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► **To cite this version:**

Moussa Labbadi, Denis Efimov. Hyperexponential Stabilization of Double Integrator with Unmatched Perturbations. IEEE Control Systems Letters, In press. hal-04812352

HAL Id: hal-04812352

<https://inria.hal.science/hal-04812352v1>

Submitted on 30 Nov 2024

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Hyperexponential Stabilization of Double Integrator with Unmatched Perturbations

Moussa Labbadi *Member, IEEE* and Denis Efimov

Abstract—In this letter, we propose linear time-varying state feedback controllers for a double integrator system subject to bounded disturbances. Under the proposed controls, the first state of the double integrator converges to zero at a hyperexponential rate (faster than any exponential decay) uniformly with respect to the disturbances. Meanwhile, the second state stays bounded or approaches the negative of an unmatched differentiable perturbation. These results are applied to design a novel differentiator.

Index Terms—Double integrator; Matched and unmatched perturbations; Hyperexponential convergence; Differentiator.

I. INTRODUCTION

GIVEN a positive integer n , the full perturbed chain of integrators is defined by the following system:

$$\dot{x}(t) = e_n x(t) + d(t) + b_n u(t), t \geq 0, \quad (1)$$
$$e_n = \begin{bmatrix} 0_{(n-1) \times 1} & I_{n-1} \\ 0 & 0_{1 \times (n-1)} \end{bmatrix}, \quad b_n = \begin{bmatrix} 0_{(n-1) \times 1} \\ 1 \end{bmatrix},$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}$ is the control, $d(t) = [d_1(t) \dots d_n(t)]^\top \in \mathbb{R}^n$ denotes matched and mismatched uncertainties.

Assumption 1: The disturbances are bounded and $\|d\|_\infty < +\infty$.

Our primary focus is to investigate control for a double integrator subject to both matched and unmatched disturbances, establishing a hyperexponential decay rate. We leave the generic problem with $n > 2$ for a future work due to its computational intricacies that we will discuss later in the paper. The backstepping method can theoretically compensate for uncertainties decreasing with the state variables [1, Sec. 7.1.2] or guarantee state boundedness by propagating input-to-state stability (ISS) through integrators [2, Cor. 2.3]. Another approach to control (1) is through time-scaling or singular perturbations [3]–[6]. While asymptotic convergence is achieved in [3], [6], prescribed-time convergence is obtained in [4], [5]. However, these studies do not address the unmatched disturbances [5] or assume decreasing in states perturbations [4].

Our goal consists in designing a time-varying state feedback control u that renders the system ISS [7] with hyperexponential rate of convergence [8]–[10], where the state component

x_1 approaches to zero and second state stays bounded, and in the case that the disturbance is absolutely continuous it goes to the opposite of the mismatched perturbation with the equilibrium trajectory $(x_1, x_2) = (0, -d_1)$. Treatment of mismatched perturbations with hyperexponential stability of the system (1) are more difficult objectives than merely conventionally considered exponential convergence with matched perturbations. Mismatched disturbances, acting on other channels than the control, appear in many applications, see for instance [11]–[18] and other publications. Note that if d_1 is continuously differentiable, then by a direct change of coordinates, introducing $\tilde{x}_2 = x_2 + d_1$, it is possible to reduce the studied problem to the regulation of a double integrator model with matched disturbances, which is significantly simpler. Therefore, the key requirement in our consideration is that we will ask only about boundedness of d_1 , as it is stated in Assumption 1.

Recently, a hyperexponentially stabilizing control design for n -th order chains of integrators with matched perturbations was proposed in [9], with the control taking the form:

$$u(t) = K\Psi(t)x(t), \quad \Psi(t) = \text{diag}\{\psi^n(t) \dots \psi(t)\}^\top, \quad (2)$$

where $K \in \mathbb{R}^{1 \times n}$ is the controller gain, $\psi(t) = 1 + t$ is a strictly growing function of time. It is shown that by tuning the gain K , the system (1), (2) with $d_1 = \dots = d_{n-1} = 0$ can be made uniformly stable in d_n having a hyperexponential convergence, which is faster than any exponential one. However, in case of unmatched uncertainties, this technique does not work. Finally, due to a rather complicated stability analysis it is not clear how to evaluate quantitatively the limitations of that feedback and to possibly improve the results.

This paper presents a control design method for (1) with $n = 2$, achieving a convergence rate that exceeds traditional exponential rates. By leveraging the inherent structure of linear time-invariant systems and their predictable dynamics, the proposed methods achieve significantly faster convergence than conventional exponential approaches, thereby enhancing system responsiveness and accuracy. This method ensures stability under both bounded matched and mismatched disturbances without requiring prior knowledge of the upper bound for tuning. This feature is particularly advantageous in practical applications where precise information about the magnitude of disturbances may not be available, simplifying the tuning process and improving the method's applicability in real-world scenarios. For the proof, we introduce new variables and employ a time-dependent scale transformation, as utilized in [8], [9]. The proposed method is used to define a new differentiator with hyperexponential convergence.

