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Manifolds of positive reach, differentiability, tangent variation, and attaining the reach

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1 — Abstract

2 This paper contains three main results.

3 Firstly, we give an elementary proof of the following statement: Let \mathcal{M} be a topological manifold
4 without boundary embedded in \mathbb{R}^d . If \mathcal{M} has positive reach, then \mathcal{M} can locally be written as the
5 graph of a $C^{1,1}$ function from the tangent space to the normal space. Conversely if \mathcal{M} can locally
6 be written as the graph of a $C^{1,1}$ function from the tangent space to the normal space, then \mathcal{M} has
7 positive reach. The result was hinted at by Federer when he introduced the reach, and proved by
8 Lytchak. Lytchak's proof relies heavily on CAT(k)-theory. The proof presented here uses only basic
9 results on homology.

10 Secondly, we give optimal Lipschitz-constants for the derivative, in other words we give an
11 optimal bound for the angle between tangent spaces in term of the distance between the points. We
12 stress that Lytchak did not provide any bound, let alone an optimal one, making his proof, although
13 interesting from a mathematical perspective, ineffectual in an algorithmic setting. To provide precise
14 and optimal bounds on the angle between tangent spaces, we formally introduce the local reach for
15 sets of positive reach, based on Aamari et al.'s discussion for C^2 manifolds. We prove that the local
16 reach of a manifold is completely characterized by the variation of tangent spaces. This improves
17 earlier results, that were either suboptimal or assumed that the manifold was C^2 .

18 Thirdly, we show that the value of the reach is equals minimum of the local reach of the set and
19 a global bottleneck for any set. This generalizes a result by Aamari et al. which explains how the
20 reach is attained for C^2 manifolds.

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21 **1** Introduction

22 In [14] Federer introduced the reach of a closed set $\mathcal{S} \subset \mathbb{R}^d$ as the minimum of the distance
23 from \mathcal{S} to the medial axis $\text{ax}(\mathcal{S})$, i.e. the set of points in \mathbb{R}^d for which the closest point in
24 \mathcal{S} is not unique. Assumptions on the reach (and its local version the local feature size [4])
25 underpin the correctness of many algorithms in computational geometry and topology. In
26 this paper we consider the differentiability class of manifolds of positive reach and give tight
27 bounds on the angle between nearby tangent spaces for general manifolds of positive reach.



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28 For arbitrary sets of positive reach we give a geometrical explanation on how the reach in
 29 attained.

30 **Previous work**

31 **Differentiability** Federer proved that the reach is stable under $C^{1,1}$ -diffeomorphisms of the
 32 ambient space. Here we write $C^{1,1}$ to indicate C^1 maps whose derivative is Lipschitz, and
 33 by a $C^{1,1}$ -diffeomorphism we mean that both the diffeomorphism and its inverse are $C^{1,1}$.
 34 Federer [14, Remark 4.20] furthermore mentioned (without going into much detail on one
 35 direction of the implication) that the graph of a function has positive reach if and only if it is
 36 $C^{1,1}$. Lytchak [21, 22] proved that a topologically embedded manifold without boundary has
 37 positive reach if and only if it is $C^{1,1}$, without a quantitative bound on the Lipschitz constant.
 38 This statement is quantified in our Theorem 1 below. We emphasize that quantifying these
 39 constants is essential for the result to allow us to formulate sampling criteria that guarantee
 40 the correctness of algorithms.

41 Lytchak’s proof requires significant background in CAT(k)-theory. Lytchak, motivated by
 42 bounds on intrinsic curvature, does not give any quantified bound on the extrinsic curvature,
 43 which play an important role in this paper.

44 Scholtes [28] also gave a proof that a hypersurface has positive reach if and only if it is
 45 $C^{1,1}$, using different techniques from both the ones employed by Lytchak and in this paper.
 46 Scholtes’ method uses the fact that the manifold has codimension one in an essential way.

47 Rataj and Zajíček [26] prove that if a Lipschitz manifold has positive reach, then it
 48 is $C^{1,1}$. They also use Lemma 17, however the assumption that the manifold is already
 49 Lipschitz simplifies the matter considerably, in particular they can skip the topological
 50 analysis. Moreover, again, they do not quantify their results in the way that we do and is
 51 essential for guarantees for algorithms. Recently, Leobacher and Steinicke [19] also reproved
 52 Lytchak’s result, but once more the result was not quantified. We further stress that the
 53 bounds that we provide on the Lipschitz constants are tight, as we’ll discuss in more detail
 54 below.

55 **Geometric bounds and tangent variation** Bounding the tangent angle variation on smooth
 56 manifolds is crucial for establishing constants in the context manifolds reconstruction and
 57 meshing [4, 11, 13, 8]. If we now focus on more recent results, some authors consider angle
 58 variation bounds for a given local feature size bound [18]. Other authors, including the
 59 ones of the present paper, give angle variation bounds based on the reach [9]. Assuming
 60 the surface to be smooth, more specifically C^2 smooth, allows using standard differential
 61 geometry tools such as second fundamental forms, curvatures and Riemannian geometry,
 62 while, as hinted at by Federer’s and Lytchak’s results, the weaker positive reach assumption
 63 is sufficient for many properties to hold.

64 For example, in [9], the angle variation bound given for C^2 manifolds are clearly optimal
 65 (the bound is attained on spheres) while only suboptimal bounds are derived for $C^{1,1}$, or
 66 positive reach, manifolds.

67 **Attaining the reach** In [1, Theorem 3.4], Aamari et al. make the following observation
 68 for C^2 manifolds: The reach of a submanifold of Euclidean space can be expressed as the
 69 minimum of a global quantity, realized at so called “bottlenecks”, and a local quantity, the
 70 inverse of the maximal extrinsic curvature. These “bottlenecks” can be formally related to a
 71 global reach and the local quantity with a local reach, which we’ll do in this paper. We stress
 72 that the concepts of the local and global reach are mentioned once in the introduction of [1],

73 no formal definition was given in that paper. Moreover, while this assumption is somewhat
 74 implicit in the proof of [1, Theorem 3.4], the analysis uses the context of Riemannian geometry,
 75 assuming the C^2 regularity of the manifold. We further emphasize that [1] focused uniquely
 76 on manifolds, which contrasts with the definition for global reach and local reach for any set
 77 of positive reach in this paper.

78 These results explain how the reach is attained, see Figure 1. This gives insight in which
 79 parts of a set of positive reach present the ‘difficult’ parts for algorithms, in the examples
 80 we’ll discuss now.

81 Motivation and related work

82 The assumption of positive reach is central to almost all triangulation algorithms, see for
 83 example [11, 13, 8], manifold learning, see for example [16, 15, 2, 29], and homotopy inference,
 84 see for example [24, 30]. Closing the gap between C^2 and positive reach manifolds significantly
 85 extends the applications domains. This extension is necessary to include for example the
 86 boundary of objects that have been designed by computer aided design software. These
 87 boundaries are generically tangent continuous and have bounded curvature (e.g. two planar
 88 faces connected through a cylindrical fillet surface), but don’t have continuous curvature.

89 The main result of this paper makes precise what it means for a topologically embedded
 90 submanifold of Euclidean space to have positive reach: The embedding is necessarily $C^{1,1}$ -
 91 smooth (Theorem 1) and we give a tight bound on the (generalized) extrinsic curvature of
 92 the submanifold (Theorem 4).

93 **Generalized tangent and normal spaces** Federer showed that although sets of positive
 94 reach are more general than smooth manifolds, they still possess generalized tangent and
 95 normal spaces, which are convex cones instead of just linear spaces. Roughly speaking, the
 96 normal cone at a point p of a set with positive reach determines the topological “link” of
 97 point p in the set, that is the local topology.

98 To prove the Theorem 1 it is essential to show that the normal cone at any point of a
 99 positive reach, topologically embedded n -manifold is a $(d - n)$ -dimensional vector space. In
 100 the proofs of [21, 22] the relation between the normal cone at a point p of a topologically
 101 embedded manifold with positive reach and the “link” of p is established using CAT(k)-theory.
 102 In contrast, the short proof of Lemma 31 is based on elementary properties of (the homology
 103 of) convex cones makes this correspondence more transparent in our opinion.

104 We further extend the results of [21, 22] by giving an optimal bound on the angle variation
 105 of tangent spaces. This also extends the same bound obtained in the particular context of
 106 C^2 manifolds [9]. We even obtain a significantly stronger result, that is, we characterize the
 107 local reach of a manifold completely by the tangent variation.

108 **Contribution** In this paper we give an elementary proof of the following characterizations
 109 of manifolds of positive reach. Moreover the statement is quantified with optimal constants.
 110 Here we write $B(p, r)$ or $B^\circ(p, r)$ for the closed or open ball centred at p with radius r .

111 **► Theorem 1.** *If \mathcal{M} is a topologically embedded manifold with positive reach and p a point
 112 in \mathcal{M} , then $\mathcal{M} \cap B^\circ(p, \sqrt{2} \operatorname{rch}(\mathcal{M})) \cap \pi_{T_p \mathcal{M}}^{-1}(B^\circ(0, \operatorname{rch}(\mathcal{M})))$ is the graph of a $C^{1,1}$ function
 113 above the open domain $B^\circ(0, \operatorname{rch}(\mathcal{M})) \subset T_p \mathcal{M}$.*

114 We also generalize the notations of local and global reach from C^2 submanifolds [1] to
 115 arbitrary compact sets (of positive reach):

XX:4 Manifolds of positive reach, differentiability, tangent variation, and attaining the reach

116 ► **Definition 2** (Local and global reach). For a closed set $\mathcal{S} \subset \mathbb{R}^d$, we define the local $\text{rch}_{loc.}(\mathcal{S})$
 117 and global $\text{rch}_{glob.}(\mathcal{S})$ reach as:

$$118 \quad \text{rch}_{glob.}(\mathcal{S}) \stackrel{\text{def.}}{=} \inf \left\{ \frac{|a-b|}{2} \mid \exists a, b \in \mathcal{S}, B^\circ \left(\frac{a+b}{2}, \frac{|a-b|}{2} \right) \cap \mathcal{S} = \emptyset \right\} \quad (1)$$

$$119 \quad \text{rch}_{loc.}(p, \mathcal{S}) \stackrel{\text{def.}}{=} \lim_{\substack{\rho \rightarrow 0 \\ \rho > 0}} \text{rch}(\mathcal{S} \cap B(p, \rho)) \in [0, +\infty] \quad (2)$$

$$121 \quad \text{rch}_{loc.}(\mathcal{S}) \stackrel{\text{def.}}{=} \inf_{p \in \mathcal{S}} \text{rch}_{loc.}(p, \mathcal{S}) \in [0, +\infty], \quad (3)$$

122 where $\inf(\emptyset) = +\infty$.

123 ► **Remark 3.** If $\text{rch}(\mathcal{S}) > 0$, then $\rho \mapsto \text{rch}(\mathcal{S} \cap B(p, \rho))$ is non-increasing (this follows from
 124 Theorem 16 because $\mathcal{S} \cap B(p, \rho)$ is geodesically convex [9, Corollary 1] for all $\rho < \text{rch}(\mathcal{S})$
 125 so that the set of bounds that need to be satisfied in (9) is larger, c.f. [6, Lemma 5]) as
 126 soon as $\rho < \text{rch}(\mathcal{S})$, so that the limit in (2) exists. Moreover, since for any p one has
 127 $\text{rch}_{loc.}(p, \mathcal{S}) \geq \text{rch}(\mathcal{S})$, we get that:

$$128 \quad \text{rch}_{loc.}(\mathcal{S}) \geq \text{rch}(\mathcal{S}). \quad (4)$$

129 To be able to give the bounds on the tangent variation/extrinsic curvature we need to
 130 introduce some notation.

131 We write $\text{Gr}(n, \mathbb{R}^d)$ for the Grassmannian, that is the space of n -linear subspaces of \mathbb{R}^d
 132 with metric given by the sine of the maximum angle. We recall that the maximum angle is
 133 given by

$$134 \quad \angle A, B \stackrel{\text{def.}}{=} \max_{a \in A \setminus \{0\}} \min_{b \in B \setminus \{0\}} \angle a, b = \max_{b \in B \setminus \{0\}} \min_{a \in A \setminus \{0\}} \angle a, b. \quad (5)$$

140 We use this maximal angle as our metric. It is not that difficult to establish that this is
 141 indeed a metric, see [31] or [25] for a far reaching generalization.¹

146 ► **Theorem 4.** If $\mathcal{M} \subset \mathbb{R}^d$ is a topologically embedded n -manifold with reach at least $R > 0$,
 147 then tangent spaces and normal spaces to \mathcal{M} are linear spaces at any point $p \in \mathcal{M}$ and the
 148 maps $\text{Tan} : \mathcal{M} \rightarrow \text{Gr}(n, \mathbb{R}^d)$ and $\text{Nor} : \mathcal{M} \rightarrow \text{Gr}(d-n, \mathbb{R}^d)$ are Lipschitz.

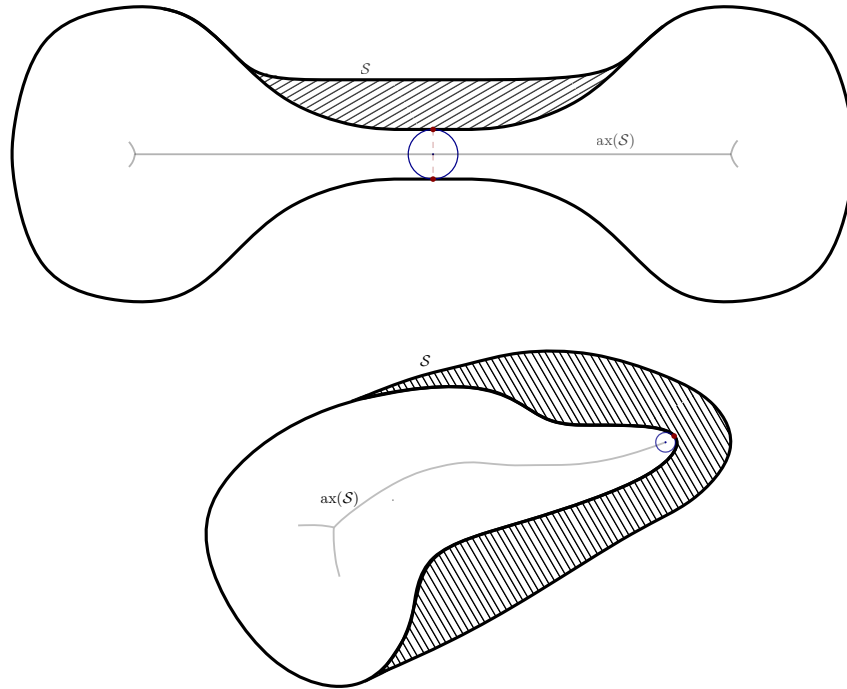
149 Moreover, the smallest Lipschitz constant satisfied by $p \mapsto \text{Tan}(p, \mathcal{S})$ is $\frac{1}{\text{rch}_{loc.}(\mathcal{M})}$:

$$150 \quad \sup_{\substack{p, q \in \mathcal{M} \\ p \neq q}} \frac{\angle \text{Tan}(p, \mathcal{M}), \text{Tan}(q, \mathcal{M})}{d_{\mathcal{M}}(p, q)} = \frac{1}{\text{rch}_{loc.}(\mathcal{M})}, \quad (6)$$

151 where $d_{\mathcal{M}}$ denotes the geodesic distance on \mathcal{M} and we say that $d_{\mathcal{M}}(p, q) = \infty$ when p and q
 152 are not connected by any path in \mathcal{M} and that $1/\infty = 0$.

153 ► **Remark 5.** If $\mathcal{M} \subset \mathbb{R}^d$ is a manifold with boundary then the results of Theorem 4 still
 154 hold as long as there is a shortest path geodesic between p and q that lies completely in the
 155 interior of \mathcal{M} .

135 ¹ The sine of this angle also defines a metric on the Grassmannian as can be found in e.g. [17, Chapter
 136 IV, Section 2.1] or [3, Section 34], see Appendix B for a more extensive description. It seems that in a
 137 number of fields the sine of the angles is a more common metric, but it is suboptimal in this context. We
 138 further stress that neither of these metrics equal the standard Riemannian metric on the Grassmannian,
 139 which is given by the sum of the squares of the principal angles, see e.g. [32].



