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# Estimation of non-separable regressions containing parameter dependent exponential functions

Rosane Ushirobira, Denis Efimov, Jose Guadalupe Romero and Romeo Ortega

**Abstract**—This paper presents a method for generating a separable regression function from a non-separable one, enabling the application of parameter estimation methods. In particular, we are interested in regressions containing parameter-dependent exponential functions — a scenario often encountered in physical systems. Our approach is based on algebraic techniques with the so-called annihilator theory, and utilizes an intermediate approximation of the nonlinear part by a polynomial function of the time. Two operators are proposed to define the annihilators: time delays and differential operators. The efficiency of the proposed approach is demonstrated in a nonlinearly parameterized fuel cell estimation problem.

## I. INTRODUCTION

Identifying values of constant parameters for models of different processes is an important fundamental mathematical and engineering problem that has been extensively studied [13], [16], [20]. Two drastically different scenarios appear when confronted with *separable* vs *non-separable* regressions. In the first case, we have a relation of the form  $y(t) = \phi^\top(t)\mathcal{G}(\theta)$ , where  $y(t) \in \mathbb{R}$  and  $\phi(t) \in \mathbb{R}^m$  are known functions of time and  $\mathcal{G} : \mathbb{R}^q \rightarrow \mathbb{R}^m$ ,  $q \leq m$ , is a known map with  $\theta \in \mathbb{R}^q$  the unknown constant parameter vector. In this case, applying (on- or off-line) parameter estimation methods to reconstruct  $\theta$ , under various assumptions regarding the map  $\mathcal{G}$ , like convexity/concavity or monotonicity (see, e.g., [1], [2], [7], [8], [12], [19], [22]). On the other hand, if the regression is non-separable, that is, of the form  $y(t) = \varphi(t, \theta)$ , where  $\varphi$  is a known map, there are no adaptation algorithms that allow us to estimate  $\theta$ .

In this work, similarly to [17], [23], we consider mixed non-separable regressions formed by a sum of linear and nonlinear parameterized terms, where the non-separable nonlinearity is given by an exponential function. Such an estimation problem is motivated by its appearance in several practical applications of great current interest, e.g., photovoltaic arrays [6], bioreactors [21], or fuel cells [5], [17]. We aim to extend the results of [23] and propose an estimation method based on an algebraic approach, the *annihilator theory*, to generate separable regressions. Two kinds of operators are considered for the annihilators: time-delays and differential operators.

For instance, the full model of the polarization curve of a fuel-cell is given by [9], [11]

$$y(t) = \theta_1 + \theta_2 \ln(u(t)) + \theta_3 u(t) + \theta_4 e^{\theta_5 u(t)}, \quad \forall t \geq 0 \quad (1)$$

where  $y(t)$  and  $u(t)$  are scalar measured quantities, while  $\theta_i$  ( $i = 1, \dots, 5$ ) are unknown constant real parameters whose values should be estimated. As we can see from this example, there is a linear regression part that contains  $\theta_1$ ,  $\theta_2$ , and  $\theta_3$ , while the dependence in  $\theta_4$  and  $\theta_5$  is nonlinear and nonseparable. This estimation problem has been considered in [5], [17], where only a simplified version of (1) has been treated. This work aims to unravel this kind of problem.

The outline of the paper is as follows. Preliminary results are introduced in Section II. The problem statement is given in Section III. Section IV establishes an auxiliary estimation algorithm. The main approach is formulated in Section V. An illustration of the fuel cell application of [5] is considered in Section VI.

### Notation

- $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} \mid x \geq 0\}$  and  $\mathbb{R}_{>0} = \mathbb{R}_{\geq 0} \setminus \{0\}$ , where  $\mathbb{R}$  is the set of real numbers, and  $\mathbb{N}$  denotes the set of nonnegative integers.
- $|\cdot|$  denotes the absolute value in  $\mathbb{R}$  and  $\|\cdot\|$  denotes the Euclidean norm on  $\mathbb{R}^n$ .
- For a linear operator  $\mathcal{O}$  defined on a vector space  $\mathcal{V}$ , its action on  $y \in \mathcal{V}$  will be denoted  $\mathcal{O}[y]$ .

## II. PRELIMINARIES

In this paper we consider the problem of parameter estimation of a non-separable regression of the form

$$y(t) = \varphi(t, \theta), \quad \forall t \in \mathbb{R}_{\geq 0},$$

where  $y(t) \in \mathbb{R}$  is the measured signal,  $\theta = (\theta_1 \dots \theta_q)^\top \in \mathbb{R}^q$  is the vector of unknown parameters to be estimated,  $\varphi : \mathbb{R}_{\geq 0} \times \mathbb{R}^q \rightarrow \mathbb{R}$  is a known nonlinear function of time. Similarly to [23], our main idea is to construct so-called *annihilators*. An annihilator is defined as a parameterized polynomial or rational function  $\mathcal{A}$  of an operator  $\mathcal{O}$  that, acting on the signal  $y$ , generates an identically zero signal:

$$\mathcal{A}(\mathcal{O}, \eta)[y] = 0,$$

where  $\eta \in \mathbb{R}^{q_\eta}$  is a vector of parameters determined by  $\theta$ . The resulting equation can be solved with respect to some  $\theta_i$ .

