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Finite-Time Input-to-State Stabilization of Discrete-Time Systems

Frédéric Mazenc

Michael Malisoff

Abstract—We prove finite-time input-to-state stability estimates for discrete-time time-varying linear systems in closed loop with sample-data control laws. The upper bounds for the norms of the states in the estimates are suprema of the uncertainties over time intervals of constant finite lengths. We cover output feedback stabilization and input delays. We combine our results with a trajectory based approach, to prove novel global exponential input-to-state stability estimates for nonlinear systems with state delays. We illustrate our work using a dynamics containing an unknown nonlinearity and an unknown state delay.

Index Terms—Finite-time, stabilization, input-to-state stability

I. INTRODUCTION

Discrete-time systems provide the basis for considerable ongoing research, thanks to their ability to accurately model discrete event dynamics in many applications [1], [2]. A key problem in the study of discrete-time systems is to quantify the effects of model or control uncertainty, which is often handled using discrete-time variants of input-to-state stability (ISS) [3], [4]. In standard ISS estimates, there are upper bounds on the norms of the current states of the systems consisting of sums of two terms, namely, terms that asymptotically converge to zero at rates that are uniform in the norms of the initial states, plus functions of the suprema of the uncertainties from the initial times through the current times. However, in practice, systems with uncertainties can enjoy a fading effect, where only values of the uncertainty on a constant finite length time interval effect the current state values. This motivated the work [5] on finite-time ISS for continuous time systems. In this letter, we provide analogs of [5] for discrete-time systems.

We construct sampled control laws [6]–[9] that ensure a finite-time ISS result for time-varying discrete-time linear systems, which has a fading property whereby the upper bound on the norm of the state at each discrete time only depends on the values that are taken by the uncertainty on an interval of a constant finite length, as long as the current time is sufficiently large; see our finite-time ISS definition below. For time-varying nonlinear systems with state delays, we also use our finite-time ISS estimates to prove a global exponential ISS result, in which the convergence rate is proportional to the log of a suitable multiple of a Lipschitz constant for the uncertain nonlinear vector field in the system. This fading effect, combined with the speed of convergence, are key

advantages of our discrete-time control methods in this letter. A key idea involves the controllability of the systems. We provide a discrete-time finite-time variant of a result in [10] to determine robust feedbacks. We also use a trajectory based approach that is a useful alternative to standard Lyapunov function methods and which can be better suited for systems with uncertain delays and uncertain vector fields; see, e.g., [11], [12], and [13] for uses of the trajectory based approach to prove asymptotic stability properties. For the problems we address in this letter, the Lyapunov approach cannot be used. We also prove a discrete-time dynamic output feedback stabilization result using finite-time observers, including measurement uncertainty.

We first provide a definition for discrete-time systems that will play a key role in this letter. To the best of our knowledge, this definition is new. A delayed system [8] of the form

$$x_{k+1} = f(k, x^{k,\tau}, \delta_k) \quad (1)$$

with x valued in \mathbb{R}^n and with a constant integer delay $\tau \geq 0$ is called finite-time input-to-state stable (which is also abbreviated as ISS) provided there are a function $\varphi \in \mathcal{K}$ and integers $T_2 \geq 0$ and $T_1 \geq T_2$ such that for each initial time $k_0 \leq T_1$, each initial function, and all choices of the sequences δ_k , the corresponding solution $x : \mathbb{Z}_0 \rightarrow \mathbb{R}^n$ of (1) satisfies

$$|x_k| \leq \varphi \left(\sup_{m \in [k-T_2, k]} |\delta_m| \right) \quad (2)$$

for all $k \geq T_1$. Here \mathcal{K} denotes the set of all strictly increasing continuous functions $\alpha : [0, +\infty) \rightarrow [0, +\infty)$ such that $\alpha(0) = 0$, and the functions $x^{k,\tau}$ in (1) are defined by $x_m^{k,\tau} = x_{k+m}$ for all $m \in \{-\tau, \dots, 0\}$ for a constant delay τ and at each discrete time k . Unlike standard ISS inequalities, in (2), only values δ_k on a time interval of a fixed finite length T_2 are present, so the values of δ_k for k outside of a time interval of a constant finite length do not affect the bound on the norm of the current value of the solution.

In Section II, we prove a foundational finite-time ISS result for time-varying discrete-time linear systems, which we combine with the trajectory based approach from [12] to prove an analog for nonlinear systems in Section III. We provide generalizations with outputs and input delays in Sections IV–V. A key ingredient for systems with outputs will be an artificial delays approach, where the original systems might not have delays, but where the control uses delayed measurements. This artificial delays approach is inspired by prior results that used delays to improve asymptotic convergence rates for linear systems, such as [14]. Then we illustrate the main ideas in an example with an unknown nonlinearity that contains an unknown state delay.

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We use the following additional notation. The dimensions of our Euclidean spaces are arbitrary unless otherwise noted, $|\cdot|$ denotes the usual Euclidean 2-norm on \mathbb{R}^n and the corresponding matrix operator norm, and $|\cdot|_S$ is the supremum over any interval S in this norm. We let $\mathbb{Z}_0 = \{0, 1, 2, \dots\}$ and $\mathbb{N} = \mathbb{Z}_0 \setminus \{0\}$, and I is the identity matrix. Our products of matrices must be understood to be from right to left, so

$$\prod_{k=i}^j A_k = A_j \cdots A_i \quad (3)$$

when $j \geq i$ and when the matrices A_k are square matrices of the same size, and we define the product on the left side of (3) to be I when $j < i$. For simplicity, we assume that the initial times for all of our solutions of our systems are $k_0 = 0$.