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NOTATION

- $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$, where \mathbb{R} represents the set of real numbers.
- The absolute value in \mathbb{R} is denoted by $|\cdot|$, and $\|\cdot\|$ represents the Euclidean norm on \mathbb{R}^n .
- For a (Lebesgue) measurable function $d : \mathbb{R}^+ \rightarrow \mathbb{R}^m$, the norm $\|d\|_\infty = \text{ess sup}_{t \geq 0} \|d(t)\|$ is defined. The set of functions d satisfying $\|d\|_\infty < +\infty$ is denoted as L_m^∞ .
- A continuous function $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ belongs to the class \mathcal{K} if $\alpha(0) = 0$ and it is strictly increasing. A function $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is in the class \mathcal{K}_∞ if $\alpha \in \mathcal{K}$ and α increases to infinity. A continuous function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is in the class \mathcal{KL} if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}^+$ and $\beta(s, \cdot)$ decreases to zero for each fixed $s > 0$.
- The identity matrix of dimension $n \times n$ is denoted by I_n .
- $\exp(1) = e$.

II. PRELIMINARIES

The standard stability concepts for non-autonomous nonlinear systems whose dynamics described by ordinary differential equations, as the asymptotic stability and the input-to-state stability, can be found in [6]. Below, less conventional notions are recalled together with the auxiliary lemma used in the paper.

A. Uniform hyperexponential stability

Consider the following non-autonomous system:

$$\frac{dx(t)}{dt} = f(t, x(t), d(t)), \quad t \geq t_0, \quad t_0 \in \mathbb{R}^+ \quad (3)$$

where $x(t) \in \mathbb{R}^n$ is the state, $d(t) \in \mathbb{R}^m$ represents the external disturbances, $d \in L_m^\infty$, and $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a continuous function with respect to x and d , and piecewise continuous with respect to t . Additionally, $f(t, 0, 0) = 0$ for all $t \in \mathbb{R}^+$. The solution of system (3) with an initial condition $x_0 \in \mathbb{R}^n$ at time $t_0 \in \mathbb{R}^+$ and a disturbance $d \in L_m^\infty$ is denoted as $\varphi(t, t_0, x_0, d)$. It is assumed that f ensures the existence of solutions $\varphi(t, t_0, x_0, d)$, at least locally in forward time. The following definition introduces the uniform stability property for (3) with a hyperexponential rate of convergence.

Definition 1: [8], [9] Given a set $D \subset L_m^\infty$, the system (3) is termed uniformly hyperexponentially stable if there exist constant $\kappa_0 \in \mathbb{R}^+$, functions $\rho, \kappa \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$ such that:

$$\|\varphi(t, t_0, x_0, d)\| \leq e^{-(\kappa(t-t_0) + \kappa_0)(t-t_0)} \rho(\|x_0\|) + \beta(\|d\|_\infty, t-t_0), \quad \forall t \geq t_0 \quad (4)$$

for all $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}^+$, and $d \in D$.

An illustrative example is:

$$\dot{x}(t) = -(1+t)x(t) + d(t), \quad t \geq 0 \quad (5)$$

where $x(t), d(t) \in \mathbb{R}$. Let $t_0 = 0$, the solutions to (5) can be bounded by:

$$|x(t)| \leq e^{-(\frac{t^2}{2} + t)} |x(0)| + \frac{2\|d\|_\infty}{1 + (\frac{t^2}{2} + t)} \quad (6)$$

for all $t \geq 0$, any $x(0) \in \mathbb{R}$, and $d \in L_1^\infty$. Thus, we can define $\rho(s) = s$, $\kappa(s) = \frac{s}{2}$, $\kappa_0 = 1$, and $\beta(s, t) = \frac{2s}{1 + (\frac{t^2}{2} + 1)t}$.

In this definition, the hyperexponential rate of convergence is required only for the initial conditions, whereas uniformity is understood as independence from the exogenous input $d \in D$ and the initial time t_0 . Although $D \subset L_m^\infty$ is assumed, any other suitable class of inputs can be considered. Similarly, the uniform hyperexponential stability can be defined with respect to an output or a state component by replacing in the left-hand side of (4) the norm of the state trajectory with the corresponding quantity.

B. Auxiliary properties

In this paper, the following results will be used.

Lemma 2: For all $\tau \geq 0$, and $a, \alpha > 0$ such that $\alpha a > 1$, the following holds:

$$\int_0^\tau e^{s-\tau} \frac{ds}{(as+1)^\alpha} \leq \frac{r_{a,\alpha}}{(a\tau+1)^\alpha},$$

where

$$r_{a,\alpha} = (a\alpha)^\alpha \int_0^{\frac{\alpha-1}{a}} e^{s-\frac{\alpha-1}{a}} \frac{ds}{(as+1)^\alpha} + (a\alpha+1) + \left(\frac{a\alpha+1}{a\alpha}\right)^\alpha \left(1 - e^{-\frac{1}{a}}\right).$$

