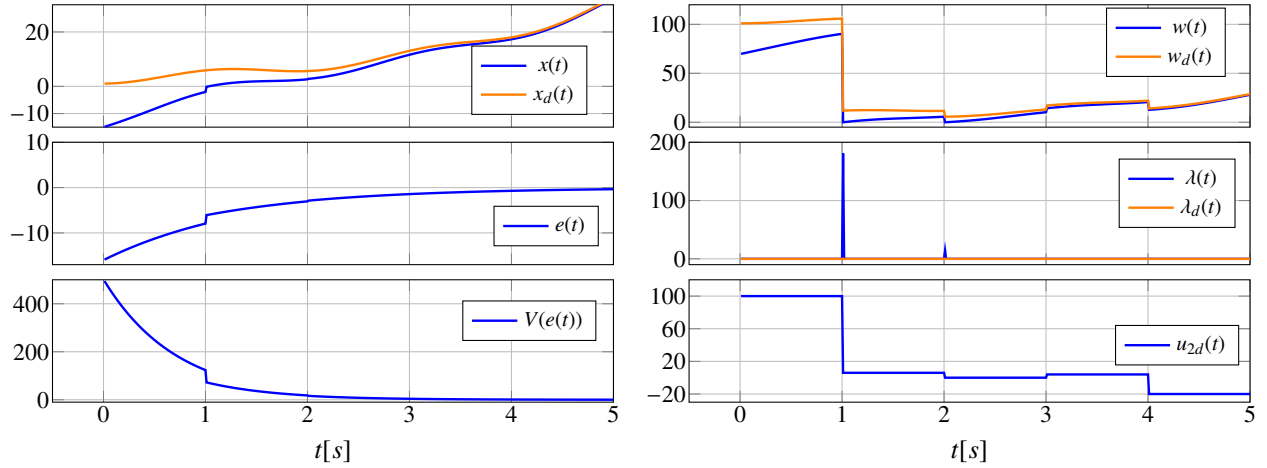


Figure 4: Numerical simulation of LCS where only  $x$  jumps

As shown in Figure 4, at the jump time  $t = 1s$ , the variation of the storage function  $\Delta V(e(1)) = V(e(1^+)) - V(e(1^-)) \leq 0$  which agrees with the result stated in Proposition 3.11.

## 6. Electrical Circuit with Parametric Uncertainties

Consider the circuit in Figure 5 with the states  $x_1$ : the charge on the capacitor  $C$  and  $x_2$ : the current passing through the inductor  $L$ . The dynamics of the electrical circuit in Figure 5 is given by the system in (50):

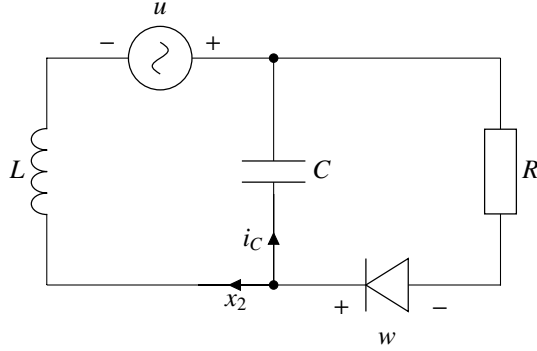


Figure 5: RLCD circuit with one ideal diode

$$\begin{cases} \dot{x}_1(t) = -x_2(t) + \lambda(t) \\ \dot{x}_2(t) = \frac{x_1(t)}{LC} + \frac{u(t)}{L} \\ 0 \leq \lambda(t) \perp w(t) = \frac{x_1(t)}{C} + R\lambda(t) \geq 0 \end{cases} \quad (50)$$

with  $L = 1$  H,  $C = 0.025$  F,  $R = 10$   $\Omega$ . Let us define the desired system as the LCS in (30) with the nominal matrices:

$$A_0 = \begin{pmatrix} 0 & -1 \\ \frac{1}{LC} & 0 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} \frac{1}{C} & 0 \end{pmatrix}, \quad D_0 = R, \quad E_0 = \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix}, \quad \text{and } F_0 = 0 \quad (51)$$

with  $x_d \in \mathbb{R}^2$ ,  $\lambda \in \mathbb{R}$  and  $u \in \mathbb{R}$ .

### 6.1. Passivity of the closed-loop system of LCS in (51) without uncertainties

Let  $u = K(x - x_d) + G(\lambda - \lambda_d) + u_d$  with  $K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}^\top$  and  $G = g_1$ . The closed-loop system is written in the form of the LCS in (12) as:

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & -1 \\ \frac{1}{LC} + \frac{k_1}{L} & \frac{k_2}{L} \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ \frac{g_1}{L} \end{pmatrix} \lambda(t) - \begin{pmatrix} 0 & 0 \\ \frac{k_1}{L} & \frac{k_2}{L} \end{pmatrix} x_d(t) - \begin{pmatrix} 0 \\ \frac{g_1}{L} \end{pmatrix} \lambda_d(t) + \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} u_d(t) \\ 0 \leq \lambda(t) \perp w(t) = \begin{pmatrix} \frac{1}{C} & 0 \end{pmatrix} x(t) + R\lambda(t) \geq 0 \end{cases} \quad (52)$$

where  $A = A_0$ ,  $B = B_0$ ,  $C = C_0$ ,  $D = D_0$  and  $E = E_0$ . In case when system (50) has no uncertainties, the control gains are calculated by solving the BMI (16), after being transformed into LMI according to Appendix C, for the closed-loop system in (52) with the minimum value of the control gain  $G$ . Then the solution is given by:

$$K = (75.0276 \quad -5.2589)^\top, \quad G = -3.9 \times 10^{-13} \quad \text{and} \quad P = \begin{pmatrix} 26.06268 & -0.99185 \\ -0.99185 & 0.25716 \end{pmatrix} \quad (53)$$

The control gain  $G$  is observed to have a negligible value, making the additional feedback from the complementarity variable  $\lambda$  useless. So, in the following, the value of  $G$  is considered null. In order to decrease the magnitude of the control gain  $K$ , the steps below are followed, knowing that  $K = NQ^{-1}$ :

- solve the LMI derived from the BMI in (16) (see Appendix C) with  $G = 0$  for the minimum Euclidean norm of the matrix  $N$  with fixed  $Q = P^{-1}$  in (53).
- Fix the value of  $N$  obtained from the previous step and solve the LMI again for the maximum trace of the matrix  $Q$ .
- repeat the first two steps until there is no change in the values of  $N$  and  $Q$  obtained.

This methodology gives the following solution:

$$K = \begin{pmatrix} 38.89363 & -2.145726 \end{pmatrix}^\top, \quad P = \begin{pmatrix} 10.911047 & -0.26867 \\ -0.26867 & 0.12605 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 3.31969 & -9.94634 \end{pmatrix}$$

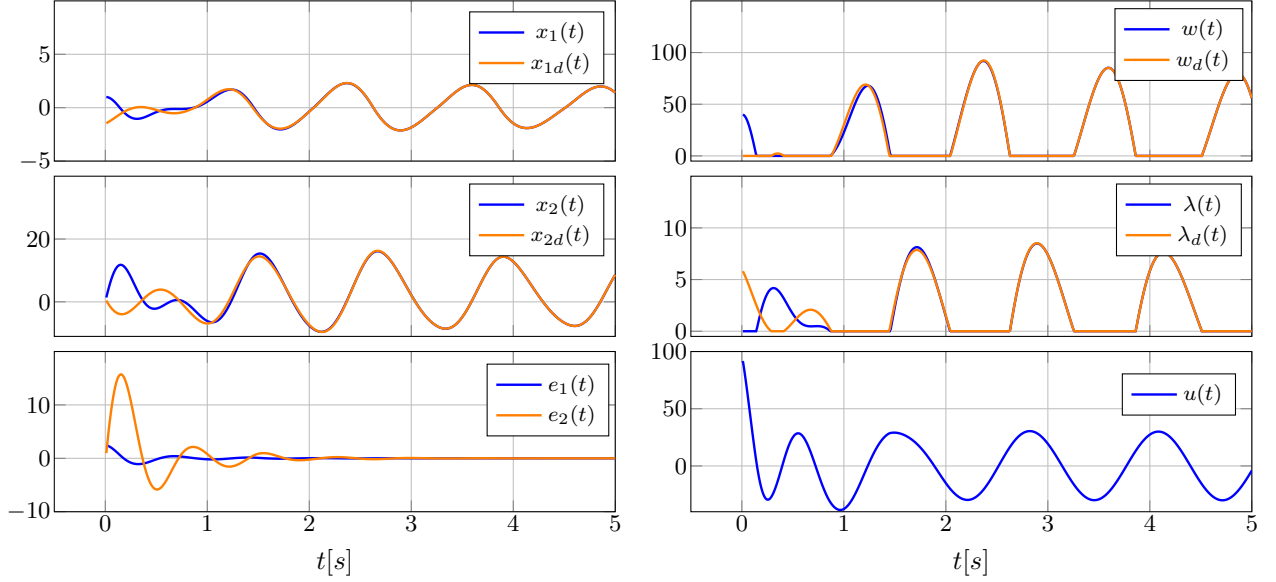


Figure 6: Numerical simulation showing the closed-loop system's trajectory  $x$ , the desired system's trajectory  $x_d$  and the error dynamics  $e = x - x_d$  of the LCS in (50), without uncertainties. The simulation also presents the controller  $u$  and the complementarity variables  $\lambda$  and  $w$ .

The plot in Figure 6 shows the numerical simulation for the desired system defined by (51), the closed-loop system in (52), the error  $e(t) = x(t) - x_d(t)$ , and the complementarity variables. The initial state vectors are  $x(0) = (1, 0)^\top$  and  $x_d(0) = (-1.5, 1)^\top$ , the time step  $h = 10^{-3}$  and the desired input is  $u_d(t) = 30 \sin 5t$ .