II. FUNDAMENTAL FINITE-TIME ISS RESULT

We first present a result for state feedback for linear time-varying discrete-time systems, which will play a key role in later sections where we cover unknown nonlinearities and outputs. Consider the time-varying discrete-time linear system

$$x_{k+1} = F_k x_k + G_k u_k + \delta_k \quad (4)$$

indexed by $k \in \mathbb{Z}_0$, where x_k and δ_k (representing the system state and uncertainty, respectively) are valued in \mathbb{R}^n , the matrices $F_k \in \mathbb{R}^{n \times n}$ and $G_k \in \mathbb{R}^{n \times p}$ are known, and the control u_k is valued in \mathbb{R}^p . In terms of the sequence

$$W_\ell = \sum_{i=\ell n}^{(\ell+1)n-1} \left(\left(\prod_{r=i+1}^{(\ell+1)n-1} F_r \right) G_i G_i^\top \left(\prod_{r=i+1}^{(\ell+1)n-1} F_r \right)^\top \right) \quad (5)$$

indexed by $\ell \in \mathbb{N}$, We make this assumption:

Assumption 1: The sequences F_k and G_k of matrices in (4) are bounded. Also, for each $\ell \in \mathbb{N}$, the matrix W_ℓ is invertible, and W_ℓ^{-1} is a bounded sequence. Also, $n \geq 2$. \square

See Remark 1 below for sufficient conditions for Assumption 1 to hold. We also require the function

$$\begin{aligned} \mathcal{B}(\ell, s) &= \prod_{r=\ell n}^{\ell n+s-1} F_r \\ &- \sum_{i=\ell n}^{\ell n+s-1} \left(\prod_{r=i+1}^{\ell n+s-1} F_r G_i G_i^\top \left(\prod_{r=i+1}^{(\ell+1)n-1} F_r \right)^\top \right) W_\ell^{-1} \prod_{r=\ell n}^{(\ell+1)n-1} F_r \end{aligned} \quad (6)$$

indexed by $\ell \in \mathbb{N}$ and $s \in \{1, \dots, n-1\}$. Then we set

$$\begin{aligned} \bar{\mathcal{B}} &= \sup \left\{ \max_{i=(\ell-1)n}^{\ell n-1} \left\{ \left| \mathcal{B}(\ell, s) \prod_{r=i+1}^{\ell n-1} F_r \right|, \left| \prod_{r=i+1}^{\ell n-1} F_r \right| \right\} \right. \\ &\quad \left. + \sum_{i=\ell n}^{\ell n+s-1} \left| \prod_{r=i+1}^{\ell n+s-1} F_r \right| : \ell \in \mathbb{N}, s \in \{1, \dots, n-1\} \right\}. \end{aligned} \quad (7)$$

Using the matrices W_ℓ from (5) and the constant (7), we prove:

Theorem 1: Let Assumption 1 hold. Then all solutions $x : \mathbb{Z}_0 \rightarrow \mathbb{R}^n$ of (4), in closed-loop with the feedback defined by

$$u_k = -G_k^\top \left(\prod_{r=k+1}^{\ell n-1} F_r \right)^\top W_{\ell-1}^{-1} \left(\prod_{r=(\ell-1)n}^{\ell n-1} F_r \right) x_{(\ell-1)n} \quad (8)$$

for all $k \in \{(\ell-1)n, \dots, \ell n-1\}$ and for all $\ell \in \mathbb{N}$, satisfy

$$|x_k| \leq \bar{\mathcal{B}} |\delta|_{[k-2n, k]} \quad (9)$$

for all $k \geq 2n$. \square

Proof: We can show by induction on $s \in \mathbb{N}$ that each solution $x : \mathbb{Z}_0 \rightarrow \mathbb{R}^n$ of (4) satisfies

$$x_{j+s} = \left(\prod_{r=j}^{j+s-1} F_r \right) x_j + \sum_{k=j}^{j+s-1} \left(\prod_{r=k+1}^{j+s-1} F_r \right) (G_k u_k + \delta_k) \quad (10)$$

for all $j \in \mathbb{Z}_0$ and $s \in \mathbb{N}$. By specializing (10) to the case where $j = (\ell-1)n$ and $s = n$, we get

$$\begin{aligned} x_{\ell n} &= \left(\prod_{r=(\ell-1)n}^{\ell n-1} F_r \right) x_{(\ell-1)n} \\ &\quad + \sum_{k=(\ell-1)n}^{\ell n-1} \left(\prod_{r=k+1}^{\ell n-1} F_r \right) (G_k u_k + \delta_k) \end{aligned} \quad (11)$$

for all $\ell \in \mathbb{N}$. By substituting our choice (8) of the control into (11), and recalling our choices (5), we get

$$x_{\ell n} = \sum_{k=(\ell-1)n}^{\ell n-1} \left(\prod_{r=k+1}^{\ell n-1} F_r \right) \delta_k. \quad (12)$$