Proof: We apply the methodology proposed in [9] to establish Lemma 2. The property to be proven can be restated as follows:

$$g_\alpha(\tau) = (a\tau+1)^\alpha \int_0^\tau \frac{e^{s-\tau}}{(as+1)^\alpha} ds \leq r_{a,\alpha}, \quad \forall \tau \geq 0, \forall \alpha a > 1. \quad (7)$$

Observe that

$$\frac{\partial g_\alpha(\tau)}{\partial \tau} = p_\alpha(\tau) g_\alpha(\tau) + 1, \quad p_\alpha(\tau) = \frac{a\alpha}{a\tau+1} - 1,$$

which is a first-order linear time-varying differential equation with respect to $g_\alpha(\tau)$ for $\tau \geq 0$ and $g_\alpha(0) = 0$. In this scenario, applying the standard solution formula for linear differential equations, we obtain:

$$g_\alpha(\tau) = \int_0^\tau e^{\int_s^\tau p_\alpha(\eta) d\eta} ds.$$

For any $\tau \geq 0$, define $\tau^* = \frac{\alpha\alpha-1}{a}$. Note that $p_\alpha(0) > 0$ since $\alpha a > 1$, then:

$$p_\alpha(\tau) \begin{cases} \geq 0 & \text{if } \tau \in [0, \tau^*], \\ \leq 0 & \text{if } \tau \in (\tau^*, \tau^* + \frac{1}{a}], \\ \leq -\frac{1}{a\alpha+1} & \text{if } \tau > \tau^* + \frac{1}{a}. \end{cases}$$

Therefore, $\frac{\partial g_\alpha(\tau)}{\partial \tau} > 0$ for $\tau \in [0, \tau^*]$, hence, $g_\alpha(\tau) \leq g_\alpha(\tau^*)$.

For $\tau \in (\tau^*, \tau^* + \frac{1}{a}]$, we have:

$$\begin{aligned} g_\alpha(\tau) &= g_\alpha(\tau^*) + \int_{\tau^*}^\tau e^{\int_s^\tau p_\alpha(\eta) d\eta} ds \\ &= g_\alpha(\tau^*) + \int_{\tau^*}^\tau e^{s-\tau} \left(\frac{a\tau+1}{as+1}\right)^\alpha ds \end{aligned}$$

$$\begin{aligned} &\leq g_\alpha(\tau^*) + \int_{\tau^*}^{\tau} e^{s-\tau} \left(\frac{a\tau+1}{a\tau^*+1} \right)^\alpha ds \\ &= g_\alpha(\tau^*) + \left(\frac{a\tau+1}{a\tau^*+1} \right)^\alpha \left(1 - e^{\tau^*-\tau} \right), \end{aligned}$$

which implies that

$$g_\alpha(\tau) \leq g_\alpha^* = g_\alpha(\tau^*) + \left(\frac{a\alpha+1}{a\alpha} \right)^\alpha \left(1 - e^{-\frac{1}{a}} \right)$$

for any $\tau \in [0, \tau^* + \frac{1}{a}]$. Finally, for $\tau > \tau^* + \frac{1}{a}$:

$$\begin{aligned} g_\alpha(\tau) &= g_\alpha\left(\tau^* + \frac{1}{a}\right) + \int_{\tau^* + \frac{1}{a}}^{\tau} e^{\int_s^{\tau} p_\alpha(\eta) d\eta} ds \\ &\leq g_\alpha^* + \int_{\tau^* + \frac{1}{a}}^{\tau} e^{-\int_s^{\tau} \frac{1}{a\alpha+1} d\eta} ds \\ &= g_\alpha^* + (a\alpha+1) \left(1 - e^{-\frac{\tau-\tau^*+a}{a\alpha+1}} \right) \leq g_\alpha^* + (a\alpha+1). \end{aligned}$$

Consequently, $g_\alpha(\tau) \leq g_\alpha^* + (a\alpha+1) = r_{a,\alpha}$ for any $\tau \geq 0$. \blacksquare

To illustrate the results of Lemma 1, we analyze the discrepancy function

$$\delta(\tau) = \frac{r_{a,\alpha}}{(a\tau+1)^\alpha} - \int_0^\tau e^{s-\tau} \frac{ds}{(as+1)^\alpha},$$

where $r_{a,\alpha}$ is given in Lemma 2. The results are depicted in Fig. 1, which presents $\delta(\tau)$ on a logarithmic scale for the following sets of parameters: $a = 3, \alpha = \frac{1}{2}$; $a = 2, \alpha = 1$; $a = 7, \alpha = \frac{1}{6}$; $a = 4, \alpha = \frac{1}{3}$; $a = 0.5, \alpha = 6$; $a = 0.2, \alpha = 5$.

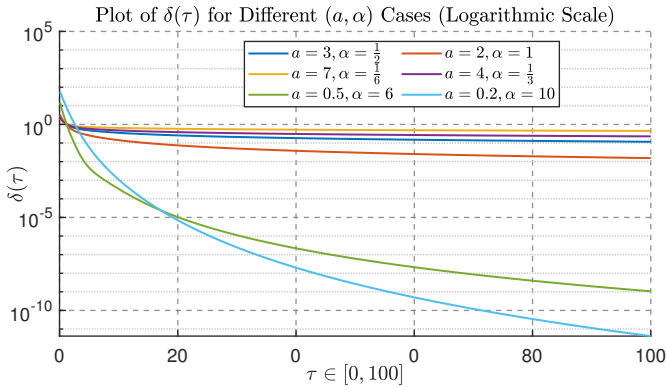


Fig. 1: Function $\delta(\tau)$ for various combinations of a and α .