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### A. The case of delay operator

In [23], we considered the case where  $\mathcal{O} = S$  is the delay operator, so  $S[y](t) = y(t - \tau)$ , with  $\tau \in \mathbb{R}_{>0}$  a constant free parameter representing the delay. By the definition of  $S$ , we have  $S^k[y] = y(t - k\tau)$ , for  $k \in \mathbb{N}$ . In this case,  $\mathcal{A}(S, \eta)$  is a polynomial function in  $S$  with coefficients parameterized by  $\eta$ .

For instance, consider

$$\varphi(t, \theta) = e^{\theta_1 t} \cos(\theta_2 t + \theta_3), \quad \forall t \in \mathbb{R}_{>0} \quad (2)$$

where  $\theta \in \mathbb{R}^3$ , and assume that  $y(t) = \varphi(t, \theta)$  is available for measurements. So choosing any delay  $\tau > 0$  in the annihilator

$$\mathcal{A}(S, \eta) = S^2 - 2\eta_1 S + \eta_2, \quad (3)$$

with  $\eta = (\cos(\theta_2 \tau) e^{-\theta_1 \tau} \quad e^{-2\theta_1 \tau})^\top$ , a direct computation shows that

$$\mathcal{A}(S, \eta)[y] = y(t - 2\tau) - 2\eta_1 y(t - \tau) + \eta_2 y(t) = 0.$$

The latter equality can be rewritten as the linear regression equation (LRE)  $\bar{y} = \bar{\phi}^\top \eta$ , where  $\bar{y}(t) \in \mathbb{R}$  and  $\bar{\phi}(t) \in \mathbb{R}^2$  are measured quantities, defined as

$$\bar{y}(t) = y(t - 2\tau), \quad \bar{\phi}^\top(t) := (2y(t - \tau) \quad -y(t)).$$

If the new regressor  $\bar{\phi}(t)$  is sufficiently excited, then the value of the vector  $\eta$  can be reconstructed using various existing approaches. Finally, it is easy to see that having the value of  $\eta$  we can obtain  $\theta_1$  and  $\theta_2$ . Once these parameters are identified, the value of the remaining parameter  $\theta_3$  can be evaluated.

Similar manipulations and computations can be performed on other cases as shown in Table I. Moreover, it has been proven in [23] that there exists a unique annihilator (up to multiplication) for any function  $\varphi$  being a combination of exponential functions of complex exponents with coefficients polynomials of time.

### B. The case of differential operator

In this subsection, we set  $p = \frac{d}{dt}$  the time-derivative operator, and we consider the differential operator  $\mathcal{O} = p$ . Then, for  $k \in \mathbb{N}$ ,  $p^k := \frac{d^k}{dt^k}$  is the usual composition. Given  $\lambda > 0$ , we define the operator  $H(p)$ , a rational function of  $p$ :

$$H(p) = \frac{\lambda}{(p + \lambda)^2}$$

(also known as a linear filter).

Consider the following non-separable nonlinear regression equation

$$y(t) = \phi(t, \theta) = \cos(\theta_1 t + \theta_2).$$

Taking the following annihilator, with  $\eta = \theta_1^2$ :

$$\mathcal{A}(p, \eta) = p^2 + \eta,$$

it yields  $\mathcal{A}(p, \eta)[y] = 0$  and we obtain an LRE. However, to obtain an implementable LRE, we will consider another annihilator, which is a multiple of the operator  $\mathcal{A}$ , namely

$$H(p)\mathcal{A}(p, \eta) = \frac{p^2 + \eta}{(p + \lambda)^2}$$

(first we act  $\mathcal{A}(p, \eta)$  on  $y$  and then  $H(p)$ ). That allows us to obtain the following LRE  $\bar{y} = \bar{\phi}\eta$  (with  $\bar{y}$  and  $\bar{\phi}$  scalar measured quantities):

$$\bar{y} = \frac{\lambda p^2 y}{(p + \lambda)^2}, \quad \bar{\phi} = \frac{\lambda y}{(p + \lambda)^2}.$$

It is easy to see that, given the knowledge of  $\theta_1$ , it is possible to estimate the parameter  $\theta_2$  using basic trigonometry.

Similar manipulations and computations can be performed on other cases, as shown in Table II. It is important to notice that, unlike the case of delay operators, for differential operators it is necessary to compose with  $H$  (in other words, to add a filtering process) to generate implementable regression equations.

**Remark II.1.** *The theory of differential operators belongs to the oldest branch of the annihilator theory, provided by the theory of ordinary differential equations (ODE). In this context, if we have a differentiable function (signal)  $y$  that satisfies an ODE with constant coefficients, it implies that the differential operator associated with the ODE (denoted by  $\mathcal{D}$ ) annihilates  $y$ , i.e.,  $\mathcal{D}[y] = 0$ .*

*A well-known class that satisfies homogeneous ODEs is the holonomic functions class. This class includes all functions listed in Table II, as well as algebraic functions (including polynomials or rational functions in  $t$ ), sine and cosine functions, exponential and logarithmic functions, orthogonal polynomials, and many more. Readers interested in learning more about this topic should refer to any book on ODEs.*

*In this paper, delay operators are used to construct annihilators, but as has been shown, the differential operators may be similarly applied to non-separable nonlinear regressions. Future research will require more detailed studies. Subsection II-B serves as a motivating example of annihilators commonly used in automatic control systems [21], which are differential operators combined with a linear filter.*

**Remark II.2.** *The annihilator theory is applied to the parameter estimation problem in the seminal paper [14]: the Laplace transform is used to obtain algebraic equations from differential ones. The framework of this work and related ones [15] is within the operational domain. The obtained annihilators live in the Weyl algebra, which provides good properties thanks to its particular algebraic structure (see [24]).*