### 6.2. Passivity of the closed-loop system of (51) in the presence of uncertainties

The uncertainties are introduced to the passive elements  $\mathbf{R}$ ,  $\mathbf{L}$  and  $\mathbf{C}$ . The plots below show the numerical simulation for the desired and closed-loop systems. It shows the trajectories of the systems, the complementarity variables  $\lambda$  and  $w$ , and the controller  $u$  with different values of bounded uncertainties.

In Figure 7, the matrix  $D$  is perturbed by  $\Delta D$  due to disturbance in the resistor  $\mathbf{R}$ . The perturbation is selected such that  $\Delta D$  is bounded according to the value of  $\Lambda_D$ , with  $\Delta D = \Delta \mathbf{R}$ . For this purpose, the following LMI

$$\begin{pmatrix} (M_{0_{in}})_{11} & (M_{0_{in}})_{12} & 0 & 0 \\ (M_{0_{in}})_{21} & (M_{0_{in}})_{22} & I_m & I_m \\ 0 & I_m & \Lambda_2^{-1} & 0 \\ 0 & I_m & 0 & \tilde{\Lambda}_2^{-1} \end{pmatrix} > 0 \quad (54)$$

which is a special case of the LMI in (38) is solved with  $\tilde{\Lambda}_2^{-1} \leq 0.25$  and  $\Lambda_2^{-1} \geq 10$ ; the solution is given as follows:

$$K_0 = \begin{pmatrix} 71.7295 & -4.626 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 26.648 & -0.8914 \\ -0.8914 & 0.2572 \end{pmatrix}, \quad \Lambda_2^{-1} = 13.36, \quad \tilde{\Lambda}_2^{-1} = 0.15$$

The numerical simulations shown in Figure 7 and Figure 8 with the initial vectors  $x(0) = (1, 0)^\top$  and  $x_d(0) = (-1.5, 1)^\top$ , the time step  $h = 10^{-3}$  and the desired input  $u_d(t) = 30 \sin 5t$ .

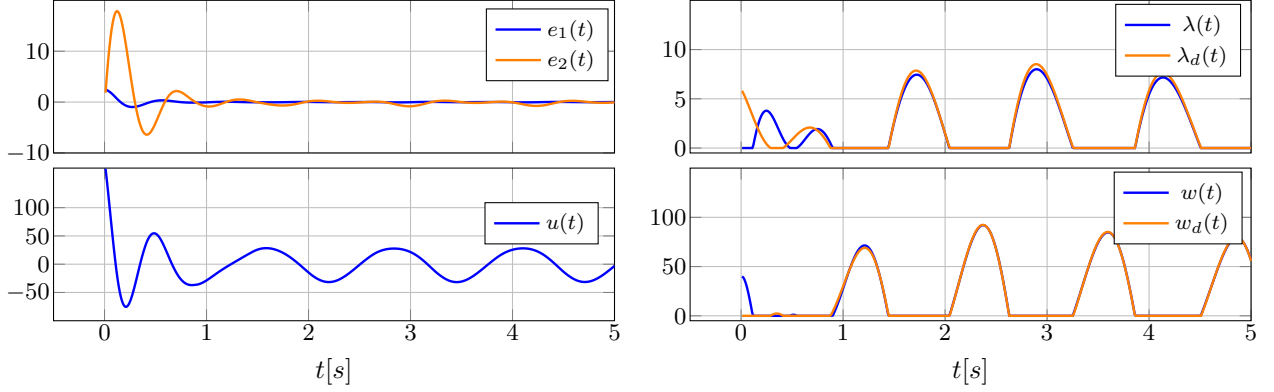


Figure 7: Numerical simulation showing the error dynamics  $e = x - x_d$  of the LCS in (50), with the uncertainty  $\Delta D = \Delta \mathbf{R} = \sin 5t$ . The simulation also presents the controller  $u$  and the complementarity variables  $\lambda$  and  $w$ .

Then  $\Delta D$  is selected such that  $|\Delta D| \leq 2.38$  as in Figure 7. It is observed that  $\|e_1\|_{[4,5],2} = 0.27$  and  $\|e_2\|_{[4,5],2} = 3.64$ . Also,  $\|e_1\|_{[4,5],\infty} = 0.054$ ,  $\|e_2\|_{[4,5],\infty} = 0.75$  and  $\|u\|_{[4,5],\infty} = 31.74$ . In order to increase the upper bound of  $\Delta D$ , the constraint on  $\tilde{\Lambda}_2^{-1}$  is modified such that  $\tilde{\Lambda}_2^{-1} \leq 0.12$ . The solutions of the LMI in (54) with a new constraint on  $\tilde{\Lambda}_2^{-1}$  are given as follows:

$$K_0 = (83.9 - 4.51), P_0 = \begin{pmatrix} 30.25 & -0.87 \\ -0.87 & 0.26 \end{pmatrix}, \Lambda_2^{-1} = 13.4, \tilde{\Lambda}_2^{-1} = 0.09.$$

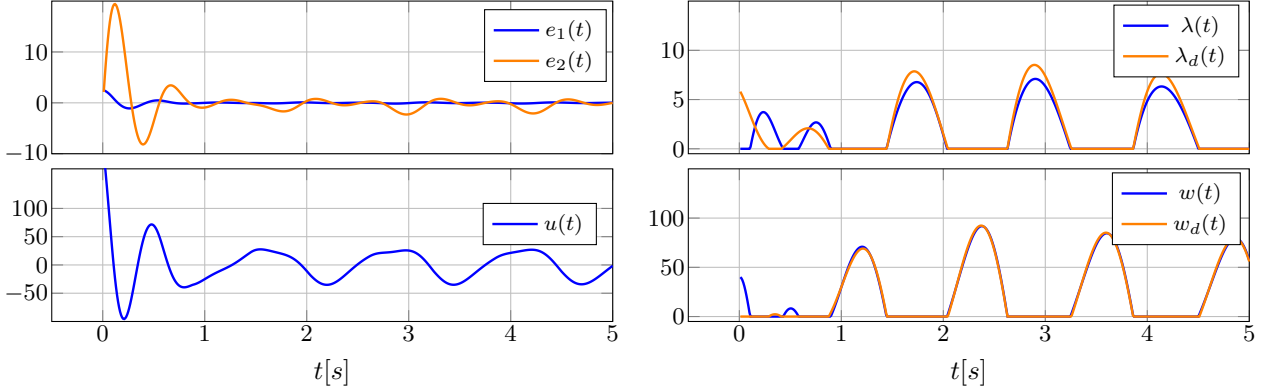


Figure 8: Numerical simulation showing the error dynamics  $e = x - x_d$  of the LCS in (50), with increasing value of uncertainty such that  $\Delta D = \Delta \mathbf{R} = 3 \sin 5t$ . The simulation also presents the controller  $u$  and the complementarity variables  $\lambda$  and  $w$ .

Then, it is possible to choose  $\Delta D$  such that  $|\Delta D| \leq 3.17$ . It is noticeable that a higher gain is obtained when the bound of the uncertainty is increased. As the value of  $\Delta D$  increases, it is noticed in Figure 8 that  $L_2$  and  $L_\infty$  norms of the error function increase such that  $\|e_1\|_{[4,5],2} = 0.67$ ,  $\|e_2\|_{[4,5],2} = 10$ ,  $\|e_1\|_{[4,5],\infty} = 0.13$  and  $\|e_2\|_{[4,5],\infty} = 2.08$  with  $\|u\|_{[2,3],\infty} = 34.2$ .

Let us introduce disturbances in the resistor, the inductor, and the capacitor as  $\Delta \mathbf{R}$ ,  $\Delta \mathbf{L}$ , and  $\Delta \mathbf{C}$  respectively. Thus, the uncertainties are present in the matrices  $A = A_0 + \Delta A$ ,  $C = C_0 + \Delta C$ ,  $D = D_0 + \Delta D$  and  $E = E_0 + \Delta E$ . The controller is designed by calculating  $K_0$  and  $G_0$  such that the quadruple  $(A_0 + E_0 K_0, B_0 + E_0 G_0, C_0, D_0)$  is strongly passive where

the control gains are given by the solution of the LMI in (38):

$$K_0 = (34.88 \quad -10.52), \quad G_0 = 2.686, \quad P_0 = \begin{pmatrix} 58.187 & -4.671 \\ -4.671 & 1.374 \end{pmatrix}, \quad \tilde{\Lambda}_1^{-1} = \begin{pmatrix} 18 & 0.195 \\ 0.195 & 7.74 \end{pmatrix}$$

$$\tilde{\Lambda}_2^{-1} = 0.101, \quad \Lambda_1^{-1} = \begin{pmatrix} 15.635 & 0.16 \\ 0.16 & 7.2 \end{pmatrix} \quad \text{and} \quad \Lambda_2^{-1} = 15.54$$

and the bounds of the uncertainties are calculated according to the values of  $\tilde{\Lambda}_1^{-1}$  and  $\tilde{\Lambda}_2^{-1}$ .