III. MAIN RESULTS

Consider the following double integrator subject to matched and mismatched perturbations:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + d_1(t), \\ \dot{x}_2(t) &= u(t) + d_2(t). \end{aligned} \quad (8)$$

Let us define a new variable as:

$$\sigma(t) = x_2(t) + \lambda\psi(t)x_1(t), \quad (9)$$

where λ is a positive parameter and $\psi(t) = 1 + t$ is a time function defined in the introduction section. It is requested

to design a controller that can handle both matched and mismatched perturbations while achieving faster convergence than any exponential rates.

Remark 3: Any continuous, strictly increasing function $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\psi(0) > 0$ and an unbounded integral can be utilized in equation (9). For instance, $\psi(t) = ae^{\alpha t}$ or $\psi(t) = bt^t$, where $a > 0, b > 0$, and $\alpha > 0$. Prescribed-time convergence can be achieved by replacing $\psi(t)$ with those functions used in [4], [5]. By applying a change of time [8], [9], all these functions can be transformed into the one used for the proof in this work.

A solution to this problem, ensuring convergence for both x_1 and x_2 , can be achieved using the following control law.

$$u(t) = -\lambda(\psi(t)x_2(t) + x_1(t)) - K\psi(t)^2\sigma(t), \quad (10)$$

where $K > 1.5\lambda$ is a control gain.

Theorem 4: Under Assumption 1, consider the double integrator (8) controlled by the algorithm (10). Then, the closed-loop system is ISS and uniformly hyperexponentially stable in the variable x_1 .

Proof: One has:

$$\begin{aligned} \dot{\sigma}(t) &= u(t) + d_2(t) + \lambda\psi(t)(x_2(t) + d_1(t)) + \lambda\psi(t)x_1(t) \\ &= -K\psi(t)^2\sigma(t) + d_2(t) + \lambda\psi(t)d_1(t). \end{aligned} \quad (11)$$

Next, let us introduce a new time variable as:

$$d\tau = K\psi(t)^2 dt, \implies \tau = \varphi(t) = Kt \left(\frac{t^2}{3} + t + 1 \right),$$

and $t = \varphi^{-1}(\tau) = \sqrt[3]{3K^{-1}\tau + 1} - 1, \psi(\tau) = \sqrt[3]{3K^{-1}\tau + 1}$.

The variables in the new time can be denoted by:

$$\sigma(\tau) = \sigma[\varphi^{-1}(\tau)], \quad d_i(\tau) = d_i[\varphi^{-1}(\tau)],$$

Then,

$$\begin{aligned} \frac{d\sigma(\tau)}{d\tau} &= \frac{d\sigma(t)}{dt} \Big|_{t=\varphi^{-1}(\tau)} \frac{d\varphi^{-1}(\tau)}{d\tau} \\ &= -\sigma(\tau) + \frac{K^{-1}d_2(\tau)}{(\sqrt[3]{3K^{-1}\tau + 1})^2} + \frac{\lambda K^{-1}d_1(\tau)}{\sqrt[3]{3K^{-1}\tau + 1}}. \end{aligned} \quad (12)$$

By transitioning to the time domain, we obtain:

$$\begin{aligned} \sigma(\tau) &= e^{-\tau} \sigma(0) \\ &+ \int_0^\tau e^{s-\tau} \left[\frac{K^{-1}d_2(s)}{(\sqrt[3]{3K^{-1}s + 1})^2} + \frac{\lambda K^{-1}d_1(s)}{\sqrt[3]{3K^{-1}s + 1}} \right] ds. \end{aligned} \quad (13)$$

The upper bound of $\sigma(\tau)$ is:

$$\begin{aligned} |\sigma(\tau)| &\leq e^{-\tau} |\sigma(0)| \\ &+ \int_0^\tau e^{s-\tau} \left[\frac{K^{-1} \|d_2\|_\infty}{(\sqrt[3]{3K^{-1}s + 1})^2} + \frac{\lambda K^{-1} \|d_1\|_\infty}{\sqrt[3]{3K^{-1}s + 1}} \right] ds. \end{aligned} \quad (14)$$

According to Lemma 2, the following holds:

$$\int_0^\tau e^{s-\tau} \frac{ds}{(as+1)^\alpha} \leq \frac{r_{a,\alpha}}{(a\tau+1)^\alpha},$$

where $r_{a,\alpha}$ are given in the formulation of the lemma with $\alpha = \{\frac{2}{3}, \frac{1}{3}\}$ and $a = \frac{3}{K}$ for both cases.