## III. PROBLEM STATEMENT

Consider a nonlinear, non-separable regression

$$y(t) = \phi_1^\top(t)\theta_1 + \phi_3(t)\theta_3 e^{\phi_2^\top(t)\theta_2}, \quad \forall t \in \mathbb{R}_{\geq 0} \quad (4)$$

Function $\varphi(t, \theta)$	Annihilator $\mathcal{A}(S, \eta)$	$\eta$
$t^n$ ( $n \in \mathbb{N}$ )	$(S - 1)^{n+1}$	
$\theta_2 e^{\theta_1 t}$ ( $\theta \in \mathbb{R}^2$ )	$S - \eta$	$e^{-\theta_1 \tau}$
$\cos(\theta_1 t + \theta_2)$ ( $\theta \in \mathbb{R}^2$ )	$S^2 - 2\theta S + 1$	$\cos(\theta_1 \tau)$
$\sin(\theta_1 t + \theta_2)$ ( $\theta \in \mathbb{R}^2$ )	$S^2 - 2\eta S + 1$	$\cos(\theta_1 \tau)$
$e^{\theta_1 t} \cos(\theta_2 t + \theta_3)$ ( $\theta \in \mathbb{R}^3$ )	$S^2 - 2\eta_1 S + \eta_2$	$\begin{pmatrix} \cos(\theta_2 \tau) e^{-\theta_1 \tau} \\ e^{-2\theta_1 \tau} \end{pmatrix}$
$t^n e^{\theta t}$ ( $\theta \in \mathbb{R}$ )	$(S - \eta)^{n+1}$	$e^{-\theta \tau}$
$t^n \cos(\theta_1 t + \theta_2)$ ( $\theta \in \mathbb{R}^2$ )	$(S^2 - 2\eta S + 1)^{n+1}$	$\cos(\theta_1 \tau)$

TABLE I  
LIST OF TRANSFORMATIONS FROM [23]

TABLE II  
ANNIHILATORS WITH DIFFERENTIAL OPERATOR

Function $\varphi(t, \theta)$	Annihilator $\mathcal{A}(p, \eta)$	$\eta$
$\theta_2 \exp^{\theta_1 t}$	$p - \eta$	$\theta_1$
$\cos(\theta_1 t + \theta_2)$	$p^2 + \eta$	$\theta_1^2$
$\sin(\theta_1 t + \theta_2)$	$p^2 + \eta$	$\theta_1^2$
$\exp^{\theta_1 t} \cos(\theta_2 t + \theta_3)$	$p^2 - 2\eta_1 p + \eta_2$	$\begin{bmatrix} \theta_1 \\ \theta_1^2 + \theta_2^2 \end{bmatrix}$
$t^n \exp^{\theta t}$	$(p - \eta_1)^{n+1}$	$\begin{bmatrix} \theta_1 \\ \theta_1^2 \\ \vdots \\ \theta_1^n \end{bmatrix}$
$t^n \cos(\theta_1 t + \theta_2)$	$(p^2 + \eta_1^2)^{n+1}$	$\begin{bmatrix} \theta_1^2 \\ \theta_1^4 \\ \vdots \\ \theta_1^{2n} \end{bmatrix}$

where  $y(t) \in \mathbb{R}$  is a measured signal,  $\theta_1 \in \mathbb{R}^{q_1}$ ,  $\theta_2 \in \mathbb{R}^{q_2}$ , and  $\theta_3 \in \mathbb{R}$  are unknown constant parameters,  $\phi_1(t) \in \mathbb{R}^{q_1}$ ,  $\phi_2(t) \in \mathbb{R}^{q_2}$ ,  $\phi_3(t) \in \mathbb{R}_{>0}$  are measured regressors, and  $v_1(t), v_2(t) \in \mathbb{R}$  are an essentially bounded noises.

We aim to develop an approach for the robust estimation of the vector of unknown parameters  $\theta = (\theta_1^\top \ \theta_2^\top \ \theta_3)^\top \in \mathbb{R}^{q_1+q_2+1}$  in (4), assuming that the level of excitation of the regressor  $\phi = (\phi_1^\top \ \phi_2^\top \ \phi_3)^\top$  is sufficient (more details will be given later).

To this end, it is enough to find a way to transform (4) into a LRE, as done in the Subsection II-A. Next, many different methods can be used, such as the DREM approach [3], [19], which allows a robust fixed-time estimation under weak excitation [4], [10], [25]. This is the idea that we will explore in this work.

#### IV. ESTIMATION OF EXPONENTIAL FUNCTIONS OF TIME POLYNOMIALS

Let us start with an intermediate result that extends the method of [23] to the following scenario:

$$y(t) = \varphi(t, \theta) = e^{P(t, \theta)}, \quad (5)$$

with  $P(t, \theta) = \sum_{i=0}^{q-1} \theta_{i+1} t^i$  a time polynomial in  $\mathbb{R}[t]$ ,  $y(t) \in \mathbb{R}_{>0}$  is the measured signal, and  $\theta \in \mathbb{R}^q$  is the vector of unknown parameters to be estimated.

