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Necessary conditions for the stability of singularly perturbed linear systems with switching slow-fast behaviors

Yacine Chitour^{*} Jamal Daafouz[†] Ihab Haidar[‡] Paolo Mason[§] Mario Sigalotti[¶]

Abstract

We consider linear singularly perturbed dynamics in which the set of fast variables is a switching parameter. We introduce auxiliary switching systems (in a single time-scale) whose instability implies the instability of the original dynamics. This translates into necessary conditions for the stability of the considered dynamics.

Keywords: Switching systems, Singular perturbation, Exponential stability.

1 Introduction

Consider the linear switching system evolving in \mathbb{R}^N

$$\mathcal{D}^\varepsilon(t)\dot{X} = \Lambda X, \quad (1)$$

where Λ is a $N \times N$ real matrix and $t \mapsto \mathcal{D}^\varepsilon(t)$ is a piecewise-constant function (the switching signal) taking values in the set of $N \times N$ diagonal matrices with diagonal entries in $\{1, \varepsilon\}$, ε being a small positive parameter. We deal in this note with the problem of understanding the asymptotic behavior of this type of systems as t goes to $+\infty$ in the regime where ε is arbitrarily small.

In the case where \mathcal{D}^ε is constant, the singular perturbation theory [5] proposes a method allowing to study the stability of the two-time-scales dynamics (1) by separating slow and fast variables. In this case, the stability of the overall system can be deduced from the stability properties of each single-scale dynamics. However, this method fails when instead of a constant matrix Λ a switching one is considered. Several stability criteria have been developed in the literature to deal with this case (see, e.g., [3, 4, 7, 10]). For example, upper and lower bounds on the limit as ε goes to 0 of the maximal Lyapunov exponent of singularly perturbed linear switching systems have been established in [3]. In [10], stability is proven to hold under a dwell time condition, and notably, this condition does not explicitly rely on the time-scale parameter. A recent contribution concerning stabilization of switched affine singularly perturbed dynamics with state-dependent switching laws is also developed in [12]. However, all these contributions do not consider changes in the nature of state variables, i.e., when \mathcal{D}^ε is switching as in (1). Analyzing the stability of systems with two time-scales and changes in the nature of state variables is very challenging. The system described by equation (1) is explored in [9] in a more general setting, that is, incorporating the possibility of switching in Λ , and admitting non-synchronizing jumps in the state. This investigation was motivated by a practical

^{*}Yacine Chitour is with Laboratoire des Signaux et Systèmes (L2S), Université Paris-Saclay, CNRS, CentraleSupélec, Gif-sur-Yvette, France, {yacine.chitour}@l2s.centralesupelec.fr.

[†]Jamal Daafouz is with CNRS-CRAN UMR 7039, Université de Lorraine, Institut Universitaire de France (IUF), Vandoeuvre-lès-Nancy, France, Jamal.Daafouz@univ-lorraine.fr.

[‡]Ihab Haidar is with Quartz, ENSEA, Cergy-Pontoise, France, and with Laboratoire Jacques-Louis Lions (LJLL), Inria, Sorbonne Université, Paris, France, ihab.haidar@ensea.fr.

[§]Paolo Mason is with Laboratoire des Signaux et Systèmes (L2S), Université Paris-Saclay, CNRS, CentraleSupélec, Gif-sur-Yvette, France, {paolo.mason}@l2s.centralesupelec.fr.

[¶]Mario Sigalotti is with Laboratoire Jacques-Louis Lions (LJLL), Inria, Sorbonne Université, Université de Paris, CNRS, Paris, France, mario.sigalotti@inria.fr.

application associated with the last phase of the rolling process known as the tail end phase. During this phase, the strip leaves the stands sequentially. Notably, the alterations in the nature of state variables are linked to the transition of a fast variable into a slow variable, occurring each time the strip leaves a stand [8].

While a Lyapunov-based stability theory for nested systems on multiple time scales was recently proposed in [11], it focuses on a different setting. The authors consider nested interconnections of nonlinear dynamical systems ordered by convergence rates (slowest to fastest). Each system is influenced only by the slower dynamics and the successive fastest one and switching behaviors are not addressed. To the best of our knowledge, the problem of switching slow/fast behaviors was first considered in [9], and as of now, there have been no further developments in this context.

In contrast to [9], where an upper bound on the minimum dwell time ensuring stability is examined, here we focus on system (1) with the aim of providing necessary conditions for its stability. To conduct our study, we first show that using a mode-dependent variable reordering one can rewrite system (1) in a form in which the variables preserve their slow or fast nature over time. The cost of preserving this conventional form of singularly perturbed systems is that the obtained system includes state jumps and that the dimensions of the slow and fast variables are time-dependent. Sufficient Lyapunov-based conditions for stability of singularly perturbed hybrid dynamics are developed in the literature (see, e.g., [1, 13]). However, the class of switching systems with state jumps that we obtain does not fall within the existing classes of hybrid singularly perturbed dynamics. This is due to the particularity of the derived system with time-dependent dimensions for the slow and fast variables. Here, starting from this new representation, we introduce two auxiliary single-scale dynamics, a linear switching system with state jumps, denoted by $\bar{\Sigma}$, and a discrete-time switching one, denoted by $\tilde{\Sigma}$. System $\bar{\Sigma}$ is obtained using the usual Tikhonov decomposition and represents the case where the switching frequency is slower than the time-scale $1/\varepsilon$, while system $\tilde{\Sigma}$ represents the case where the switching frequency is slower than the time-scale $1/\varepsilon$ but sufficiently large in such a way that the slow manifolds are left almost instantaneously. The modes of $\tilde{\Sigma}$ actually correspond to the jump dynamics of $\bar{\Sigma}$. We show that, if for some dwell time $\tau_0 > 0$ the auxiliary system $\bar{\Sigma}$ is exponentially unstable then there exist $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ and every $\tau \in [0, \tau_0]$ the initial dynamics (1) is exponentially unstable as well. Knowing that $\tilde{\Sigma}$ represents the jump part of $\bar{\Sigma}$, a similar result is given based on $\tilde{\Sigma}$.