And in the original time:

$$|\sigma(t)| \leq e^{-Kt\left(\frac{t^2}{3}+t+1\right)}|\sigma(0)| + \frac{r_{\frac{3}{K},\frac{2}{3}}K^{-1}\|d_2\|_\infty}{\psi(t)^2} + \lambda \frac{r_{\frac{3}{K},\frac{1}{3}}K^{-1}\|d_1\|_\infty}{\psi(t)}. \quad (15)$$

The upper bound on $\sigma(t)$ calculated in (15) implies that it is bounded and exhibits a hyperexponential decay rate in initial conditions. This decay rate is further enhanced by the compensation applied to both perturbations, ensuring a rapid convergence of the system's state to $\sigma = 0$.

On this surface, we have:

$$\dot{x}_1 = -\lambda\psi(t)x_1(t) + d_1(t) + \sigma(t). \quad (16)$$

Next, let us introduce another new time variable as:

$$d\tau = \lambda\psi(t)dt, \implies \tau = \varphi(t) = \lambda\frac{t}{2}(t+2),$$

$$\text{and } t = \varphi^{-1}(\tau) = \sqrt{2\lambda^{-1}\tau + 1} - 1, \psi(\tau) = \sqrt{2\lambda^{-1}\tau + 1}.$$

The state and the disturbance in the new time can be denoted as previously (with a slight ambiguity):

$$x_1(\tau) = x_1[\varphi^{-1}(\tau)], \quad d_i(\tau) = d_i[\varphi^{-1}(\tau)], \text{ and}$$

$$\frac{dx_1(\tau)}{d\tau} = \frac{dx_1(t)}{dt} \Big|_{t=\varphi^{-1}(\tau)} \frac{d\varphi^{-1}(\tau)}{d\tau} = -x_1(\tau) + \frac{d_1(\tau) + \sigma(\tau)}{\lambda\sqrt{2\lambda^{-1}\tau + 1}}. \quad (17)$$

Upon transitioning to the time domain, we derive:

$$x_1(\tau) = e^{-\tau}x_1(0) + \int_0^\tau e^{s-\tau} \left[\frac{d_1(s) + \sigma(s)}{\lambda\sqrt{2\lambda^{-1}s + 1}} \right] ds, \quad (18)$$

and the upper bound of x_1 is given by:

$$|x_1(\tau)| \leq e^{-\tau}|x_1(0)| + \int_0^\tau e^{s-\tau} \left[\frac{\|d_1\|_\infty + |\sigma(s)|}{\lambda\sqrt{2\lambda^{-1}s + 1}} \right] ds. \quad (19)$$

Using (13), one obtains:

$$|x_1(\tau)| \leq \int_0^\tau e^{s-\tau} \left[\frac{\|d_1\|_\infty}{\lambda\psi(s)} + \frac{e^{-\frac{K}{3}\left[\sqrt{\frac{2s}{\lambda}+1}\left(\frac{2s}{\lambda}+1\right)-1\right]}|\sigma(0)|}{\lambda\psi(s)} + \frac{r_{\frac{3}{K},\frac{2}{3}}K^{-1}\|d_2\|_\infty}{\lambda\psi(s)^3} + \frac{r_{\frac{3}{K},\frac{1}{3}}K^{-1}\|d_1\|_\infty}{\psi(s)^2} \right] ds + e^{-\tau}|x_1(0)|. \quad (20)$$

The first term admits the following upper bound

$$\int_0^\tau e^{s-\tau} \frac{e^{-\frac{K}{3}\left[\sqrt{\frac{2s}{\lambda}+1}\left(\frac{2s}{\lambda}+1\right)-1\right]}}{\lambda\sqrt{1+2s\lambda^{-1}}} ds \leq \int_0^\tau e^{s-\tau} \frac{e^{-\frac{K}{3}\frac{2s}{\lambda}}}{\lambda} ds = 3 \frac{e^{-\tau} - e^{-\frac{2K}{3\lambda}\tau}}{2K - 3\lambda}, \text{ with } 2K > 3\lambda.$$

According to Lemma 2, the upper bound on the state of x_1 satisfies:

$$|x_1(\tau)| \leq e^{-\tau}|x_1(0)| + 3 \frac{e^{-\tau} - e^{-\frac{2K}{3\lambda}\tau}}{2K - 3\lambda} |\sigma(0)| + \frac{r_{\frac{2}{\lambda},\frac{1}{2}}\|d_1\|_\infty}{\lambda\left(\frac{2\tau}{\lambda} + 1\right)^{\frac{1}{2}}} + \frac{r_{\frac{2}{\lambda},\frac{3}{2}}r_{\frac{3}{K},\frac{2}{3}}K^{-1}\|d_2\|_\infty}{\lambda\left(\frac{2\tau}{\lambda} + 1\right)^{\frac{3}{2}}} + \frac{r_{\frac{2}{\lambda},1}r_{\frac{3}{K},\frac{1}{3}}K^{-1}\|d_1\|_\infty}{\frac{2\tau}{\lambda} + 1},$$

and in the original time:

$$|x_1(t)| \leq e^{-\lambda\frac{t}{2}(t+2)}|x_1(0)| + 3 \frac{e^{-\lambda\frac{t}{2}(t+2)} - e^{-\frac{K}{3}t(t+2)}}{2K - 3\lambda} |\sigma(0)| + \left(\frac{r_{\frac{2}{\lambda},\frac{1}{2}}}{\lambda\psi(t)} + \frac{r_{\frac{2}{\lambda},1}r_{\frac{3}{K},\frac{1}{3}}K^{-1}}{\psi(t)^2} \right) \|d_1\|_\infty + \frac{r_{\frac{2}{\lambda},\frac{3}{2}}r_{\frac{3}{K},\frac{2}{3}}K^{-1}\|d_2\|_\infty}{\lambda\psi(t)^{\frac{3}{2}}}.$$