Note that by definition  $y(t) \in \mathbb{R}_{>0}$ , hence it has an inverse. For a delay  $\tau > 0$ , let us apply the operator  $S$

on the output:

$$S[y] = y(t - \tau) = e^{P(t-\tau, \theta)} = y(t) e^{\sum_{i=0}^{q-2} \theta_{i+1}^1 (t-\tau)^i},$$

which can be rearranged as follows:

$$y^1(t) = e^{\sum_{i=0}^{q-2} \theta_{i+1}^1 (t-\tau)^i}$$

for  $y^1(t) = \frac{y(t-\tau)}{y(t)} \in \mathbb{R}_{>0}$  and properly computed  $\theta^1 = (\theta_1^1 \ \dots \ \theta_{q-1}^1)^\top \in \mathbb{R}^{q-1}$  (being dependent linearly on  $\theta$  and nonlinearly on  $\tau$ , with exact expressions conditioned by the degree  $q$ ). Therefore, applying the delay operator to (5), we can represent it in the same form, but for a new vector of unknown parameters  $\theta^1$ , whose dimension is decreased by one, and a new measured output  $y^1$ . Consequently, repeating this transformation  $q - 1$  times, we will obtain

$$y^{q-1}(t) = e^{\theta_1^{q-1}}$$

for some iteratively defined  $y^{q-1}(t) \in \mathbb{R}$  and  $\theta_1^{q-1} \in \mathbb{R}$ . This equation can be solved for  $\theta_1^{q-1}$ , and next, we proceed backward, finding all intermediate coefficients, and finally reconstructing  $\theta$ . In other words, we have an annihilator for (5).

The proposed approach allows a particular class of nonlinear regression estimation problems to be reduced to a set of linear ones.

Note that for (5), the condition of persistence of excitation is satisfied by construction since the signal is decomposed on a polynomial basis.

**Example IV.1.** To illustrate the procedure described above, we consider the case of  $q = 3$ :

$$y(t) = e^{\theta_3 t^2 + \theta_2 t + \theta_1} + v(t),$$

where the uncertain term  $v(t) \in \mathbb{R}$ , representing additive noise. Applying the suggested approach, we obtain:

$$\begin{aligned} y(t - \tau) &= e^{\theta_3 t^2 + \theta_2 t + \theta_1} e^{\theta_3 (\tau - 2t) \tau - \theta_2 \tau} + v(t - \tau) \\ &= (y(t) - v(t)) e^{\theta_3 (\tau - 2t) \tau - \theta_2 \tau} + v(t - \tau), \\ y(t - 2\tau) &= (y(t - \tau) - v(t - \tau)) e^{\theta_3 (\tau - 2t) \tau - \theta_2 \tau} e^{2\theta_3 \tau^2} \\ &\quad + v(t - 2\tau) \\ &= \frac{(y(t - \tau) - v(t - \tau))^2}{y(t) - v(t)} e^{2\theta_3 \tau^2}. \end{aligned}$$

Note that  $y - v$  equals an exponential function. Hence, it is always separated from zero. These equations lead to the following expressions for  $\theta$ :

$$\begin{aligned}\theta_3 &= \frac{1}{2\tau^2} \ln \left( \frac{y(t-2\tau)(y(t)-v(t))}{(y(t-\tau)-v(t-\tau))^2} \right), \\ \theta_2 &= \frac{1}{\tau} \ln \left( \frac{y(t)-v(t)}{y(t-\tau)-v(t-\tau)} \right) \\ &+ \frac{\tau-2t}{2\tau^2} \ln \left( \frac{y(t-2\tau)(y(t)-v(t))}{(y(t-\tau)-v(t-\tau))^2} \right), \\ \theta_1 &= \ln(y(t)-v(t)) - \frac{t}{\tau} \ln \left( \frac{y(t)-v(t)}{y(t-\tau)-v(t-\tau)} \right) \\ &+ \frac{t^2-\tau t}{2\tau^2} \ln \left( \frac{y(t-2\tau)(y(t)-v(t))}{(y(t-\tau)-v(t-\tau))^2} \right),\end{aligned}$$

from which we can get the required estimates:

$$\begin{aligned}\hat{\theta}_3(t) &= \frac{1}{2\tau^2} \ln \left( \frac{y(t-2\tau)y(t)}{y^2(t-\tau)} \right), \\ \hat{\theta}_2(t) &= \frac{1}{\tau} \ln \left( \frac{y(t)}{y(t-\tau)} \right) + \frac{\tau-2t}{2\tau^2} \ln \left( \frac{y(t-2\tau)y(t)}{y^2(t-\tau)} \right), \\ \hat{\theta}_1(t) &= \ln(y(t)) - \frac{t}{\tau} \ln \left( \frac{y(t)}{y(t-\tau)} \right) \\ &+ \frac{t^2-\tau t}{2\tau^2} \ln \left( \frac{y(t-2\tau)y(t)}{y^2(t-\tau)} \right).\end{aligned}$$

The noise dependence for these estimates can be easily evaluated. The method gives a fixed-time converging solution (the estimates are derived for  $t > 2\tau$ ).

#### V. ESTIMATION IN THE NONLINEAR REGRESSION (4)

We will use the same idea as in the previous section by showing that under mild restrictions, a delay-based transformation can be iteratively applied to (4) that reduces the initial problem to a simpler one, which, under a finite number of iterations, degenerates to a linear regression.