If we now specialize (10) to the case where $j = \ell n$ and then substitute our formula (8) for the control into this special case of the result, and then factor out $x_{\ell n}$ from the right side of the result, then we obtain

$$\begin{aligned} x_{\ell n+s} &= \left(\prod_{r=\ell n}^{\ell n+s-1} F_r \right) x_{\ell n} \\ &\quad + \sum_{i=\ell n}^{\ell n+s-1} \left(\prod_{r=i+1}^{\ell n+s-1} F_r \right) (G_i u_i + \delta_i) \\ &= \left\{ \prod_{r=\ell n}^{\ell n+s-1} F_r \right. \\ &\quad \left. - \sum_{i=\ell n}^{\ell n+s-1} \left(\prod_{r=i+1}^{\ell n+s-1} F_r G_i G_i^\top \left(\prod_{r=i+1}^{(\ell+1)n-1} F_r \right)^\top \right) W_\ell^\# \right\} x_{\ell n} \\ &\quad + \sum_{i=\ell n}^{\ell n+s-1} \left(\prod_{r=i+1}^{\ell n+s-1} F_r \right) \delta_i \end{aligned} \quad (13)$$

for $s = 1, \dots, n-1$, where

$$W_\ell^\# = W_\ell^{-1} \prod_{r=\ell n}^{(\ell+1)n-1} F_r. \quad (14)$$

If we now substitute our formula (12) for $x_{\ell n}$ into (13) and recall are choices of $\mathcal{B}(\ell, s)$ and $\bar{\mathcal{B}}$ in (6)-(7), then we conclude from (12)-(13) that

$$|x_{\ell n+s}| \leq \bar{\mathcal{B}} |\delta|_{[(\ell-1)n, \ell n+s]} \quad (15)$$

holds for $s = 0, \dots, n-1$ and all $\ell \in \mathbb{N}$, where we used the maximum inside the supremum in (7) in order to cover the case (12) which corresponds to choosing $s = 0$ in (15). By setting $k = \ell n + s$, it follows from (15) that our conclusion (9) holds. \blacksquare

Remark 1: In the special case where F and G are constant matrices and $j = 0$ and $s = n$, our formula (10) becomes

$$x_n = F^n x_0 + \sum_{i=0}^{n-1} F^i (G u_{n-1-i} + \delta_{n-1-i}). \quad (16)$$

Hence, if (F, G) is a constant controllable pair, then the linear operator $L : \mathbb{R}^{np} \rightarrow \mathbb{R}^n$ defined by

$$L(u) = \sum_{i=0}^{n-1} F^i G u_{n-1-i} \quad (17)$$

is onto. It follows from [15, Section 3.5] that in this case, the sequence W_ℓ from (5), which takes the constant value

$$W_* = \sum_{i=0}^{n-1} F^i G G^\top (F^\top)^i, \quad (18)$$

is invertible. Therefore, when (F, G) is a constant controllable pair, Assumption 1 holds. Hence, Assumption 1 also holds if the sequence (F_k, G_k) of coefficient matrices have the form

$$F_k = F_* + \Delta_{F_k} \text{ and } G_k = G_* + \Delta_{G_k} \quad (19)$$

for a constant controllable pair (F_*, G_*) when the sequences Δ_{F_k} and Δ_{G_k} have small enough norms. \square

III. NONLINEAR SYSTEMS

We can use the method from Theorem 1 to prove an analog for systems

$$x_{k+1} = F_k x_k + G_k u_k + f(k, x^{k,\tau}) + \epsilon_k \quad (20)$$

that can contain nonlinearities in the state and state delays, where x_k is again valued in \mathbb{R}^n , the delay systems notation $x^{k,\tau}$ was defined in Section I, Assumption 1 is satisfied, u is valued in \mathbb{R}^p , the sequence ϵ_k represents uncertainty, and $\tau \in \mathbb{Z}_0$ represents the delay. Our main strategy is to apply the method from Theorem 1 with the choice

$$\delta_k = f(k, x^{k,\tau}) + \epsilon_k, \quad (21)$$

under the following assumption, where \bar{B} is from (7) as before:

Assumption 2: There is a known constant $f_0 > 0$ such that

$$\bar{B}f_0 \in (0, 1) \quad (22)$$

holds, and such that

$$|f(k, \phi)| \leq f_0 \sup_{m \in [-\tau, 0]} |\phi(m)| \quad (23)$$

holds for all functions $\phi : \{-\tau, \dots, 0\} \rightarrow \mathbb{R}^n$ and $k \geq 0$. \square

A key advantage of our Assumption 2 will be that we allow f to be unknown, as long as we know a constant f_0 satisfying Assumption 2. We then prove the following, which we refer to as an almost finite-time ISS result because, instead of finite-time convergence as we had in Theorem 1 with the convergence times $k = 2n$, the rate of exponential convergence

$$R(f_0) = -\frac{\ln(\bar{B}f_0)}{2n+\tau} \quad (24)$$

in the ISS estimate (25) to zero is such that $\lim_{f_0 \rightarrow 0} R(f_0) = +\infty$, so smaller growth constants f_0 on f (for given values of \bar{B}) imply faster convergence in the exponential ISS estimate:

Theorem 2: Let the system (20) satisfy Assumptions 1-2, and choose the control u in (8). Then for each solution $x : \mathbb{Z}_0 \rightarrow \mathbb{R}^n$ of the closed loop system given by (20) in closed loop with (8) for all initial conditions, we have

$$|x_k| \leq |x|_{[0, 2n+\tau]} e^{\frac{\ln(\bar{B}f_0)(k-2n-\nu)}{2n+\tau}} + \frac{\bar{B}|\epsilon|_{[0, k]}}{(1-\bar{B}f_0)^2} \quad (25)$$

for all $k \geq 2n + \tau$. \square

Proof: Since the feedback defined in (8) is used, the inequality (9) applied with $\delta_\ell = f(\ell, x^{\ell,\tau}) + \epsilon_\ell$ gives

$$|x_k| \leq \bar{B} \sup_{\ell \in [k-2n, k]} |f(\ell, x^{\ell,\tau}) + \epsilon_\ell| \quad (26)$$

for all $k \geq 2n + \tau$. It follows from Assumption 2 that

$$|x_k| \leq \bar{B}f_0 \sup_{m \in [k-2n-\tau, k]} |x_m| + \bar{B} \sup_{\ell \in [k-2n, k]} |\epsilon_\ell| \quad (27)$$