142 **Figure 1** In this figure we see the two ways in which the reach can be attained for a general set
 143 of positive reach, by means of the global reach (top) or local reach (bottom), for two sets of positive
 144 reach in the plane. The shading indicates a solid set. The points (red) and the circle (blue) in which
 145 the reach is attained are indicated.

156 Next theorem is a generalization of [1, Theorem 3.4] which was formulated in the particular
 157 situation where \mathcal{S} is a C^2 embedded manifold, in which case $\text{rch}_{loc.}(\mathcal{S})$ is also the supremum
 158 of extrinsic curvature.

159 **Theorem 6.** *If \mathcal{S} is a compact subset of \mathbb{R}^d , then:*

160
$$\text{rch}(\mathcal{S}) = \min(\text{rch}_{glob.}(\mathcal{S}), \text{rch}_{loc.}(\mathcal{S}))$$

161 We refer to Figure 1 for an illustration.

162 **2 Preliminaries**

163 In this section we first recall some results and definitions concerning sets of positive reach.
 164 The results concerning the reach are mainly taken from [14] and [9].

165 **2.1 Sets of positive reach**

166 Throughout this paper we will write \mathcal{S} for a general closed set (mostly with positive
 167 reach) and \mathcal{M} for a manifold (mostly without boundary) and in some (rare) cases
 168 $\partial\mathcal{M}$ for its boundary (if it exists).

169 The concept of a tangent space of a smoothly embedded manifold has been generalized
 170 by Federer to tangent cones in the following manner:

XX:6 Manifolds of positive reach, differentiability, tangent variation, and attaining the reach

171 ► **Definition 7** (Generalized tangent spaces, Definitions 4.3 and 4.4 of [14]). *If $\mathcal{S} \subset \mathbb{R}^d$ and*
 172 *$p \in \mathcal{S}$, then the generalized tangent space*

173
$$\text{Tan}(p, \mathcal{S})$$

174 *is the set of all tangent vectors of \mathcal{S} at p consists of all those $u \in \mathbb{R}^d$, such that either $u = 0$*
 175 *or for every $\epsilon > 0$ there exists a point $q \in \mathcal{S}$ with*

176
$$0 < |q - p| < \epsilon \quad \text{and} \quad \left| \frac{q - p}{|q - p|} - \frac{u}{|u|} \right| < \epsilon.$$

177 *The set*

178
$$\text{Nor}(p, \mathcal{S})$$

179 *of all normal vectors of \mathcal{S} at p consists of all those $v \in \mathbb{R}^d$ such that $v \cdot u \leq 0$ for all*
 180 *$u \in \text{Tan}(p, \mathcal{S})$.*

181 ► **Definition 8.** *We write $\pi_{\mathcal{S}}$ for the closest point projection on \mathcal{S} .*

182 Although it is possible to define a set valued version of $\pi_{\mathcal{S}}$, we only use the projection map
 183 outside the medial axis, that is at those points where there is a single closest point.

184 For the remainder of this subsection we will recall the properties of this map and sets of
 185 positive reach that are essential for this paper.

186 ► **Lemma 9** (Angle with tangent space, Theorem 4.8(7) of [14]). *Suppose that \mathcal{S} is a set*
 187 *positive reach $\text{rch}(\mathcal{S})$. Let $p, q \in \mathcal{S} \subset \mathbb{R}^d$ such that $|p - q| < \text{rch}(\mathcal{S})$. We have*

188
$$\sin \angle(q - p, \text{Tan}(p, \mathcal{S})) \leq \frac{|p - q|}{2 \text{rch}(\mathcal{S})}. \quad (7)$$

189

190 Moreover, a variant of the previous lemma characterizes the reach for arbitrary closed
 191 sets, that is:

192 ► **Lemma 10** (Distance to tangent space characterizes the reach, Theorem 4.18 of [14]). *For a*
 193 *closed subset \mathcal{S} of Euclidean space $\mathbb{E} = \mathbb{R}^d$, one has:*

194
$$\text{rch}(\mathcal{S}) = \inf_{\substack{p, q \in \mathcal{S} \\ p \neq q}} \frac{|p - q|^2}{2d_{\mathbb{E}}(q, p + \text{Tan}(p, \mathcal{M}))} \quad (8)$$

195 ► **Lemma 11** (Theorem 4.8 (4) of [14]). *$\pi_{\mathcal{S}}$ is continuous on $\mathbb{R}^d \setminus \text{ax}(\mathcal{S})$.*

196 In fact the following lemma strengthens the previous.

197 ► **Lemma 12** (Theorem 4.8 (8) of [14]). *If $x, y \in \mathbb{R}^d$, and*

198
$$\max\{|x - \pi_{\mathcal{S}}(x)|, |y - \pi_{\mathcal{S}}(y)|\} = \mu \leq \text{rch}(\mathcal{S}),$$

199 *then*

200
$$|\pi_{\mathcal{S}}(x) - \pi_{\mathcal{S}}(y)| \leq \frac{\text{rch}(\mathcal{S})}{\text{rch}(\mathcal{S}) - \mu} |x - y|$$

201 ▶ **Lemma 13** (Theorem 4.8 (12) of [14]). *If $p \in \mathcal{S}$ and $\text{lfs}(p) > \rho > 0$, then*

202
$$\text{Nor}(p, \mathcal{S}) = \{\lambda v \mid \lambda \geq 0, |v| = \rho, \pi_{\mathcal{S}}(p + v) = p\}.$$

203 $\text{Tan}(p, \mathcal{S})$ is the convex cone dual to $\text{Nor}(p, \mathcal{S})$, and

204
$$\lim_{t \rightarrow 0^+} t^{-1} d(p + tu, \mathcal{M}) = 0,$$

205 for $u \in \text{Tan}(p, \mathcal{S})$, where d is the Euclidean distance.

206 ▶ **Lemma 14** (Remark 4.15 (1) of [14]). *If $p \in \mathcal{S}$ and $\text{lfs}(p) > \rho > 0$, then $\mathcal{S} \cap B(p, \rho)$ is*
 207 *contractible.*

208 **2.1.1 Geodesics**

209 We also recall a result from the second paragraph of Part III, Section 1: “Die Existenz
 210 geodätischer Bogen in metrischen Räumen” in [23]:

211 ▶ **Lemma 15** (Menger’s existence of geodesics). *For a closed set $\mathcal{S} \subset \mathbb{R}^d$, $d_{\mathcal{S}}$ denotes the*
 212 *geodesic distance in \mathcal{S} , i.e. $d_{\mathcal{S}}(a, b)$ is the infimum of lengths of paths in \mathcal{S} between a and*
 213 *b . If there is at least one path between a and b with finite length, then it is known that a*
 214 *minimizing geodesic, i.e. a path with minimal length connecting a to b exists. When no such*
 215 *path between a and b with finite length exists, we write $d_{\mathcal{S}}(a, b) = \infty$.*

216 We’ll be using the metric characterization of the reach from [9] quite often in this paper.

217 ▶ **Theorem 16** (Theorem 1 of [9]). *If $\mathcal{S} \subset \mathbb{R}^d$ is a closed set, then*

218
$$\text{rch } \mathcal{S} = \sup \left\{ r > 0, \forall a, b \in \mathcal{S}, |a - b| < 2r \Rightarrow d_{\mathcal{S}}(a, b) \leq 2r \arcsin \frac{|a - b|}{2r} \right\}, \quad (9)$$

219 where the sup over the empty set is 0.

220 **2.2 Embeddings and atlases**

221 We consider below an embedded, compact n -manifold $\mathcal{M} \subset \mathbb{R}^d$ with an atlas associated with
 222 an open cover $(U_i)_{i \in I}$:

223
$$\mathcal{M} = \bigcup_{i \in I} U_i$$

224 and charts $f_i : U_i \rightarrow \mathbb{R}^n$ that are homeomorphisms on their images, where the topology on
 225 \mathcal{M} is the one induced by the ambient metric of \mathbb{R}^d .

229 Note that, topologically² embedded manifold does not exclude wild embeddings [12] but,
 230 as seen below, the positive reach assumption does, because the embedding is proven to be
 231 $C^{1,1}$.

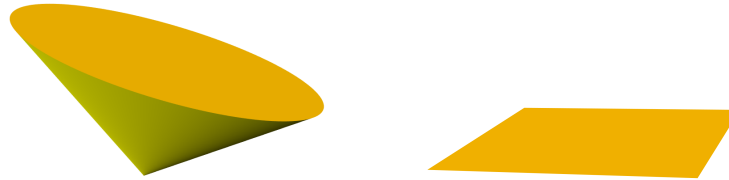
226 ² We say that an embedding is topological to stress that we don’t make any assumption beyond the
 227 assumption that it is a homeomorphism onto its image, in particular no assumptions on differentiability
 228 are made.

232 **3 Overview of the first two main results: Manifolds of positive reach,**
 233 **differentiability, and tangent variation**

234 In this section we provide an overview of the proof of the first two main results of this paper
 235 Theorems 1 and 4. We discuss this in complete detail with full proofs of the statements in
 236 the appendix.

237 **A topologically embedded manifold with positive reach is differentiably embedded**

238 Consider a topologically embedded manifold with positive reach. That is, we don't assume
 239 anything on the embedding of the manifold except that it is a homeomorphism with its image.
 240 However, the assumption that the manifold has positive reach yields that the generalized
 241 tangent spaces (see Definition 7) are cones.



242 **Figure 2** Convex cones come in two variants, a tapering object or a half space on the one hand
 243 and vector spaces on the other.

244 A characterization of convex cones by topological means can be found in Section A.1 of
 245 the appendix. We recall that a convex cone is a subset of \mathbb{R}^d that is convex and for each
 246 v in the set we have that λv is also in the set for $\lambda \in \mathbb{R}_{\geq 0}$. By this we mean that, roughly
 247 speaking, convex cones A come into two classes:

- 248 1. The convex cone is defined by

249
$$A = \{\lambda a \mid \lambda \geq 0, a \in \tilde{A}\},$$

250 where \tilde{A} is a contractible subset of \mathbb{S}^{d-1} . Note that $\tilde{A} = A \cap \mathbb{S}^{d-1}$. In this case A is
 251 (again roughly speaking) either a tapering object or a half space.

- 252 2. The convex cone is defined by

253
$$A = \{\lambda a \mid \lambda \geq 0, a \in \tilde{A}\},$$

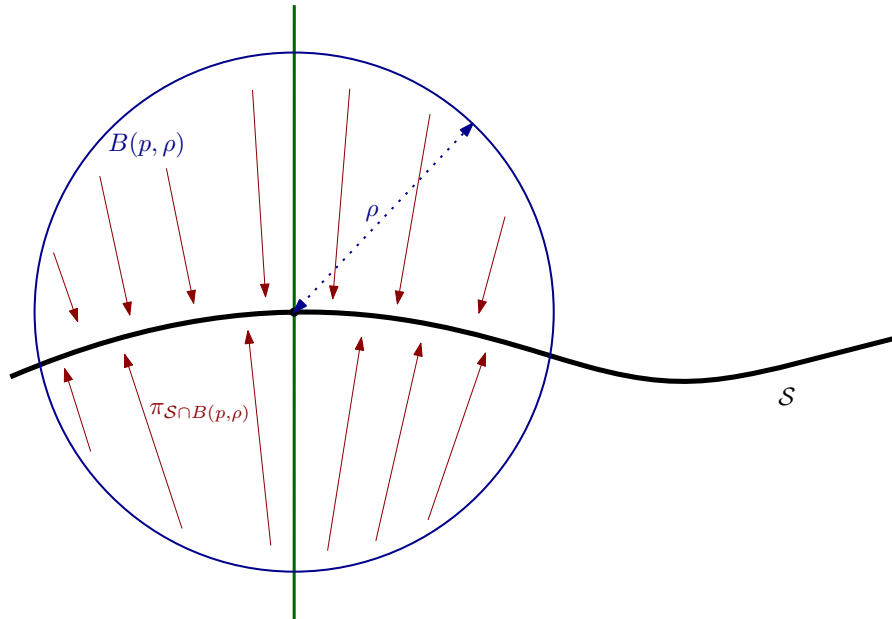
254 where \tilde{A} is a (k dimensional, with $0 \leq k \leq d - 1$) geodesic sphere in \mathbb{S}^{d-1} . In this case A
 255 is a vector space.

256 These two classes are characterized by the homology of $B(0, 1) \setminus A^\perp$, see Lemma 30. The
 257 contractibility of \tilde{A} is the key ingredient of proof of Lemma 30. More precisely we'll use that
 258 \tilde{A} is contractible on the sphere if and only if A isn't a vector space.

259 In Section A.2 of the appendix we'll establish, based on the characterization of Section
 260 A.1 of the appendix, that for topologically embedded manifolds the tangent cones are in fact
 261 tangent spaces. The proof proceeds in two steps:

- 262 ■ In Lemma 31, we'll see that $B(p, \rho) \setminus (p + \text{Nor}(p, \mathcal{S}))$ has the same homology as $\mathcal{S} \cap$
 263 $B(p, \rho) \setminus \{p\}$, see Figure 3. This result will rely on the deformation retract along the
 264 closest point projection of sets of positive reach and its continuity as established by
 265 Federer [14]. We stress that the normal cone $\text{Nor}(p, \mathcal{S})$ is the dual of the tangent cone
 266 $\text{Tan}(p, \mathcal{S})$, which explains the link with Lemma 30.

267 ■ In the proof of Lemma 32, we'll see that if \mathcal{M} is manifold, then $\mathcal{M} \cap B(p, \rho) \setminus \{p\}$ has
 268 the homology of a sphere of dimension $n - 1$, which with the previous lemma allows us
 269 to conclude that $\text{Tan}(p, \mathcal{M})$ is a vector space. The proof combines topological reasoning
 270 with arguments based on the charts of the manifold \mathcal{M} .



271 ■ **Figure 3** Sketch of the proof of Lemma 31. We establish a homotopy between $B(p, \rho) \setminus (p +$
 272 $\text{Nor}(p, \mathcal{S}))$ and $\mathcal{S} \cap B(p, \rho) \setminus \{p\}$ by deformation retract induced by the map $\pi_{\mathcal{S} \cap B(p, \rho)}$.

273 By definition we have that if a manifold has a tangent space it is differentially embedded,
 274 but that does not mean that the manifold has to be continuously differentiable,
 275 this is what will be the topic we'll address next.

276 **Semi-continuity of normal cones and C^1 embeddings** In Section A.3 of the appendix
 277 we will first prove a more general statement (Lemma 33), namely semi-continuity of the
 278 normal cones for general sets of positive reach. By this we mean that given a point p and its
 279 normal cone $\text{Nor}(p, \mathcal{S})$, for all ϵ there exists a neighbourhood of p such that for all p' in this
 280 neighbourhood, we have that the normal cone $\text{Nor}(p', \mathcal{S})$ makes only an ϵ angle with some
 281 subset of $\text{Nor}(p, \mathcal{S})$. The proof is elementary, but rather technical, and relies on compactness
 282 arguments and the manipulation of offsets.

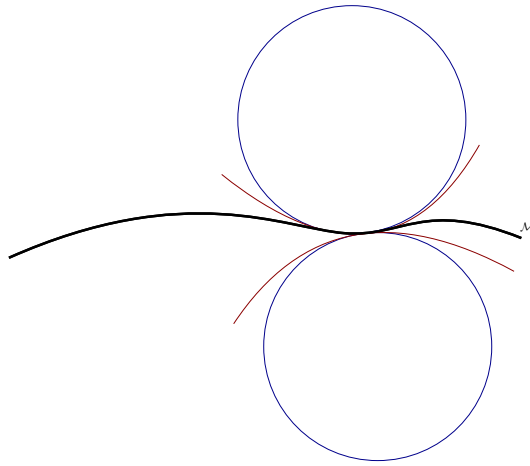
283 Lemma 33 together with the observation that the tangent spaces are k -dimensional vector
 284 spaces the one sided bound on the angle between the normal cones above is in fact a two-sided
 285 bound, see Lemma 34, yields that the manifold is C^1 , that is continuously differentiable.
 286 Lemma 35 strengthens this even further and explains that \mathcal{M} can locally be written as the
 287 graph of a function. To put it differently, using some geometrical and analytical arguments
 288 we see that, locally, the manifold projects injectively on its tangent cone (which is an affine
 289 space by Lemma 32). This then allows us to identify it, locally, as the graph of a differentiable
 290 map from the tangent space to the normal space.