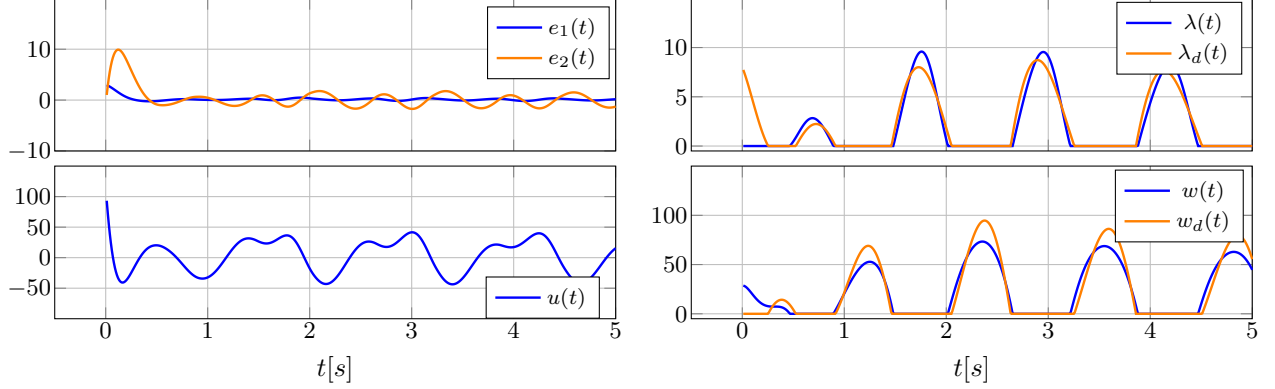


Figure 9: Numerical simulation showing the error dynamics  $e = x - x_d$  of the LCS in (50), with the parametric uncertainties  $\Delta \mathbf{R} = 2 \sin 5t$ ,  $\Delta \mathbf{L} = 0.02$ , and  $\Delta \mathbf{C} = 0.01 \cos 5t$ . The simulation also presents the controller  $u$  and the complementarity variables  $\lambda$  and  $w$ .

The plot in Figure 9 shows the numerical simulation of the error  $e(t) = x(t) - x_d(t)$ , and the complementarity variables. The initial state vectors are  $x(0) = (1, 0)^\top$  and  $x_d(0) = (-2, 1)^\top$ , the time step  $h = 10^{-3}$  and the desired input is  $u_d(t) = 30 \sin 5t$ . The values of  $L_2$  norm of the error function are  $\|e_1\|_{[4,5],2} = 1.65$  and  $\|e_2\|_{[4,5],2} = 10.7$  and that of  $L_\infty$  norm are  $\|e_1\|_{[4,5],\infty} = 0.34$  and  $\|e_2\|_{[4,5],\infty} = 1.6$ .

**Remark 6.1.** *More circuits examples where state jumps occur, and an academic example where a nonzero  $G$  matrix is needed, are presented in [60].*

## 7. Networks with Unilateral Interactions

Let us present an example of a network with unilateral interactions [61]. These systems can be treated as LCS [1], and by adding exogenous signals, they can be formulated as LCS with external inputs as in (2). They find applications in various fields such as sensor network, robotics, and game theory. Let us consider the following dynamics:

$$\begin{cases} \dot{x}_1(t) = \max(0, x_3(t) - x_1(t) - u_1(t)) + u_2(t) \\ \dot{x}_2(t) = x_1(t) - x_2(t) + \min(0, x_3(t) - x_2(t) + u_3(t)) + u_4(t) \\ \dot{x}_3(t) = \max(0, x_1(t) - x_3(t) - u_5(t)) + u_6(t) \end{cases} \quad (55)$$

which is written in the form of the LCS in (2) with [1, Equation (3.16)]:

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (56)$$

$$\text{and } F = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The desired system is defined by the LCS in (10) with the same plant matrices in (56).

**Remark 7.1.** In this analysis, the most general form of inputs is chosen (they appear in both the linear and the nonsmooth parts of the dynamics). It is out of the scope of this analysis to justify the feasibility of such inputs. The well-posedness of this LCS easily follows from [Appendix B](#) item 1.

### 7.1. First case: Complete controller

Let  $u = (u_1 \ u_2 \ u_3 \ u_4 \ u_5 \ u_6)^\top$ . The closed-loop system with  $u(t) = K[x(t) - x_d(t)] + u_d(t)$  where  $K = \begin{pmatrix} k_{11} & k_{21} & k_{31} & k_{41} & k_{51} & k_{61} \\ k_{12} & k_{22} & k_{32} & k_{42} & k_{52} & k_{62} \\ k_{13} & k_{23} & k_{33} & k_{43} & k_{53} & k_{63} \end{pmatrix}^\top$  is written in the form of the LCS in (12) with

$$\left\{ \begin{array}{l} \dot{x}(t) = \begin{pmatrix} k_{21} & k_{22} & k_{23} \\ 1+k_{41} & -1+k_{42} & k_{43} \\ k_{61} & k_{62} & k_{63} \end{pmatrix} x(t) + \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \lambda(t) - \begin{pmatrix} k_{21} & k_{22} & k_{23} \\ k_{41} & k_{42} & k_{43} \\ k_{61} & k_{62} & k_{63} \end{pmatrix} x_d(t) + \begin{pmatrix} u_{2d}(t) \\ u_{4d}(t) \\ u_{6d}(t) \end{pmatrix} \\ 0 \leq \lambda(t) \perp w(t) = \begin{pmatrix} 1+k_{11} & k_{12} & -1+k_{13} \\ k_{31} & -1+k_{32} & 1+k_{33} \\ -1+k_{51} & k_{52} & 1+k_{53} \end{pmatrix} x(t) + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \lambda(t) - \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{31} & k_{32} & k_{33} \\ k_{51} & k_{51} & k_{53} \end{pmatrix} x_d(t) + \begin{pmatrix} u_{1d}(t) \\ u_{3d}(t) \\ u_{5d}(t) \end{pmatrix} \geq 0 \end{array} \right. \quad (57)$$

The solution of the BMI in (15) which is transformed to LMI (see [Appendix C](#)) with  $\epsilon = 0.001$  for the closed-loop system in (57)

$$K = \begin{pmatrix} -0.43 & -0.497 & 0.003 & -0.97 & 0.996 & 0.002 \\ -0.003 & -0.017 & 0.431 & 0.5 & -0.003 & 0.009 \\ 0.996 & 0.0029 & -0.996 & -0.003 & -0.43 & -0.497 \end{pmatrix}^\top \quad \text{and} \quad P = \begin{pmatrix} 0.57 & -0.003 & -0.003 \\ -0.0032 & 0.57 & -0.003 \\ -0.003 & -0.003 & 0.57 \end{pmatrix}$$

Hence, the quadruple of the closed-loop system  $(A+EK, B, C+FK, D)$  is strictly state passive. Take  $x(0) = (-1, 1, 0)^\top$ ,  $x_d(0) = (1, 2, -1)^\top$  and  $u_d = (\sin t \ \cos t \ \sin 2t \ \cos 2t \ \sin 3t \ \cos 3t)^\top$ . The numerical simulation is shown below with time step  $h = 0.01s$ .

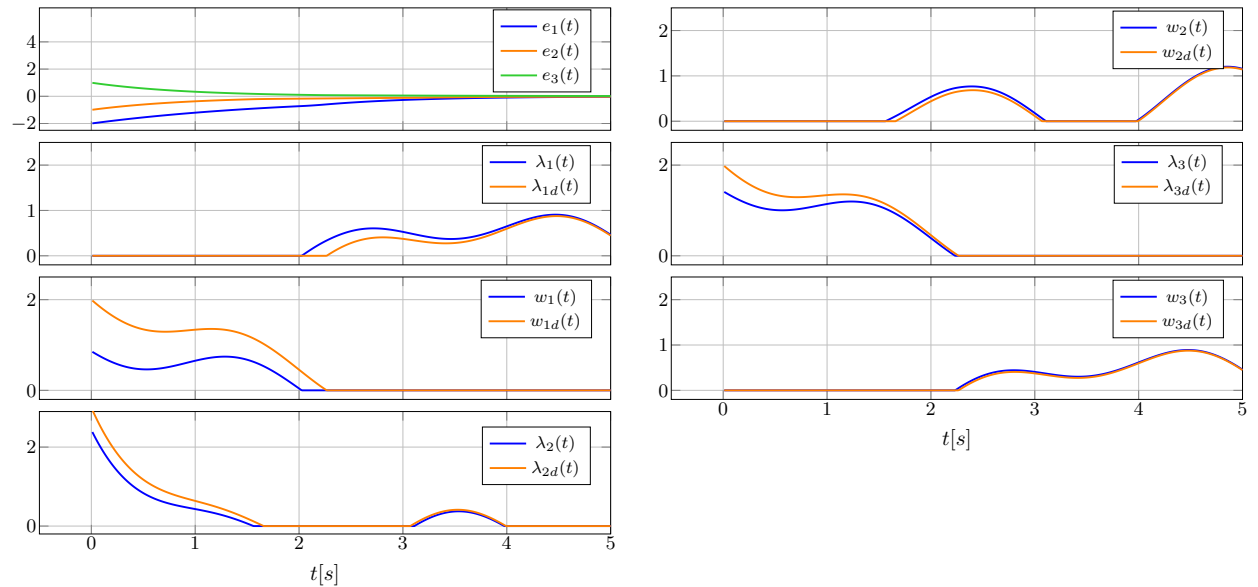


Figure 10: Numerical simulation showing the tracking error  $e = x - x_d$  of the Network dynamics in (55) controlled by the complete controller  $u$ . The simulation also shows the behaviour of the complementarity variables  $\lambda$  and  $w$ .

In this case, the closed-loop quadruple  $(A + EK, B, C + FK, D)$  is strongly passive. Thus, the system can be studied within the framework of section 4 which allows the consideration of uncertainties.

7.2. *Second case:*  $u = (u_2 \quad u_4 \quad u_6)^\top$

The solution is:

$$K = \begin{pmatrix} 0 & -0.87 & 0 & -0.16 & 0 & 0.26 \\ 0 & -0.42 & 0 & 0.327 & 0 & 0.233 \\ 0 & 0.18 & 0 & -0.0001 & 0 & -1.27 \end{pmatrix}^\top \quad \text{and} \quad P = \begin{pmatrix} 0.99 & -0.085 & -0.26 \\ -0.085 & 0.72 & -0.2 \\ -0.26 & -0.2 & 1.18 \end{pmatrix}$$

Hence, the closed-loop system  $(A + EK, B, C, D)$  is strictly state passive. Take  $x(0) = (-2, 1, 1)^\top$ ,  $x_d(0) = (2, -5, -1)^\top$  and  $u_d = (\sin t \quad \cos t \quad \sin 2t \quad \cos 2t \quad \sin 3t \quad \cos 3t)^\top$ . The numerical simulation is shown below with time step  $h = 0.01s$ .