2 Notations

By \mathbb{R} we denote the set of real numbers and by \mathbb{R}_+ the set of non-negative real numbers. We use \mathbb{N} for the set of nonnegative integers and we write $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. We use $M_{n,m}(\mathbb{R})$ to denote the set of $n \times m$ real matrices and simply $M_n(\mathbb{R})$ if $n = m$. The $n \times n$ identity matrix is denoted by I_n . The spectral radius of a square matrix M (i.e., the maximal modulus of its eigenvalues) is denoted by $\rho(M)$. For $Q \in M_{n,m}(\mathbb{R})$ and $\ell \leq n$, we denote by $(Q)_\ell$ the $\ell \times m$ matrix obtained by truncating Q and keeping only its first ℓ lines.

Given a set \mathcal{Z} , we denote by $\mathcal{S}_{\mathcal{Z}}$ the set of (right-continuous) piecewise-constant functions from \mathbb{R}_+ to \mathcal{Z} . Given $\tau \geq 0$, we denote by $\mathcal{S}_{\mathcal{Z}}^\tau$ the set of (right-continuous) piecewise-constant signals from \mathbb{R}_+ to \mathcal{Z} with dwell time $\tau \geq 0$. Note that $\mathcal{S}_{\mathcal{Z}}^\tau \subset \mathcal{S}_{\mathcal{Z}}^0 = \mathcal{S}_{\mathcal{Z}}$ for every $\tau \geq 0$. We use $(t_i)_{i \in \mathbb{N}}$, with $t_0 = 0$ and $t_i \rightarrow +\infty$, to denote the increasing sequence of switching times of an element of $\mathcal{S}_{\mathcal{Z}}$.

For every positive integer N , we use $\llbracket 1, N \rrbracket$ to denote the set of integers between 1 and N and $\mathcal{P}(\llbracket 1, N \rrbracket)$ for the set of subsets of $\llbracket 1, N \rrbracket$. For every $J \in \mathcal{P}(\llbracket 1, N \rrbracket)$ we use J^c to denote the set $\llbracket 1, N \rrbracket \setminus J$. We denote by $\#J$ the cardinal of J . For every $J \in \mathcal{P}(\llbracket 1, N \rrbracket)$ and $\varepsilon \geq 0$, we use $\mathcal{D}_J^\varepsilon$ to denote the $N \times N$ diagonal matrix with diagonal coefficients equal to ε for indices in J and equal to 1 otherwise. Note that, for $\varepsilon > 0$, $\varepsilon(\mathcal{D}_J^\varepsilon)^{-1} = \mathcal{D}_{J^c}^\varepsilon$.

3 Problem statement and main results

3.1 Systems under consideration and stability notions

Fix a positive integer N , a matrix $\Lambda \in M_N(\mathbb{R})$, and a subset \mathcal{J} of $\mathcal{P}(\llbracket 1, N \rrbracket) \setminus \llbracket 1, N \rrbracket$. For every $\tau \geq 0$, we define the family of switching systems $\Sigma_{\mathcal{J}}^{\tau}$ given by $(\Sigma_{\mathcal{J}}^{\varepsilon, \tau})_{\varepsilon > 0}$ where, for every $\varepsilon > 0$, $\Sigma_{\mathcal{J}}^{\varepsilon, \tau}$ is defined as

$$\Sigma_{\mathcal{J}}^{\varepsilon, \tau} : \quad \mathcal{D}_J^{\varepsilon} \dot{X} = \Lambda X, \quad J \in \mathcal{S}_{\mathcal{J}}^{\tau}. \quad (2)$$

We simply denote by $\Sigma_{\mathcal{J}}^{\varepsilon}$ and $\Sigma_{\mathcal{J}}$ the system $\Sigma_{\mathcal{J}}^{\varepsilon, 0}$ and the family of systems $\Sigma_{\mathcal{J}}^0$. The switching signal $J \in \mathcal{S}_{\mathcal{J}}^{\tau}$ identifies at each time the slow and fast variables of the system. Note that we do not exclude the possibility that all variables are slow, which may happen if $\emptyset \in \mathcal{J}$, but we do exclude the case with only fast variables.

We write $\Phi_J^{\varepsilon}(t, 0)$ to denote the flow from time 0 to time t of $\Sigma_{\mathcal{J}}^{\varepsilon, \tau}$ associated with a signal $J \in \mathcal{S}_{\mathcal{J}}^{\tau}$.

Given $\varepsilon, \tau > 0$, the usual stability notions, recalled in the following definition, apply to the linear switching system $\Sigma_{\mathcal{J}}^{\varepsilon, \tau}$.