for all $k \geq 2n + \tau$. Hence, by replacing k by $k + 2n + \tau$ in (27), it follows that the function $V_k = x_{k+2n+\tau}$ satisfies

$$|V_k| \leq \bar{B}f_0 |V|_{[k-2n-\tau, k]} + d_k \quad (28)$$

for all $k \in \mathbb{Z}_0$, where $d_k = \bar{B}|\epsilon|_{[k+\tau, k+2n+\tau]}$. Since $\bar{B}f_0 \in (0, 1)$, it follows from applying [12, Theorem 1] to the function V that

$$|V_k| \leq |V|_{[-2n-\tau, 0]} e^{\frac{\ln(\bar{B}f_0)k}{2n+\tau}} + \frac{\bar{B}|\epsilon|_{[0, k+2n+\tau]}}{(1-\bar{B}f_0)^2} \quad (29)$$

for all $k \in \mathbb{Z}_0$. Then (25) follows from our choice of V and from replacing k by $k - 2n - \tau$ in (29). \blacksquare

IV. SYSTEMS WITH OUTPUTS

Our finite-time ISS feedback (8) calls for the sampled state measurements $x_{(\ell-1)n}$, which might not always be available in practice. Therefore, in this section, we provide a finite-time ISS extension of Theorem 1 for systems

$$\begin{cases} x_{k+1} = F_k x_k + G_k u_k + \delta_k \\ y_k = C_k x_k + \mu_k \end{cases} \quad (30)$$

with outputs, where the unknown sequences δ_k and μ_k represent model and measurement uncertainty, respectively (but see Remark 3 below, for generalized versions with nonlinearities in the state variable). To this end, we introduce the sequence

$$E_k = \sum_{i=k}^{k+n-1} \left(\left(\prod_{r=k}^{i-1} F_r \right)^\top C_i^\top C_i \prod_{r=k}^{i-1} F_r \right) \quad (31)$$

indexed by $k \in \mathbb{N}$ where we continue our notation convention that products where the lower limit is larger than the upper one are defined to be I , and we introduce this assumption, which we assume in addition to Assumption 1 in this section:

Assumption 3: The sequence C_k is bounded. Also, E_k is invertible for each $k \in \mathbb{N}$, and E_k^{-1} is a bounded sequence. \square

See Remark 2 below for ways to check that the preceding assumption holds. Assumptions 1 and 3 ensure that the matrix

$$\mathcal{L}_k = \begin{bmatrix} C_k \\ C_{k+1}F_k \\ \vdots \\ C_{k+n-1} \prod_{r=k}^{k+n-2} F_r \end{bmatrix} \quad (32)$$

admits the left inverse

$$\mathcal{L}_k^\sharp = E_k^{-1} \mathcal{L}_k^\top \quad (33)$$

for each $k \in \mathbb{N}$, and that \mathcal{L}_k^\sharp is a bounded sequence, because $E_k = \mathcal{L}_k^\top \mathcal{L}_k$ for each $k \in \mathbb{N}$. To construct the required functions in our estimates, we use the sequence

$$\begin{aligned} \mathcal{H}_k &= \left| \left(\prod_{r=k-n+1}^{k-1} F_r \right) \mathcal{L}_{k-n+1}^\sharp \right| \\ &\times \left(\sum_{r=1}^{n-1} \left| C_{k-n+1+r} \sum_{i=k-n+1}^{k-n+r} \left(\prod_{s=i+1}^{k-n+r} F_s \right) \right| + n \right) \\ &+ \sum_{i=k-n+1}^{k-1} \left| \prod_{r=i+1}^{k-1} F_r \right| \end{aligned} \quad (34)$$

which is defined for all integers $k \geq n-1$, and the constants

$$\bar{C} = \sup_{k \geq n-1} \mathcal{H}_k \quad (35)$$

and

$$\bar{D} = \sup \left\{ \left| G_k G_k^\top \left(\prod_{r=k+1}^{\ell n-1} F_r \right)^\top W_{\ell-1}^{-1} \prod_{r=(\ell-1)n}^{\ell n-1} F_r \right| : \ell \in \mathbb{N}, (\ell-1)n \leq k \leq \ell n-1 \right\} \quad (36)$$

which are both finite because of our Assumptions 1 and 3.

Since the state values x_k are no longer assumed to be available for measurement, we next define the new feedback

$$\hat{u}_k = -G_k^\top \left(\prod_{r=k+1}^{\ell n-1} F_r \right)^\top W_{\ell-1}^{-1} \prod_{r=(\ell-1)n}^{\ell n-1} F_r \hat{x}_{(\ell-1)n} \quad (37)$$

for all $k \in \{(\ell-1)n, \dots, \ell n-1\}$ and $\ell \in \mathbb{N}$, where the observer states \hat{x}_k for $k \geq n-1$ are defined by

$$\hat{x}_k = x_{k,*} + \left(\prod_{r=k-n+1}^{k-1} F_r \right) \mathcal{L}_{k-n+1}^\# \begin{bmatrix} y_{k-n+1}^b \\ y_{k-n+2}^b \\ \vdots \\ y_k^b \end{bmatrix} \quad (38)$$

where

$$y_r^b = y_r - C_r x_{r,*} \text{ for } r = k-n+1, \dots, k \quad (39)$$

and

$$x_{k+1,*} = F_k x_{k,*} + G_k \hat{u}_k \quad (40)$$

for all $k \in \mathbb{Z}_0$, and where $x_{0,*} = 0$ and $\hat{x}_k = 0$ for $k < n-1$. Using the constants \bar{B} , \bar{C} , and \bar{D} from (7), (35), and (36), we then prove this result:

Theorem 3: Let Assumptions 1 and 3 hold. Then the following two conclusions hold:

(i) For each integer $k \geq n-1$, the inequality

$$|x_k - \hat{x}_k| \leq \bar{C} (|\mu|_{[k-n+1,k]} + |\delta|_{[k-n+1,k]}) \quad (41)$$

is satisfied.