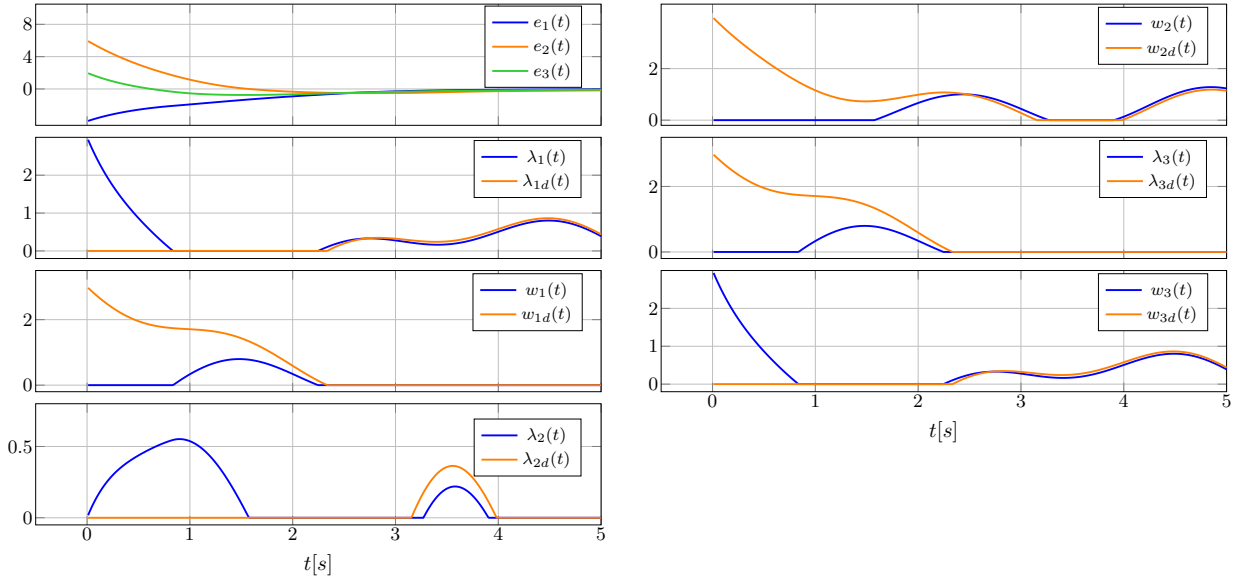


Figure 11: Numerical simulation showing the tracking error  $e = x - x_d$  of the Network dynamics in (55) with the active controller  $u = (u_2, u_4, u_6)^\top$  included in the linear part of the LCS in (55). The simulation also shows the behaviour of the complementarity variables  $\lambda$  and  $w$ .

It is important to note that, in this case, the closed-loop quadruple  $(A + EK, B, C, D)$  is strongly passive. This property allows us to analyse the problem of trajectory tracking in the presence of uncertainties (see section 4). However, whether or not considering such uncertainties in networks with unilateral interactions makes sense, is out of the scope of this article.

7.3. *Third case:*  $u = (u_1 \quad u_3 \quad u_5)^\top$

The BMI in (15) has no solution. Hence, the quadruple of the closed-loop system  $(A, B, C + FK, D)$  cannot be made strictly state passive with such a set of inputs and the chosen feedback, and the framework developed in the foregoing sections does not apply. It is noteworthy that the conclusions are still valid even if we consider  $G \neq 0$ .

## 8. Nonsmooth Mechanical Systems with Unilateral Springs

It is known that unilateral spring/dashpot contact/impact models, can be written in a complementarity framework [28, 1, 62, 20]. This class of contact/impact models significantly differs from unilateral constraints which yield complementarity constraints and impact models as in Remark 8.1.



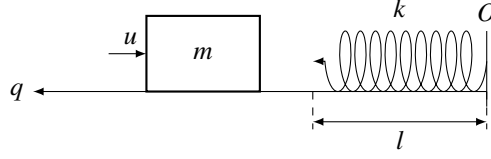


Figure 12: Mechanical system with unilateral spring.

Consider the mechanical system in Figure 12 which possesses the following dynamics:

$$\begin{cases} m\ddot{q}(t) = u(t) + \lambda(t) \\ 0 \leq \lambda(t) \perp w(t) = \lambda(t) + k(q(t) - l) \geq 0 \end{cases} \quad (58)$$

where  $q - l$  is the deformation of the spring,  $k > 0$  is the stiffness of the spring. The complementarity constraint is written between the contact force and the signed distance between the spring and the mass. Clearly the dynamics (58) fits with our general framework. Let  $x = (x_1, x_2)^\top = (q - l, \dot{q})^\top$ , (58) is rewritten equivalently as the LCS:

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} \lambda(t) + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u(t) \\ 0 \leq \lambda(t) \perp w(t) = \lambda(t) + kx_1(t) \geq 0 \end{cases} \quad (59)$$

Here  $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ ,  $B = E = \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}$ ,  $C = (k, 0)$ ,  $D = 1$ ,  $F = 0$ , and the system has AC solutions with uniqueness, see Appendix B item 1. The quadruple  $(A, B, C, D)$  can be made strongly passive with state feedback  $u = Kx = k_1x_1 + k_2x_2$ . This means that there exists a control gain  $K = (k_1, k_2)$  such that the quadruple  $(A + EK, B, C, D)$  of the closed-loop system is strongly passive. Equivalently, the BMI in (16) has a solution after being transformed into an LMI, as detailed in Appendix C and the solution is given by:

$$P = \begin{pmatrix} 7.863 & 0.538 \\ 0.538 & 0.337 \end{pmatrix} \quad \text{and} \quad K = (-21.38 \quad -2.148)$$

with  $m = 1\text{kg}$  and  $k = 5\text{N/m}$ . Thus, the results in section 4.2 apply to this example. As the stiffness of the spring  $k$  increases, the control gain gives very large numerical values like  $k_1 = -1.2 \times 10^7$  for  $k = 100\text{N/m}$ . This is explained analytically by explicitly writing the matrix inequality in (16) and verifying the positive definiteness conditions. Following the matrix  $M$  in (15), the matrix inequality is written as:

$$\begin{pmatrix} -\frac{2p_{12}k_1}{m} & -p_{11} - \frac{p_{12}k_2}{m} - \frac{p_{22}k_1}{m} & -\frac{p_{12}}{m} + k \\ -p_{11} - \frac{p_{12}k_2}{m} - \frac{p_{22}k_1}{m} & -2p_{12} - \frac{2p_{22}k_2}{m} & -\frac{p_{22}}{m} \\ -\frac{p_{12}}{m} + k & -\frac{p_{22}}{m} & \frac{2}{m} \end{pmatrix} > 0$$

with  $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} > 0$ . One of the necessary conditions to be satisfied is:

$$2 \times \begin{pmatrix} -\frac{2p_{12}k_1}{m} & -p_{11} - \frac{p_{12}k_2}{m} - \frac{p_{22}k_1}{m} \\ -p_{11} - \frac{p_{12}k_2}{m} - \frac{p_{22}k_1}{m} & -2p_{12} - \frac{2p_{22}k_2}{m} \end{pmatrix} - \begin{pmatrix} \left(-\frac{2p_{12}}{m} + k\right)^2 & 0 \\ 0 & 0 \end{pmatrix} > 0$$

Given that  $-\frac{p_{12}k_1}{m} > 0$  which is a necessary condition for positive definiteness, thus for strong passivity of the quadruple  $(A + EK, B, C, D)$ , it is required to prove that:  $-\frac{4p_{12}k_1}{m} - \left(-\frac{2p_{12}}{m} + k\right)^2 > 0$ . This condition implies that if  $k \rightarrow +\infty$ , then  $k_1 \rightarrow +\infty$ . This conclusion is similar to the result in [28, section 7.6] [63] which deals with quadratic stabilization of the same system. It demonstrates the limitation of the stabilization approach which certainly provides conservative sufficient conditions.

It is seen that in a certain sense this class of mechanical systems (with  $D > 0$  and  $F = 0$ ) lies in-between the unilaterally constrained mechanical systems (with always  $D = 0$  and  $F = 0$ ), and LCS (with possibly both  $D$  and  $F$  nonzero).

**Remark 8.1** (Linear Complementarity Lagrangian Systems). *Linear Lagrangian systems with unilateral constraints have the nonlinear nonsmooth dynamics:*

$$\begin{cases} M\ddot{q}(t) + R\dot{q}(t) + Sq(t) = u(t) + C_q^\top \lambda(t) \\ 0 \leq \lambda(t) \perp w(t) = C_q q(t) + F_q \geq 0 \\ C_q \dot{q}(t^+) = -e_n C_q \dot{q}(t) \text{ if } C_q q(t) = 0 \text{ and } C_q \dot{q}(t^-) \leq 0, \end{cases} \quad (60)$$

where  $q(t) \in \mathbf{R}^n$ ,  $M = M^\top > 0$ , the restitution coefficient  $e_n \in [0, 1]$ ,  $R \succcurlyeq 0$  is a Rayleigh dissipation matrix [33, Definition 6.12],  $S \succcurlyeq 0$  is a stiffness matrix, and it is assumed that  $\dot{q}$  has left and right limits. For simplicity we also assume that there is a unique unilateral constraint, i.e.,  $C_q \in \mathbf{R}^{1 \times n}$ ,  $\lambda \in \mathbf{R}$  and  $F_q \in \mathbf{R}$  is a constant (the system's admissible domain in the configuration space, is a convex polyhedral set, assumed to be nonempty). Denoting  $x = (q^\top, \dot{q}^\top)^\top$ ,  $x_1 = q$ ,  $x_2 = \dot{q}$ , we obtain:

$$\begin{cases} \dot{x}(t) = \begin{pmatrix} 0 & I_n \\ -M^{-1}S & -M^{-1}R \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ M^{-1}C_q^\top \end{pmatrix} \lambda(t) \\ 0 \leq \lambda(t) \perp w(t) = Cx(t) + F_q = (C_q, 0)x(t) + F_q \geq 0 \\ C\dot{x}(t^+) = -e_n C\dot{x}(t^-) \text{ if } Cx(t) = 0 \text{ and } C\dot{x}(t^-) \leq 0. \end{cases} \quad (61)$$

Therefore  $A = \begin{pmatrix} 0 & I_n \\ -M^{-1}S & -M^{-1}R \end{pmatrix}$ ,  $B = \begin{pmatrix} 0 \\ M^{-1}C_q^\top \end{pmatrix}$ ,  $E = \begin{pmatrix} 0 \\ M^{-1} \end{pmatrix}$ ,  $F = 0$ ,  $C = (C_q, 0)$ ,  $D = 0$ . Since  $D = 0$  and no control acts in the complementarity constraint, passivity implies that  $PB = C^\top$  whatever the controller  $u(x) = Kx$ . It can be checked that this is not possible with  $P > 0$ . Adding a multiplier feedback  $u(x, \lambda) = Kx + G\lambda$  does not change the conclusion. Fundamentally, systems as (61) have a relative degree 2 when  $\lambda$  is seen as the input and  $w$  is seen as the output [28] (hence hampering passivity [33]), while systems as in (59) have a relative degree 0 between the same input/output variables.