Definition 1. Let $n \in \mathbb{N}^*$ and \mathcal{Z} be a bounded subset of $M_n(\mathbb{R})$. Consider the linear switching system

$$\Sigma_{\mathcal{Z}}^{\tau} : \quad \dot{x} = Zx, \quad Z \in \mathcal{S}_{\mathcal{Z}}^{\tau}, \quad (3)$$

and denote by $\Phi_Z(t, 0)$ the flow from time 0 to time t of $\Sigma_{\mathcal{Z}}^{\tau}$ associated with the switching signal Z . System $\Sigma_{\mathcal{Z}}^{\tau}$ is said to be

1. exponentially stable (ES, for short) if there exist $c > 0$ and $\delta > 0$ such that we have

$$\|\Phi_Z(t, 0)\| \leq ce^{-\delta t}, \quad \forall t \geq 0, \forall Z \in \mathcal{S}_{\mathcal{Z}}^{\tau}; \quad (4)$$

2. exponentially unstable (EU, for short) if there exist $c > 0$, $\delta > 0$, and $x_0 \in \mathbb{R}^n \setminus \{0\}$ such that

$$|\Phi_Z(t, 0)x_0| \geq ce^{\delta t}|x_0|, \quad \forall t \geq 0.$$

The maximal Lyapunov exponent of $\Sigma_{\mathcal{Z}}^{\tau}$ is defined as

$$\lambda(\Sigma_{\mathcal{Z}}^{\tau}) = \limsup_{t \rightarrow +\infty} \frac{1}{t} \sup_{Z \in \mathcal{S}_{\mathcal{Z}}^{\tau}} \log(\|\Phi_Z(t, 0)\|).$$

Remark 2. For every $t > 0$ and every $Z \in \mathcal{S}_{\mathcal{Z}}^{\tau}$, by Gelfand's formula,

$$\rho(\Phi_Z(t, 0)) = \lim_{k \rightarrow +\infty} \|\Phi_Z(t, 0)^k\|^{\frac{1}{k}} = \lim_{k \rightarrow +\infty} \|\Phi_{\tilde{Z}}(kt, 0)\|^{\frac{1}{k}}$$

where \tilde{Z} denotes the t -periodic signal obtained by periodization of $Z|_{[0, t]}$. Notice that $\tilde{Z} \in \mathcal{S}_{\mathcal{Z}}^{\tau}$ if t is a switching time of Z . As a consequence, if t is a switching time of Z , then

$$\rho(\Phi_Z(t, 0)) \leq e^{t\lambda(\Sigma_{\mathcal{Z}}^{\tau})}. \quad (5)$$

In particular, if $\rho(\Phi_Z(t, 0)) > 1$ (with t a switching time of Z) then $\Sigma_{\mathcal{Z}}^{\tau}$ is EU. The converse implication is actually true, since it is well known (see, e.g., [14]) that an equivalent definition of the maximal Lyapunov exponent is

$$\lambda(\Sigma_{\mathcal{Z}}^{\tau}) = \limsup_{t \rightarrow +\infty} \frac{1}{t} \sup_{Z \in \mathcal{S}_{\mathcal{Z}}^{\tau}} \log \rho(\Phi_Z(t, 0)), \quad (6)$$

where the sup is made among all switching signals Z for which t is a switching time for Z .

Remark 3. It follows from Definition 1 and by (6) that system $\Sigma_{\mathcal{Z}}^{\tau}$ is EU if and only if $\lambda(\Sigma_{\mathcal{Z}}^{\tau}) > 0$.

Remark 4. Notice that if $\Sigma_{\mathcal{J}}^{\varepsilon, \tau}$ is EU for some $\tau \geq 0$ and some $\varepsilon > 0$, then the same is true for $\Sigma_{\mathcal{J}}^{\varepsilon, \rho}$ for every $\rho \in [0, \tau]$. This simply follows from the inclusion $\mathcal{S}_{\mathcal{J}}^{\tau} \subset \mathcal{S}_{\mathcal{J}}^{\rho}$.

3.2 Equivalent representation and statement of the main result

For each $J \in \mathcal{J} \setminus \emptyset$, we fix a permutation matrix $P_J \in M_N(\mathbb{R})$ such that

$$P_J \mathcal{D}_J^0 P_J^{-1} = \begin{pmatrix} I_{N-\#J} & 0 \\ 0 & 0 \end{pmatrix}.$$

If $\emptyset \in \mathcal{J}$ we set $P_\emptyset = I_N$. Given $J \in \mathcal{J}$ we introduce the matrices A_J , B_J , C_J , and D_J through the relation

$$\begin{pmatrix} A_J & B_J \\ C_J & D_J \end{pmatrix} = P_J \Lambda P_J^{-1}. \quad (7)$$

Notice that $A_\emptyset = \Lambda$ and that B_\emptyset , C_\emptyset , and D_\emptyset are empty matrices. In order to represent system (2) in the usual form of singularly perturbed systems where slow and fast variables do not change roles over time, we define the time-dependent change of variables

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = P_{J(t)} X(t), \quad J \in \mathcal{S}_\mathcal{J}, \quad t \geq 0. \quad (8)$$

Note that the dimensions of $x(t)$ and $y(t)$ are time-varying (piecewise-constant) functions. We denote by $\ell(t)$ the dimension of $x(t)$, for $t \geq 0$.

Using (7) and (8), we can equivalently represent system $\Sigma_{\mathcal{J}}^{\varepsilon, \tau}$ as the linear switching system with jumps

$$\begin{cases} \begin{pmatrix} \dot{x}(t) \\ \varepsilon \dot{y}(t) \end{pmatrix} = \begin{pmatrix} A_{J_i} & B_{J_i} \\ C_{J_i} & D_{J_i} \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, & i \in \mathbb{N}, \\ \begin{pmatrix} x(t_i) \\ y(t_i) \end{pmatrix} = P_{J_i} P_{J_{i-1}}^{-1} \lim_{t \nearrow t_i} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, & i \in \mathbb{N}^*, \end{cases} \quad (9)$$

where J_i is the constant value of $J \in \mathcal{S}_\mathcal{J}$ on the interval $[t_i, t_{i+1})$, for $i \geq 0$.

For the rest of the paper we make the following working assumption.

Assumption 5. For each $J \in \mathcal{J} \setminus \emptyset$, the matrix D_J is Hurwitz.