(ii) For each initial state $x_0 \in \mathbb{R}^n$ and all choices of the sequences δ_k and μ_k , the corresponding solution of

$$x_{k+1} = F_k x_k + G_k \hat{u}_k + \delta_k \quad (42)$$

in closed loop with the control \hat{u}_k from (37) satisfies

$$|x_k| \leq \bar{B} |\delta|_{[k-2n,k]} + \bar{B} \bar{C} \bar{D} |\mu|_{[(\ell-2)n+1,k]} + \bar{B} \bar{C} \bar{D} |\delta|_{[(\ell-2)n+1,k]} \quad (43)$$

for all $k \in \{(\ell-1)n, \dots, \ell n-1\}$ and all integers $\ell \geq 3$. \square

Proof: The proof has three parts. In the first part, we use the new variable

$$z_k = x_k - x_{k,*} \quad (44)$$

to eliminate the effects of the $G_k \hat{u}_k$ terms from (42), and we derive a formula for z_k in terms of the values of y_{k+r}^b for $r = 0, \dots, n-1$ and all uncertainties δ . In the second part, we use a change of variable to eliminate the dependence of the expression for z_k on future values of the output. In the third part, we use the formula for z_k from the second part, in order

to prove part (i) of the theorem, which we then combine with the method from Theorem 1 to prove part (ii) of the theorem.

First Part. Since (40), (42), and (44) give

$$z_{k+1} = F_k z_k + \delta_k \quad (45)$$

for all $k \in \mathbb{Z}_0$, we can use an induction argument on $j \in \mathbb{N}$ that is analogous to the one that we used to derive (10), to prove that

$$z_{k+j} = \left(\prod_{r=k}^{k+j-1} F_r \right) z_k + \sum_{i=k}^{k+j-1} \left(\prod_{s=i+1}^{k+j-1} F_s \right) \delta_i \quad (46)$$

for $j = 1, \dots, n-1$ and all $k \in \mathbb{Z}_0$. Therefore,

$$y_{k+j}^b = C_{k+j} \left(\prod_{r=k}^{k+j-1} F_r \right) z_k + C_{k+j} \sum_{i=k}^{k+j-1} \left(\prod_{s=i+1}^{k+j-1} F_s \right) \delta_i + \mu_{k+j} \quad (47)$$

for $j = 1, \dots, n-1$ and all $k \in \mathbb{Z}_0$, by (30), (39), and (44).

Collecting the equations (39) and (47) in matrix form gives

$$\begin{bmatrix} y_k^b \\ y_{k+1}^b \\ \vdots \\ y_{k+n-1}^b \end{bmatrix} = \mathcal{L}_k z_k + \begin{bmatrix} \mu_k \\ C_1^\#(k) + \mu_{k+1} \\ \vdots \\ C_{n-1}^\#(k) + \mu_{k+n-1} \end{bmatrix} \quad (48)$$

where \mathcal{L}_k is defined in (32) and

$$C_r^\#(k) = C_{k+r} \sum_{i=k}^{k+r-1} \left(\prod_{s=i+1}^{k+r-1} F_s \right) \delta_i \quad (49)$$

for $r = 1, \dots, n-1$. Then we can solve for z_k in (48) using our formulas (33) for the left inverses $\mathcal{L}_k^\#$ to obtain

$$z_k = \mathcal{L}_k^\# \begin{bmatrix} y_k^b \\ y_{k+1}^b \\ \vdots \\ y_{k+n-1}^b \end{bmatrix} - \mathcal{L}_k^\# \begin{bmatrix} \mu_k \\ C_1^\#(k) + \mu_{k+1} \\ \vdots \\ C_{n-1}^\#(k) + \mu_{k+n-1} \end{bmatrix}. \quad (50)$$

The formulas (50) cannot be applied directly, because they depend on future values of the measurement at each time $k \in \mathbb{Z}_0$. We address this challenge in the next part of the proof.

Second Part. To eliminate the dependence of the z_k formulas on future measurement values, first note that (50) implies that

$$z_{k-n+1} = \mathcal{L}_{k-n+1}^\# \begin{bmatrix} y_{k-n+1}^b \\ y_{k-n+2}^b \\ \vdots \\ y_k^b \end{bmatrix} - \mathcal{L}_{k-n+1}^\# \begin{bmatrix} \mu_{k-n+1} \\ C_1^\#(k-n+1) + \mu_{k-n+2} \\ \vdots \\ C_{n-1}^\#(k-n+1) + \mu_k \end{bmatrix} \quad (51)$$

for all $k \geq n-1$. On the other hand, by replacing j and k in (46) by $n-1$ and $k-n+1$ respectively, we obtain

$$z_k = \left(\prod_{r=k-n+1}^{k-1} F_r \right) z_{k-n+1} + \sum_{i=k-n+1}^{k-1} \left(\prod_{s=i+1}^{k-1} F_s \right) \delta_i \quad (52)$$

for all $k \geq n - 1$. By substituting (51) into (52), we obtain

$$z_k = \left(\prod_{r=k-n+1}^{k-1} F_r \right) \mathcal{L}_{k-n+1}^\# \begin{bmatrix} y_{k-n+1}^b \\ y_{k-n+2}^b \\ \vdots \\ y_k^b \end{bmatrix} - \left(\prod_{r=k-n+1}^{k-1} F_r \right) \mathcal{L}_{k-n+1}^\# \begin{bmatrix} \mu_{k-n+1} \\ C_1^\#(k-n+1) + \mu_{k-n+2} \\ \vdots \\ C_{n-1}^\#(k-n+1) + \mu_k \end{bmatrix} + \sum_{i=k-n+1}^{k-1} \left(\prod_{s=i+1}^{k-1} F_s \right) \delta_i \quad (53)$$

for all $k \geq n - 1$. The expression (53) has the advantage of having its right side not depending on future measurements.