It is clear that the upper estimate of x_1 is bounded and exhibits uniform hyperexponential convergence, which is faster than any standard exponential one. Since $x_1(t)$ and $\sigma(t)$ uniformly converge to zero, from (9) we obtain the estimate for the behavior of $x_2(t)$:

$$|x_2(t)| = |\sigma(t) - \lambda\psi(t)x_1(t)| \leq |\sigma(t)| + \lambda\psi(t)|x_1(t)| \leq \left[e^{-Kt\left(\frac{t^2}{3}+t+1\right)} + \frac{e^{-\lambda\frac{t}{2}(t+2)} - e^{-\frac{K}{3}t(t+2)}}{2K - 3\lambda} \right] \cdot 3\lambda\psi(t) \left[|\sigma(0)| + \lambda\psi(t)e^{-\lambda\frac{t}{2}(t+2)}|x_1(0)| \right] + \left[r_{\frac{2}{\lambda},\frac{1}{2}} + \left(1 + r_{\frac{2}{\lambda},1}\right) \frac{r_{\frac{3}{K},\frac{1}{3}}\lambda K^{-1}}{\psi(t)} \right] \|d_1\|_\infty + r_{\frac{3}{K},\frac{2}{3}}K^{-1} \left[\frac{r_{\frac{2}{\lambda},\frac{3}{2}}}{\sqrt{\psi(t)}} + \frac{1}{\psi(t)^2} \right] \|d_2\|_\infty,$$

which implies boundedness of this variable, with the asymptotic bound proportional to d_1 with the gain $r_{\frac{2}{\lambda},\frac{1}{2}}$, and hyperexponential decay of vanishing of initial conditions. If d_1 is absolutely continuous (differentiable almost everywhere), then due to boundedness and continuity of all variables, and convergence of x_1 to zero, it implies that $\dot{x}_1(t) \rightarrow 0$ as $t \rightarrow +\infty$, hence, $x_2(t) \rightarrow -d_1(t)$ for $t \rightarrow +\infty$. This completes the proof. ■

Remark 5: From Theorem 4, it follows that $x_2(t)$ approaches $-d_1(t)$ within a hyperexponential decay rate. Consequently, a new estimator for the mismatched disturbance is obtained, eliminating the need for a disturbance observer.

Now, consider the following simple controller:

$$u(t) = -\psi(t)x_2(t) - x_1(t) - \psi(t)\sigma(t) = -(1 + \psi(t)^2)x_1(t) - 2\psi(t)x_2(t), \quad (21)$$

with $\psi(t) = t + \lambda, \forall \lambda > 0$.

Theorem 6: Under Assumption 1, consider the double integrator (8) controlled by the algorithm (21). Then, the closed-loop system is ISS and uniformly hyperexponentially stable in the variable x_1 .

Proof: We have:

$$\dot{\sigma}(t) = -\psi(t)\sigma(t) + \psi(t)d_1(t) + d_2(t),$$

and

$$\dot{x}_1(t) = -\psi(t)x_1(t) + d_1(t) + \sigma(t).$$

Let us introduce a new time variable τ as:

$$d\tau = \psi(t) dt \implies \tau = \frac{t^2}{2} + \lambda t, \text{ thus, } t = \sqrt{2\tau + \lambda^2} - \lambda.$$

We obtain the following dynamics for both the state x_1 and the variable σ in the new time frame:

$$\frac{dx_1(\tau)}{d\tau} = -x_1(\tau) + \frac{d_1(\tau) + \sigma(\tau)}{\psi(\tau)},$$

and

$$\frac{d\sigma(\tau)}{d\tau} = -\sigma(\tau) + d_1(\tau) + \frac{d_2(\tau)}{\psi(\tau)}.$$

The solutions of these systems are:

$$x_1(\tau) = e^{-\tau} x_1(0) + \int_0^\tau e^{s-\tau} \frac{d_1(s) + \sigma(s)}{\psi(s)} ds,$$

and

$$\sigma(\tau) = e^{-\tau} \sigma(0) + \int_0^\tau e^{s-\tau} \left(d_1(s) + \frac{d_2(s)}{\psi(s)} \right) ds.$$

The upper bounds of both variables are:

$$|x_1(\tau)| \leq e^{-\tau} |x_1(0)| + \int_0^\tau e^{s-\tau} \frac{\|d_1\|_\infty + |\sigma(s)|}{\psi(s)} ds,$$

and

$$|\sigma(\tau)| \leq e^{-\tau} |\sigma(0)| + \int_0^\tau e^{s-\tau} \left(\|d_1\|_\infty + \frac{\|d_2\|_\infty}{\psi(s)} \right) ds.$$