## 9. Conclusions

This article focuses on the trajectory tracking for linear complementarity systems. Passification by feedback and maximal monotonicity are central tools for stability analysis. Systems with and without state jumps, with and without parameter uncertainties, are analysed. Numerical simulations on circuits, networks with unilateral interactions and mechanical systems with unilateral springs illustrate the theoretical developments. Future research should focus on tracking with output feedback or state observation.

### Appendix A. Linear Complementarity Problems

**Definition A.1.** A matrix  $M \in \mathbf{R}^{n \times n}$  is said to be *P-matrix* if and only if all its principal minors are positive.

The complementarity between two vectors  $\lambda, w \in \mathbf{R}^n$  is denoted as  $0 \leq \lambda \perp w \geq 0$  where the notation  $\lambda \perp w$  means that  $\lambda^\top w = 0$ , and nonnegativeness is understood componentwise. When  $w = M\lambda + q$ , this gives rise to a Linear Complementarity Problem (LCP), denoted as  $\text{LCP}(M, q)$ .

**Theorem A.2.** [32] A matrix  $M \in \mathbf{R}^{n \times n}$  is a *P-matrix* if and only if  $\text{LCP}(M, q)$  has a unique solution for all vectors  $q \in \mathbf{R}^n$ .

### Appendix B. LCS's Well-posedness

Let us provide a summary of well-posedness results in [18, 22, 35, 36, 37, 38, 23, 21]. Most of them apply to larger classes of nonsmooth systems than LCS. Let us remind that a right-continuous function  $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$  of locally bounded variation (RCLBV), has a countable set of discontinuities on  $I$  (allowing for left accumulations of jump instants), and possesses a differential measure  $df$ , see [42]. The conditions associated with the results recalled below, are sufficient. However they are well fitted with the conditions imposed in this article for stability purposes, i.e., passivity.

1. (Well-posedness based on reduction to an ODE)

- Assume that  $D$  is a P-matrix. Then from the fundamental theorem of Complementarity Theory [32], it follows that  $\lambda$  is a piecewise continuous, Lipschitz single-valued function of  $x$  and  $u$ . Hence the LCS in (2) is the ODE:

$$\dot{x}(t) = Ax(t) + B\lambda(x(t), u(t)) + Eu(t).$$

Provided  $u(\cdot)$  satisfies Lebesgue integrability conditions, an AC solution exists on  $\mathbf{R}_+$  for any bounded initial condition, with uniqueness and continuous dependence on initial data [64, 65].

- Assume that  $D > 0$  (not necessarily symmetric), then using the formalism in (3), the result can be extended to systems with operators  $(D + \mathcal{M}_t)^{-1}(\cdot)$  where  $\mathcal{M}_t(\cdot)$  is a maximal monotone operator for each fixed  $t$ . Using [23, Proposition 1] or [66], the mapping  $(D + \mathcal{M}_t)^{-1}(\cdot)$  is then single-valued, defined everywhere, and Lipschitz continuous with constant  $\frac{1}{\lambda_{\min}(D + D^\top)}$  for each fixed  $t$ . Therefore the mapping  $H(t, x)$  in (3) is single-valued Lipschitz continuous in  $x$  and classical results on ODE's well-posedness apply.

2. (Well-posedness based on passivity and relaxations of passivity) Let us summarize the results presented in [18, 21, 35, 39, 23, 36], which deal with the case with external inputs.

(a) [18, Theorem 7.5] Assume that:

- i)  $(B^\top, D + D^\top)^\top$  has full column rank,
- ii)  $(A, B, C, D)$  is passive with  $P = P^\top > 0$ , and minimal,
- iii)  $u(\cdot)$  is piecewise continuous with rational Laplace transform.

Then for any  $x(0) = x_0$ , there exists a unique global solution to (2) such that  $(\lambda, x, w) \in \mathcal{L}_\delta^2(\mathbf{R}_+; \mathbf{R}^{m+n+m})$ , the space of Schwartz' distributions, with regular parts in  $\mathcal{L}_{loc}^2 \subseteq \mathcal{L}_{loc}^1$ , and atomic part with isolated atoms (instants of state jumps) that is represented by sums of Dirac measures (higher degree distributions do not occur).

(b) [35, Theorems 7,8, 9] Assume that:

- i)  $(A, B, C, D)$  is passive with  $P = P^\top > 0$ ,
- ii)  $Fu(t) \in Q_D^* + \text{Im}(C)$  for all  $t \geq [0, T)$ ,  $T > 0$  (see section 2.1.2 for the definition of  $Q_D$  and of its dual cone),
- iii)  $u(\cdot)$  is a Bohl function (*i.e.*,  $u(t) = M \exp(Nt)R$  for some constant matrices  $M, N, R$ ).

Then  $x(0^+)$  is calculated as in (7), there exists a unique AC solution  $x : (0, T) \rightarrow \mathbf{R}^n$ , and  $\lambda(\cdot)$  is locally Lebesgue integrable. The solution is a forward solution, *i.e.*, it is the concatenation of Bohl functions defined between state jumps. Left accumulations of state jumps are allowed.

(c) [21, Corollary 2] [1, Corollary 5.9] Assume that:

- i)  $D \succcurlyeq 0$ ,  $\text{Ker}(D + D^\top) \subseteq \text{Ker}(PB - C^\top)$  for some  $P = P^\top > 0$ ,
- ii)  $\text{rint}(\mathbf{R}_+^m - Fu(t)) \subseteq \text{rint}(\text{Im}(\partial\sigma_{\mathbf{R}_+^m - Fu(t)} + D))$ ,
- iii)  $D\mathbf{R}_+^m \subseteq \text{Im}(C)$ ,
- iv)  $\text{Im}(C) - \mathbf{R}_+^m = \mathbf{R}^m$ ,
- v) for each  $x \in \mathbf{R}^n$  and each  $t \geq 0$ , if the set  $\Lambda = \{\lambda \in \mathbf{R}_+^m \mid v = Cx + D\lambda + Fu(t) \geq 0, \lambda^\top v = 0\}$  has a nonzero element, and  $\Lambda \cap \text{Im}(D + D^\top) \neq \emptyset$ .

Then if  $Fu(\cdot)$  is AC (resp. RCLBV), there exists a unique AC (resp. RCLBV) solution to (2) for any  $x(0)$  such that  $Cx(0) \in \text{Im}(\partial\sigma_{\mathbf{R}_+^m - Fu(0)} + D)$ .

(d) [36, Theorems 11, 26] Let:

- i)  $(A, B, C, D)$  be passive,
- ii)  $\text{Im}(C) \cap \text{rint}(\text{Im}(D + \mathcal{N}_{\mathbf{R}_+^m}) - Fu(t)) \neq \emptyset$ ,
- iii)  $Fu(\cdot)$  be AC.

Then there exists a unique AC solution for any  $x(0) = x_0 \in \text{Dom}(H(0, \cdot))$ .

(e) [23, Theorem 1, Corollaries 2, 3] Assume that:

- i)  $D = \begin{pmatrix} 0 & 0 \\ 0 & D_2 \end{pmatrix} \in \mathbf{R}^{m_2 \times m_2}$ ,  $m_2 \leq m$
- ii)  $F = 0$ ,

- iii)  $u(\cdot)$  is continuous with  $\dot{u} \in \mathcal{L}_{loc}^1(\mathbf{R}_+; \mathbf{R}^p)$ ,
- iv) there exists a full-rank  $R = R^\top \in \mathbf{R}^{n \times n}$  such that  $R^2 B_1 = C_1^\top$ ,  $B = (B_1 \ B_2)$ ,  $C = (C_1^\top \ C_2^\top)^\top$ ,
- v) there exists  $w_0 \in \mathbf{R}^{m_1}$  at which the operator  $w \mapsto \partial \Psi_{\mathbf{R}_+^{m_1}}(C_1 R^{-1} w)$  is continuous,
- vi) the operator  $w \mapsto RB_2(D_2 + \partial \Psi_{\mathbf{R}_+^{m_2}})^{-1}(-C_2 R^{-1} w)$  is well-defined, single-valued and Lipschitz continuous.

Then there exists a unique continuous right-differentiable solution for any  $x(0) \in \text{Dom}(\partial \Psi_{\mathbf{R}_+^{m_1}}(C_1 R^{-1} \cdot)) = \{w \in \mathbf{R}^n \mid C_1 R^{-1} w \geq 0\}$ , with  $\dot{x} \in \mathcal{L}^\infty(\mathbf{R}_+; \mathbf{R}^n)$ . The result holds if  $m_2 = 0$  and i-v) are verified, or if  $m_2 = m$  and vi) is verified (then item 1 applies). Condition vi) holds if  $D_2$  is a P-matrix [23, Proposition 1, Corollary 1].