First, we decouple the estimation problem for  $\theta_1$ ,  $\theta_3$ , and  $\theta_2$  appearing in the exponent. To this end, assume that there exists  $\theta_4 \in \mathbb{R}^{q_4}$  such that

$$\phi_2^\top(t)\theta_2 + v_2(t) = \theta_{4,0} + \sum_{i=1}^{q_4} \theta_{4,i}t^i, \quad (6)$$

where  $\theta_4 = (\theta_{4,1} \ \dots \ \theta_{4,q_4})^\top$  is unknown and should be identified, while the constant term  $\theta_{4,0}$  is fixed due to the presence of  $\theta_3$  (for simplicity, we can take  $\theta_{4,0} = 1$ , and if the sign of  $\theta_3$  is known, say positive, then  $\theta_{4,0} = \ln(\theta_3)$  can be considered unknown). Once the parameters in  $\theta_4$  are estimated, (6) is just a linear regression with respect to  $\theta_2$  that can be used to identify its values provided that the regressor  $\phi_2$  is (at least) intervally or persistently excited.

**Remark V.1.** Note that we do not need the relation (6) to be satisfied for all  $t \geq 0$ , it can be valid only locally, on a time window  $[iT, (i+1)T)$  with some period  $T$  and  $i \in \mathbb{N}$ , and  $\theta_4$  must be constant only on an interval,

and it may change its value next. It is sufficient for identification if we use finite-time estimation schemes based on DREM [10], [18], [19], [25], for example, which can evaluate  $\theta_2$  rapidly once the estimate for  $\theta_4$  has been also quickly derived using the measurements from a part of the interval.

Therefore, (4) can be rewritten as follows:

$$y(t) = \phi_1^\top(t)\theta_1 + \phi_3(t)\theta_3 e^{\sum_{i=0}^{q_4} \theta_{4,i+1}t^i} + v_1(t), \quad (7)$$

where we need to estimate  $\theta_1$ ,  $\theta_3$  and  $\theta_4$ . Let us apply the delay operator on this output:

$$\begin{aligned}S[y] &= y(t-\tau) = \phi_1^\top(t-\tau)\theta_1 + v_1(t-\tau) \\ &+ \phi_3(t-\tau)\theta_3 e^{\sum_{i=0}^{q_4} \theta_{4,i+1}(t-\tau)^i} \\ &= \phi_1^\top(t-\tau)\theta_1 + v_1(t-\tau) \\ &+ \phi_3(t-\tau)\theta_3 e^{\sum_{i=0}^{q_4} \theta_{4,i+1}t^i} e^{\sum_{i=0}^{q_4-1} \theta_{4,i+1}^1 t^i} \\ &= \phi_1^\top(t-\tau)\theta_1 + v_1(t-\tau) \\ &+ \frac{\phi_3(t-\tau)}{\phi_3(t)} (y(t) - \phi_1^\top(t)\theta_1 - v_1(t)) e^{\sum_{i=0}^{q_4-1} \theta_{4,i+1}^1 t^i},\end{aligned}$$

where  $\theta_4^1 \in \mathbb{R}^{q_4}$  is some new vector of unknown parameters linearly dependent on  $\theta_4$  and nonlinearly on  $\tau$  (on this step, the known element  $\theta_{4,0}$  and  $\theta_3$  have been eliminated). Recall that  $\phi_3(t) \in \mathbb{R}_{>0}$  by standing assumptions, hence, we can divide by it. Applying the delay operation to this new expression we obtain:

$$\begin{aligned}y(t-2\tau) &= \phi_1^\top(t-2\tau)\theta_1 + v_1(t-2\tau) + \frac{\phi_3(t-2\tau)}{\phi_3(t-\tau)} \\ &\times (y(t-\tau) - \phi_1^\top(t-\tau)\theta_1 - v_1(t-\tau)) e^{\sum_{i=0}^{q_4-1} \theta_{4,i+1}^1 (t-\tau)^i} \\ &= \phi_1^\top(t-2\tau)\theta_1 + v_1(t-2\tau) + \frac{\phi_3(t-2\tau)}{\phi_3(t-\tau)} e^{\sum_{i=0}^{q_4-2} \theta_{4,i+1}^2 t^i} \\ &\times (y(t-\tau) - \phi_1^\top(t-\tau)\theta_1 - v_1(t-\tau)) e^{\sum_{i=0}^{q_4-1} \theta_{4,i+1}^1 t^i} \\ &= \phi_1^\top(t-2\tau)\theta_1 + v_1(t-2\tau) + \frac{\phi_3(t)\phi_3(t-2\tau)}{\phi_3^2(t-\tau)} \\ &\times \frac{(y(t-\tau) - \phi_1^\top(t-\tau)\theta_1 - v_1(t-\tau))^2}{y(t) - \phi_1^\top(t)\theta_1 - v_1(t)} e^{\sum_{i=0}^{q_4-2} \theta_{4,i+1}^2 t^i},\end{aligned}$$

where  $\theta_4^2 \in \mathbb{R}^{q_4-1}$  is another new vector of unknown parameters with similar properties, having a reduced dimension with respect to  $\theta_4$ .

Clearly, repeating this operation  $q_4 - 1$  times, as in the previous section, the order of the polynomial in the exponent can be brought to zero, then the remaining parameter  $\theta_{4,1}^{q_4-1}$  can be identified from the last equation, which can be rewritten as an extended linear regression.