Third Part. In terms of our observer and difference values \hat{x}_k and z_k from (38) and (44), we conclude from (53) that

$$x_k = \hat{x}_k + \sum_{i=k-n+1}^{k-1} \left(\prod_{s=i+1}^{k-1} F_s \right) \delta_i - \left(\prod_{r=k-n+1}^{k-1} F_r \right) \mathcal{L}_{k-n+1}^\# \begin{bmatrix} \mu_{k-n+1} \\ C_1^\#(k-n+1) + \mu_{k-n+2} \\ \vdots \\ C_{n-1}^\#(k-n+1) + \mu_k \end{bmatrix} \quad (54)$$

for all $k \geq n - 1$. Then our first conclusion (41) of the theorem follows from moving the \hat{x}_k from the right side of (54) to the left side of (54), then calculating the norm of the right side of the result, and then recalling our choice (35) of \bar{C} . To prove (43), first note that (42) coincides with (4), except with δ_k in (4) replaced by $\delta_k + G_k(\hat{u}_k - u_k)$. Hence,

$$|x_k| \leq \bar{B}|\delta|_{[k-2n,k]} + \bar{B}\bar{D}|x_{(\ell-1)n} - \hat{x}_{(\ell-1)n}| \quad (55)$$

for all $k \geq 2n$, by applying the argument from Theorem 1 with δ_k replaced by $\delta_k + G_k(\hat{u}_k - u_k)$ where $\ell \in \mathbb{N}$ is such that $k \in [(\ell-1)n, \ell n]$, \bar{D} is defined in (36), and \bar{B} is from (7). Then the second conclusion (43) follows by using (41) to upper bound the second right side term in (55). ■

Remark 2: In the special case where F_k , G_k , and C_k are constant, our formula (31) gives $E_k = E_*$ for all $k \in \mathbb{Z}_0$, where

$$E_* = \sum_{i=0}^{n-1} (F^\top)^i C^\top C F^i. \quad (56)$$

By Remark 1 above, it follows that in this special case, E_* will be invertible if (F^\top, C^\top) is a controllable pair, i.e., if (F, C) is an observable pair. Hence, Assumption 3 will hold if the sequence (F_k, C_k) of coefficient matrices have the form $F_k = F_* + \Delta_{F_k}$ and $C_k = C_* + \Delta_{C_k}$ for an observable pair (F_*, C_*) of matrices when the time-varying parts Δ_{F_k} and Δ_{C_k} have small enough sup norms. □

Remark 3: Our conclusion (43) of Theorem 3 implies that

$$|x_k| \leq \bar{B}|\delta|_{[k-2n,k]} + \bar{B}\bar{C}\bar{D}|\mu|_{[k-2n+1,k]} + \bar{B}\bar{C}\bar{D}|\delta|_{[k-2n+1,k]} \quad (57)$$

for all $k \geq 2n$. We can combine (57) with the approach that we used to prove Theorem 2, to generalize Theorem 3 to systems

$$\begin{cases} x_{k+1} = F_k x_k + G_k u_k + f(k, x^{k,\tau}) + \delta_k \\ y_k = C_k x_k + \mu_k \end{cases} \quad (58)$$

whose nonlinearities satisfy (23) for some constant $f_0 \geq 0$ such that $\bar{B}(1 + \bar{C}\bar{D})f_0 \in (0, 1)$. This is done by replacing δ in (57) by $\delta_k + f(k, x^{k,\tau})$, which gives

$$|x_k| \leq \bar{B}(1 + \bar{C}\bar{D})|\delta|_{[k-2n,k]} + \bar{B}\bar{C}\bar{D}|\mu|_{[k-2n+1,k]} + \bar{B}(1 + \bar{C}\bar{D})f_0|x|_{[k-2n-\tau,k]} \quad (59)$$

for all $k \geq 2n + \tau$. Then, instead of finite-time ISS, we can apply [12, Theorem 1], to prove that

$$|x_k| \leq \bar{B}(1 + \bar{C}\bar{D})e^{\frac{\ln(\bar{B}(1 + \bar{C}\bar{D})f_0)}{2n+\tau}(k-2n-\tau)}|x|_{[0,2n+\tau]} + \frac{\bar{B}(1 + \bar{C}\bar{D})}{(1 - \bar{B}(1 + \bar{C}\bar{D})f_0)^2} (|\mu|_{[0,k]} + |\delta|_{[0,k]}) \quad (60)$$

for all $k \geq 2n + \tau$. □

V. INPUT DELAYS

Sections III-IV cover systems with state delays, using the nonlinear terms $f(k, x^{k,\tau})$ in the dynamics (20) and (58) for the cases of state and output feedback, respectively. On the other hand, the control (8) requires measurements $x_{(\ell-1)n}$ to compute the value of the control on the interval $[(\ell-1)n, \ell n]$ for each $\ell \in \mathbb{N}$. In some cases the sample values $x_{(\ell-1)n}$ might not be immediately available, and then we must instead use sample values $x_{(\ell-1)n-h}$, where $h \in \mathbb{N}$ is an input delay.