(f) [50, Theorems 5.3, 5.4] and [56, Theorems 1, 2] deal with DIs of the form:

$$\dot{x}(t) \in g(t, x(t)) - B(D + F_{t,x}^{-1})^{-1}(Cx(t)),$$

where  $F_{t,x} : \mathbf{R}^m \rightrightarrows \mathbf{R}^m$  is maximal monotone for each  $(t, x)$ . For brevity we focus on  $F_{t,x}(\cdot) = \mathcal{N}_{K(t,x)}(\cdot)$  [56]. Assume that:

- i)  $D \succcurlyeq 0$ ,  $\text{Im}(D) \subset \text{Im}(C)$ ,  $\text{Ker}(D + D^\top) \subseteq \text{Ker}(PB - C^\top)$  for some  $P = P^\top > 0$ ,
- ii)  $K(t, x)$  has closed convex values,  $K(t, x) \cap \text{Im}(C) \neq \emptyset$ ,  $d_H(K(t, x) \cap \text{Im}(C), K(s, y) \cap \text{Im}(C)) \leq k_1|t - s| + l_2\|x - y\|$  for all  $t, s, x, y$ ,  $l_1 \geq 0$ ,  $0 \leq l_2 \leq \frac{\lambda_2}{\|C\|}$ ,  $\lambda_2$  the smallest positive eigenvalue of  $CC^\top$ ,
- iii)  $(\mathcal{N}_{K(t,x)}^{-1} + D)^{-1}(Cx) \neq \emptyset \Rightarrow \text{Im}(D + D^\top) \cap (\mathcal{N}_{K(t,x)}^{-1} + D)^{-1}(Cx) \neq \emptyset$ ,
- iv)  $\text{Im}(C) \cap \text{rint}(\text{Im}(\mathcal{N}_{K(t,x)}^{-1} + D)) \neq \emptyset$ ,
- v)  $g(\cdot, \cdot)$  is continuous in  $t$  and Lipschitz continuous in  $x$ .

Then existence and uniqueness of Lipschitz continuous solutions is guaranteed for each initial condition such that  $(\mathcal{N}_{K(0,x_0)}^{-1} + D)^{-1}(Cx_0) \neq \emptyset$ . Moreover,  $\|\dot{x}(t)\| \leq \alpha + \beta\|x(0)\|$  for some  $\alpha > 0$  and  $\beta > 0$ .

3. (Well-posedness by transformation to a sweeping process [37, 1, 40]) Assume that  $D = 0$ , and that:

- i)  $PB = C^\top$  for some  $P = P^\top > 0$ ,
- ii)  $u \in \mathcal{L}_{loc}^1(\mathbf{R}_+; \mathbf{R}^p)$ ,
- iii)  $\text{Im}(C) - \mathbf{R}_+^m = \mathbf{R}^m$ .

Then the LCS in (2) can be rewritten equivalently as a FOSwP (see Definition 1.4, and [37, 1] for details on the transformation, and [42, 1] for details on FOSwP). The basic condition i) is implied by, but does not imply, the passivity of  $(A, B, C, 0)$ . Depending on the signal  $u(\cdot)$  being AC (resp. RCLBV), the solution is shown to be AC (resp. RCLBV), defined on  $\mathbf{R}_+$ , and uniqueness holds. For AC to hold it is needed that  $x(0) \in S(0)$  (no initial jump).

The various assumptions made in items 2 and 3 (which may be thought as constraint qualifications in many instances, a concept which is familiar in Optimization) have different meanings:

- property of the plant's model: item 2 (a) ii), (b) i), (c) i), (d) i), (e) i) ii) iv), item 3 i),
- constraints qualifications: item 2 (a) i), (b) ii), (c) ii), iii), iv), v), (e) v), (f) ii), iii), iv), item 3 iii),
- exogenous signals properties: item 2 (a) iii), (b) iii), conclusion of item 2 (c), item 2 (e) iii), (f) v), conclusion of item 3.

The condition in item 2 (b) ii) stems from a fundamental result in Complementarity Theory [32, Theorem 3.8.6, Corollary 3.8.10], which guarantees the existence of solutions to the LCP in (2). This assumption thus secures that there exists a bounded multiplier  $\lambda(t)$  which solves the LCP (notice that at the end time  $T$ , there may be a state jump if this assumption does not hold). This is an assumption quite similar to item 2 (c) ii), and item 2 (d) ii). Roughly speaking, they all guarantee that the various operators have nonempty domains so that the problem makes sense. They stem from [31, Theorem 12.43] to guarantee maximal monotonicity (see [23, section 3.2.1]). Such CQ may not be easy to check and may require some developments [67]. Condition in item 2 (c) iii) and (f) i) ( $\text{Im}(D) \subset \text{Im}(C)$ ) are used to compute Vladimirov's pseudo distance (see [55] for the definition, which is beyond the scope of this article) between time-varying and/or state-dependent sets.

### Appendix C. Transformation of BMI to an LMI

The inequalities in (16) and (31) are bilinear matrix inequalities (BMI). In order to solve these inequalities, the BMI must be reformulated to an LMI (following a classical technique, *e.g.*, [68, Example 5.5 p.136]). First, the left-hand and the right-hand sides of (16) are multiplied by  $\begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}$ , where  $Q \triangleq P^{-1}$ , and let  $N \triangleq KQ$ . It follows that

$$\begin{pmatrix} QA^\top + AQ + N^\top E^\top + EN + \epsilon Q & B + EG - QC^\top - N^\top F^\top \\ B^\top + G^\top E^\top - CQ - FN & -D - FG - (D + FG)^\top \end{pmatrix} \preceq 0 \quad (\text{C.1})$$

This LMI is a feasible problem and it is solved in the new variables  $Q = Q^\top > 0$ ,  $G$ , and  $N$  where  $K = NQ^{-1}$ .

### Appendix D. Uniform and Ultimate Boundedness

The following Lyapunov-like theorem is for showing uniform boundedness and ultimate boundedness.

**Theorem D.1.** [57, Theorem 4.18] *Let  $D \subset \mathbf{R}^n$  be a domain that contains the origin and  $V : [0, \infty) \times D \rightarrow \mathbf{R}$  be a continuously differentiable function such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad (\text{D.1})$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \frac{\partial x}{\partial t} \leq -W_3(x), \quad \forall \|x\| \geq \mu > 0 \quad (\text{D.2})$$

for all  $t \geq 0$ , for all  $x \in D$ , where  $\alpha_1$  and  $\alpha_2$  are class  $\mathcal{K}$  functions and  $W_3(\cdot)$  is a continuous positive definite function. Take  $r > 0$  such that  $B_r \subset D$  and suppose that,

$$\mu < \alpha_2^{-1}(\alpha_1(r)) \quad (\text{D.3})$$

Then, there exists a class  $\mathcal{KL}$  function  $\beta$  and for every initial state  $x(t_0)$ , satisfying  $\|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r))$ , there exists  $T > 0$  (dependent on  $x(t_0)$  and  $\mu$ ) such that the solution satisfies

$$\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(r)), \quad \forall t \geq t_0 + T \quad (\text{D.4})$$

Moreover, if  $D = \mathbf{R}^n$  and  $\alpha_1$  belongs to class  $\mathcal{K}_\infty$ , then (D.4) holds for any initial state  $x(t_0)$ , with no restrictions on how large  $\mu$  is.

### Appendix E. Conditions for Positive Semidefiniteness

**Lemma E.1.** [54, Proposition 8.2.4] *Consider the matrix  $M = \begin{pmatrix} Q & S \\ S^\top & R \end{pmatrix}$ . Assume that  $Q = Q^\top$  and  $R = R^\top$ . Then  $M \succeq 0$  if and only if:*

1.  $R \succeq 0$ ,
2.  $SR^\dagger R = S \Leftrightarrow \text{Im}(S^\top) \subseteq \text{Im}(R)$ ,
3.  $Q \succeq SR^\dagger S^\top$ .

The equivalence in item 2 follows from [54, Remark after Fact 6.4.5]. The next lemma provides some necessary conditions for item 3 to hold.

**Lemma E.2.** *Assume that  $Q = Q^\top \succeq 0$  and  $R = R^\top \succeq 0$ . Then  $Q \succeq SR^\dagger S^\top$ :*

1.  $\Rightarrow \text{Im}(Q) \supseteq \text{Im}(SR^\dagger S^\top) \Leftrightarrow \text{Im}(Q) \supseteq \text{Im}(S)$ .
2.  $\Rightarrow \lambda_i(Q) \geq \lambda_i(SR^\dagger S^\top)$  for all  $i \in \{1, \dots, n\}$ , and  $\text{tr}(Q) \geq \text{tr}(SR^\dagger S^\top)$  and  $\det(Q) \geq \det(SR^\dagger S^\top) \geq 0$ .

- Proof.* 1. From [59, Fact 7.17.24], there exists matrices  $L$  such that  $Q = LL^\top$  and  $N$  such that  $SR^\dagger S^\top = NN^\top$ , since  $R^\dagger$  is symmetric [54, Proposition 6.1.6]. From [59, Theorem 10.6.1] it follows that  $LL^\top \succcurlyeq NN^\top \Rightarrow \text{Im}(N) \subseteq \text{Im}(L)$ . Using [54, Theorem 2.4.3] we have that  $\text{Im}(N) = \text{Im}(NN^\top)$  and  $\text{Im}(L) = \text{Im}(LL^\top)$ . The first implication is proved. Using [54, Lemma 2.4.1] we have  $\text{Im}(SR^\dagger S^\top) \subseteq \text{Im}(S)$ . The second implication is proved.
2. From [59, Theorem 10.4.9, Corollary 10.4.10]. □

Notice that  $\lambda_i(SR^\dagger S^\top) = \lambda_i(NN^\top) = \sigma_i^2(N)$  (because  $R^\dagger \succcurlyeq 0$ ), and similarly  $\lambda_i(Q) = \sigma_i^2(L)$ . The matrix inequality in Lemma E.1 item 3 can therefore be transformed into a singular values inequality, provided the matrices  $N$  and  $L$  are computed (Cholesky decomposition [59, Fact 10.10.42], or the Gram matrix decomposition [59, Fact 10.10.41] can be chosen).