For $J \in \mathcal{J}$, let

$$M_J = \begin{cases} \Lambda & \text{if } J = \emptyset, \\ A_J - B_J D_J^{-1} C_J & \text{if } J \neq \emptyset, \end{cases} \quad (10)$$

where A_J , B_J , C_J , and D_J are given by (7).

For every $\tau \geq 0$, we introduce the linear switching system with state jumps

$$\bar{\Sigma}_\tau : \begin{cases} \dot{\bar{x}}(t) = M_{J_i} \bar{x}(t), & t \in [t_i, t_{i+1}), \quad i \in \mathbb{N}, \\ \bar{x}(t_i) = R(i) \lim_{t \nearrow t_i} \bar{x}(t), & i \in \mathbb{N}^*, \end{cases}$$

where J_i is the constant value of $J \in \mathcal{S}_\mathcal{J}$ on the interval $[t_i, t_{i+1})$ and

$$R(i) = \begin{cases} (P_{J_i})_{\ell_i} & J_{i-1} = \emptyset, \\ \left(P_{J_i} P_{J_{i-1}}^{-1} \begin{pmatrix} I_{\ell_{i-1}} \\ -D_{J_{i-1}}^{-1} C_{J_{i-1}} \end{pmatrix} \right)_{\ell_i} & J_{i-1} \neq \emptyset, \end{cases} \quad (11)$$

with $\ell_i = N - \#J_i$ the size of the square matrix A_{J_i} . We also write $\bar{\Sigma}$ for $\bar{\Sigma}_0$. The stability notions introduced in Definition 1 directly extend to systems such as $\bar{\Sigma}_\tau$, in which the state undergoes jumps and has time-dependent dimension.

We also introduce the discrete-time switching system

$$\check{\Sigma} : \check{x}(i) = R(i) \check{x}(i-1), \quad i \in \mathbb{N}^*,$$

where $R(i)$ is given by (11). Note that the dimension of $\check{x}(i)$ in $\check{\Sigma}$ is not constant and is equal to ℓ_i for $i \in \mathbb{N}$. We say that $\check{\Sigma}$ is *exponentially unstable (EU)* if it admits a trajectory whose norm tends to infinity exponentially.

Remark 6. System $\tilde{\Sigma}$ is exponentially unstable if and only if the following holds true: there exist $J \in \mathcal{S}_{\mathcal{J}}$ and $K \in \mathbb{N}^*$ such that $J_K = J_0$ and the product $R(K) \cdots R(1)$ (which is a square matrix of size ℓ_0) has spectral radius larger than one.

Since $\tilde{\Sigma}$ corresponds to the jump part of $\bar{\Sigma}$, one deduces the following result, whose proof is given in Section 4.4.

Lemma 7. If $\tilde{\Sigma}$ is EU then $\bar{\Sigma}$ is EU. In turns, if $\bar{\Sigma}$ is EU, then there exists $\tau > 0$ such that $\bar{\Sigma}_\tau$ is EU.

We now state our main result.

Theorem 8. Suppose that Assumption 5 holds. Let $\tau \geq 0$. If $\bar{\Sigma}_\tau$ is EU, then there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, $\Sigma_{\mathcal{J}}^{\varepsilon, \tau}$ is EU.

The proof of Theorem 8 is given in Section 4.

Remark 9. By Remark 4, we could equivalently restate the conclusion of Theorem 8 as follows: There exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$ and every $\theta \in [0, \tau]$, $\Sigma_{\mathcal{J}}^{\varepsilon, \theta}$ is EU.

A direct corollary of Lemma 7, Theorem 8, and Remark 9 is the following.

Corollary 10. Suppose that Assumption 5 holds. If $\tilde{\Sigma}$ is EU then there exist $\tau_0 > 0$ and $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$ and every $\tau \in [0, \tau_0]$, $\Sigma_{\mathcal{J}}^{\varepsilon, \tau}$ is EU.

4 Proof of the necessary stability conditions

For simplicity of notations, we will prove the main results assuming that $\emptyset \notin \mathcal{J}$. The case $J = \emptyset$ can be studied either by direct adaptation of the results presented below or by considering an additional fast variable X_{N+1} in (2) satisfying, e.g., $\varepsilon \dot{X}_{N+1} = -X_{N+1}$. The condition $\emptyset \notin \mathcal{J}$ is then satisfied for the extended system, and its instability is equivalent to the instability of the original system.

Following a classical approach (see e.g. [6]), we introduce on each interval $[t_i, t_{i+1})$ the variable

$$z(t) = y(t) + D_{J_i}^{-1} C_{J_i} x(t) + \varepsilon Q_{J_i}^\varepsilon x(t), \quad (12)$$

where $Q_{J_i}^\varepsilon$ is chosen in such a way that

(q1) System (2) can be equivalently represented in triangular form on each interval $[t_i, t_{i+1})$, for $i \in \mathbb{N}$, as

$$\begin{cases} \dot{x}(t) = (M_{J_i} - \varepsilon B_{J_i} Q_{J_i}^\varepsilon) x(t) + B_{J_i} z(t), \\ \varepsilon \dot{z}(t) = (D_{J_i} + \varepsilon \tilde{Q}_{J_i}^\varepsilon B_{J_i}) z(t), \\ \begin{pmatrix} x(t_i) \\ z(t_i) \end{pmatrix} = T_{J_i}^\varepsilon (T_{J_{i-1}}^\varepsilon)^{-1} \lim_{t \nearrow t_i} \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}, \end{cases} \quad (13)$$

where $\tilde{Q}_{J_i}^\varepsilon = D_{J_i}^{-1} C_{J_i} + \varepsilon Q_{J_i}^\varepsilon$ and

$$T_J^\varepsilon = \begin{pmatrix} I_{N-\#J} & 0 \\ D_J^{-1} C_J + \varepsilon Q_J^\varepsilon & I_{\#J} \end{pmatrix} P_J, \quad J \in \mathcal{J}; \quad (14)$$

(q2) $\|Q_J^\varepsilon\|$ is upper bounded uniformly with respect to $J \in \mathcal{J}$ and ε small enough.