To cover these cases, we can replace the feedback (8) by

$$u_k = -\mathcal{M}(k, \ell) \left[\left(\prod_{r=(\ell-1)n-h}^{(\ell-1)n-1} F_r \right) x_{(\ell-1)n-h} + \sum_{s=(\ell-1)n-h}^{(\ell-1)n-1} \left(\prod_{r=s+1}^{(\ell-1)n-1} F_r \right) G_s u_s \right] \quad (61)$$

for all $k \in [(\ell-1)n, \ell n]$, $\ell \in \mathbb{N}$, and $\ell \geq 1 + h/n$, where

$$\mathcal{M}(k, \ell) = G_k^\top \left(\prod_{r=k+1}^{\ell n-1} F_r \right)^\top W_{\ell-1}^{-1} \prod_{r=(\ell-1)n}^{\ell n-1} F_r \quad (62)$$

and where we define $u_k = 0$ otherwise. Using the constant

$$\bar{M} = \sup \left\{ \sum_{s=(\ell-1)n-h}^{(\ell-1)n-1} \left| G_k \mathcal{M}(k, \ell) \prod_{r=s+1}^{(\ell-1)n-1} F_r \right| : k \in [(\ell-1)n, \ell n] : \ell \geq 1 + h/n, \ell \geq 3, \ell \in \mathbb{N} \right\} \quad (63)$$

and our \bar{B} from (7), we can then prove the following:

Theorem 4: Let Assumption 1 hold and $h \in \mathbb{N}$. Then each solution $x : \mathbb{Z}_0 \rightarrow \mathbb{R}^n$ of (4), in closed loop with the control (61), satisfies

$$|x_k| \leq \bar{B}(1 + \bar{M})|\delta|_{[k-3n-h,k]} \quad (64)$$

for all $k \geq 3n + h$. □

Proof: The proof is based on comparing solutions x_k obtained by using the original control (8) with the solutions that would instead be obtained by using the delay compensating control (61). To this end, first notice that we can choose $j = (\ell-1)n - h$ and $s = h$ in (10) to get

$$x_{(\ell-1)n} = \left(\prod_{r=(\ell-1)n-h}^{(\ell-1)n-1} F_r \right) x_{(\ell-1)n-h} + \sum_{s=(\ell-1)n-h}^{(\ell-1)n-1} \left(\prod_{r=s+1}^{(\ell-1)n-1} F_r \right) (G_s u_s + \delta_s) \quad (65)$$

for all integers $\ell \geq 1 + h/n$. By moving the terms containing δ values from the right side of (65) to the left side of (65), and then using the result to replace the quantity in squared brackets in (61), we can rewrite (61) as

$$u_k = -\mathcal{M}(k, \ell) \left[x_{(\ell-1)n} - \sum_{s=(\ell-1)n-h}^{(\ell-1)n-1} \left(\prod_{r=s+1}^{(\ell-1)n-1} F_r \right) \delta_s \right] \quad (66)$$

for all $k \in [(\ell-1)n, \ell n]$ and $\ell \geq 1 + h/n$. Since the first portion $-\mathcal{M}(k, \ell)x_{(\ell-1)n}$ of the control (66) is the same as the control from Theorem 1, the proof of Theorem 1 except with δ_k replaced by the new disturbance values

$$\delta_{k,\text{new}} = \delta_k + G_k \mathcal{M}(k, \ell) \sum_{s=(\ell-1)n-h}^{(\ell-1)n-1} \left(\prod_{r=s+1}^{(\ell-1)n-1} F_r \right) \delta_s \quad (67)$$

gives $|x_k| \leq \bar{B} |\delta_{\text{new}}|_{[k-2n-h, k]}$ for all $k \geq 2n + h$ such that $k \in [(\ell-1)n, \ell n]$ and $\ell \geq 1 + h/n$. Then the conclusion (64) follows from our choice of \bar{M} in (63). ■

VI. ILLUSTRATION

We apply Theorem 2 to the dynamics

$$x_{k+1} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.97 \end{bmatrix} x_k + \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix} u + f(k, x^k, \tau) + \epsilon_k \quad (68)$$

where the nonlinear function f , its state delay value $\tau \in \mathbb{Z}_0$, and the sequence ϵ_k are all unknown. In [2, Section IV], [16, Section 6.1.3.2], and [17], the system (68) was studied in the special case where f was zero, and where $u_k = x_{k-d_k}$ under suitable conditions on the time-varying delay d_k . Here we study a complementary problem of understanding how the nonlinearities f affect the rate of convergence.

To this end, notice that in this case, $n = 2$, so our function $\mathcal{B}(\ell, s)$ from (6) takes the constant value $F - GG^T W_*^{-1} F^2$ for all $\ell \in \mathbb{N}$ and $s \in \{1, \dots, n-1\}$, where $W_* = GG^T + FGG^T F^T$ is the constant value of the matrices W_ℓ as defined in (18). Using the Mathematica computer program, we found that in this case, the constant \bar{B} in (7) is

$$\begin{aligned} \bar{B} &= \max\{|F - GG^T W_*^{-1} F^2|, 1\} \\ &\quad + \max\{|(F - GG^T W_*^{-1} F^2)F|, |F|\} + 1 \\ &= 2.97. \end{aligned} \quad (69)$$