An interesting result follows from [59, Theorem 10.6.2] and the same reasoning as in the proof of Lemma E.2 item 1: there exists  $\alpha > 0$  such that  $\alpha Q \succcurlyeq SR^\dagger S^\top \iff \text{Im}(Q) \supseteq \text{Im}(SR^\dagger S^\top)$ . A first step may be to check the ranges inclusion, then to calculate an  $\alpha > 0$ , following the reasoning in the proof of [59, Theorem 10.6.2]: compute the matrices  $N$  and  $L$ , compute the matrix  $T$  such that  $N = LT$ , and compute  $\alpha = \lambda_{\max}(TT^\top)$ . If  $\alpha \leq 1$  then Lemma E.1 item 3 holds true. Following the proof of [59, Theorem 10.6.2],  $T$  can be calculated.

Let us provide an excerpt of [69, Theorem 2.11], and a corollary of it. Let us recall that for a given  $M \in \mathbf{R}^{n \times n}$ ,  $\|M\|_{2,2}$  is the induced matricial norm such that  $\|M\|_{2,2} = \sigma_{\max}(M)$  (the largest singular value) [54, Proposition 9.4.9], it is a submultiplicative norm [54, Corollary 9.4.12].

**Theorem E.3.** [69] *Let  $M \in \mathbf{R}^{n \times n}$  be a positive definite matrix. Then every matrix*

$$A \in \{A \in \mathbf{R}^{n \times n} \mid \left\| \left( \frac{M+M^\top}{2} \right)^{-1} \right\|_{2,2} \|M - A\|_{2,2} < 1\}$$

*is positive definite.*

**Corollary E.4.** [70] *Let  $D = P + N$ , where  $D, P$  and  $N$  are  $n \times n$  real matrices, and  $P > 0$ , not necessarily symmetric. If*

$$\|N\|_{2,2} < \frac{1}{\left\| \left( \frac{P+P^\top}{2} \right)^{-1} \right\|_{2,2}} \tag{E.1}$$

*then  $D > 0$ .*

## Appendix F. Optimization problem in (7) for the example in section 5

In this section, the optimization problem in (7) is solved using KKT conditions. In this case, KKT conditions are necessary and sufficient conditions for the optimal value because the following optimization problem

$$x(t^+) = \operatorname{argmin}_{x \in \mathcal{K}} \frac{1}{2} (x - x(t^-))^\top P (x - x(t^-)).$$

is convex knowing that  $P = P^\top > 0$ ,  $x(t)$  and  $x(t^-) \in \mathbf{R}$ . The optimization problem is solved for the two systems.

### Appendix F.1. Desired system in (43)

It is required to find  $x_d(t^+)$  which is the optimal value to minimize the following

$$\begin{aligned} f_d(x_d) &= \min \frac{1}{2} (x_d - x_d^-)^2 \\ \text{s.t. } x_d &\in \mathcal{K}_d \end{aligned}$$

where  $P = 1$  and  $\mathcal{K}_d = \{x_d \in \mathbf{R} \mid cx_d + u_{2d}(t^+) \geq 0\} = \{x_d \in \mathbf{R} \mid -cx_d - u_{2d}^+ \leq 0\}$ . Let us write the Lagrangian function of the minimization problem

$$\mathcal{L}(x_d, \mu) = \frac{1}{2} (x_d - x_d^-)^2 - \mu (-cx_d - u_{2d}^+)$$

where  $\mu$  is the lagrangian multiplier. Let us state KKT conditions to be satisfied:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x_d} = 0 \Leftrightarrow x_d - x_d^- + c\mu = 0 \Leftrightarrow x_d = x_d^- - c\mu \\ \mu(-cx_d - u_{2d}^+) = 0 \Leftrightarrow \mu(c^2\mu - cx_d^- - u_{2d}^+) = 0 \\ \mu \leq 0 \\ -cx_d - u_{2d}^+ \leq 0 \Leftrightarrow c^2\mu - cx_d^- - u_{2d}^+ \leq 0 \end{cases}$$

KKT conditions are written in the form of a LCP:

$$0 \leq -\mu \perp -c^2\mu + cx_d^- + u_{2d}^+ \geq 0$$

The LCP( $c^2, cx_d^- + u_{2d}^+$ ) has a unique solution defined by:

$$-\mu = \begin{cases} 0 & \text{if } cx_d^- + u_{2d}^+ \geq 0 \\ -\frac{1}{c}x_d^- - \frac{1}{c^2}u_{2d}^+ & \text{if } cx_d^- + u_{2d}^+ < 0 \end{cases}$$

Let us substitute the values of the solution in the KKT condition  $\frac{\partial \mathcal{L}}{\partial x_d} = 0$  to calculate  $x_d$  as follows:

$$x_d = \begin{cases} x_d^- & \text{if } cx_d^- + u_{2d}^+ \geq 0 \\ -\frac{1}{c}u_{2d}^+ & \text{if } cx_d^- + u_{2d}^+ < 0 \end{cases} \quad (\text{F.1})$$

*Appendix F.2. Closed-loop system in (45)*

It is required to find the optimal value  $x(t^+)$  in order to minimize the following

$$\begin{aligned} f(x) &= \min \frac{1}{2}(x - x^-)^2 \\ \text{s.t. } &x \in \mathcal{K} \end{aligned}$$

where  $\mathcal{K} = \{x \in \mathbf{R} \mid (c + k_3)x - k_3x_d(t^+) + u_{2d}(t^+) \geq 0\} = \{x \in \mathbf{R} \mid -(c + k_3)x + k_3x_d(t^+) - u_{2d}(t^+) \leq 0\}$ . Let us write the Lagrangian function

$$\mathcal{L}(x, \mu) = \frac{1}{2}(x - x^-)^2 - \mu(-(c + k_3)x + k_3x_d^+ - u_{2d}^+)$$

The necessary and sufficient conditions to be satisfied are the following:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = 0 \Leftrightarrow x - x^- + \mu(c + k_3) = 0 \Leftrightarrow x = x^- - \mu(c + k_3) \\ \mu(-(c + k_3)x + k_3x_d^+ - u_{2d}^+) = 0 \Leftrightarrow \mu(-(c + k_3)x^- + \mu(c + k_3)^2 + k_3x_d^+ - u_{2d}^+) = 0 \\ \mu \leq 0 \\ -(c + k_3)x + k_3x_d^+ - u_{2d}^+ \leq 0 \Leftrightarrow -(c + k_3)x^- + \mu(c + k_3)^2 + k_3x_d^+ - u_{2d}^+ = 0 \end{cases}$$

KKT conditions are written in the form of a LCP as:

$$0 \leq -\mu \perp -(c + k_3)^2\mu + (c + k_3)x^- - k_3x_d^+ + u_{2d}^+ \geq 0$$

The LCP( $(c + k_3)^2, (c + k_3)x^- - k_3x_d^+ + u_{2d}^+$ ) has a unique solution defined by:

$$-\mu = \begin{cases} 0 & \text{if } (c + k_3)x^- - k_3x_d^+ + u_{2d}^+ \geq 0 \\ -\frac{1}{(c+k_3)}x^- + \frac{k_3}{(c+k_3)^2}x_d^+ - \frac{1}{(c+k_3)^2}u_{2d}^+ & \text{if } (c + k_3)x^- - k_3x_d^+ + u_{2d}^+ < 0 \end{cases}$$

Let us substitute the values of the solution in the KKT condition  $\frac{\partial \mathcal{L}}{\partial x} = 0$  to calculate  $x$  as follows:

$$x = \begin{cases} x^- & \text{if } (c + k_3)x^- - k_3x_d^+ + u_{2d}^+ \geq 0 \\ -\frac{k_3x_d^+ - u_{2d}^+}{(c+k_3)} & \text{if } (c + k_3)x^- - k_3x_d^+ + u_{2d}^+ < 0 \end{cases}$$

## Appendix G. Proofs for the Example in section 5

### Appendix G.1. Proof of Lemma 5.2

Consider the Lyapunov candidate function  $V(e(t)) = e(t)^\top P e(t)$ . Recall from section 3.2, Proposition 3.8, that the error dynamics in (48) has a globally exponentially stable equilibrium point  $e^* = 0$  when both the states  $x(\cdot)$  and  $x_d(\cdot)$  are continuous. This result is proved by showing the variation of the storage function of the error dynamics  $\dot{V}(e(t)) < 0$  for all  $e \neq 0$ .

In view of the desired dynamics (43), the closed-loop dynamics (45) and the controllers (44), let  $t_k$  where  $k > 0$  be the set of time instants at which  $u_{2d}(\cdot)$  is discontinuous. The states  $x$  and  $x_d$  undergoes a jump at  $t \in \{0\} \cup \{t_k\}$ . The goal is to study the sign of  $\Delta V(e(t))$  at the jump time which is written as follows in scalar case:

$$\begin{aligned} \Delta V(e(t)) &= V(e(t^+)) - V(e(t^-)) \\ &= (x(t^+) - x_d(t^+))^\top p (x(t^+) - x_d(t^+)) - (x(t^-) - x_d(t^-))^\top p (x(t^-) - x_d(t^-)) \\ &= p (x(t^+) - x_d(t^+))^2 - p (x(t^-) - x_d(t^-))^2 \end{aligned} \quad (\text{G.1})$$

There are three cases to consider when analyzing the sign of  $\Delta V(e(t))$ . To lighten notations we denote  $f^+ = f(t^+)$  and  $f^- = f(t^-)$ .