According to Lemma 2, we have:

$$|\sigma(\tau)| \leq e^{-\tau} |\sigma(0)| + \|d_1\|_\infty + r_{\frac{2}{\lambda^2}, \frac{1}{2}} \frac{\lambda^{-1} \|d_2\|_\infty}{\left(1 + \frac{2}{\lambda^2} \tau\right)^{\frac{1}{2}}},$$

and

$$\begin{aligned} |x_1(\tau)| &\leq e^{-\tau} (|x_1(0)| + |\sigma(0)|) \\ &+ \int_0^\tau e^{s-\tau} \left(\frac{2\|d_1\|_\infty}{\psi(s)} + \frac{r_{\frac{2}{\lambda^2}, \frac{1}{2}} \lambda^{-1} \|d_2\|_\infty}{\psi(s) \left(1 + \frac{2}{\lambda^2} s\right)^{\frac{1}{2}}} \right) ds, \\ &\leq e^{-\tau} (|x_1(0)| + |\sigma(0)|) \\ &+ \frac{2r_{\frac{2}{\lambda^2}, \frac{1}{2}} \lambda^{-1} \|d_1\|_\infty}{\left(1 + \frac{2}{\lambda^2} \tau\right)^{\frac{1}{2}}} + \frac{r_{\frac{2}{\lambda^2}, \frac{1}{2}} r_{\frac{2}{\lambda^2}, 1} \lambda^{-2} \|d_2\|_\infty}{1 + \frac{2}{\lambda^2} \tau}. \end{aligned}$$

Returning to the original time, we have the following estimations of both variables:

$$\begin{aligned} |x_1(t)| &\leq e^{-\frac{t}{2}(t+2\lambda)} (|x_1(0)| + |\sigma(0)|) \\ &+ \frac{2r_{\frac{2}{\lambda^2}, \frac{1}{2}} \|d_1\|_\infty}{\psi(t)} + \frac{r_{\frac{2}{\lambda^2}, \frac{1}{2}} r_{\frac{2}{\lambda^2}, 1} \|d_2\|_\infty}{\psi(t)^2}, \end{aligned}$$

and

$$|\sigma(t)| \leq e^{-\frac{t}{2}(t+2\lambda)} |\sigma(0)| + \|d_1\|_\infty + r_{\frac{2}{\lambda^2}, \frac{1}{2}} \frac{\|d_2\|_\infty}{\psi(t)}.$$

Using (9), the estimation of the state x_2 is:

$$\begin{aligned} |x_2(t)| &\leq e^{-\frac{t}{2}(t+2\lambda)} [(1 + \psi(t)) |\sigma(0)| + \psi(t) |x_1(0)|] \\ &+ (2r_{\frac{2}{\lambda^2}, \frac{1}{2}} + 1) \|d_1\|_\infty + \frac{r_{\frac{2}{\lambda^2}, \frac{1}{2}} \left(r_{\frac{2}{\lambda^2}, 1} + 1 \right)}{\psi(t)} \|d_2\|_\infty. \end{aligned}$$

The remaining steps and arguments are the same as in the proof of Theorem 4. ■

The difference between controls (10) and (21) is that the former guarantees convergence of σ to zero, while the latter does not, and both control laws provide similar qualitative behavior for the state of (8).

Remark 7: The closed-loop system is a linear time-varying system with external inputs. Thus, for simulation or implementation, as the time-varying gain $\psi(t)$ grows unbounded, the explicit Euler method cannot discretize it effectively, whereas the implicit method is applicable [19].

Remark 8: For a chain of integrators of order $n > 2$, let the auxiliary variables for (1) be chosen recursively in a linear time-varying form:

$$\begin{aligned} \sigma_1(t) &= x_1(t), \\ \sigma_2(t) &= \omega_1(t) + \lambda_1 \sigma_1(t) \psi(t), \\ \sigma_3(t) &= \omega_2(t) + \lambda_2 \sigma_2(t) \psi^2(t), \\ &\vdots \\ \sigma_n(t) &= \omega_{n-1}(t) + \lambda_{n-1} \sigma_{n-1}(t) \psi^{n-1}(t), \end{aligned} \quad (22)$$

where $\lambda_i, i = 1, \dots, n-1$ are tuning gains such that $\lambda_{i+1} > \lambda_i$, and $\psi(t) = 1 + t$. Each recursive term $\sigma_i(t)$ is constructed to reject unmatched perturbations by progressively increasing the gains, as discussed in [20]. The term $\omega_i(t)$ represents the dynamics of each $\sigma_i(t)$, such that $\omega_i(t) = \dot{\sigma}_i(t)$ under the condition $d_i \equiv 0$.

We introduce the desired control law for (1) as follows:

$$u(t) = -\Omega(t, x(t)) - K \sigma_n(t) \psi^m(t), \quad (23)$$

where $\Omega(t, x(t))$ contains terms to be compensated in the final step, $K > 0$, and $m \geq n$.