**Example V.2.** To demonstrate how this method works, let us consider  $q_4 = 2$ , then the expressions above take

the form:

$$\begin{aligned}
y(t) &= \phi_1^\top(t)\theta_1 + \phi_3(t)\theta_3 e^{\theta_{4,2}t^2 + \theta_{4,1}t + \theta_{4,0}} + v_1(t), \\
y(t-\tau) &= \phi_1^\top(t-\tau)\theta_1 + v_1(t-\tau) \\
&+ \frac{\phi_3(t-\tau)}{\phi_3(t)} (y(t) - \phi_1^\top(t)\theta_1 - v_1(t)) e^{\theta_{4,2}(\tau-2t)\tau - \theta_{4,1}\tau}, \\
y(t-2\tau) &= \phi_1^\top(t-2\tau)\theta_1 + v_1(t-2\tau) + \frac{\phi_3(t)\phi_3(t-2\tau)}{\phi_3^2(t-\tau)} \\
&\times \frac{(y(t-\tau) - \phi_1^\top(t-\tau)\theta_1 - v_1(t-\tau))^2}{y(t) - \phi_1^\top(t)\theta_1 - v_1(t)} e^{2\theta_{4,2}\tau^2},
\end{aligned}$$

which can be used to write linear regressions to estimate  $\theta_4$  (for brevity we write only the case for  $v_1 \equiv 0$ ):

$$\begin{aligned}
&y(t-2\tau)y(t) - \rho_3(t)y^2(t-\tau) \\
&= (y(t-2\tau)\phi_1^\top(t) + y(t)\phi_1^\top(t-2\tau))\theta_1 \\
&- \phi_1^\top(t)\theta_1\phi_1^\top(t-2\tau)\theta_1 + \rho_3(t)[\phi_1^\top(t-\tau)\theta_1\phi_1^\top(t-\tau)\theta_1]
\end{aligned} \tag{8}$$

$$\begin{aligned}
&-2y(t-\tau)\phi_1^\top(t-\tau)\theta_1]e^{2\theta_{4,2}\tau^2}, \\
y(t-\tau) &= \phi_1^\top(t-\tau)\theta_1 + \rho_2(t)(y(t) - \phi_1^\top(t)\theta_1) e^{-\theta_{4,1}\tau},
\end{aligned} \tag{9}$$

$$y(t) = \phi_1^\top(t)\theta_1 + \rho_1(t)\theta_3, \tag{10}$$

where

$$\begin{aligned}
\rho_1(t) &= \phi_3(t)e^{\hat{\theta}_{4,2}(t)t^2 + \hat{\theta}_{4,1}(t)t + \theta_{4,0}}, \\
\rho_2(t) &= \frac{\phi_3(t-\tau)}{\phi_3(t)} e^{\hat{\theta}_{4,2}(t)(\tau-2t)\tau}, \quad \rho_3(t) = \frac{\phi_3(t)\phi_3(t-2\tau)}{\phi_3^2(t-\tau)}
\end{aligned}$$

are known time-dependent functions, and  $\hat{\theta}_{4,2}(t) \in \mathbb{R}$ ,  $\hat{\theta}_{4,1}(t) \in \mathbb{R}$  are the estimates of the respective components on  $\theta_4$ , which have to be derived first to proceed with estimation of  $\theta_{4,1}$  and  $\theta_3$ , respectively. Note that the term  $\phi_1^\top(t)\theta_1\phi_1^\top(t-2\tau)\theta_1$  and similar ones in (8) can be presented as a linear regression of extended dimension with respect to the vector of parameters  $\bar{\theta}_1 = [\theta_{1,1}^2 \ \theta_{1,1}\theta_{1,2} \ \dots \ \theta_{1,1}\theta_{1,q_1} \ \theta_{1,2}^2 \ \theta_{1,2}\theta_{1,3} \ \dots \ \theta_{1,q_1}^2]^\top \in \mathbb{R}^{\frac{q_1+1}{2}q_1}$ . So, the total linear regression in (8) has components  $\theta_1$ ,  $\bar{\theta}_1$ ,  $\theta_1 e^{2\theta_{4,2}\tau^2}$  and  $\theta_1 e^{2\theta_{4,1}\tau}$ , which allows us to estimate  $\theta_1$  and  $e^{2\theta_{4,2}\tau^2}$  (hence,  $\theta_{4,2}$ ). Next, in (9) we have a linear regression with respect to  $\theta_1$ ,  $e^{-\theta_{4,1}\tau}$  and  $\theta_1 e^{-\theta_{4,1}\tau}$ , which provides estimates for  $\theta_1$  and  $\theta_{4,1}$ . Finally, (10) is a linear regression in  $\theta_1$  and  $\theta_3$  (where  $\theta_{4,0}$  is fixed by a designer), hence the value of  $\theta_3$  can be derived together with an auxiliary estimate of  $\theta_1$ .

Therefore, estimating  $\theta_1$ ,  $\theta_3$  and  $\theta_4$  may require a rather high excitation level of the regressor. On the other side, since  $\theta_1$  is identified several times (it appears as a part of all linear regressions), these estimates can be fused to improve their evaluation accuracy or utilized to detect the convergence of the algorithms. Another way is just to substitute the previously obtained estimates of  $\theta_1$  in upcoming regressions, which may simplify the algorithm significantly.

Let us investigate the efficiency of the proposed approach in the motivating application (1).

## VI. APPLICATION TO THE FUEL CELL EXAMPLE

Clearly, the nonlinear regression (1) from [5], [17] can be rewritten in the form (4):

$$y(t) = \phi_1^\top(t)\theta_1 + \theta_3 e^{\phi_2(t)\theta_2 + v_2(t)} + v_1(t),$$

where  $q_1 = 3$ ,  $q_2 = 1$ ,

$$\begin{aligned}
\theta_1 &= (\vartheta_1 \ \vartheta_2 \ \vartheta_3)^\top, \quad \phi_1^\top(t) = (1 \ \ln(u(t)) \ u(t)); \\
\theta_2 &= \vartheta_5, \quad \phi_2(t) = u(t), \quad \theta_3 = \vartheta_4, \quad \phi_3(t) = 1,
\end{aligned}$$

and we assume that the measurements of  $y$  and  $u$  are available with noises that can be represented by  $v \in \mathcal{L}_\infty^2$ . The frequency of measurements is 100Hz.