We simply denote by T_J the matrix T_J^0 .

4.1 Preliminary results and useful lemmas

For every $J \in \mathcal{J}$ and $\varepsilon > 0$, we introduce the matrix

$$\Gamma_J^\varepsilon = \begin{pmatrix} M_J - \varepsilon B_J Q_J^\varepsilon & B_J \\ 0 & \frac{1}{\varepsilon} D_J + \tilde{Q}_J^\varepsilon B_J \end{pmatrix}, \quad (15)$$

where M_J is given by (10) and $Q_J^\varepsilon, \tilde{Q}_J^\varepsilon$ are as in (12)-(13).

In the following lemma we clarify the relation between the flows of system (2) and system (13).

Lemma 11. *Given $J \in \mathcal{S}_J$, we have*

$$\Phi_J^\varepsilon(t_n, 0) = \prod_{i=0}^{n-1} (T_{J_i}^\varepsilon)^{-1} e^{(t_{i+1}-t_i)\Gamma_{J_i}^\varepsilon} T_{J_i}^\varepsilon, \quad \forall n \geq 1.$$

Proof. The proof follows directly from (2) and (13) together with relation (8). \square

A useful result is given by the following lemma.

Lemma 12. *Suppose that Assumption 5 holds. For each $t > 0$, there exists $K > 0$ such that for $J \in \mathcal{J}$ and $\varepsilon > 0$ small enough,*

$$\left\| e^{t\Gamma_J^\varepsilon} - \begin{pmatrix} e^{tM_J} & 0 \\ 0 & e^{\frac{t}{\varepsilon}D_J} \end{pmatrix} \right\| \leq K\varepsilon. \quad (16)$$

Proof. Consider $t > 0$, $J \in \mathcal{J}$ and $(x_0, z_0) \in \mathbb{R}^\ell \times \mathbb{R}^{N-\ell}$ be fixed. Consider the trajectory $t \mapsto (x(t), z(t))^T = e^{t\Gamma_J^\varepsilon}(x_0, z_0)^T$. By Lemma 20 (in appendix), we have that

$$|z(t)| \leq K e^{-\frac{\alpha}{\varepsilon}t} |z_0| \quad \text{and} \quad |z(t) - e^{\frac{t}{\varepsilon}D_J} z_0| \leq K\varepsilon \quad (17)$$

for some $K, \alpha > 0$ independent of t, J and ε small enough. By a slight abuse of notation, in what follows we still use K to denote possibly larger constants independent of J and ε . Using estimate (17) in the dynamics of x , we deduce by a simple application of Gronwall's lemma that

$$|x(t)| \leq K |(x_0, z_0)|. \quad (18)$$

By applying the variation of constant formula, we have

$$x(t) = e^{M_J t} x_0 - \varepsilon \int_0^t e^{M_J(t-s)} B_J Q_J^\varepsilon x(s) ds + \int_0^t e^{M_J(t-s)} B_J z(s) ds.$$

Notice that, by (18),

$$\left| \varepsilon \int_0^t e^{M_J(t-s)} B_J Q_J^\varepsilon x(s) ds \right| \leq K\varepsilon t |(x_0, z_0)|$$

and, thanks to (17),

$$\left| \int_0^t e^{M_J(t-s)} B_J z(s) ds \right| \leq K\varepsilon |z_0|. \quad (19)$$

Hence, inequality (16) holds. \square

The following lemma is useful for the proof of the main result and will be used in Section 4.2.

Lemma 13. *Let $J \in \mathcal{J}$ and $t > 0$ be fixed. There exists $K > 0$ such that, for every $\varepsilon > 0$ small enough,*

$$\|(T_J^\varepsilon)^{-1} e^{t\Gamma_J^\varepsilon} T_J^\varepsilon - T_J^{-1} e^{t\Gamma_J} T_J\| \leq K\varepsilon,$$

where T_J^ε is given by (14) and $T_J = T_J^0$.

Proof. It follows from (q2) that $\|T_J^\varepsilon - T_J\| \leq K\varepsilon$ and $\|(T_J^\varepsilon)^{-1} - T_J^{-1}\| \leq K\varepsilon$, for some $K > 0$ independent of ε small enough. The proof is then a direct consequence of these inequalities and of the uniform boundedness of $e^{t\Gamma_J^\varepsilon}$ for ε small enough. \square

4.2 Auxiliary weighted discrete-time switching systems

For $\tau \geq 0$, we introduce the following subset of $M_N(\mathbb{R}) \times \mathbb{R}_+$

$$\mathcal{N}_\tau = \left\{ \left(T_J^{-1} \begin{pmatrix} e^{tM_J} & 0 \\ 0 & 0 \end{pmatrix} T_J, t \right) \mid J \in \mathcal{J}, t \geq \tau \right\}.$$

Notice that $\mathcal{N}_\tau \subset \mathcal{N}_0$ for every $\tau \geq 0$.

We recall the following definition of stability for weighted discrete-time switching systems (see [2]).