To establish the stability of the closed-loop system under this control law, new time variables should be defined starting with the variable $\sigma_n(t)$.

IV. SIMULATION RESULTS

Example 1: Consider the double integrator system (8) with the time-varying controller (10), and initial conditions $x(0) = [2 \ 3]^\top$. The perturbations are set as $d_1(t) = \sin(2t)$ and $d_2(t) = 2 \cos(2t) + 0.02 \text{rnd}(1)$. The control gains are $\lambda = 4$ and $K = 7$. Theorem 4 is illustrated in Fig. 2. After a brief transient phase, the state x_1 and an auxiliary variable σ converge to zero, while x_2 tracks the disturbance $-d_1(t)$. The input remains smooth without chattering, due to the differentiability of $d_1(t)$.

Example 2: Consider a differentiable signal $f(t)$ defined on $[0, \infty)$ with a bounded time derivative, i.e., $\|\dot{f}\|_\infty < +\infty$. The objective is to estimate the time derivative of this signal f . To achieve this, we use the following time-varying differentiator:

$$\dot{\hat{x}}_1 = \hat{x}_2, \quad \dot{\hat{x}}_2 = u(f, \hat{x}),$$

where $u(f, \hat{x})$ is defined in (10) or (21), with the substitutions $\hat{x}_1 - f \rightarrow x_1$ and $\hat{x}_2 \rightarrow x_2$. The tracking error is $e(t) = \hat{x}_1(t) - f(t)$, and its derivative is $\dot{e}(t) = \hat{x}_2(t) - \dot{f}(t)$, thus \hat{x}_2 can estimate $\dot{f}(t)$. This leads to the system (8) with $d_1(t) = -\dot{f}(t)$. The signal is chosen as $f(t) = \frac{1}{5} \cos(5t)$, and the control parameters are $\lambda = 10$ and $K = 20$ for (10), satisfying Theorem 4. The simulation results with $\hat{x}(0) = [1, 1]^\top$ are

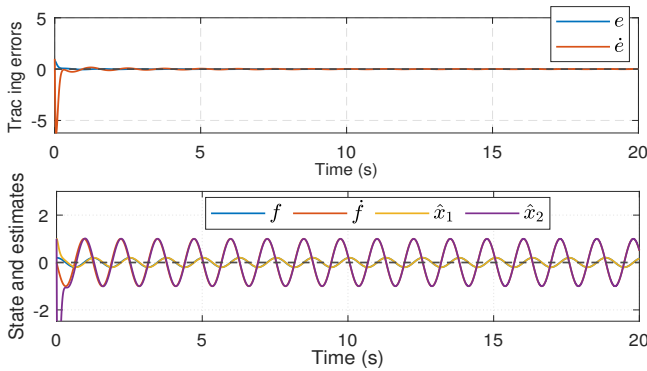


Fig. 2: Double integrator with controller (10).

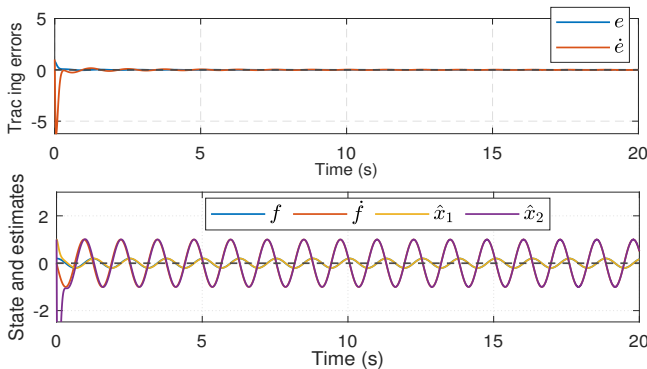


Fig. 3: Proposed differentiable with scheme (10).

shown in Fig. 3. After a reaching phase, $\hat{x}_1(t)$ converges to $f(t)$, and \hat{x}_2 converges to $\dot{f}(t)$.

According to the results of both *Examples 1 and 2*, the proposed control approach is validated. The drawback of these results is that the control may become unbounded if the disturbance d_1 is not differentiable, hence, it may be difficult to use this control for all $t \geq 0$. However, taking into account its remarkable ability to reject perturbations (we do not need any information about perturbations except their qualitative boundedness) and fast convergence, it can be used as an auxiliary or intermediate solution applied on a finite interval of time to an uncertain plant while more sophisticated approaches (including adaptive controls, identifiers or disturbance observers) are converging.

V. CONCLUSIONS

In this paper, we propose novel linear feedback state control methods with time-varying gains for a double integrator affected by both matched and unmatched perturbations. We demonstrate a hyperexponential decay rate of convergence for the state, along with the estimation of unmatched perturbations. These methods are applied to develop a new time-varying differentiator. The controllers do not require

any disturbance observer, simplifying the overall solution. In future work, we plan to extend these strategies to a chain of integrators, including their discretization, and to expand the class of admissible perturbations to include those proportional to the state vector, particularly for nonholonomic systems like perturbed unicycle-type mobile robots.

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