*First case.* In this case, both  $x$  and  $x_d$  jump at the same time  $t$ . The values of the state jump for the desired and the closed-loop system are given by  $x_d^+ = -\frac{1}{c}u_{2d}^+$  and  $x^+ = \frac{k_2x_d^+ - u_{2d}^+}{c+k_2}$  respectively by referring to Appendix F. If we substitute the value of  $x_d^+$  in  $x^+$ , then  $x^+ = -\frac{1}{c}u_{2d}^+$ . The variation of the storage function at the jump time is:

$$\begin{aligned} \Delta V(e(t)) &= p (x(t^+) - x_d(t^+))^2 - p (x(t^-) - x_d(t^-))^2 \\ &= p \left( -\frac{1}{c}u_{2d}^+ + \frac{1}{c}u_{2d}^+ \right) - p (x(t^-) - x_d(t^-))^2 \\ &= -p (x(t^-) - x_d(t^-))^2 \leq 0 \end{aligned}$$

Thus, when both  $x$  and  $x_d$  jump, the variation of the storage function of the error dynamics  $\Delta V(e) < 0$  for all  $e \neq 0$ .

*Second case.* In this case, the closed-loop system's state  $x$  jumps and the desired system's state  $x_d$  is continuous such that  $x_d^+ = x_d^- = x_d$ . The value of the closed-loop system's state jump is given by  $x^+ = \frac{k_2x_d - u_{2d}^+}{c+k_2}$  (see Appendix F). If these values are substituted in  $\Delta V(e(t))$  in (G.1), then

$$\begin{aligned} \Delta V(e(t)) &= p \left( \frac{k_2x_d - u_{2d}^+}{c+k_2} - x_d \right)^2 - p (x^- - x_d)^2 = p \left( \left( \frac{-cx_d - u_{2d}^+}{c+k_2} \right)^2 - (x^- - x_d)^2 \right) \\ &= p \left( \left( \frac{-cx_d - u_{2d}^+}{c+k_2} - x^- + x_d \right) \left( \frac{-cx_d - u_{2d}^+}{c+k_2} + x^- - x_d \right) \right) = p \left( \left( \frac{-(c+k_2)x^- + k_2x_d - u_{2d}^+}{c+k_2} \right) \left( \frac{(c+k_2)x^- - (2c+k_2)x_d - u_{2d}^+}{c+k_2} \right) \right) \\ &= \frac{p}{(c+k_2)^2} \left( \left( -(c+k_2)x^- + k_2x_d - u_{2d}^+ \right) \left( (c+k_2)x^- - (2c+k_2)x_d - u_{2d}^+ \right) \right) \end{aligned}$$

Let  $h(x) \triangleq -(c+k_2)x^- + k_2x_d - u_{2d}^+$  and  $r(x) \triangleq (c+k_2)x^- - (2c+k_2)x_d - u_{2d}^+$ . Let us study the signs of  $h(x)$  and  $r(x)$ .

Knowing that  $x^- \notin \mathcal{K}$ , the following inequality holds:

$$(c+k_2)x^- - k_2x_d + u_{2d}^+ < 0 \Leftrightarrow -(c+k_2)x^- + k_2x_d - u_{2d}^+ > 0 \Leftrightarrow h(x) > 0$$



Now, let us check the sign of  $r(x)$ . If we add and subtract  $u_{2d}^+$ , then

$$\begin{aligned} r(x) \pm u_{2d}^+ &\stackrel{\Delta}{=} (c + k_2)x^- - k_2x_d + u_{2d}^+ - 2cx_d - 2u_{2d}^+ \\ &\stackrel{\Delta}{=} -h(x) - 2cx_d - 2u_{2d}^+ \end{aligned}$$

Provided that  $-h(x) < 0$  and  $cx_d + u_{2d}^+ \geq 0$  (i.e.,  $x_d \in \mathcal{K}_d$ )  $\Leftrightarrow -2cx_d - 2u_{2d}^+ \leq 0$ . So,  $r(x) < 0$ .

Thus,  $h(x) > 0$  and  $r(x) < 0$ . Therefore,  $\Delta V(e(t)) \stackrel{\Delta}{=} \frac{p}{(c+k_2)^2} (h(x)r(x)) < 0$ .

*Third case.* In this case, the desired system's state  $x_d$  jumps and the closed-loop system's state  $x$  is continuous such that  $x^+ = x^- = x(t)$ . The value of the desired system's state jump is  $x_d^+ = -\frac{1}{c}u_{2d}^+$  (check [Appendix F](#)). If these values are substituted in  $\Delta V(e(t))$  in [\(G.1\)](#), then

$$\Delta V(e(t)) = p \left( x + \frac{u_{2d}^+}{c} \right)^2 - p(x - x_d^-)^2 = p \left( \left( x + \frac{u_{2d}^+}{c} + x - x_d^- \right) \left( \frac{u_{2d}^+}{c} + x_d^- \right) \right)$$

Let  $h(x) \stackrel{\Delta}{=} x + \frac{u_{2d}^+}{c} + x - x_d^-$  and  $r(x) \stackrel{\Delta}{=} \frac{u_{2d}^+}{c} + x_d^-$ . Let us study the signs of  $h(x)$  and  $r(x)$ . Knowing that  $x_d^- \notin \mathcal{K}_d$ , then

$$cx_d^- + u_{2d}^+ < 0 \Leftrightarrow x_d^- + \frac{u_{2d}^+}{c} < 0 \Leftrightarrow r(x) < 0$$

Now, let us check the sign of  $h(x)$  by adding and subtracting  $u_{2d}^+$ . Then,

$$h(x) \pm u_{2d}^+ \stackrel{\Delta}{=} 2(cx + u_{2d}^+) - cx_d^- - u_{2d}^+ \stackrel{\Delta}{=} 2(cx + u_{2d}^+) - r(x)$$

The following holds:

$$x \in \mathcal{K} \Leftrightarrow (c + k_2)x - k_2x_d^+ + u_{2d}^+ \geq 0 \Leftrightarrow (c + k_2)x + \frac{k_2}{c}u_{2d}^+ + u_{2d}^+ \geq 0 \Leftrightarrow \frac{(c+k_2)}{c}(cx + u_{2d}^+) \geq 0.$$

According to [Remark 5.1](#),  $\frac{c+k_2}{c} > 0$ . So,  $cx + u_{2d}^+ \geq 0 \Leftrightarrow 2cx + 2u_{2d}^+ \geq 0$  and  $-r(x) > 0$ , and  $h(x) > 0$ . Thus  $h(x) > 0$  and  $r(x) < 0$  and  $\Delta V(e(t)) \stackrel{\Delta}{=} p(h(x)r(x)) < 0$ . Therefore, the storage function is strictly decreasing (i.e.,  $\dot{V} < 0$  when  $t \neq t_k$  and  $\Delta V < 0$  when  $t = t_k$  for all  $e \neq 0$  where  $t_k$  are the time instants of jumps) on  $\mathbb{R}_+$ .

### Appendix G.2. Proof of Corollary 5.3

Let the time instants at which  $u_{2d}(\cdot)$  is discontinuous be denoted by  $t_k$  with  $k = \{1, 2, \dots, n\}$ . The goal is to show that if the input  $u_{2d}(t_k^+) < \min\{-cx(t_k^-), -cx_d(t_k^-)\}$ , then both  $x$  and  $x_d$  jump to the same value at  $t = t_k$ . According to the set  $\mathcal{K}_d$  in [\(46\)](#), the state  $x_d$  performs a jump at  $t = t_k$  if and only if  $x_d(t_k^-) \notin \mathcal{K}_d$ , which is written as follows:

$$x_d(t_k^-) \notin \mathcal{K}_d \Leftrightarrow cx_d(t_k^-) + u_{2d}(t_k^+) < 0 \Leftrightarrow u_{2d}(t_k^+) < -cx_d(t_k^-)$$

Also, the state  $x$  in [\(45\)](#) performs a jump at  $t = t_k$  if and only if  $x(t_k^-) \notin \mathcal{K}$  in [\(47\)](#). Let us write the following:

$$x(t_k^-) \notin \mathcal{K} \Leftrightarrow (c + k_2)x(t_k^-) - k_2x_d(t_k^+) + u_{2d}(t_k^+) < 0 \tag{G.2}$$

Given that  $x_d$  jumps at  $t = t_k$ , let us substitute  $x_d(t_k^+) = -\frac{1}{c}u_{2d}(t_k^+)$  (see [\(F.1\)](#) in [Appendix F](#)) in [\(G.2\)](#). Then,

$$(c + k_2)x(t_k^-) + \frac{k_2}{c}u_{2d}(t_k^+) + u_{2d}(t_k^+) < 0 \Leftrightarrow \frac{c+k_2}{c} \left( cx(t_k^-) + u_{2d}(t_k^+) \right) < 0$$

Let  $c$  and  $c + k_2$  have the same sign (see [Remark 5.1](#)). It is inferred that equivalently

$$u_{2d}(t_k^+) < -cx(t_k^-).$$

According to the solution of the optimization problem in [\(7\)](#) in [Appendix F](#), the closed-loop system's state  $x$  jumps at  $t_k$  such that  $x(t_k^+) = \frac{k_2x_d(t_k^-) - u_{2d}(t_k^+)}{c+k_2} = -\frac{1}{c}u_{2d}(t_k^+)$  where  $x_d(t_k^+) = -\frac{1}{c}u_{2d}(t_k^+)$ . Therefore, both  $x$  and  $x_d$  jumps at  $t_k$  to the same value  $x_d(t_k^+) = x(t_k^+) = -\frac{1}{c}u_{2d}(t_k^+)$  (see [Appendix F](#)) if and only if  $u_{2d}(t_k^+) < \mu$ , where  $\mu$  is given by  $\mu = \min\{-cx_d(t_k^-), -cx(t_k^-)\}$ .

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