Similarly to [5], for simulations we assume that

$$\begin{aligned}
\vartheta_1 &= 39.3543, \quad \vartheta_2 = -2.582, \quad \vartheta_3 = -0.1808, \\
\vartheta_4 &= -1.261, \quad \vartheta_5 = 0.0046,
\end{aligned}$$

and  $u(t) = 5 \sin(0.2\pi t) + 25$ . The harmonic signal  $\phi_2(t)\theta_2$  can be approximated locally, for  $t \in [(i-1)T, iT)$  with  $i \in \mathbb{N}$ , where  $T = 0.1$  for simulation, by a time polynomial of degree  $q_4 = 1$  (we do not use a higher approximation due to a weak excitation of the regressors):

$$\phi_2(t)\theta_2 + v_2(t) = \theta_{4,1}t + \theta_{4,0},$$

where  $\theta_{4,0}$  is randomly chosen at the instant  $iT$  in the admissible range of  $\vartheta_5 u(t) \pm 50\%$ . Following our approach, it is better to select  $u$  as a piecewise linear in-time function, then the parameter  $\theta_2$  can be found directly skipping the intermediate identification of  $\theta_4$ .

The linear regressions for estimating  $\theta_4$  take the form:

$$\begin{aligned}
\tilde{y}(t) &= (\phi_1^\top(t-\tau) \ -\phi_1^\top(t)) \begin{pmatrix} \theta_1 \\ \theta_1 e^{-\theta_{4,1}\tau} \end{pmatrix} + \tilde{v}_1(t) \\
&= \tilde{\phi}^\top(t)\tilde{\theta} + \tilde{v}_1(t),
\end{aligned}$$

$$y(t) = \left( \phi_1^\top(t) \ e^{\hat{\theta}_{4,1}(t)t + \theta_{4,0}} \right) \begin{pmatrix} \theta_1 \\ \theta_3 \end{pmatrix} + v_1(t),$$

where

$$\begin{aligned}
\tilde{y}(t) &= y(t-\tau) - y(t), \\
\tilde{\phi}^\top(t) &= (1 \ \ln(u(t-\tau)) \ u(t-\tau) \ -\ln(u(t)) \ -u(t)), \\
\tilde{\theta} &= ((1 - e^{-\theta_{4,1}\tau})\theta_{1,1} \ \theta_{1,2} \ \theta_{1,3} \ \theta_{1,2}e^{-\theta_{4,1}\tau} \ \theta_{1,3}e^{-\theta_{4,1}\tau})^\top \\
&\quad \tilde{v}_1(t) = v_1(t-\tau) - e^{-\theta_{4,1}\tau}v_1(t),
\end{aligned}$$

where we have grouped the terms with the same weights. Applying the DREM method [4], [10] to these regressions, where we use the delay-based extension filters, we get the estimates  $\hat{\theta}_1(t)$ ,  $\hat{\theta}_3(t)$  and  $\hat{\theta}_{4,1}(t)$  after a finite number of steps. Since  $\phi_2(t)$  is separated from zero, the value of  $\theta_2$  can be directly found:

$$\hat{\theta}_2(t) = \frac{\hat{\theta}_{4,1}(t)t + \theta_{4,0}}{\phi_2(t)}.$$

The results of estimation for  $\tau = 0.5$  and  $v_1$  a uniformly distributed noise with amplitude  $10^{-5}$  are shown in Fig. 1. Due to a weak excitation in the system, it is difficult to counteract the effect of measurement noise,

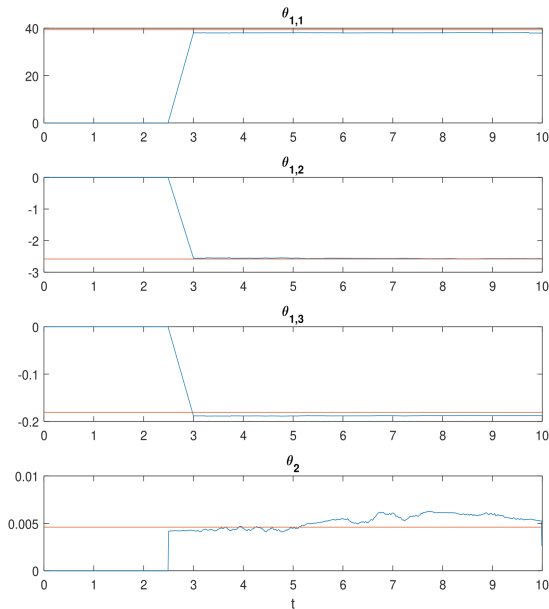


Fig. 1. The results of estimation for  $\theta_1$  and  $\theta_2$

and the estimation quality for  $\theta_3$  is not satisfactory. A similar problem was encountered in [5], where it is thoroughly discussed.

## VII. CONCLUSION

This paper addressed the parameter identification problem in a non-separable regression composed of linear and nonlinear terms, where the nonlinearity is presented by an exponential function. Such a challenging problem was solved in two steps: first, the regression in the nonlinear term was approximated by a time series. Second, applying delay-based annihilators, it was shown that the unknown coefficients from a time series in the exponent could be reduced to a linear regression. The first part of the presentation was given for the case of noise-free measurements. As shown in the second part, the same manipulations can be repeated for the noisy case, which would lead to a new perturbation in the derived linear regression, depending on the original noise's delayed (or the filtered) values. The efficiency of the proposed approach was illustrated in a fuel cell application.