XX:10 Manifolds of positive reach, differentiability, tangent variation, and attaining the reach

291 **From C^1 to $C^{1,1}$ embeddings** Section A.4 of the appendix proves the majority of the main
292 statements of the paper, namely that manifolds with positive reach are $C^{1,1}$ and characterizes
293 the Lipschitz constants of the derivative in terms of the local reach. It is the most technical
294 section and relies heavily on the theory of semiconcave functions. Most crucially we'll use
295 the following result, namely Corollary 3.3.8 of [10]:

296 ► **Lemma 17.** *Let $F : U \rightarrow \mathbb{R}$ be a function defined on a convex open set $U \subset \mathbb{R}^n$. Then the*
297 *following properties are equivalent:*

- 298 ■ *F is of class $C^{1,1}$ with 1-Lipschitz derivative.*
- 299 ■ *both $x \rightarrow 1/2 x^2 - F(x)$ and $x \rightarrow 1/2 x^2 + F(x)$ are convex functions.*



300 ■ **Figure 4** A manifold with empty tangent balls (blue) and (locally) empty tangent parabola
301 (red).

302 Although that the precise technical way that this result is used is too intricate to explain
303 in a paragraph or two, the link between sets of positive reach and Lemma 17 can be explained
304 in geometrical terms. Assume for the moment that \mathcal{M} has reach 1. As a consequence of
305 Lemma 13 the interior of every ball that is tangent to a manifold \mathcal{M} , that has radius less or
306 equal to 1 (the reach of \mathcal{M}) does not intersect \mathcal{M} itself. Geometrically speaking the fact
307 that $x \rightarrow 1/2 x^2 - F(x)$ and $x \rightarrow 1/2 x^2 + F(x)$ are convex functions can be expressed by the
308 fact that (locally) the interior of parabolae that are tangent to the manifold have an empty
309 interior, see Figure 4. Of course these parabolae are just second order approximations of
310 the spheres. Because we are just interested in the Lipschitz constants of the derivative this
311 approximation suffices.

312 Using Lemma 17 we do not directly prove our bound on the angle between nearby tangent
313 spaces, instead we first establish the bound

$$314 \quad \|D\phi(y_2) - D\phi(y_1)\| \leq \frac{1}{(\text{rch}_{loc.}(p, \mathcal{M}) - \epsilon)} |y_2 - y_1|,$$

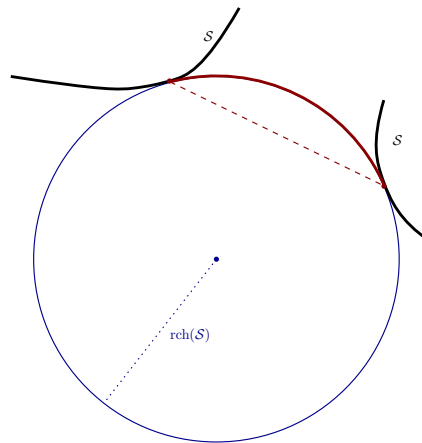
315 see Lemma 39, where ϕ is the local function (from the tangent to normal space) of which \mathcal{M}
316 is locally the graph and $\|\cdot\|$ denotes the operator two norm, in other words we regard the
317 Jacobians as matrices depending on y and bound their Lipschitz constant in a way that is ϵ
318 close to optimal.

319 From the result of Lemma 39 it is not very difficult to derive a local bound between
 320 tangent spaces, which is done in Lemma 40. This local bound can then be extended (Lemma
 321 41) to a non-local statement by gluing the local estimates on the angle between tangent
 322 spaces along a geodesic, which exists thanks to Lemma 15. This allows us to establish one of
 323 the main results, the characterization of the local reach in terms of the tangent variation in
 324 Theorem 4, whose proof requires a significant amount of computation. The optimal size of
 325 the domain (in a fixed tangent space) such that \mathcal{M} can be written as the graph of a function
 326 from a (fixed) tangent space to its normal space is then given in Theorem 1. The proof can
 327 be sketched as follows: By combining Theorems 4 and 16 with the submersion theorem we
 328 can establish a local diffeomorphism. The global diffeomorphism is established by proving
 329 that the covering number is one.

330 **4 Global and local reach**

331 In this section we establish the third main result of the paper, that is Theorem 6. This
 332 theorem generalizes the results of [1] on how the reach is attained from C^2 manifolds to
 333 arbitrary sets of positive reach. It states that the reach is a consequence of either curvature
 334 maxima or so-called bottlenecks. In our result we replace the curvature by the local reach,
 335 which in turn is defined using local metric distortion, while the bottlenecks are similar in
 336 nature. In a number of cases we will have to assume that the set is bounded/compact.

337 Although some of the details of the proof of Theorem 6 are a bit technical, the main idea
 338 is not that complicated. In fact we provide a sketch of the proof in Figure 5 and its caption.



339 **Figure 5** To derive a contradiction assume that $\text{rch}(\mathcal{S}) < \text{rch}_{loc}(\mathcal{S})$ and \mathcal{S} is tangent to the ball
 340 with radius $\text{rch}(\mathcal{S})$ (blue) in two distinct points (red) that are closer than $2\text{rch}(\mathcal{S})$, and therefore not
 341 antipodal. Consider the cord connecting these two points (dashed red), then Theorem 16 yields that
 342 the length of the geodesic is at most equal to part of the circle indicated in red, however that is
 343 the only curve of that length (or less) that connects the two red points that does not intersect the
 344 interior of the ball of radius $\text{rch}(\mathcal{S})$. This implies that this curve has to lie inside \mathcal{S} , moreover this
 345 curve has local reach $\text{rch}(\mathcal{S})$. Combining these observations yields that $\text{rch}(\mathcal{S}) \geq \text{rch}_{loc}(\mathcal{S})$

346 We recall the definition from the introduction:

347 **Definition 2** (Local and global reach). For a closed set $\mathcal{S} \subset \mathbb{R}^d$, we define the local $\text{rch}_{loc}(\mathcal{S})$

XX:12 Manifolds of positive reach, differentiability, tangent variation, and attaining the reach

348 and global $\text{rch}_{glob.}(\mathcal{S})$ reach as:

$$349 \quad \text{rch}_{glob.}(\mathcal{S}) \stackrel{\text{def.}}{=} \inf \left\{ \frac{|a-b|}{2} \mid \exists a, b \in \mathcal{S}, B^\circ \left(\frac{a+b}{2}, \frac{|a-b|}{2} \right) \cap \mathcal{S} = \emptyset \right\} \quad (1)$$

$$350 \quad \text{rch}_{loc.}(p, \mathcal{S}) \stackrel{\text{def.}}{=} \lim_{\substack{\rho \rightarrow 0 \\ \rho > 0}} \text{rch}(\mathcal{S} \cap B(p, \rho)) \in [0, +\infty] \quad (2)$$

$$352 \quad \text{rch}_{loc.}(\mathcal{S}) \stackrel{\text{def.}}{=} \inf_{p \in \mathcal{S}} \text{rch}_{loc.}(p, \mathcal{S}) \in [0, +\infty], \quad (3)$$

353 where $\inf(\emptyset) = +\infty$.

354 ► **Lemma 18.** *The map $p \mapsto \text{rch}_{loc.}(p, \mathcal{S})$ defined over \mathcal{S} is lower semi-continuous, in other*
 355 *words:*

$$356 \quad \text{rch}_{loc.}(p, \mathcal{S}) = \lim_{\substack{\rho \rightarrow 0 \\ \rho > 0}} \inf_{q \in \mathcal{S} \cap B^\circ(p, \rho)} \text{rch}_{loc.}(q, \mathcal{S}).$$

357 **Proof.** By definition of $\text{rch}_{loc.}(p, \mathcal{S})$, we have that for any $\epsilon > 0$ there is $\rho > 0$ such that:

$$358 \quad \text{rch}(p, \mathcal{S} \cap B(p, \rho)) > \text{rch}_{loc.}(p, \mathcal{S}) - \epsilon. \quad (10)$$

359 One gets

$$\begin{aligned} 360 \quad \text{rch}_{loc.}(p, \mathcal{S}) &\geq \inf_{q \in \mathcal{S} \cap B^\circ(p, \rho)} \text{rch}_{loc.}(q, \mathcal{S}) \\ 361 \quad &= \inf_{q \in \mathcal{S} \cap B^\circ(p, \rho)} \text{rch}_{loc.}(q, \mathcal{S} \cap B(p, \rho)) \\ 362 \quad &\geq \text{rch}_{loc.}(\mathcal{S} \cap B(p, \rho)) && \text{(by (3) applied to the set } \mathcal{S} \cap B(p, \rho)) \\ 363 \quad &\geq \text{rch}(\mathcal{S} \cap B(p, \rho)) && \text{(by (4))} \\ 364 \quad &> \text{rch}_{loc.}(p, \mathcal{S}) - \epsilon. && \text{(by (10))} \end{aligned}$$

365 Because this inequality holds for any positive ϵ , the equality follows. ◀

366 ► **Remark 19.** As a consequence, when \mathcal{S} is compact, the inf in (3) is a min, in other words
 367 there exists at least one point $p \in \mathcal{S}$ such that $\text{rch}_{loc.}(p, \mathcal{S}) = \text{rch}_{loc.}(\mathcal{S})$.

368 ► **Definition 20.** *A bottleneck is a pair of points $(a, b) \in \mathcal{S}^2$ such that*

$$369 \quad B^\circ \left(\frac{a+b}{2}, \frac{|a-b|}{2} \right) \cap \mathcal{S} = \emptyset.$$

370 We have the following variant of Lemma 12 of [9], which in turn is a relatively straight-
 371 forward generalization to arbitrary dimension of Property I of [5]:

372 ► **Lemma 21.** *If a geodesic (or more generally a curve) γ parametrized according to arc*
 373 *length on the interval $[0, \ell]$ satisfies*

$$374 \quad \angle \dot{\gamma}(s_1), \dot{\gamma}(s_2) \leq \frac{1}{R} |s_2 - s_1|, \quad (11)$$

375 then as long as $\ell \leq 2\pi R$,

$$376 \quad |q - p| \geq 2 \text{rch}(\mathcal{M}) \sin \left(\frac{\ell}{2R} \right),$$

377 where $p = \gamma(0)$ and $q = \gamma(\ell)$.

378 **Proof.** The length of γ in the direction $\dot{\gamma}(\frac{\ell}{2})$ is

$$\begin{aligned}
 379 \quad \left\langle q - p, \dot{\gamma}\left(\frac{\ell}{2}\right) \right\rangle &= \int_0^\ell \langle \dot{\gamma}(s), \dot{\gamma}(\ell/2) \rangle ds \\
 380 \quad &= \int_0^{\ell/2} \langle \dot{\gamma}(s), \dot{\gamma}(\ell/2) \rangle ds + \int_{\ell/2}^\ell \langle \dot{\gamma}(s), \dot{\gamma}(\ell/2) \rangle ds \\
 381 \quad &\geq \int_0^{\ell/2} \cos \frac{|s - \ell/2|}{R} ds + \int_{\ell/2}^\ell \cos \frac{|s - \ell/2|}{R} ds \\
 382 \quad &= 2R \sin\left(\frac{\ell}{2R}\right).
 \end{aligned}$$

383 where, in the above inequality, we use the assumption $\ell \leq 2\pi R$ and the fact that $\theta \mapsto \cos \theta$
 384 is decreasing on $[0, \pi]$.

385 Because $|q - p| \geq \langle q - p, \dot{\gamma}(\frac{\ell}{2}) \rangle$, we see that

$$386 \quad |q - p| \geq 2R \sin\left(\frac{\ell}{2R}\right).$$

387

388 Combining Lemmas 42 and 21 we get:

389 ► **Corollary 22.** *If γ is a geodesic in \mathcal{S} with length $\ell \leq 2\pi \text{rch}_{loc.}(\mathcal{S})$, then*

$$390 \quad \ell \leq 2\text{rch}_{loc.}(\mathcal{S}) \arcsin \frac{|p - q|}{2\text{rch}_{loc.}(\mathcal{S})}.$$

391 The Lemma 23 below is a special case of [20, Lemma 4.6], we still provide a proof for
 392 completeness.

393 ► **Lemma 23.** *Let $x \notin \text{ax}(\mathcal{S})$ and denote by \tilde{x} the unique point closest to x in \mathcal{S} .*

394 *Then, for any $\epsilon > 0$, there is $\alpha > 0$ such that,*

$$395 \quad |y - x| < \alpha \Rightarrow \{z \in \mathcal{S} \mid d(y, z) = d(y, \mathcal{S})\} \subset B^\circ(\tilde{x}, \epsilon). \tag{12}$$

396 **Proof.** Consider the closed set $\mathcal{S}' := \mathcal{S} \setminus B^\circ(\tilde{x}, \epsilon)$. The minimum distance $d(x, \mathcal{S}')$ to x is
 397 attained at least at some point z in \mathcal{S}' so that

$$398 \quad d(x, \mathcal{S}') = d(x, z) > d(x, \tilde{x}) = d(x, \mathcal{S}).$$

399 Taking

$$400 \quad \alpha := \frac{d(x, \mathcal{S}') - d(x, \mathcal{S})}{2},$$

401 we get, for any y such that $|y - x| < \alpha$ and $z \in \mathcal{S}'$ that:

$$\begin{aligned}
 402 \quad d(y, z) &\geq d(y, \mathcal{S}') \geq d(x, \mathcal{S}') - d(y, x) \\
 403 \quad &= d(x, \mathcal{S}) + 2\alpha - d(y, x) = d(x, \tilde{x}) + 2\alpha - d(y, x) \\
 404 \quad &> d(x, \tilde{x}) + d(y, x) \geq d(y, \tilde{x}) \geq d(y, \mathcal{S}).
 \end{aligned}$$

405 We have shown that if $z \in \mathcal{S}'$, then $d(y, z) > d(y, \mathcal{S})$. It follows that any closest point to y in
 406 \mathcal{S} is in $\mathcal{S} \setminus \mathcal{S}' = \mathcal{S} \cap B^\circ(\tilde{x}, \epsilon)$ which is (12). ◀

407 ► **Theorem 6.** *If \mathcal{S} is a compact subset of \mathbb{R}^d , then:*

408
$$\text{rch}(\mathcal{S}) = \min(\text{rch}_{\text{glob.}}(\mathcal{S}), \text{rch}_{\text{loc.}}(\mathcal{S}))$$

409 **Proof.** We consider first the situation where $\text{rch}(\mathcal{S}) = 0$. In this case, since \mathcal{S} is bounded,
 410 there exists some point $p \in \overline{\text{ax}(\mathcal{S})} \cap \mathcal{S}$. Then, for any $\rho > 0$, there is some $p' \in \text{ax}(\mathcal{S})$ such
 411 that $d(p', p) < \rho/2$, so that there are at least two points in $\mathcal{S} \cap B(p, \rho)$ closest to p' . Thus,
 412 $\text{rch}(\mathcal{S} \cap B(p, \rho)) < \rho/2$, and, since this holds for any $\rho > 0$, we get that $\text{rch}_{\text{loc.}}(p, \mathcal{S}) = 0$ and
 413 as a direct consequence $\text{rch}_{\text{loc.}}(\mathcal{S}) = \inf_{q \in \mathcal{S}} \text{rch}_{\text{loc.}}(q, \mathcal{S}) = 0$.

414 We assume now that $\text{rch}(\mathcal{S}) > 0$. As we have seen in Equation (4) of Remark 3, we
 415 have that $\text{rch}(\mathcal{S}) \leq \text{rch}_{\text{loc.}}(\mathcal{S})$. Therefore it suffices to prove that, if $\text{rch}(\mathcal{S}) < \text{rch}_{\text{loc.}}(\mathcal{S})$, then
 416 $\text{rch}(\mathcal{S}) = \text{rch}_{\text{glob.}}(\mathcal{S})$. So let us assume now that $\text{rch}(\mathcal{S}) < \text{rch}_{\text{loc.}}(\mathcal{S})$.

417 Since \mathcal{S} is compact and thus bounded, which implies that a closed $\text{rch}(\mathcal{S})$ -neighbourhood
 418 of \mathcal{S} is also bounded, there must exist $x \in \overline{\text{ax}(\mathcal{S})}$, where $\overline{\text{ax}(\mathcal{S})}$ is the closure of the medial
 419 axis $\text{ax}(\mathcal{S})$, such that $d(x, \mathcal{S}) = \text{rch}(\mathcal{S})$.