Definition 14. Let $n \in \mathbb{N}^*$ and \mathcal{A} be a subset of $M_n(\mathbb{R}) \times \mathbb{R}_+$. We associate with \mathcal{A} the weighted discrete-time switching system

$$\Xi_{\mathcal{A}} : x(k) = A(k)x(k-1), (A(k), \tau(k)) \in \mathcal{A}, k \geq 1,$$

in which the transfer from $x(k-1)$ to $x(k)$ takes time $\tau(k)$. We say that system $\Xi_{\mathcal{A}}$ is

1. exponentially stable (ES, for short) if there exist $c > 0$ and $\delta > 0$ such that

$$\|A(k) \cdots A(1)\| \leq ce^{-\delta(\tau(k) + \cdots + \tau(1))},$$

for all $k \geq 1$ and $(A(1), \tau(1)), \dots, (A(k), \tau(k)) \in \mathcal{A}$;

2. exponentially unstable (EU, for short) if there exist $c > 0$, $\delta > 0$, $x_0 \in \mathbb{R}^n \setminus \{0\}$ and a sequence $\{(A(k), \tau(k))\}_{k \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ such that $\sum_{k \geq 1} \tau(k) = +\infty$ and

$$|A(k) \cdots A(1)x_0| \geq ce^{\delta(\tau(k) + \cdots + \tau(1))}|x_0|$$

for all $k \geq 1$.

4.3 Stability comparison with the auxiliary system $\bar{\Sigma}_\tau$

Lemma 15. For every $(N, t) \in \mathcal{N}_0$ there exist $J \in \mathcal{J}$ and $K > 0$ such that

$$\|(T_J^\varepsilon)^{-1} e^{t\Gamma_J^\varepsilon} T_J^\varepsilon - N\| \leq K\varepsilon$$

for every $\varepsilon > 0$ small enough.

Proof. If $(N, t) \in \mathcal{N}_0$ then $N = T_J^{-1} \begin{pmatrix} e^{tM_J} & 0 \\ 0 & 0 \end{pmatrix} T_J$ for some $J \in \mathcal{J}$. The conclusion follows from Lemmas 12 and 13. \square

We have the following proposition.

Proposition 16. Suppose that Assumption 5 holds and that $\Xi_{\mathcal{N}_\tau}$ is EU for some $\tau \geq 0$. Then the same is true for $\Sigma_{\mathcal{J}}^{\varepsilon, \tau}$ for ε small enough.

Proof. Let $\tau \geq 0$ and suppose that the system $\Xi_{\mathcal{N}_\tau}$ is EU. Following [2], it follows that there exist $n \geq 1$ and $(N_0, \tau_0), \dots, (N_{n-1}, \tau_{n-1}) \in \mathcal{N}_\tau$ such that the spectral radius of $\prod_{i=0}^{n-1} N_i$ is greater than one. Let J_i be the element of \mathcal{J} associated with N_i for $i = 0, \dots, n-1$. Notice that, by definition, $\tau_0, \dots, \tau_{n-1} \geq \tau$.

Now, let t_0, \dots, t_n be defined by $t_0 = 0$ and $t_{i+1} - t_i = \tau_i$ for $i = 0, \dots, n-1$. Let $J \in \mathcal{S}_{\mathcal{J}}^\tau$ be the t_n -periodic piecewise-constant switching signal defined by $J(t) = J_i$ for each $t \in [t_i, t_{i+1})$, $i = 0, \dots, n-1$.

It follows from Lemma 15 that for every $\delta > 0$ and every $\varepsilon > 0$ small enough we have

$$\left\| \prod_{i=0}^{n-1} (T_{J_i}^\varepsilon)^{-1} e^{(t_{i+1}-t_i)\Gamma_{J_i}^\varepsilon} T_{J_i}^\varepsilon - \prod_{i=0}^{n-1} N_i \right\| \leq \delta.$$

From Lemma 11 together with the continuity of the spectral radius, it follows that for ε small enough we have $\rho(\Phi_J^\varepsilon(t_n, 0)) > 1$. By consequence, according to Remarks 2 and 3, system $\Sigma_{\mathcal{J}}^{\varepsilon, \tau}$ is EU for ε small enough. \square

Proof of Theorem 8. Assume that $\bar{\Sigma}_\tau$ is EU. Then there exists a trajectory $\bar{x}(\cdot)$ of $\bar{\Sigma}_\tau$ such that $\|\bar{x}(t)\|$ goes exponentially to infinity as t tends to $+\infty$. Denote by $J \in \mathcal{S}_\mathcal{J}^\tau$ the signal associated with $\bar{x}(\cdot)$. Denoting by t_i the switching times of J , set

$$X(0) = T_{J_0}^{-1} \begin{pmatrix} \bar{x}(0) \\ 0 \end{pmatrix}$$

and

$$X(i) = P_{J_{i-1}}^{-1} \lim_{t \nearrow t_i} \begin{pmatrix} \bar{x}(t) \\ -D_{J_{i-1}}^{-1} C_{J_{i-1}} \bar{x}(t) \end{pmatrix}$$

for $i \in \mathbb{N}^*$. Then $X(\cdot)$ is a solution of $\Xi_{\mathcal{N}_\tau}$ corresponding to the discrete-time signal $((N(i), t_i - t_{i-1}))_{i \in \mathbb{N}^*}$ with

$$N(i) = T_{J_{i-1}}^{-1} \begin{pmatrix} e^{(t_i - t_{i-1})M_{J_{i-1}}} & 0 \\ 0 & 0 \end{pmatrix} T_{J_{i-1}}.$$

Moreover, there exists $c > 0$ independent of i such that

$$\|X(i)\| \geq c \|\bar{x}(t_i)\|$$

for every $i \in \mathbb{N}$. Hence $\Xi_{\mathcal{N}_\tau}$ is EU. The conclusion of the theorem follows from Proposition 16. \square