## REFERENCES

- [1] V. Adetola, M. Guay, and D. Lehrer. Adaptive estimation for a class of nonlinearly parameterized dynamical systems. *IEEE Transactions on Automatic Control*, 59(10):2818–2824, 2014.
- [2] A. M. Annaswamy, F. P. Skantze, and A. Loh. Adaptive control of continuous time systems with convex/concave parametrization. *Automatica*, 34(1):33–49, 1998.
- [3] S. Aranovskiy, A. Bobtsov, R. Ortega, and A. Pyrkin. Performance enhancement of parameter estimators via dynamic regressor extension and mixing. *IEEE Transactions on Automatic Control*, 62(7):3546–3550, 2017.
- [4] S. Aranovskiy, R. Ushirobira, D. Efimov, and J. Wang. Accelerated convergence with improved robustness for discrete-time parameter estimation. *Systems & Control Letters*, 168, 2022.
- [5] C. Beltran, A. Bobtsov, R. Ortega, D. Langarica-Cordoba, R. Cisneros, and L. H. Diaz-Saldierna. On-line parameter estimation of the polarization curve of a fuel cell with guaranteed convergence properties: Theoretical and experimental results. *IEEE Transactions on Industrial Electronics (Early Access)*, pages 1–8, 2024.
- [6] A. Bobtsov, F. Mancilla-David, S. Aranovskiy, and R. Ortega. On-line identification of photovoltaic arrays' dynamic model parameters. *arXiv:2209.07246v1*, 2022.
- [7] V. Fomin, A. Fradkov, and V. Yakubovich. *Adaptive control of dynamical systems*. Nauka, Moscow, 1981. In Russian.
- [8] A. Fradkov, R. Ortega, and G. Bastin. Semi-adaptive control of convexly parametrized systems with application to temperature regulation of chemical reactors. *International Journal of Adaptive Control and Signal Processing*, 15(4):415–426, 2001.
- [9] L. Guzzella. Control-oriented modeling of fuel-cell based vehicles. In *NSF Workshop on the Integration of Modeling and Control for Automotive Systems*, Santa Barbara, USA, 1999.
- [10] M. Korotina, S. Aranovskiy, R. Ushirobira, D. Efimov, and J. Wang. A note on fixed- and discrete-time estimation via the DREM method. *IEEE Transactions on Automatic Control*, pages 1–6, 2024.
- [11] J Larminie and A. Dicks. *Fuel Cell Systems Explained*. John Wiley and Sons, England, second edition, 2003.
- [12] X. Liu, R. Ortega, H. Su, and J. Chu. Immersion and invariance adaptive control of nonlinearly parameterized nonlinear systems. *IEEE Transactions on Automatic Control*, 55(9):2209–2214, 2010.
- [13] L. Ljung. *System identification: theory for the user*. Prentice Hall PTR, 1999.
- [14] M. Fliess and H. Sira-Ramírez. An algebraic framework for linear identification. *ESAIM: COCV*, 9, 2003.
- [15] A. Neves, Miranda M. D., and Mboup M. Algebraic parameter estimation of damped exponentials. In *Proc. 15th IEEE European Signal Processing Conference*, September 2007.
- [16] J.P. Norton. *An introduction to identification*. Dover, NY, 1986.
- [17] R. Ortega, A. Bobtsov, N. Nikolaev, and R. Costa-Castelló. Parameter estimation of two classes of nonlinear systems with non-separable nonlinear parameterizations. *Automatica*, 163:111559, 2024.
- [18] R. Ortega, V. Nikiforov, and D. Gerasimov. On modified parameter estimators for identification and adaptive control. a unified framework and some new schemes. *Annual Reviews in Control*, 50:278–293, 2020.
- [19] R. Ortega, J. G. Romero, and S. Aranovskiy. A new least squares parameter estimator for nonlinear regression equations with relaxed excitation conditions and forgetting factor. *Syst and Control Letters*, 169(1):105377, 2022.
- [20] R. Pintelon and J. Schoukens. *System Identification: A Frequency Domain Approach*. Wiley, IEEE Press, 2nd edition, 2012.
- [21] A. Pyrkin, A. Bobtsov, R. Ortega, J. G. Romero, and D. Dochain. A globally convergent estimator of the parameters of the classical model of a continuous stirred tank reactor. *Journal of Process Control*, 132:103129, 2023.
- [22] I. Y. Tyukin and D. V. Prokhorov nd van C. Leeuwen. Adaptation and parameter estimation in systems with unstable target dynamics and nonlinear parametrization. *IEEE Transactions on Automatic Control*, 52(9):1543–1559, 2007.
- [23] R. Ushirobira and D. Efimov. Constructing annihilators for parameter estimation in nonlinearly parameterized signals. *IFAC-PapersOnLine*, 56(2), 2023.
- [24] R. Ushirobira, W. Perruquetti, and M. Mboup. An algebraic continuous time parameter estimation for a sum of sinusoidal waveform signals. *Int. J. Adapt. Control Signal Process.*, 30(12), 2016.
- [25] J. Wang, D. Efimov, S. Aranovskiy, and A. Bobtsov. Fixed-time estimation of parameters for non-persistent excitation. *European Journal of Control*, 55:24–32, 2019.