420 If $x \in \overline{\text{ax}(\mathcal{S})} \setminus \text{ax}(\mathcal{S})$, in particular $x \notin \text{ax}(\mathcal{S})$ and there is a unique point \tilde{x} closest to x in
 421 \mathcal{S} . Choose some $0 < \epsilon < \text{rch}(\mathcal{S})$ and, as in Lemma 23, take $\alpha > 0$ so that (12) holds and,
 422 moreover, such that

423
$$\alpha < \text{rch}_{\text{loc.}}(\mathcal{S}) - \text{rch}(\mathcal{S}).$$

424 Since $x \in \overline{\text{ax}(\mathcal{S})}$, there exists $y \in \text{ax}(\mathcal{S}) \cap B^\circ(x, \alpha)$, and therefore Lemma 23 yields that
 425 there exists at least two points $p, q \in B^\circ(\tilde{x}, \epsilon)$ such that $d(y, p) = d(y, q) = d(y, \mathcal{S})$.

426 Because $d(p, q) < 2\epsilon < 2\text{rch}(\mathcal{S})$, by Theorem 16, there is a geodesic from p to q of length
 427 at most $\pi\text{rch}(\mathcal{S})$, which is less than $2\pi\text{rch}_{\text{loc.}}(\mathcal{S})$. But then one can apply Corollary 22, which
 428 implies that the length of a geodesic from p to q is at most

429
$$\ell_{\max} = 2\text{rch}_{\text{loc.}}(\mathcal{S}) \arcsin \frac{|p - q|}{2\text{rch}_{\text{loc.}}(\mathcal{S})}.$$

430 However, since projecting a curve outside a given sphere onto the given sphere only decreases
 431 its length, a curve from p to q outside the ball centered at y whose radius is at most
 432 $\text{rch}(\mathcal{S}) + |y - x| < \text{rch}(\mathcal{S}) + \alpha < \text{rch}_{\text{loc.}}(\mathcal{S})$ must have length strictly greater than ℓ_{\max} , a
 433 contradiction.

434 We have so far proven that when $\text{rch}(\mathcal{S}) < \text{rch}_{\text{loc.}}(\mathcal{S})$, then the reach cannot be realized
 435 in $\overline{\text{ax}(\mathcal{S})} \setminus \text{ax}(\mathcal{S})$. One must then have $x \in \text{ax}(\mathcal{S})$ and there are at least two points $p \neq q$
 436 closest points to x on \mathcal{S} .

437 If $|p - q| < 2\text{rch}(\mathcal{S})$ then by Theorem 16 one has that $d_{\mathcal{S}}(p, q) \leq 2\text{rch}(\mathcal{S}) \arcsin \frac{|p - q|}{2\text{rch}(\mathcal{S})}$
 438 which is less than $2\pi\text{rch}_{\text{loc.}}(\mathcal{S})$. Again, Corollary 22 implies that the length of a geodesic from
 439 p to q is at most $2\text{rch}_{\text{loc.}}(\mathcal{S}) \arcsin \frac{|p - q|}{2\text{rch}_{\text{loc.}}(\mathcal{S})}$, which, again, is impossible without entering the
 440 ball centered at x with radius $\text{rch}(\mathcal{S}) < \text{rch}_{\text{loc.}}(\mathcal{S})$.

441 Since $|p - q| \leq 2\text{rch}(\mathcal{S})$, the only possibility is that $|p - q| = 2\text{rch}(\mathcal{S})$, i.e. $x = (p + q)/2$,
 442 a bottleneck. ◀

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521 **A Manifolds of positive reach, differentiability, and tangent variation**

522 **A.1 Convex cones**

523 In this section we recall the definition of a convex cone, and some known consequences. We
 524 follow Chapter 2 of [27] and refer to this book for further reading. After the short recap, we
 525 characterize vector spaces among convex cones by means of their homology, which we have
 526 not been able to find in the literature.

527 In one of the equivalent definitions of a convex cone we use geodesic convexity on the
 528 sphere \mathbb{S}^{d-1} , which we’ll recall first.

529 **► Definition 24.** *A set of directions $S \subset \mathbb{S}^{d-1}$ is said geodesically convex if, given $p, p' \in S$
 530 such that $\angle p, p' < \pi$, the unique minimizing geodesic joining p and p' , i.e. the arc pp' of a
 531 great circle strictly shorter than π , is contained in S .*

532 The notion of convex cone is standard. We use the following definitions, the first of which is
 533 standard and the second is easy to verify.

534 **► Definition 25 (Convex cone).** *For a subset $A \subset \mathbb{R}^d$, the following properties are equivalent
 535 (and define convex cones):*

- 536 (1) $\forall a_1, a_2 \in A, \forall \lambda_1, \lambda_2 \geq 0, \lambda_1 a_1 + \lambda_2 a_2 \in A$
 537 (2) *Both*

538
$$\forall a \in A, \forall \lambda \geq 0, \lambda a \in A$$

539 *and*

540
$$A \cap \mathbb{S}^{d-1} \text{ is geodesically convex in } \mathbb{S}^{d-1}$$

541 If $A \subset \mathbb{R}^d$, A^\perp , its dual convex cone is

542
$$A^\perp \stackrel{\text{def.}}{=} \{x \in \mathbb{R}^d, \forall a \in A, \langle a, x \rangle \leq 0\}.$$

543 We have the following, see [27].

544 ► **Lemma 26.** *If A is a closed convex cone it is the dual of its dual: $A = (A^\perp)^\perp$.*

545 Also, from Definition 25, item (1), we see that:

546 ► **Observation 27.** *For a convex cone A , we have that $A = -A$ if and only if A is a vector*
 547 *space.*

548 ► **Observation 28.** *A convex cone A is a k -dimensional vector space if and only if A^\perp is a*
 549 *$(n - k)$ -dimensional vector space.*

550 Moreover vector spaces can be distinguished among convex cones by the homology of
 551 their intersection with the unit sphere \mathbb{S}^{d-1} . More precisely, we have:

552 ► **Lemma 29.** *For a convex cone $A \subset \mathbb{R}^d$ and a non negative integer $k \leq d$, the following*
 553 *are equivalent:*

- 554 ■ *A is a k -dimensional vector space,*
- 555 ■ *$k \geq 1$ and the $(k - 1)$ -homology of $A \cap \mathbb{S}^{d-1}$ is non-trivial or $k = 0$ and $A \cap \mathbb{S}^{d-1} = \emptyset$.*

556 **Proof.** If A is a k -dimensional vector space, then $A \cap \mathbb{S}^{d-1} = \mathbb{S}^{k-1}$ whose $(k - 1)$ -homology is
 557 non-trivial and $(k' - 1)$ -homology for $k' \neq k$ is trivial. Of course, if $k = 0$, then $A \cap \mathbb{S}^{d-1} = \emptyset$.

558 If A is not a vector space, then by Observation 27, $A \neq -A$. This implies that there
 559 exists $a_0 \in A \cap \mathbb{S}^{d-1}$ with $-a_0 \notin A$. This in turn gives that for any $a \in A \cap \mathbb{S}^{d-1}$, one has
 560 $\angle a_0, a < \pi$ so that the (unique) minimal geodesic from a to a_0 remains in $A \cap \mathbb{S}^{d-1}$. It follows
 561 that there is a deformation retract from $A \cap \mathbb{S}^{d-1}$ to a_0 along geodesics. Hence that A is
 562 contractible and has therefore trivial homology. ◀

563 Vector spaces can also be distinguished among convex cones by the homology of the
 564 complement of their dual cone in the unit ball. More precisely, we have:

565 ► **Lemma 30.** *Let $A \subset \mathbb{R}^d$ be a closed convex cone and let $k \leq d$ be a non negative integer.*
 566 *The following are equivalent:*

- 567 ■ *The cone A is a k -dimensional vector space.*
- 568 ■ *If $k > 0$, the $(k - 1)$ -homology of $B(0, 1) \setminus A^\perp$ is non-trivial and if $k = 0$, $B(0, 1) \setminus A^\perp$ is*
 569 *empty.*

570 **Proof.** Since $\{0\}^\perp = \mathbb{R}^d$, the case $k = 0$ is trivial. We assume now $k > 0$. Since $0 \in A^\perp$,
 571 $B(0, 1) \setminus A^\perp$ can be written as

$$572 \quad \bigcup_{u \in \mathbb{S}^{d-1} \setminus A^\perp} \{\lambda u \mid \lambda \in (0, 1]\},$$

573 where \mathbb{S}^{d-1} denotes the unit sphere centred at the origin. This means that there is a
 574 deformation retract from $B(0, 1) \setminus A^\perp$ to $\mathbb{S}^{d-1} \setminus A^\perp$.

575 Now we use the same argument as in Lemma 29. If $k \geq 1$ and A is a k -dimensional vector
 576 space, then A^\perp is a $d - k$ dimensional vector space and $\mathbb{S}^{d-1} \setminus A^\perp$ is homeomorphic to the
 577 open cylinder $\mathbb{S}^{k-1} \times (-1, 1)^{d-k}$, whose $(k - 1)$ -homology is non-trivial.

578 Conversely, $k \geq 1$ and A is not a k -dimensional vector space, then by Observation 28
 579 neither is A^\perp . As in the proof of Lemma 29 we note that by Observation 27, $A^\perp \neq -A^\perp$.
 580 This implies that there exists $a_1 \in A^\perp \cap \mathbb{S}^{d-1}$ with $-a_1 \notin A^\perp$. This in turn gives that for
 581 any $\tilde{a} \in A^\perp \cap \mathbb{S}^{d-1}$, one has $\angle a_1, \tilde{a} < \pi$ so that the (unique) minimal geodesic from \tilde{a} to a_1
 582 remains in $A^\perp \cap \mathbb{S}^{d-1}$. It follows that there is a deformation retract from $A^\perp \cap \mathbb{S}^{d-1}$ to a_1
 583 along geodesics as well as a homotopy from $\mathbb{S}^{d-1} \setminus A^\perp$ to $\mathbb{S}^{d-1} \setminus a_1 \simeq B^{d-1}$, where B^{d-1}

584 denotes the open $d - 1$ -dimensional ball. Hence $\mathbb{S}^{d-1} \setminus A^\perp$ is contractible and its homology
 585 is therefore trivial.

586 Finally, we consider the case where $k = 0$. If A is a 0-dimensional vector space, that is a
 587 point, then A^\perp is the entire Euclidean space, and therefore $B(0, 1) \setminus A^\perp$ is empty. Conversely
 588 if $B(0, 1) \setminus A^\perp$ is empty then A^\perp is the entire Euclidean space and its dual is therefore a
 589 point. ◀

590 **A.2 The (generalized) tangent spaces of embedded manifolds**

591 In this section we establish that the tangent spaces of topologically embedded manifolds
 592 are vector spaces of the right dimension. The argument is based on homotopy theory and
 593 deformations retract in particular, which uses a more general statement that we prove first.

594 **Closest point projection in a neighborhood gives a deformation retract** We know (Lemma
 595 14 above and Lemmas 5 and 9 in [6]), that, if $\rho < \text{rch}(\mathcal{S})$, then $\text{rch}(\mathcal{S} \cap B(p, \rho)) \geq \text{rch}(\mathcal{S})$
 596 and that $\mathcal{S} \cap B(p, \rho)$ is contractible. Moreover one has:

597 ▶ **Lemma 31.** *If $0 < \rho < \text{rch}(\mathcal{S})$, the projection $\pi_{\mathcal{S} \cap B(p, \rho)}$ defines a deformation retract from*
 598 *$B(p, \rho) \setminus (p + \text{Nor}(p, \mathcal{S}))$ to $\mathcal{S} \cap B(p, \rho) \setminus \{p\}$.*

599 **Proof.** Since $\text{rch}(\mathcal{S} \cap B(p, \rho)) \geq \text{rch}(\mathcal{S})$, we know that $\pi_{\mathcal{S} \cap B(p, \rho)}$ is continuous (Lemma 11
 600 or 12). The homotopy $(t, x) \mapsto (1 - t)x + t\pi_{\mathcal{S} \cap B(p, \rho)}(x)$ is then a deformation retract from
 601 $B(p, \rho)$ to $\mathcal{S} \cap B(p, \rho)$. Therefore, the only thing we have to prove is that, for $y \in B(p, \rho)$:

$$602 \quad y \in p + \text{Nor}(p, \mathcal{S}) \iff \pi_{\mathcal{S} \cap B(p, \rho)}(y) = p.$$

603 We note that thanks to Definition 7, we have $\text{Nor}(p, \mathcal{S}) = \text{Nor}(p, \mathcal{S} \cap B(p, \rho))$. The result
 604 now follows by Lemma 13 and the fact that $\rho < \text{rch}(\mathcal{S}) \leq \text{rch}(\mathcal{S} \cap B(p, \rho))$. ◀

605 **Tangent cones of embedded n-manifolds with positive reach are n-vector spaces**

606 ▶ **Lemma 32.** *Let $\mathcal{M} \subset \mathbb{R}^d$ be a topologically embedded n -manifold in \mathbb{R}^d , with positive*
 607 *reach. Then for any $p \in \mathcal{M}$, $\text{Tan}(p, \mathcal{M})$ is a n -dimensional vector space.*

608 **Proof.** By definition of an embedded manifold, \mathcal{M} has the induced topology. Let $p \in U$,
 609 where $U \subset \mathcal{M}$ is the open domain of a chart in the atlas of \mathcal{M} . This means that there exists
 610 a map $f : U \rightarrow \mathbb{R}^n$ that is a homeomorphism on its image. We note that because \mathcal{M} is
 611 compact, or equivalently closed and bounded, $\mathcal{M} \setminus U$ is compact too and thus $d(p, \mathcal{M} \setminus U) > 0$.
 612 We now choose ρ , so that

$$613 \quad 0 < \rho < \min(\text{rch}(\mathcal{M}), d(p, \mathcal{M} \setminus U)).$$

614 Note that in particular $\mathcal{M} \cap B(p, \rho) \subset U$.

615 By Lemma 14 the set $\mathcal{M} \cap B(p, \rho)$ is contractible. Because f is an homeomorphism on
 616 its image $f(\mathcal{M} \cap B(p, \rho))$ is a contractible subset of \mathbb{R}^n containing a neighborhood of $f(p)$.
 617 Therefore, we know that the $n - 1$ homology of $f(\mathcal{M} \cap B(p, \rho)) \setminus \{f(p)\}$ is non-trivial. Using
 618 again that f is an homeomorphism on its image, we see that the set $f(\mathcal{M} \cap B(p, \rho)) \setminus \{f(p)\}$
 619 is homeomorphic to $\mathcal{M} \cap B(p, \rho) \setminus \{p\}$. By Lemma 31, $\mathcal{M} \cap B(p, \rho) \setminus \{p\}$ has the same
 620 homotopy type as $B(p, \rho) \setminus (p + \text{Nor}(p, \mathcal{M}))$.

621 We get that the $n - 1$ homology of $B(p, \rho) \setminus (p + \text{Nor}(p, \mathcal{M}))$ is non-trivial, which, by
 622 Lemma 30 gives that $\text{Tan}(p, \mathcal{M})$ is a n -dimensional vector space. ◀

623 **A.3 Semi-continuity of normal cones and C^1 embeddings**

624 In this section we first discuss a weak form of continuity for tangent and normal cones of
 625 general sets of positive reach before focusing on manifolds.