4.4 Comparison between $\check{\Sigma}$ and $\bar{\Sigma}$

Proof of Lemma 7. Assume that $\check{\Sigma}$ is EU. According to Remark 6, fix $J \in \mathcal{S}_\mathcal{J}$ and $K \in \mathbb{N}^*$ such that $J_K = J_0$ and $R(K) \cdots R(1)$ (which is a square matrix of size ℓ_0) has spectral radius larger than one. Let us modify if necessary the signal J in order to impose, on the one hand, all the switching times to be $t_i = i\tau$ for some $\tau > 0$ to be chosen and, on the other hand, $i \mapsto J_i$ to be K -periodic. Hence, as τ tends to zero, the flow Ψ_τ of $\bar{\Sigma}$ from time 0 to time $t_K = K\tau$ converges to $R(K) \cdots R(1)$. By the continuity of the spectral radius, for τ small we have $\rho(\Psi_\tau) > 1$, yielding that $\bar{\Sigma}_\tau$ is EU.

Assume now that $\bar{\Sigma}$ is EU. In accordance with Remark 2, there exist a signal $J \in \mathcal{S}_\mathcal{J}$ and a switching time t for J such that the flow up to time t for $\bar{\Sigma}$ has spectral radius larger than one. The same is true if we replace J by the signal \tilde{J} obtained by t -periodization of $J|_{[0,t]}$. Notice now that there exists $\tau > 0$ such that \tilde{J} is in $\mathcal{S}_\mathcal{J}^\tau$. Hence $\bar{\Sigma}_\tau$ is also EU. \square

5 The complementary case

Here we consider the *complementary case*, namely, we assume that $\mathcal{J} = \{J, J^c\}$ for some nontrivial subset $J \subsetneq \llbracket 1, N \rrbracket$. With no loss of generality we can assume that $J = \llbracket 1, \ell \rrbracket$ with $1 \leq \ell < N$. We have the following result.

Proposition 17. *Consider the family of switching systems $\Sigma_\mathcal{J}$ given by (2) in the complementary case, i.e., when the dynamics switches among the two modes*

$$\begin{cases} \dot{x} = Ax + By, \\ \varepsilon \dot{y} = Cx + Dy, \end{cases} \quad \begin{cases} \varepsilon \dot{x} = Ax + By, \\ \dot{y} = Cx + Dy, \end{cases} \quad (20)$$

where $x \in \mathbb{R}^\ell$ and $y \in \mathbb{R}^{N-\ell}$ with $\ell \in \llbracket 1, N-1 \rrbracket$ and the matrices A, B, C, D have appropriate sizes. Assume that either A or D admits an eigenvalue of positive real part or, in the case where A and D are both Hurwitz, that the spectral radius of $D^{-1}CA^{-1}B$ is larger than one. Then there exist $\tau_0 > 0$ and $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$ and every $\tau \in [0, \tau_0]$, $\Sigma_\mathcal{J}^{\varepsilon, \tau}$ is EU.

Proof. Thanks to Corollary 10, we know that a sufficient condition for the instability of $\Sigma_\mathcal{J}$ is that $\check{\Sigma}$ is EU. The permutation matrices in this case can be taken as $P = \begin{pmatrix} I_\ell & 0 \\ 0 & I_{N-\ell} \end{pmatrix}$ and $\tilde{P} = \begin{pmatrix} 0 & I_{N-\ell} \\ I_\ell & 0 \end{pmatrix}$. One can easily check that the discrete-time switching system $\check{\Sigma}$ is given by $\check{x}(i+1) = R(i)\check{x}(i)$, with $R(i) \in \{A^{-1}B, D^{-1}C\}$, for $i \in \mathbb{N}$. A sufficient condition for the instability of $\check{\Sigma}$ is that the spectral radius of $D^{-1}CA^{-1}B$ is larger than one. \square

Remark 18. Note that in general the condition that the spectral radius of $D^{-1}CA^{-1}B$ is smaller than one is not a sufficient condition for $\Sigma_{\mathcal{J}}$ to be exponentially stable. Here follows an example in the case $N = 4$ and $\ell = 2$ where $\Sigma_{\mathcal{J}}$ is actually exponentially unstable even though the spectral radius of $D^{-1}CA^{-1}B$ is smaller than one.

Example 19. Consider system (20) with $A = \begin{pmatrix} -1 & a \\ 0 & -a \end{pmatrix}$, $B = C = I_2$, $D = \begin{pmatrix} -1 & 0 \\ -a & -a \end{pmatrix}$, where a is a constant real number. In this case we have $D^{-1}CA^{-1}B = \begin{pmatrix} 1 & 1 \\ -1 & \frac{1-a^2}{a^2} \end{pmatrix}$.

- If we take $a = 0.5$, we have $\rho(D^{-1}CA^{-1}B) = 2 > 1$. By consequence, from Proposition 17, there exist $\tau_0, \varepsilon_0 > 0$ such that $\Sigma_{\mathcal{J}}^{\varepsilon, \tau}$ is EU for every $\tau \in [0, \tau_0]$ and $\varepsilon \in (0, \varepsilon_0)$.
- If we take $a = 1.4$, in this case we have $\rho(D^{-1}CA^{-1}B) \sim 0.7 < 1$. However, system $\bar{\Sigma}$ associated with (20) is EU. This can be clearly seen from Figure 1 obtained with the periodic piecewise-constant switching signal $t \mapsto J(t)$ given by

$$J(t) = \begin{cases} \{3, 4\} & \text{if } t \in [0, 1) \\ \{1, 2\} & \text{if } t \in [1, 2). \end{cases} \quad (21)$$

By consequence, from Lemma 7 and Theorem 8, it follows that $\Sigma_{\mathcal{J}}^{\varepsilon, \tau}$ is EU as well for ε and τ small enough. In Figure 1 we plot the trajectories of (20) with the same periodic piecewise-constant switching signal given by (21) and $\varepsilon = 0.01$.