It follows from Theorem 2 above that, if we use the sampled controller (8), then all solutions $x : \mathbb{Z}_0 \rightarrow \mathbb{R}^2$ of the closed loop system satisfy the exponential ISS estimate

$$|x_k| \leq |x|_{[0, 4+\tau]} e^{\frac{\ln(2.97f_0)(k-4-\nu)}{4+\tau}} + \frac{2.97|\epsilon|_{[0, k]}}{(1-2.97f_0)^2} \quad (70)$$

for all $k \geq 4 + \tau$ provided there is a constant $\omega_0 > 0$ such that the nonlinearity f and its delay τ satisfy the requirement

$$|f(k, \phi)| \leq \frac{1}{2.97 + \omega_0} \sup_{m \in [-\tau, 0]} |\phi(m)| \quad (71)$$

for all functions $\phi : \{-\tau, \dots, 0\} \rightarrow \mathbb{R}^2$ and $k \geq 0$ (where the constant $\omega_0 > 0$ was required because of the strictness of the inequality in the requirement $\bar{B}f_0 < 1$ from (22)). In the special case where f is the zero function, we can instead use Theorem 1 to obtain finite-time ISS, where the fixed convergence time is $k = 4$. We believe that these results were beyond the scope of earlier treatments of the dynamics.

VII. CONCLUSIONS

We advanced the state of the art for robust control for discrete-time time-varying systems. Our new finite-time ISS approach used sampled feedbacks, and it applied to systems with outputs and input delays. We covered unknown nonlinearities that contain unknown state delays, using a trajectory based approach. Our output feedback control used an artificial delays method, where delays were used in the feedback control even when no delays were present in the original system, and it allowed measurement uncertainty. Since quantized and switched systems [6], [7] are also important in applications, we aim to prove generalizations for data-driven [18] switched systems with quantization that identify uncertainties in the coefficient matrices.

REFERENCES

- [1] R. Geiselhart and F. Wirth, "Relaxed ISS small-gain theorems for discrete-time systems," *SIAM Journal on Control and Optimization*, vol. 54, no. 2, pp. 423–449, 2016.
- [2] P. Pepe, "Discrete-time systems with constrained time delays and delay-dependent Lyapunov functions," *IEEE Transactions on Automatic Control*, vol. 65, no. 4, pp. 1724–1730, 2020.
- [3] E. Sontag, "Input to state stability: Basic concepts and results," in *Nonlinear and Optimal Control Theory*, P. Nistri and G. Stefani, Eds. Berlin, Germany: Springer, 2008, pp. 163–220.
- [4] A. Mironchenko, *Input-to-State Stability: Theory and Applications*. New York, NY: Springer, 2023.
- [5] F. Mazenc and M. Malisoff, "Finite-time stabilization of continuous-time systems with sampled control," *IEEE Control Systems Letters*, 2024, submitted, <http://www.math.lsu.edu/~malisoff/>.
- [6] M. Di Ferdinando, G. Pola, S. Di Gennaro, and P. Pepe, "On sampled-data control of nonlinear asynchronous switched systems," *IEEE Control Systems Letters*, vol. 8, pp. 1691–1696, 2024.
- [7] M. Di Ferdinando, S. Di Gennaro, D. Bianchi, and P. Pepe, "On robust quantized sampled-data tracking control of nonlinear systems," *IEEE Transactions on Automatic Control*, 2024, to appear, <https://doi.org/10.1109/TAC.2024.3393119>.
- [8] M. Di Ferdinando, S. Di Gennaro, and P. Pepe, "On Sontag's formula for the sampled-data observer-based stabilization of nonlinear time-delay systems," *Automatica*, vol. 153, no. 111052, 2023.
- [9] M. Di Ferdinando, P. Pepe, and S. Di Gennaro, "On robustification of sampled-data dynamic output-feedback stabilizers for control-affine nonlinear systems," *IEEE Control Systems Letters*, vol. 6, pp. 3277–3282, 2022.
- [10] J. Klamka, *Controllability and Minimum Energy Control*. New York, NY: Springer, 2019.
- [11] F. Mazenc and M. Malisoff, "Trajectory based approach for the stability analysis of nonlinear systems with time delays," *IEEE Transactions on Automatic Control*, vol. 60, no. 6, pp. 1716–1721, 2015.
- [12] F. Mazenc, M. Malisoff, and J. Knox, "Discrete-time trajectory-based stabilization approach using interval observer," *IEEE Control Systems Letters*, vol. 7, pp. 3175–3180, 2023.
- [13] F. Mazenc, M. Malisoff, and S.-I. Niculescu, "Stability and control design for time-varying systems with time-varying delays using a trajectory-based approach," *SIAM Journal on Control and Optimization*, vol. 55, no. 1, pp. 533–556, 2017.
- [14] S. Fueyo, G. Mazanti, I. Boussaada, Y. Chitour, and S.-I. Niculescu, "On the pole placement of scalar linear delay systems with two delays," *IMA Journal of Mathematical Control and Information*, vol. 40, no. 1, pp. 81–105, 2023.
- [15] E. Sontag, *Mathematical Control Theory, Second Edition*. New York, NY: Springer, 1998.
- [16] E. Fridman, *Introduction to Time-Delay Systems: Analysis and Control*. Boston, MA: Springer, 2014.
- [17] L. Hetel, J. Daafouz, and C. Iung, "Equivalence between the Lyapunov-Krasovskii functionals approach for discrete delay systems and that of the stability conditions for switched systems," *Nonlinear Analysis: Hybrid Systems*, vol. 2, no. 3, pp. 697–705, 2008.
- [18] J. Rueda-Escobedo, E. Fridman, and J. Schiffer, "Data-driven control for linear discrete-time delay systems," *IEEE Transactions on Automatic Control*, vol. 67, no. 7, pp. 3321–3336, 2021.