626 **Semi-continuity of $p \mapsto \text{Nor}(p, \mathcal{S})$ and $p \mapsto \text{Tan}(p, \mathcal{S})$ for sets of positive reach** For a cone
 627 X , $X^{\angle \epsilon}$ is the set of vectors making an angle less than ϵ with some vector in X . For sets with
 628 positive reach, we have a semi-continuity of the normal and tangent cones, more precisely,

629 **► Lemma 33.** *If $\text{rch}(\mathcal{S}) > 0$, then*

630
$$\forall p \in \mathcal{S}, \forall \epsilon > 0, \exists \alpha > 0 \mid p' \in \mathcal{S} \cap B(p, \alpha) \Rightarrow \text{Nor}(p', \mathcal{S}) \subset \text{Nor}(p, \mathcal{S})^{\angle \epsilon}. \tag{13}$$

631 **Proof.** Take ρ such that $0 < \rho < \text{rch}(\mathcal{S})$ and denote by $\mathcal{S}^{\partial \oplus \rho}$ the boundary of the ρ -offset of
 632 \mathcal{S} , in other words,

633
$$\mathcal{S}^{\partial \oplus \rho} \stackrel{\text{def.}}{=} \{x \in \mathbb{R}^d \mid d(x, \mathcal{S}) = \rho\}.$$

634 For $p \in \mathcal{S}$, we write $N(p, \mathcal{S}, \rho)$ for the set of points of $\mathcal{S}^{\partial \oplus \rho}$ whose projection is p , that is

635
$$N(p, \mathcal{S}, \rho) \stackrel{\text{def.}}{=} \{q \in \mathcal{S}^{\partial \oplus \rho} \mid \pi_{\mathcal{S}}(q) = p\}.$$

636 Lemma 13 yields that if $p \in \mathcal{S}$, then

637
$$\text{Nor}(p, \mathcal{S}) = \{\lambda(q - p) \mid \lambda \geq 0, q \in N(p, \mathcal{S}, \rho)\}. \tag{14}$$

638 For $\eta > 0$ we denote by $N(p, \mathcal{S}, \rho)^\eta$ the open offset of $N(p, \mathcal{S}, \rho)$ in $\mathcal{S}^{\partial \oplus \rho}$, that is

639
$$N(p, \mathcal{S}, \rho)^\eta \stackrel{\text{def.}}{=} \left\{ q \in \mathcal{S}^{\partial \oplus \rho}, d(q, N(p, \mathcal{S}, \rho)) < \eta \right\}.$$

640 For any integer $j \geq 1$, the set

641
$$\mathcal{S}^{\partial \oplus \rho} \cap \pi_{\mathcal{S}}^{-1}(B(p, 1/j) \cap \mathcal{S})$$

642 is compact, being bounded and the inverse image of a closed set by the continuous map $\pi_{\mathcal{S}}$.

643 Because $\mathcal{S}^{\partial \oplus \rho} \setminus N(p, \mathcal{S}, \rho)^\eta$ is a closed set, the set

644
$$K_j \stackrel{\text{def.}}{=} \mathcal{S}^{\partial \oplus \rho} \cap \pi_{\mathcal{S}}^{-1}(B(p, 1/j) \cap \mathcal{S}) \cap (\mathcal{S}^{\partial \oplus \rho} \setminus N(p, \mathcal{S}, \rho)^\eta)$$

 645
$$= (\mathcal{S}^{\partial \oplus \rho} \setminus N(p, \mathcal{S}, \rho)^\eta) \cap \pi_{\mathcal{S}}^{-1}(B(p, 1/j) \cap \mathcal{S})$$

646 is compact, but may be empty. We have that

647
$$\bigcap_{j \geq 1} \pi_{\mathcal{S}}^{-1}(B(p, 1/j) \cap \mathcal{S}) = \pi_{\mathcal{S}}^{-1}(\{p\}),$$

648 because if $a \notin \pi_{\mathcal{S}}^{-1}(\{p\})$, then $\pi_{\mathcal{S}}(a) = q \neq p$ and thus $a \notin \bigcap_{j \geq 1} \pi_{\mathcal{S}}^{-1}(B(p, 1/j) \cap \mathcal{S})$ for
 649 some j . Hence, since

650
$$\bigcap_{j \geq 1} K_j = \mathcal{S}^{\partial \oplus \rho} \cap \pi_{\mathcal{S}}^{-1}(\{p\}) \cap (\mathcal{S}^{\partial \oplus \rho} \setminus N(p, \mathcal{S}, \rho)^\eta) = N(p, \mathcal{S}, \rho) \cap (\mathcal{S}^{\partial \oplus \rho} \setminus N(p, \mathcal{S}, \rho)^\eta) = \emptyset,$$

XX:20 Manifolds of positive reach, differentiability, tangent variation, and attaining the reach

651 we have that $(K_j)_{(j \geq 1)}$ is a nested (in inverse order) sequence of compact sets whose
 652 intersection is empty. Therefore, there is some integer number $n \geq 1$ such that $K_n = \emptyset$, such
 653 that

$$654 \quad \mathcal{S}^{\partial \oplus \rho} \cap \pi_{\mathcal{S}}^{-1}(B(p, 1/n) \cap \mathcal{S}) \subset N(p, \mathcal{S}, \rho)^\eta.$$

655 We note that such an n exists for any $\eta > 0$. This implies that for any η such that
 656 $0 < \eta < \rho \sin \frac{\epsilon}{2}$, there exists an n large enough such that both:

$$657 \quad \frac{1}{n} < \frac{1}{2} \rho \sin \epsilon \tag{15}$$

658 and

$$659 \quad \mathcal{S}^{\partial \oplus \rho} \cap \pi_{\mathcal{S}}^{-1}(B(p, 1/n) \cap \mathcal{S}) \subset N(p, \mathcal{S}, \rho)^{\rho \sin \frac{\epsilon}{2}}.$$

660 Using this n we pick $\alpha = \frac{1}{n}$, and we claim that this α satisfies the assertion in the lemma.
 661 Indeed, if $p' \in \mathcal{S} \cap B(p, \alpha)$, then $N(p', \mathcal{S}, \rho) \subset N(p, \mathcal{S}, \rho)^{\rho \sin \frac{\epsilon}{2}}$. Hence by definition, if
 662 $q' \in N(p', \mathcal{S}, \rho)$, then there exists a $q \in N(p, \mathcal{S}, \rho)$ such that $d(q, q') < \rho \sin \frac{\epsilon}{2}$, while by (15),
 663 $d(p, p') < \rho \sin \frac{\epsilon}{2}$. Since $|q - p| = |q' - p'| = \rho$ and $|p - p'|, |q - q'| < \rho \sin \frac{\epsilon}{2}$, we get

$$664 \quad |(q - p) - (q' - p')| < 2\rho \sin \frac{\epsilon}{2},$$

665 so that

$$666 \quad \angle(q' - p', q - p) < \epsilon.$$

667 This holds for any $q' \in N(p', \mathcal{S}, \rho)$ and since $q \in N(p, \mathcal{S}, \rho)$, the claim is proved, thanks to
 668 (14). ◀

669 The semi-continuity of normal cones translates into a symmetric result for tangent cones.
 670 This is easy to see: For set \mathcal{S} with positive reach, $\text{Tan}(p, \mathcal{S}) = \text{Nor}(p, \mathcal{S})^\perp$, and, if X and Y
 671 are convex cones, then:

$$672 \quad X \subset Y^{\angle \epsilon} \iff \forall x \in X \setminus \{0\}, \forall y' \in Y^\perp \setminus \{0\}, \angle x, y' > \frac{\pi}{2} - \epsilon$$

$$673 \quad \iff Y^\perp \subset (X^\perp)^{\angle \epsilon}.$$

674 Lemma 33 therefore similarly (but inclusion reversed) gives semi-continuity of $p \mapsto \text{Tan}(p, \mathcal{S})$
 675

$$676 \quad \forall p \in \mathcal{S}, \forall \epsilon > 0, \exists \alpha > 0 \mid p' \in \mathcal{S} \cap B(p, \alpha) \Rightarrow \text{Tan}(p, \mathcal{S}) \subset \text{Tan}(p', \mathcal{S})^{\angle \epsilon}. \tag{16}$$

677 **Continuity if the tangent spaces are vector spaces** Of course, in cases where Tan and Nor
 678 are vector spaces, this semi-continuity gives continuity. This results from a straightforward
 679 lemma:

680 **► Lemma 34.** *If A and B are k -dimensional vector subspaces of \mathbb{R}^d , with $1 \leq k \leq d - 1$,
 681 then*

$$682 \quad A \subset B^{\angle \epsilon} \iff B \subset A^{\angle \epsilon}.$$

683 **Proof.** If $A = B$ the statement is trivially true. If $A \neq B$, the bisector of A and B contains
 684 at least one hyperplane P and B is the image of A by the isometric mirror symmetry with
 685 respect to P . ◀

686 From Lemma 34, we get that, in case of manifolds, Lemma 33 and equation (16) gives us
 687 the continuity of tangent spaces. More precisely, we have:

688 ► **Lemma 35.** *If \mathcal{M} is a topologically embedded n -manifold in \mathbb{R}^d , with positive reach,
 689 then \mathcal{M} is C^1 embedded in \mathbb{R}^d . More precisely for any $p \in \mathcal{M}$, there exists some open
 690 $p \in U_p \subset \text{Tan}(p, \mathcal{M})$ and a map $\phi : U_p \rightarrow \text{Nor}(p, \mathcal{M})$, such that $x \mapsto \Phi(x) \stackrel{\text{def.}}{=} p + x + \phi(x)$
 691 is a C^1 map from U_p to \mathcal{M} .*

692 **Proof.** Lemma 32 says that $\text{Tan}(q, \mathcal{M})$ is a n -dimensional vector space for any $q \in \mathcal{M}$.

693 Regarding $\text{Tan}(q, \mathcal{M})$ as an element of the Grassmannian $\text{Gr}(n, \mathbb{R}^d)$, Lemmas 33 and 34
 694 give that the map defined from \mathcal{M} to $\text{Gr}(n, \mathbb{R}^d)$ by $q \mapsto \text{Tan}(q, \mathcal{M})$ is continuous. It follows
 695 that, for $p \in \mathcal{M}$, there is some ϵ , with $0 < \epsilon < \text{rch}(\mathcal{M})/2$, such that

$$696 \quad \forall q \in \mathcal{M} \cap B(p, \epsilon), \angle \text{Tan}(q, \mathcal{M}), \text{Tan}(p, \mathcal{M}) < \pi/4. \tag{17}$$

697 We claim that for this ϵ , the restriction to $\mathcal{M} \cap B(p, \epsilon)$ of the orthogonal projection on
 698 $p + \text{Tan}(p, \mathcal{M})$ is injective.

699 Indeed, assume (to derive a contradiction) that there exist $q_1, q_2 \in \mathcal{M} \cap B(p, \epsilon)$ such
 700 that $q_1 \neq q_2$ and $\pi_{p+\text{Tan}(p, \mathcal{M})}(q_1) = \pi_{p+\text{Tan}(p, \mathcal{M})}(q_2)$. Then, because $|q_2 - q_1| < \text{rch}(\mathcal{M})$,
 701 inequality (7) from Lemma 9 yields that $\sin \angle(q_1 - q_2, \text{Tan}(q_1, \mathcal{M})) \leq \frac{1}{2}$. Combining this
 702 with (17),

$$703 \quad \begin{aligned} \angle(q_1 - q_2, \text{Tan}(p, \mathcal{M})) &\leq \angle(\text{Tan}(p, \mathcal{M}), \text{Tan}(q_1, \mathcal{M})) + \angle(q_1 - q_2, \text{Tan}(q_1, \mathcal{M})) \\ 704 &< \frac{\pi}{4} + \frac{\pi}{6} < \frac{\pi}{2}, \end{aligned}$$

705 which cannot be since $q_2 - q_1$ is orthogonal to $\text{Tan}(p, \mathcal{M})$. Hence, the claim is proved.

706 Since the restriction to $\mathcal{M} \cap B(p, \epsilon)$ of the orthogonal projection on $p + \text{Tan}(p, \mathcal{M})$ is
 707 an injective continuous map, from an open n -manifold to the n -manifold $p + \text{Tan}(p, \mathcal{M})$,
 708 invariance of domain implies that it is a homeomorphism on its image. This means that
 709 $\mathcal{M} \cap B(p, \epsilon)$ is the graph of a map $x \mapsto \phi(x)$ from some open set $U_p \subset \text{Tan}(p, \mathcal{M})$ to
 710 $\text{Nor}(p, \mathcal{M})$, i.e. $x \mapsto p + x + \phi(x)$ is a local homeomorphism from U_p to \mathcal{M} . Moreover, these
 711 maps induce an atlas for \mathcal{M} .

712 Definition 7 and the fact that $\text{Tan}(p, \mathcal{M})$ is a n -dimensional linear space together imply
 713 that ϕ is differentiable at 0 with $D\Phi(0) = 0$, and more generally that ϕ is differentiable at
 714 any $x \in U_p$. The continuity (16) of $q \mapsto \text{Tan}(q, \mathcal{M})$ for the Grassmannians metric yields that
 715 ϕ is of class C^1 . ◀

716 A.4 From C^1 to $C^{1,1}$ embeddings

717 In this section we prove the first main result of this paper, namely that manifolds of positive
 718 reach are $C^{1,1}$. We distinguish ourselves from the previous methods by making all constants
 719 explicit.

720 In this paper we view derivatives of a $C^{1,1}$ function F as linear operators (matrices). The
 721 norm we use on the derivatives of F is the operator 2-norm, that is

$$722 \quad \|DF(x)\| = \max_{|u|=1} \frac{|DF(x)u|}{|u|}.$$

723 For example, F having a λ -Lipschitz derivative means that the difference of Jacobians viewed
 724 as matrices (or linear operators) satisfies :

$$725 \quad \|DF(x) - DF(y)\| \leq \lambda|x - y| \text{ for all } x, y. \tag{18}$$

726 The following Lemma on $C^{1,1}$ functions is essential to our proof and can be found in a
 727 number of texts concerning semiconcave functions as well as [21, 22], albeit without constants,
 728 the version with constants we recall here can be found as Corollary 3.3.8 in [10].

729 ► **Lemma 17.** *Let $F : U \rightarrow \mathbb{R}$ be a function defined on a convex open set $U \subset \mathbb{R}^n$. Then the*
 730 *following properties are equivalent:*

- 731 ■ F is of class $C^{1,1}$ with 1-Lipschitz derivative.
- 732 ■ both $x \rightarrow 1/2 x^2 - F(x)$ and $x \rightarrow 1/2 x^2 + F(x)$ are convex functions.

733 Following two lemmas are of use in the proof of Lemma 38.

734 ► **Lemma 36.** *Assume that $\mathcal{M} \subset \mathbb{R}^d$ is a topologically embedded n -manifold, with reach*
 735 *larger or equal to $R > 0$. Suppose that $p \in \mathcal{M}$, and let $U_p \subset \text{Tan}(p, \mathcal{M})$, $\phi : U_p \rightarrow \text{Nor}(p, \mathcal{M})$,*
 736 *and Φ be as in Lemma 35.*

737 *Then, for all $\theta > 0$, there is an $\alpha > 0$ such that:*

$$738 \quad |y|, |y'| \leq \alpha \Rightarrow \begin{cases} \angle \text{Tan}(\Phi(y), \mathcal{M}), \text{Tan}(\Phi(y'), \mathcal{M}) < \theta \\ \text{and} \\ \angle \Phi(y') - \Phi(y), \text{Tan}(p, \mathcal{M}) < \theta. \end{cases} \quad (19)$$

739 **Proof.** Because $\phi(0) = 0$, $D\phi(0) = 0$ and, according to Lemma 35, ϕ is C^1 , there exists an
 740 $\alpha > 0$ such that:

$$741 \quad |y| \leq \alpha \Rightarrow \|D\phi(y)\| < \tan \frac{\theta}{2}$$

$$742 \quad \Rightarrow \angle \text{Tan}(p, \mathcal{M}), \text{Tan}(\Phi(y), \mathcal{M}) < \frac{\theta}{2},$$

743 which gives us the first implication in (19), thanks to the triangle inequality on Grassmannians.
 744 For the second implication, observe that:

$$745 \quad \phi(y') - \phi(y) = \int_0^1 D\phi((1-t)y + ty') \cdot (y' - y) dt$$

$$746 \quad < |y' - y| \tan \frac{\theta}{2},$$

747 so that

$$748 \quad \angle \Phi(y') - \Phi(y), \text{Tan}(p, \mathcal{M}) < \frac{\theta}{2} < \theta.$$

749 ◀

750 ► **Lemma 37.** *For any $p \in \mathcal{M}$, there is $\alpha > 0$ such that for any α' such that $0 < \alpha' \leq \alpha$,*
 751 *for any $q \in \mathcal{M} \cap \partial(B(p, \alpha'))$ and $q' \in \mathcal{M} \cap B(p, \alpha')$ the closest point projection of q' on*
 752 *$q + \text{Tan}(q, \mathcal{M})$ belongs to $q + \text{Tan}(q, \mathcal{M} \cap B(p, \alpha'))$, in particular, one has:*

$$753 \quad d_{\mathbb{E}}(q', q + \text{Tan}(q, \mathcal{M} \cap B(p, \alpha'))) = d_{\mathbb{E}}(q', q + \text{Tan}(q, \mathcal{M})) \quad (20)$$

754 **Proof.** Thanks to Lemma 36 we have that for sufficiently α' and $q \in \mathcal{M} \cap \partial B(p, \alpha')$, the
 755 bound $\angle q - p, \text{Tan}(q, \mathcal{M}) < \pi/2$ holds. Since $\text{Nor}(q, B(p, \alpha')) = \{\lambda(q - p) \mid \lambda \geq 0\}$, this
 756 means in particular that one has:

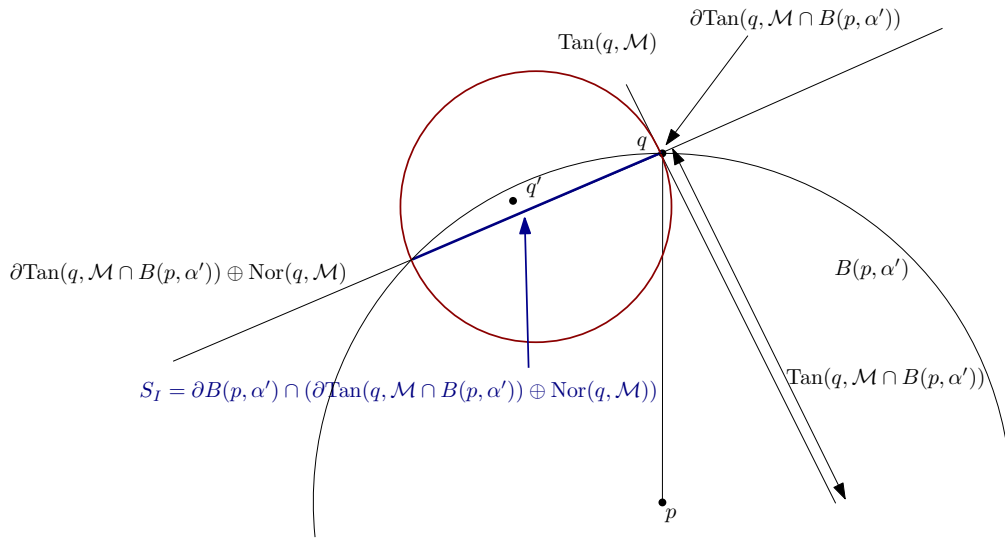
$$757 \quad \text{Nor}(q, B(p, \alpha')) \cap (-\text{Nor}(q, \mathcal{M})) = \text{Nor}(q, B(p, \alpha')) \cap \text{Nor}(q, \mathcal{M}) = \{0\}.$$

758 This means that one can apply [14, Theorem 4.10 (3)], which gives us

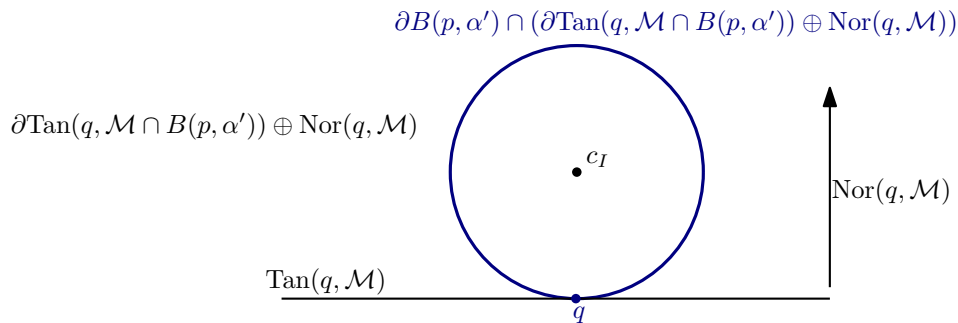
$$\begin{aligned}
 759 \quad \text{Nor}(q, \mathcal{M} \cap B(p, \alpha')) &= \text{Nor}(q, \mathcal{M}) \oplus \text{Nor}(q, B(p, \alpha')) \\
 760 \quad &= \text{Nor}(q, \mathcal{M}) \oplus \{\lambda(q - p) \mid \lambda \geq 0\}.
 \end{aligned}$$

761 Because normal and tangent cones are dual we see that

$$\begin{aligned}
 762 \quad \text{Tan}(q, \mathcal{M} \cap B(p, \alpha')) &= (\text{Nor}(q, \mathcal{M}) \oplus \{\lambda(q - p) \mid \lambda \geq 0\})^\perp \\
 763 \quad &= \text{Nor}(q, \mathcal{M})^\perp \cap \{\lambda(q - p) \mid \lambda \geq 0\}^\perp \\
 764 \quad &= \text{Tan}(q, \mathcal{M}) \cap \{u \in \mathbb{R}^d \mid \langle u, q - p \rangle \leq 0\}.
 \end{aligned} \tag{21}$$



765 ■ **Figure 6** Notation used in the proof of Lemma 37 (side view).



766 ■ **Figure 7** Notation used in the proof of Lemma 37 (the restriction to the $d - 1$ -plane $\partial \text{Tan}(q, \mathcal{M} \cap B(p, \alpha')) \oplus \text{Nor}(q, \mathcal{M})$).

768 Consider the $d - 1$ dimensional plane

$$\begin{aligned}
 769 \quad P &= \partial \text{Tan}(q, \mathcal{M} \cap B(p, \alpha')) \oplus \text{Nor}(q, \mathcal{M}). \\
 770 \quad &= \partial (\text{Tan}(q, \mathcal{M}) \cap \{u \in \mathbb{R}^d \mid \langle u, q - p \rangle \leq 0\}) \oplus \text{Nor}(q, \mathcal{M}) && \text{(by (21))} \\
 771 \quad &= (\text{Tan}(q, \mathcal{M}) \cap \{u \in \mathbb{R}^d \mid \langle u, q - p \rangle = 0\}) \oplus \text{Nor}(q, \mathcal{M}),
 \end{aligned}$$

772 see Figure 6. We stress that $\{u \in \mathbb{R}^d \mid \langle u, q - p \rangle = 0\}$ is the tangent plane of the sphere
 773 $\partial B(p, \alpha')$ at q . Moreover the plane P separates exactly those points q' for which the closest
 774 point projection of q' on $q + \text{Tan}(q, \mathcal{M})$ belongs to $q + \text{Tan}(q, \mathcal{M} \cap B(p, \alpha'))$ from the points
 775 q' for which this does not hold. The plane P intersects the sphere $\partial B(p, \alpha')$ in q , more
 776 precisely, the intersection $S_I = q + P \cap \partial B(p, \alpha')$ is a $d - 2$ -sphere that is tangent to q .
 777 These two observations mean that the centre c_I of S_I satisfies $c_I - q \in \text{Nor}(q, \mathcal{M})$, see
 778 Figure 7. On the other hand any point q' that does not project onto a point that belongs in
 779 $q + \text{Tan}(q, \mathcal{M} \cap B(p, \alpha'))$ will lie in the centred at c_I , now seen as a point in \mathbb{R}^d , with radius
 780 $|c_I - q|$, that is the red sphere in Figure 6. However, by assumption $\alpha < \text{rch}(\mathcal{M})$, which
 781 means that such a ball cannot contain points of \mathcal{M} . ◀

782 ▶ **Lemma 38.** *Assume that $\mathcal{M} \subset \mathbb{R}^d$ is a topologically embedded n -manifold with positive reach.
 783 Suppose that $p \in \mathcal{M}$, with $\text{rch}_{loc.}(p, \mathcal{M}) > 0$ and let $U_p \subset \text{Tan}(p, \mathcal{M})$, $\phi : U_p \rightarrow \text{Nor}(p, \mathcal{M})$,
 784 and Φ be as in Lemma 35.*

785 *Then, for all $\epsilon > 0$ and $v \in \text{Nor}(p, \mathcal{M})$, with $|v| = 1$, there is an $\alpha > 0$ such that the
 786 restriction to $U_p \cap B(0, \alpha)^\circ$ of*

$$787 \quad x \mapsto 1/2 x^2 - (\text{rch}_{loc.}(p, \mathcal{M}) - \epsilon) \langle v, \phi(x) \rangle$$

788 *is convex. If $\text{rch}_{loc.}(p, \mathcal{M}) = \infty$, then the statement holds with $\text{rch}_{loc.}(p, \mathcal{M})$ replaced by any
 789 $R > \epsilon$.*

790 *Moreover, if $r > \text{rch}_{loc.}(p, \mathcal{M})$ then, for any $\eta > 0$, there is $v \in \text{Nor}(p, \mathcal{M})$, with $|v| = 1$,
 791 such that the restriction of*

$$792 \quad x \mapsto 1/2 x^2 - r \langle v, \phi(x) \rangle$$

793 *to $B(0, \eta)$ is not convex.*

794 **Proof.** Let us choose $\alpha > 0$ as in Lemma 36 and 37, so that (19) and (20) hold. From the
 795 definition of $\text{rch}_{loc.}(p, \mathcal{M})$, $\alpha > 0$ can moreover be chosen small enough such that

$$796 \quad \text{rch}(\mathcal{M} \cap B(p, \alpha)) > R := \text{rch}_{loc.}(p, \mathcal{M}) - \epsilon/2. \tag{22}$$

797 Let $y \in U_p$ and write $q = p + y + \phi(y) = \Phi(y) \in \mathcal{M}$. Choose $\theta > 0$ such that:

$$798 \quad \cos^3 \theta > \frac{R - \epsilon/2}{R}. \tag{23}$$

799 For $y, y' \in U_p$ we denote the images under Φ , that is the points on \mathcal{M} , by $q = \Phi(y)$, $q' = \Phi(y')$.
 800 We now take $y, y' \in U_p$, such that one has $q, q' \in B(p, \alpha)$. This means in particular that
 801 $|y|, |y'| \leq \alpha$.

802 Thanks to Equation (8) in Lemma 10 and the fact that $R \leq \text{rch}(B(p, \alpha))$, one has

$$803 \quad d(q', q + \text{Tan}(q, \mathcal{M})) \leq \frac{|q' - q|^2}{2R}.$$

804 Combining this with (20) yields

$$805 \quad d(q', q + \text{Tan}(q, \mathcal{M} \cap B(p, \alpha))) = d(q', q + \text{Tan}(q, \mathcal{M})) \leq \frac{|q' - q|^2}{2R}. \tag{24}$$

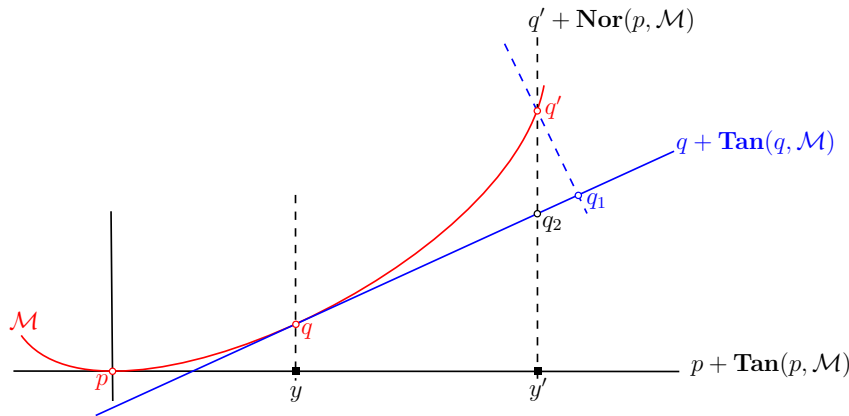
806 The first order Taylor expansion of Φ near y is given by

$$807 \quad z \mapsto \Phi(y) + D\Phi(y)(z - y) = (p + y + \phi(y)) + (z - y) + D\phi(y)(z - y) \\ 808 \quad = p + \phi(y) + z + D\phi(y)(z - y),$$

809 and its graph coincides with the affine tangent space $q + \text{Tan}(q, \mathcal{M})$ of \mathcal{M} at $q = \Phi(y)$.

810 Write q_1 for orthogonal projection of q' on $q + \text{Tan}(q, \mathcal{M})$, that is $q_1 = \pi_{q + \text{Tan}(q, \mathcal{M})}(q')$.
 811 We note that $q_1 - q' \in \text{Nor}(q, \mathcal{M})$. Since, $q' + \text{Nor}(p, \mathcal{M})$ and $q + \text{Tan}(q, \mathcal{M})$ are transversal
 812 and their dimension sum to d , they have a unique intersection point q_2 . The point q_2 , see
 813 Figure 8, is also the point above y' on the graph of the first order Taylor expansion of ϕ
 814 near y :

815
$$q_2 = p + y' + \phi(y) + D\phi(y)(y' - y).$$



816 ■ **Figure 8** Notation used in the proof of Lemma 38.

817 Since $q_1, q_2 \in q + \text{Tan}(q, \mathcal{M})$, one has that $q_2 - q_1 \in \text{Tan}(q, \mathcal{M})$. As we concluded above
 818 $q_1 - q' \in \text{Nor}(q, \mathcal{M})$. Combining these observations gives that the triangle $q'q_1q_2$ has a right
 819 angle at q_1 .

820 Because $q_2 - q_1 \in \text{Tan}(q, \mathcal{M})$ and $\angle \text{Tan}(q, \mathcal{M}) \text{Tan}(p, \mathcal{M}) < \theta$, there is a vector $w \in$
 821 $\text{Tan}(p, \mathcal{M})$ such that $\angle w, q_2 - q_1 < \theta$. As $q_2 - q' \in \text{Nor}(p, \mathcal{M})$, one has $\angle w, q_2 - q' = \pi/2$,
 822 and therefore the triangle inequality on the sphere yields $\angle q_2 - q', q_2 - q_1 > \pi/2 - \theta$. This
 823 is one angle of the triangle $q'q_1q_2$ and we already established that this triangle has a right
 824 angle at q_1 and hence,

825
$$\angle q_2 - q', q_1 - q' < \theta.$$

826 This angle bound translates to the first inequality between the edge lengths of the triangle
 827 $q'q_1q_2$, because $q_1 = \pi_{q + \text{Tan}(q, \mathcal{M})}(q')$ and (24) the second inequality below follows,

828
$$|q' - q_2| \leq \frac{1}{\cos \theta} |q' - q_1| \leq \frac{1}{\cos \theta} \frac{|q' - q|^2}{2R}. \tag{25}$$

829 Since $\angle q' - q, \text{Tan}(p, \mathcal{M}) < \theta$ by (19), and $y' - y$ is the projection of $q' - q$ on $p + \text{Tan}(p, \mathcal{M})$,
 830 we have

831
$$|q' - q| \leq \frac{1}{\cos \theta} |y' - y|.$$

832 and (25) gives

833
$$|q' - q_2| \leq \frac{|y' - y|^2}{2R \cos^3 \theta}.$$

834 Because $q' = \Phi(y') = p + y' + \phi(y')$, $|q' - q_2|$ is the difference between $\phi(y')$ and the value,
 835 at y' , of the first order Taylor expansion of ϕ near y , so the last inequality gives

$$836 \quad \left| \phi(y') - \left(\phi(y) + D\phi(y)(y' - y) \right) \right| \leq \frac{|y' - y|^2}{2R \cos^3 \theta}$$

$$837 \quad < \frac{|y' - y|^2}{2(R - \epsilon/2)}. \quad (\text{by (23)})$$

838 Denoting by $f : U_p \rightarrow \mathbb{R}$ the map defined by $f(y) \stackrel{\text{def.}}{=} \langle v, \phi(y) \rangle$ this together with $|v| = 1$
 839 gives

$$840 \quad \left| f(y') - \left(f(y) + Df(y)(y' - y) \right) \right| < \frac{|y' - y|^2}{2(R - \epsilon/2)},$$

841 so that

$$842 \quad \frac{|y' - y|^2}{2(R - \epsilon/2)} + f(y') > f(y) + Df(y)(y' - y). \quad (26)$$

843 Since both $y' \mapsto |y' - y|^2$ and its derivative vanish at y , the right-hand side in (26) is the first
 844 order Taylor expansion at y of the left-hand side $y' \mapsto \frac{|y' - y|^2}{2(R - \epsilon/2)} + f(y')$. We have shown that
 845 the map $y' \mapsto \frac{|y' - y|^2}{2(R - \epsilon/2)} + f(y')$ is greater or equal to its first order Taylor expansion at y .

846 Because $y' \mapsto (y' - y)^2 - y'^2$ is a linear function, this property extends to the map
 847 $g : U_p \cap B(0, \alpha)^\circ \rightarrow \mathbb{R}$ defined by

$$848 \quad g(y') \stackrel{\text{def.}}{=} \frac{y'^2}{2(R - \epsilon/2)} + f(y') = \frac{y'^2}{2(R - \epsilon/2)} + \langle v, \phi(y) \rangle = \frac{y'^2}{2(\text{rch}_{loc.}(p, \mathcal{S}) - \epsilon)} + \langle v, \phi(y) \rangle,$$

849 where the last equality follows from (22). This holds for any $y \in U_p$ such that $\Phi(y) \in \cap B(p, \alpha)$,
 850 so that for some $\alpha' > 0$ small enough, it holds for any $y \in U_p \cap B(0, \alpha')$. There, the function
 851 g is C^1 and greater or equal to its first order expansion at any point, and is therefore convex.
 852 We have proven the first statement of the lemma.