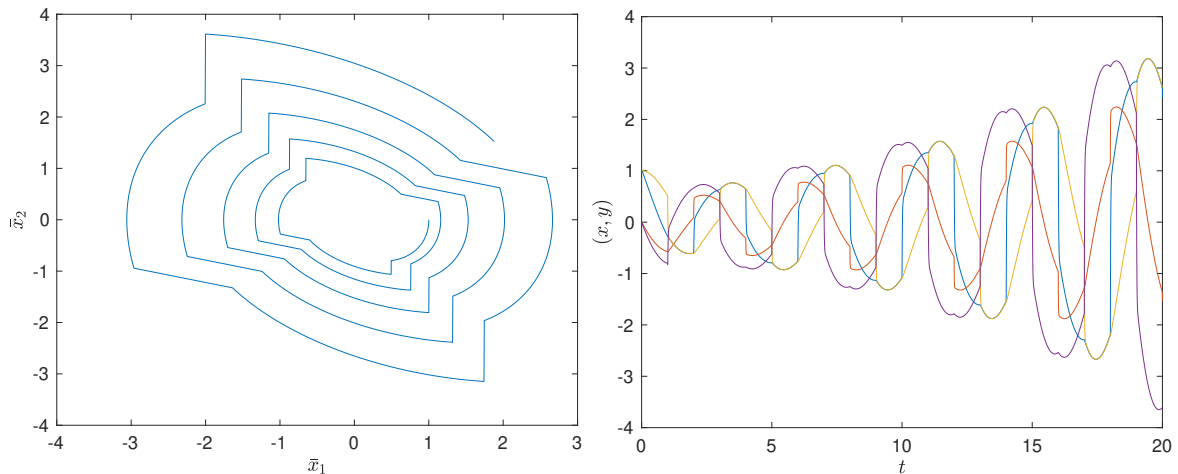


Figure 1: (Left) Trajectory of system $\bar{\Sigma}$ from Section 5 with $a = 1.4$, starting from $(1, 0)$. (Right) Trajectory of system $\Sigma_{\mathcal{J}}^{\varepsilon}$ for the same example, with $a = 1.4$, $\varepsilon = 0.01$, starting from $(1, 0, 1, 0)$.

6 Conclusion

In this paper, we consider a family of linear singularly perturbed dynamics $(\Sigma_{\varepsilon})_{\varepsilon > 0}$ in which the set of fast variables is a switching parameter. Using a Tikhonov approach, we introduce two auxiliary single-scale dynamics, a linear switching system with state jumps $\bar{\Sigma}$ and a discrete-time switching system $\check{\Sigma}$. These auxiliary systems provide necessary conditions for the exponential stability of the initial singularly perturbed dynamics. More precisely, if $\bar{\Sigma}$ is exponentially unstable for some dwell time $\tau \geq 0$ then Σ_{ε} is exponentially unstable for the same dwell time and for every $\varepsilon \ll 1$. Similarly, if $\check{\Sigma}$ is exponentially unstable then Σ_{ε} is exponentially unstable for every dwell time $\tau \ll 1$ and every $\varepsilon \ll 1$. More importantly, system $\check{\Sigma}$ offers an explicit criterion for assessing the instability of $(\Sigma_{\varepsilon})_{\varepsilon > 0}$. Future works will aim to

introduce additional auxiliary systems and establish necessary and sufficient conditions for the stability of linear singularly perturbed dynamics with switching slow-fast behaviors.

7 Appendix

7.1 A useful Gronwall's type result

Lemma 20. *Let $A \in M_n(\mathbb{R})$ be Hurwitz and $B \in L^\infty([t_0, \infty), M_n(\mathbb{R}))$. Then there exist $c, \alpha > 0$ and $k > 0$ depending on A and $\|B\|_\infty$ such that for every $\varepsilon > 0$ the solution of*

$$\dot{z}(t) = \left(\frac{A}{\varepsilon} + B(t) \right) z(t), \quad \forall t \geq t_0, \quad (22)$$

satisfies the inequalities $|z(t)| \leq ke^{-\frac{\alpha}{\varepsilon}(t-t_0)}|z(t_0)|$ and $|z(t) - e^{\frac{A}{\varepsilon}(t-t_0)}z(t_0)| \leq k\varepsilon|z(t_0)|$.

Proof. By applying the variation of constants formula to (22), we obtain

$$z(t) = e^{\frac{A}{\varepsilon}(t-t_0)}z(t_0) + \int_{t_0}^t e^{\frac{A}{\varepsilon}(t-s)}B(s)z(s)ds. \quad (23)$$

Thanks to the fact that A is Hurwitz, there exist $c, \alpha > 0$ such that and every $0 \leq t_0 < t$, we have

$$\|e^{\frac{A}{\varepsilon}(t-t_0)}\| \leq ce^{-\frac{2\alpha}{\varepsilon}(t-t_0)}. \quad (24)$$

From (24) together with (23) we get the following inequality

$$|z(t)| \leq ce^{-\frac{2\alpha}{\varepsilon}(t-t_0)}|z(t_0)| + c\|B\|_\infty \int_{t_0}^t e^{-\frac{2\alpha}{\varepsilon}(t-s)}|z(s)|ds, \quad (25)$$

from which, by using the Gronwall's inequality we get for sufficiently small ε

$$|z(t)| \leq ke^{-\frac{\alpha}{\varepsilon}(t-t_0)}|z(t_0)|. \quad (26)$$

The second inequality in the statement hence follows by bounding the integral term in (23) using (24) and (26). \square

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