853 We now concentrate on the second statement.

854 As we have seen in Remark 3, $\rho \mapsto \text{rch}(\mathcal{M} \cap B(p, \rho))$ is non-increasing and hence

$$855 \quad \text{rch}(\mathcal{M} \cap B(p, \eta)) < \text{rch}_{loc.}(p, \mathcal{S}) \quad (\text{because non-increasing})$$

$$856 \quad < r. \quad (\text{by assumption})$$

857 Reusing the same construction (see Figure 8) as for the proof of the first statement of
 858 the lemma, we see that by (8), there exists $q, q' \in \mathcal{M} \cap B(p, \eta)$ with $q' \neq q$ such that:

$$859 \quad r > \frac{|q - q'|^2}{2d_{\mathbb{E}}(q', q + \text{Tan}(q, \mathcal{M} \cap B(p, \eta)))},$$

860 that is

$$861 \quad d_{\mathbb{E}}(q', q + \text{Tan}(q, \mathcal{M} \cap B(p, \eta))) > \frac{|q - q'|^2}{2r}. \quad (27)$$

862 By Lemma 37, one has

$$863 \quad d_{\mathbb{E}}(q', q + \text{Tan}(q, \mathcal{M})) > \frac{|q - q'|^2}{2r}. \quad (28)$$

864 Looking at Figure 8, we see that (28) translates into

$$865 \quad |q' - q_1| > \frac{|q - q'|^2}{2r},$$

866 which is equivalent to

$$867 \quad \frac{|q - q'|^2}{2r|q' - q_1|} < 1.$$

868 Because the left hand side is strictly less than 1, there exists a $\theta' > 0$ such that $\cos \theta' >$
 869 $\frac{|q - q'|^2}{2r|q' - q_1|}$. This means that in the context of Lemma 36, one can choose η (where we take
 870 $\alpha = \eta$ in Lemma 36) small enough to get $\cos \theta > \frac{|q - q'|^2}{2r|q' - q_1|}$. Because the angle between
 871 $\text{Tan}(p, \mathcal{M})$ and $\text{Tan}(q, \mathcal{M})$ is upper bounded by θ examining the triangle $q'q_1q_2$ in Figure 8
 872 yields that $|q' - q_2| \geq |q' - q_1| \cos \theta$, which combined with the previous result yields

$$873 \quad |q' - q_2| > \frac{|q - q'|^2}{2r}.$$

874 Writing

$$875 \quad v = \frac{q' - q_2}{|q' - q_2|} \in \text{Nor}(p, \mathcal{M}),$$

876 so that $|q' - q_2| = \langle v, q' - q_2 \rangle$ and using $|q - q'| > |y' - y|$, this can be reformulated into

$$877 \quad \langle v, q' - q_2 \rangle \geq \frac{|y' - y|^2}{2r}. \tag{29}$$

878 Because

$$879 \quad q' - q_2 = \left(0, \phi(y') - (\phi(y) + D\phi(y)(y' - y)) \right)_{\text{Tan}_p \mathcal{M} \times \text{Nor}_p \mathcal{M}},$$

880 and $f(y) \stackrel{\text{def.}}{=} \langle v, \phi(y) \rangle$, we may rewrite (29) as

$$881 \quad \frac{|y' - y|^2}{2r} + f(y') < f(y) + Df(y)(y' - y). \tag{30}$$

882 The function $x \mapsto g(x) = \frac{|x - y|^2}{2r} + f(x)$ is not convex since (30) says that $g(y')$ is below
 883 the first order Taylor expansion of g at y . ◀

884 A relatively straightforward consequence of Lemmas 17 and 38 is the following:

885 ▶ **Lemma 39.** *Consider a n -manifold $\mathcal{M} \subset \mathbb{R}^d$ topologically embedded in \mathbb{R}^d , with positive*
 886 *reach, a point $p \in \mathcal{M}$, $U_p \subset \text{Tan}(p, \mathcal{M})$ a neighborhood of p in $\text{Tan}(p, \mathcal{M})$, and $\phi : U_p \rightarrow$*
 887 *$\text{Nor}(p, \mathcal{M})$, the a C^1 map defined in Lemma 35, such that $x \mapsto \Phi(x) = p + x + \phi(x)$ is a*
 888 *chart from U_p to \mathcal{M} .*

889 *Then, for all $\epsilon > 0$, there is $\alpha > 0$ such that the derivative of the restriction of ϕ to*
 890 *$U_p \cap B(0, \alpha)^\circ$ is $\frac{1}{(\text{rch}_{\text{loc.}}(p, \mathcal{M}) - \epsilon)}$ -Lipschitz, where the distance on derivatives is the operator*
 891 *norm of the difference, that is*

$$892 \quad \|D\phi(y_2) - D\phi(y_1)\| \leq \frac{1}{(\text{rch}_{\text{loc.}}(p, \mathcal{M}) - \epsilon)} |y_2 - y_1|.$$

893 *If $\text{rch}_{\text{loc.}}(p, \mathcal{M}) = \infty$, then the statement holds with $\text{rch}_{\text{loc.}}(p, \mathcal{M})$ replaced by any $R > \epsilon$.*

894 **Proof.** We get from Lemmas 17 and 38, and reusing the same ϵ and α selection as in
 895 Lemma 38, we get that for any $v \in \text{Nor}(p, \mathcal{M})$ such that $|v| = 1$, the map $y \mapsto \langle v, \phi(y) \rangle$ is
 896 $\frac{1}{(\text{rch}_{loc.}(p, \mathcal{S}) - \epsilon)}$ -Lipschitz in $U_p \cap B(0, \alpha)^\circ$.

897 It follows that if $y_1, y_2 \in U_p \cap B(0, \alpha)^\circ$, then, for any $w \in \text{Tan}(p, \mathcal{M})$ and any $v \in$
 898 $\text{Nor}(p, \mathcal{M})$ such that $|w| = |v| = 1$:

$$899 \quad \langle v, D\phi(y_2)w \rangle - \langle v, D\phi(y_1)w \rangle \leq \frac{1}{(\text{rch}_{loc.}(p, \mathcal{M}) - \epsilon)} |y_2 - y_1|$$

900 and this precisely means that, for the operator norm:

$$901 \quad \|D\phi(y_2) - D\phi(y_1)\| \leq \frac{1}{(\text{rch}_{loc.}(p, \mathcal{M}) - \epsilon)} |y_2 - y_1|.$$

902 ◀

903 Next lemma is instrumental in the proof of Lemma 41:

904 ▶ **Lemma 40.** For F_1, F_2 linear maps from \mathbb{R}^n to \mathbb{R}^m , denote by L_1 and L_2 the corresponding
 905 graphs in the product space $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$ equipped with the canonical dot product:
 906 $L_i = \{(x, F_i(x)) \in \mathbb{R}^{n+m}\}$.

$$907 \quad 2 \sin \frac{\angle L_1, L_2}{2} \leq \|F_2 - F_1\|$$

908 where $\|\cdot\|$ is the L^2 operator norm.

909 **Proof.** The angle between L_1 and L_2 is attained in a pair of vectors, that is there exists
 910 $v_1 \in L_1$ and $v_2 \in L_2$ such that

$$911 \quad \angle L_1, L_2 = \angle v_1, v_2 = \min_{v'_2 \in L_2} \angle v_1, v'_2 = \max_{v'_1 \in L_1} \min_{v'_2 \in L_2} \angle v'_1, v'_2 = \max_{v'_2 \in L_2} \min_{v'_1 \in L_1} \angle v'_1, v'_2,$$

912 where the last equality follows from Lemma 34. This explains the asymmetrical expression.

913 For some $v_0 \in \mathbb{R}^n$, with $|v_0| = 1$, and $\lambda > 0$, we have $v_1 = \lambda(v_0, F_1(v_0))$ and therefore

$$914 \quad \angle L_1, L_2 = \angle(v_0, F_1(v_0)), v_2 \leq \angle(v_0, F_1(v_0)), (v_0, F_2(v_0)), \tag{31}$$

915 because, by definition of F_2 , we have $(v_0, F_2(v_0)) \in L_2$.

916 Since $|(v_0, F_1(v_0))|, |(v_0, F_2(v_0))| \geq 1$, one has:

$$917 \quad 2 \sin \frac{\angle(v_0, F_1(v_0)), (v_0, F_2(v_0))}{2} \leq |(v_0, F_2(v_0)) - (v_0, F_1(v_0))|$$

$$918 \quad \quad \quad = |F_2(v_0) - F_1(v_0)| \leq \|F_2 - F_1\|.$$

919 This with (31) gives the inequality of the lemma. ◀

920 ▶ **Lemma 41.** If $\mathcal{M} \subset \mathbb{R}^d$ is a n -manifold topologically embedded in \mathbb{R}^d , with positive reach,
 921 then \mathcal{M} is $C^{1,1}$ embedded and if $p_1, p_2 \in \mathcal{M}$ then

$$922 \quad \angle \text{Tan}(p_1, \mathcal{M}), \text{Tan}(p_2, \mathcal{M}) \leq \frac{1}{\text{rch}_{loc.}(\mathcal{M})} d_{\mathcal{M}}(p_1, p_2),$$

923 where $d_{\mathcal{M}}$ denotes the geodesic distance inside \mathcal{M} and we say that $1/\infty = 0$.

924 If now \mathcal{M} is an n -manifold with boundary, topologically embedded in \mathbb{R}^d , the same property
 925 holds when the shortest geodesic between p_1 and p_2 avoids the boundary.

926 **Proof.** If \mathcal{M} has local reach $\text{rch}_{loc.}(\mathcal{M}) = \infty$, (9) implies that all geodesics are straight lines,
 927 that is every connected component of \mathcal{M} is convex. Therefore, because \mathcal{M} does not contain
 928 a boundary by assumption, every connected component of \mathcal{M} is affine and the statement
 929 holds. This means that we can concentrate on the case where $\text{rch}_{loc.}(\mathcal{M}) < \infty$.

930 Lemma 39 already says that \mathcal{M} is $C^{1,1}$ embedded by considering the atlas defined by
 931 projections on local tangent spaces. Now, (if p_1 and p_2 lies in the same connected component
 932 of \mathcal{M}) considers (thanks to Lemma 15) a geodesic γ of length $L = d_{\mathcal{M}}(p_1, p_2)$ joining p_1 and
 933 p_2 . Take $\epsilon > 0$ and $\theta > 0$. By application of Lemma 39 there is at each point $q \in \gamma([0, L])$ an
 934 open neighborhood V_q of q in \mathcal{M} , such that V_q is the graph of a map ϕ_q from $\text{Tan}(q, \mathcal{M})$ to
 935 $\text{Nor}(q, \mathcal{M})$ whose derivative is bounded by $\tan \theta$ and, if $R \leq \text{rch}_{loc.}(\mathcal{M})$, is $\frac{1}{(R-\epsilon)}$ -Lipschitz
 936 (in the sense of Lemma 39). Consider for each $t \in [0, L]$ an open interval (t^-, t^+) such that
 937 $\gamma((t^-, t^+) \cap [0, L]) \subset V_{\gamma(t)}$. By compactness, one can extract a finite open cover of $[0, L]$ by
 938 intervals in the form $(t_i^-, t_i^+)_{i=0, N}$, with $t_i \in (t_i^-, t_i^+)$, $t_0 = 0$ and $t_N = L$. One can therefore
 939 find a finite sequence $0 = t'_0 < \dots < t'_{N'} = L$ such that for any j such that $0 \leq j < N'$, there
 940 is some i such that $[t'_j, t'_{j+1}] \subset (t_i^-, t_i^+)$. Since $\gamma(t'_0) = \gamma(0) = p_1$ and $\gamma(t'_{N'}) = \gamma(L) = p_2$,
 941 one has:

$$942 \quad \angle \text{Tan}(p_1, \mathcal{M}), \text{Tan}(p_2, \mathcal{M}) \leq \sum_{j=0}^{N'-1} \angle \text{Tan}(\gamma(t'_j), \mathcal{M}), \text{Tan}(\gamma(t'_{j+1}), \mathcal{M}) \quad (32)$$

943 The segment of geodesic $\gamma([t'_j, t'_{j+1}])$ belongs to the graph of the map from $\text{Tan}(\gamma(t_i), \mathcal{M})$
 944 which derivative is $\frac{1}{(R-\epsilon)}$ -Lipschitz for the operator norm, we get from Lemma 40 that:

$$945 \quad 2 \sin \frac{\angle \text{Tan}(\gamma(t'_j), \mathcal{M}), \text{Tan}(\gamma(t'_{j+1}), \mathcal{M})}{2} \leq \frac{1}{(R-\epsilon)} \left| \pi_{\text{Tan}(\gamma(t_i), \mathcal{M})}(\gamma(t'_{j+1})) - \pi_{\text{Tan}(\gamma(t_i), \mathcal{M})}(\gamma(t'_j)) \right|$$

946 and, since:

$$947 \quad \left| \pi_{\text{Tan}(\gamma(t_i), \mathcal{M})}(\gamma(t'_{j+1})) - \pi_{\text{Tan}(\gamma(t_i), \mathcal{M})}(\gamma(t'_j)) \right| \leq |\gamma(t'_{j+1}) - \gamma(t'_j)|$$

948 we get:

$$949 \quad 2 \sin \frac{\angle \text{Tan}(\gamma(t'_j), \mathcal{M}), \text{Tan}(\gamma(t'_{j+1}), \mathcal{M})}{2} \leq \frac{1}{(R-\epsilon)} |\gamma(t'_{j+1}) - \gamma(t'_j)|$$

950 Since $\angle \text{Tan}(\gamma(t'_j), \mathcal{M}), \text{Tan}(\gamma(t'_{j+1}), \mathcal{M}) < 2\theta$ and $\theta \mapsto \frac{\sin \theta}{\theta}$ is decreasing for positive small
 951 θ , one has:

$$952 \quad 2 \sin \frac{\angle \text{Tan}(\gamma(t'_j), \mathcal{M}), \text{Tan}(\gamma(t'_{j+1}), \mathcal{M})}{2} > \frac{\sin \theta}{\theta} \angle \text{Tan}(\gamma(t'_j), \mathcal{M}), \text{Tan}(\gamma(t'_{j+1}), \mathcal{M})$$

953 so that (32) gives:

$$954 \quad \angle \text{Tan}(p_1, \mathcal{M}), \text{Tan}(p_2, \mathcal{M}) \leq \sum_{j=0}^{N'-1} \frac{\theta}{(R-\epsilon) \sin \theta} |\gamma(t'_{j+1}) - \gamma(t'_j)| \leq \frac{\theta}{(R-\epsilon) \sin \theta} d_{\mathcal{M}}(p_1, p_2)$$

955 Since the last inequality holds for any $\theta > 0$ and $\epsilon > 0$, the lemma is proved. ◀

956 We can now prove one of our main theorems:

957 **► Theorem 4.** *If $\mathcal{M} \subset \mathbb{R}^d$ is a topologically embedded n -manifold with reach at least $R > 0$,*
 958 *then tangent spaces and normal spaces to \mathcal{M} are linear spaces at any point $p \in \mathcal{M}$ and the*
 959 *maps $\text{Tan} : \mathcal{M} \rightarrow \text{Gr}(n, \mathbb{R}^d)$ and $\text{Nor} : \mathcal{M} \rightarrow \text{Gr}(d-n, \mathbb{R}^d)$ are Lipschitz.*

XX:30 Manifolds of positive reach, differentiability, tangent variation, and attaining the reach

960 Moreover, the smallest Lipschitz constant satisfied by $p \mapsto \text{Tan}(p, \mathcal{S})$ is $\frac{1}{\text{rch}_{loc.}(\mathcal{M})}$:

$$961 \sup_{\substack{p, q \in \mathcal{M} \\ p \neq q}} \frac{\angle \text{Tan}(p, \mathcal{M}), \text{Tan}(q, \mathcal{M})}{d_{\mathcal{M}}(p, q)} = \frac{1}{\text{rch}_{loc.}(\mathcal{M})}, \quad (6)$$

962 where $d_{\mathcal{M}}$ denotes the geodesic distance on \mathcal{M} and we say that $d_{\mathcal{M}}(p, q) = \infty$ when p and q
963 are not connected by any path in \mathcal{M} and that $1/\infty = 0$.

964 **Proof.** One direction is given by Lemma 41. For the other direction, we distinguish two
965 cases, $\text{rch}_{loc.}(\mathcal{M}) = \infty$ and $\text{rch}_{loc.}(\mathcal{M}) < \infty$. If

$$966 \sup_{\substack{p, q \in \mathcal{M} \\ p \neq q}} \frac{\angle \text{Tan}(p, \mathcal{M}), \text{Tan}(q, \mathcal{M})}{d_{\mathcal{M}}(p, q)} = 0,$$

967 then \mathcal{M} is an affine space, because it is a closed manifold without boundary. This means that
968 each connected component of \mathcal{M} has infinite reach and by extension the local reach is also
969 infinite. This means that we can concentrate on the case where $\text{rch}_{loc.}(\mathcal{M}) < \infty$. It now
970 suffices to prove that, if $R > \text{rch}_{loc.}(\mathcal{M})$, then there exists distinct $q_1, q_2 \in \mathcal{M}$ (i.e. $q_1 \neq q_2$)
971 such that

$$972 \frac{\angle \text{Tan}(q_1, \mathcal{M}), \text{Tan}(q_2, \mathcal{M})}{d_{\mathcal{M}}(q_1, q_2)} > \frac{1}{R}. \quad (33)$$

973 Let $\theta > 0$ be such that

$$974 \cos^3 \theta > \frac{\text{rch}_{loc.}(\mathcal{M})}{R}.$$

975 From the definition of $\text{rch}_{loc.}(\mathcal{M})$ it follows that there exists $p \in \mathcal{M}$ such that

$$976 \text{rch}_{loc.}(\mathcal{M}) \leq \text{rch}_{loc.}(p, \mathcal{M}) < r := R \cos^3 \theta. \quad (34)$$

977 Lemma 36 implies that there exists $\alpha > 0$ such that (19) holds in the neighbourhood of p .
978 By taking $\eta = \alpha$ in the second statement of Lemma 38 we have that the restriction of

$$979 x \mapsto f(x) := 1/2 x^2 - r \langle v, \phi(x) \rangle$$

980 to $B(0, \eta)$ is not convex. We stress that as in Lemma 35 (and by extension Lemma 38)
981 p has coordinate 0. This implies that there exists $y, y' \in B(0, \eta)$ such that the segment
982 $[(y, f(y)), (y', f(y'))]$ is not always above the graph of f . Let $t_1, t_2 \in (0, 1)$, with $t_1 < t_2$
983 be such that the open interval (t_1, t_2) is a connected component of the non empty open
984 set $\{t \in [0, 1] \mid f((1-t)y + ty') > (1-t)f(y) + tf(y')\}$. This means that if we define
985 $y_i = (1-t_i)y + t_i y'$ we have that

$$986 \frac{d}{dt} \Big|_{t=0} f((1-t)y_1 + ty_2) \geq \frac{f(y_2) - f(y_1)}{t_2 - t_1}$$

$$987 \frac{d}{dt} \Big|_{t=1} f((1-t)y_1 + ty_2) \leq \frac{f(y_2) - f(y_1)}{t_2 - t_1}.$$

988 Combining these equations yields

$$989 \frac{d}{dt} \Big|_{t=1} f((1-t)y_1 + ty_2) - \frac{d}{dt} \Big|_{t=0} f((1-t)y_1 + ty_2) \leq 0.$$

990 Using the definition of f given above, this translates into

$$991 \quad (y_2 - y_1)^2 - r \left(\langle v, (D\phi(y_2) - D\phi(y_1))(y_2 - y_1) \rangle \right) \leq 0. \tag{35}$$

992 We define $q_i = \Phi(y_i) = (y_i; \phi(y_i))$, for $i = 1, 2$ and the vectors $v_i := D\Phi(y_i)(y_2 - y_1) \in$
 993 $\text{Tan}_{q_i} \mathcal{M}$. Note that $\angle \text{Tan}_p \mathcal{M}, \text{Tan}_{q_1} \mathcal{M} < \theta$. Further writing v'_2 for the projection of v_2 on
 994 $\text{Tan}_{q_1} \mathcal{M}$, we consider the triangle v_2, v'_2, v_1 .

995 Since $v'_2 - v_2$ is orthogonal to $\text{Tan}_{q_1} \mathcal{M}$, and $v'_2 - v_1 \subset \text{Tan}_{q_1} \mathcal{M}$, the triangle v_2, v'_2, v_1 is
 996 orthogonal at v'_2 and $\angle v'_2 - v_2, v_1 - v_2 < \theta$. One has then

$$997 \quad |v_2| \sin \angle v_2, \text{Tan}_{q_1} \mathcal{M} = |v_2 - v'_2| \geq \cos \theta |v_2 - v_1|.$$

998 Moreover, because

$$999 \quad \angle \text{Tan}_{q_2} \mathcal{M}, \text{Tan}_{q_1} \mathcal{M} \geq \angle v_2, \text{Tan}_{q_1} \mathcal{M} \geq \sin \angle v_2, \text{Tan}_{q_1} \mathcal{M},$$

1000 and $|v_2| < \frac{1}{\cos \theta} |y_2 - y_1|$, one has

$$1001 \quad \angle \text{Tan}_{q_2} \mathcal{M}, \text{Tan}_{q_1} \mathcal{M} > \cos^2 \theta \frac{|v_2 - v_1|}{|y_2 - y_1|}.$$

1002 By (35) one has

$$1003 \quad |v_2 - v_1| \geq \langle v, v_2 - v_1 \rangle \geq \frac{1}{r} (y_2 - y_1)^2.$$

1004 Combining the last two equations yields

$$1005 \quad \angle \text{Tan}_{q_2} \mathcal{M}, \text{Tan}_{q_1} \mathcal{M} > \frac{\cos^2 \theta}{r} |y_2 - y_1| \geq \frac{\cos^3 \theta}{r} d_{\mathcal{M}}(q_1, q_2),$$

1006 where the last inequality follows from the fact that $d_{\mathcal{M}}(q_1, q_2)$ is upper bounded by the
 1007 length of $\Phi([y_1, y_2])$. This with (34) gives us (33). ◀

1008 ▶ **Lemma 42.** *If $\mathcal{M} \subset \mathbb{R}^d$ is a n -manifold topologically embedded in \mathbb{R}^d , and γ a geodesic in*
 1009 *\mathcal{M} of length ℓ , the local reach of $\gamma([0, \ell])$ is upper bounded by the local reach of \mathcal{M} and in*
 1010 *particular:*

$$1011 \quad \forall s_1, s_2 \in [0, \ell], \quad \angle \dot{\gamma}(s_1), \dot{\gamma}(s_2) \leq \frac{1}{\text{rch}_{loc}(\mathcal{M})} |s_2 - s_1| \tag{36}$$

1012 **Proof.** Applying Theorem 16 to the set $\mathcal{M} \cap B(p, \rho)$, for arbitrary small ρ , shows that the
 1013 local reach of \mathcal{M} is the infimum on all the local reach on all geodesics on \mathcal{M} . Then one can
 1014 apply Lemma 41 to $\gamma([0, \ell])$ which gives (36). ◀

1015 ▶ **Theorem 1.** *If \mathcal{M} is a topologically embedded manifold with positive reach and p a point*
 1016 *in \mathcal{M} , then $\mathcal{M} \cap B^\circ(p, \sqrt{2} \text{rch}(\mathcal{M})) \cap \pi_{T_p \mathcal{M}}^{-1}(B^\circ(0, \text{rch}(\mathcal{M})))$ is the graph of a $C^{1,1}$ function*
 1017 *above the open domain $B^\circ(0, \text{rch}(\mathcal{M})) \subset T_p \mathcal{M}$.*

1018 **Proof.** Thanks to Theorem 16 we have that for $q \in \mathcal{M} \cap B^\circ(p, \sqrt{2} \text{rch}(\mathcal{M}))$

$$1019 \quad d_{\mathcal{M}}(q, p) \leq 2 \text{rch}(\mathcal{M}) \arcsin \frac{|q - p|}{2 \text{rch}(\mathcal{M})} < 2 \text{rch}(\mathcal{M}) \arcsin \frac{\sqrt{2} \text{rch}(\mathcal{M})}{2 \text{rch}(\mathcal{M})} = \frac{\pi}{2} \text{rch}(\mathcal{M}).$$

1020 Now using Theorem 4 or Lemma 41, this implies that

$$1021 \quad \angle \text{Tan}(p, \mathcal{M}), \text{Tan}(q, \mathcal{M}) \leq \frac{1}{\text{rch}(\mathcal{M})} d_{\mathcal{M}}(p, q) < \frac{\pi}{2}.$$

XX:32 Manifolds of positive reach, differentiability, tangent variation, and attaining the reach

1022 The submersion theorem now yields that the orthogonal projection $\pi_{T_p\mathcal{M}}$ from $\mathcal{M} \cap$
 1023 $B^\circ(p, \sqrt{2} \operatorname{rch}(\mathcal{M}))$ onto $\operatorname{Tan}(p, \mathcal{M}) = T_p\mathcal{M}$ is a local diffeomorphism.

1024 We recall that the covering number of a point in the image of a map is the number of
 1025 points in the preimage above that point. Because the orthogonal projection $\pi_{T_p\mathcal{M}}$ is a local
 1026 diffeomorphism, the covering number is constant on $B^\circ(0, \operatorname{rch}(\mathcal{M})) \subset T_p\mathcal{M}$ except when one
 1027 crosses the boundary $\overline{\partial(\mathcal{M} \cap B^\circ(p, \sqrt{2} \operatorname{rch}(\mathcal{M})))}$. However, we claim that

$$1028 \quad \overline{\partial(\mathcal{M} \cap B^\circ(p, \sqrt{2} \operatorname{rch}(\mathcal{M})))} \cap \pi_{T_p\mathcal{M}}^{-1}(B^\circ(0, \operatorname{rch}(\mathcal{M}))) = \emptyset.$$

1029 This claim can be established as follows: To derive a contradiction we assume that
 1030 $q \in \overline{\partial(\mathcal{M} \cap B^\circ(p, \sqrt{2} \operatorname{rch}(\mathcal{M})))}$, and write $q = x + n$, with $x \in B^\circ(0, \operatorname{rch}(\mathcal{M})) \subset T_p\mathcal{M}$
 1031 and $n \in N_p\mathcal{M}$. Then, by Pythagoras, we have that $n > \operatorname{rch}(\mathcal{M})$. This contradicts the
 1032 characterization of the reach given in Lemma 10.

1033 We therefore conclude that the covering number is constant on $B^\circ(0, \operatorname{rch}(\mathcal{M})) \subset T_p\mathcal{M}$.
 1034 However, thanks to Lemma 35 the covering number is one at 0 and in fact in a neighbourhood
 1035 of zero. Hence the covering number is 1 on $B^\circ(0, \operatorname{rch}(\mathcal{M})) \subset T_p\mathcal{M}$. Because \mathcal{M} and the
 1036 graph is locally $C^{1,1}$ it is globally $C^{1,1}$ and the result follows. \blacktriangleleft

1037 Lemma 43 provides us with an explicit bound on α in Lemmas 38 and 39. Such a bound
 1038 is useful for explicit calculations, see e.g. [7].

1039 **► Lemma 43.** *As long as*

$$1040 \quad \frac{R - \epsilon}{R} \geq 1/2,$$

1041 *or equivalently $\epsilon/R \leq 1/2$, it suffices for α , as defined in first statement of Lemmas 38 and*
 1042 *39, to satisfy*

$$1043 \quad \alpha \leq \sqrt{\epsilon R}.$$

1044 **Proof.** In the proof of Lemma 38 we pick $\theta \in [0, \pi/2]$ such that

$$1045 \quad \cos^3 \theta > \frac{R - \epsilon}{R}. \tag{23}$$

1046 Given θ we then choose an $\alpha > 0$ so that

$$1047 \quad |y| \leq \alpha \Rightarrow \angle \operatorname{Tan}(p, \mathcal{M}), \operatorname{Tan}(q, \mathcal{M}) < \theta. \tag{19}$$

1048 Because $\frac{d^3}{d\theta^3} \cos^3(\theta) = \sin(\theta)(21 \cos^2(\theta) - 6 \sin^2(\theta)) \geq 0$ if $\theta \in [0, \pi/4]$, Taylor's Theorem
 1049 with integral remainder yields that remainder the $R(\theta)$ in

$$1050 \quad \cos^3(\theta) = 1 - \frac{3}{2}\theta^2 + R(\theta)$$

1051 is positive, if $\theta \in [0, \pi/4]$. As $\cos^3(\theta)$ is monotone on $[0, \pi/2]$ and

$$1052 \quad \cos\left(\frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}} < 1/2 \leq \frac{R - \epsilon}{R},$$

1053 and hence $\theta \in [0, \pi/4]$, Therefore it follows that

$$1054 \quad \cos^3(\theta) \leq 1 - \frac{3}{2}\theta^2 \leq 1 - \theta^2,$$

1055 which in combination with (23) gives that $\theta^2 \leq \frac{\epsilon}{R}$. We need to bound α such that
 1056 $\angle \text{Tan}(p, \mathcal{M}), \text{Tan}(q, \mathcal{M}) < \theta$, but thanks to Lemma 41 we know that

1057
$$\angle \text{Tan}(p, \mathcal{M}), \text{Tan}(q, \mathcal{M}) \leq \frac{1}{R} d_{\mathcal{M}}(p, q),$$

1058 so that we need to impose

1059
$$\frac{1}{R} d_{\mathcal{M}}(p, q) \leq \sqrt{\frac{\epsilon}{R}}.$$

1060 Because $q = p + y + \phi(y)$, we have that $|y| \leq d_{\mathcal{M}}(p, q)$, which results in

1061
$$|y| \leq \sqrt{R\epsilon}.$$

1062

1063 **B The metric on Grassmannians**

1064 In this appendix we recall some results on the metric on Grassmannians.

1065 The distance on Grassmannians is intricately linked with projections. More precisely it is
 1066 shown in [3, Section 34] that

1067
$$d_G(A, B) = \max\left\{ \sup_{u \in S_A} d(u, B), \sup_{u \in S_B} d(u, A) \right\} = \|P_A - P_B\|,$$

1068 where S_A is the unit sphere restricted to A , $d(u, A)$ denotes the distance from u to A (as
 1069 above), and P_A denotes the linear (orthogonal) projection operator on A . We write d_G to
 1070 emphasize its use in the context of Grassmannians, but it is defined for arbitrary Hilbert
 1071 spaces. With these definitions we have $d_G(A, B) = \sin(\angle A, B)$. Because d_G is now written
 1072 in terms of an operator norm the d_G is a metric. The fact that the maximal angle is a metric
 1073 can also be seen as follows. By taking the intersection of an n -dimensional linear subspace
 1074 with the unit sphere in \mathbb{R}^d , one finds an $n - 1$ -dimensional sphere. The Hausdorff distance
 1075 between two such sphere is precisely the angle between the two linear subspaces. Because we
 1076 already know that the Hausdorff distance is a metric, the result follows.

1077 The fact that it is a metric can also be proven directly. The fact that the angle is zero if
 1078 the spaces coincide follows immediately from the definition, and the same holds for positively,
 1079 while symmetry was proven in Lemma 34. The only thing that remains to verify is the
 1080 triangle inequality. By definition $\angle A, B$ and $\angle B, C$ are the smallest number such that

1081
$$\forall a \in A \quad \exists b \in B \quad \text{such that} \quad \angle a, b \leq \angle A, B$$

 1082
$$\forall b \in B \quad \exists c \in C \quad \text{such that} \quad \angle b, c \leq \angle B, C$$

1083 so that by combining these

1084
$$\forall a \in A \quad \exists b \in B, c \in C \quad \text{such that} \quad \angle a, b + \angle b, c \leq \angle A, B + \angle B, C.$$

1085 by the triangle inequality between vectors $\angle a, c \leq \angle a, b + \angle b, c$ so that

1086
$$\forall a \in A \quad \exists c \in C \quad \text{such that} \quad \angle a, c \leq \angle A, B + \angle B, C.$$

1087 This gives $\angle A, C \leq \angle A, B + \angle B, C$, that is the triangle inequality, by definition.