



HAL
open science

A priori and a posteriori error estimates of a DG-CG method for the wave equation in second order formulation

Zhaonan Dong, Lorenzo Mascotto, Zuodong Wang

► **To cite this version:**

Zhaonan Dong, Lorenzo Mascotto, Zuodong Wang. A priori and a posteriori error estimates of a DG-CG method for the wave equation in second order formulation. 2024. hal-04768144

HAL Id: hal-04768144

<https://inria.hal.science/hal-04768144v1>

Preprint submitted on 5 Nov 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

A priori and a posteriori error estimates of a DG-CG method for the wave equation in second order formulation

Z. Dong^{*†}, L. Mascotto^{‡§¶}, Z. Wang^{*†}

Abstract

We establish fully-discrete a priori and semi-discrete in time a posteriori error estimates for a discontinuous-continuous Galerkin discretization of the wave equation in second order formulation; the resulting method is a Petrov-Galerkin scheme based on piecewise and piecewise continuous polynomial in time test and trial spaces, respectively. Crucial tools in the a priori analysis for the fully-discrete formulation are the design of suitable projection and interpolation operators extending those used in the parabolic setting, and stability estimates based on a nonstandard choice of the test function; a priori estimates are shown, which are measured in L^∞ -type norms in time. For the semi-discrete in time formulation, we exhibit constant-free, reliable a posteriori error estimates for the error measured in the $L^\infty(L^2)$ norm; to this aim, we design a reconstruction operator into C^1 piecewise polynomials over the time grid with optimal approximation properties in terms of the polynomial degree distribution and the time steps. Numerical examples illustrate the theoretical findings.

AMS subject classification: 65M50; 65M60; 65J10.

Keywords: wave equation; discontinuous Galerkin method; adaptive algorithm; a posteriori error analysis.

1 Introduction

We establish fully-discrete a priori and semi-discrete in time a posteriori error estimates for a DG-CG (discontinuous Galerkin - continuous Galerkin) method, see, e.g., [32], approximating solutions to the wave equation in second order formulation, which are explicit in the spatial mesh size, the time steps, and the polynomial degrees.

Formulation, a priori estimates, and minimal literature. The DG-CG method we are interested in is based on a second order formulation of the wave equation. Compared to several references where first order systems are considered [9, 10, 18, 20], the dimension of the corresponding discrete spaces is smaller.

The method lies in between a fully DG and a C^1 schemes in time: it employs piecewise and piecewise continuous polynomial in time and space; upwind terms involving first time derivatives in time of the trial functions are included in the formulation. The polynomial degree in time of the trial functions is larger by 1 than that for the test functions, leading to square systems for each time interval, which are solved sequentially as a time marching scheme.

A key tool in deriving stability and a priori error estimates is the choice of an appropriate test function. For instance, in [32], a higher order fully-discrete version of the test function in [3] is used, leading to stability estimates in the L^∞ -type norms in time.

^{*}Inria, 48 rue Barrault, 75013 Paris, France, zhaonan.dong@inria.fr, zuodong.wang@inria.fr

[†]CERMICS, Ecole des Ponts, 77455 Marne-la-Vallée 2, France

[‡]Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, 20125 Milan, Italy, lorenzo.mascotto@unimib.it

[§]IMATI-CNR, 27100, Pavia, Italy

[¶]Fakultät für Mathematik, Universität Wien, 1090 Vienna, Austria

A posteriori estimates and minimal literature. A posteriori error estimates are well established for elliptic problems and a considerable amount of work is available for parabolic problems as well; on the other hand, hyperbolic (and in particular wave) problems are less explored.

A posteriori error estimates for wave problems in second order formulation are studied in [1] and later rigorously analysed in [4]; first order systems are instead the topic of [20, 23, 29]; a posteriori error estimates measured in the $L^\infty(L^2)$ norm are investigated for different time stepping schemes in [11, 12, 14]. A posteriori error estimates in other norms are instead investigated, among others, in [6, 7].

Features of the DG-CG method. Compared to fully DG schemes in time, the DG-CG method involves fewer unknowns; in view of the a posteriori error analysis for the semi-discrete in time version, one can employ tools from the parabolic setting and deduce a posteriori bounds that are fully explicit with respect to the polynomial degree.

On the other hand, compared to schemes with C^1 in time trial functions, the DG-CG approach seems to be more suited to allow for mesh changes in space [21]; we are not aware of works in the literature with spatial mesh changes for C^1 in time discretizations.

First main contribution of the manuscript. We modify the analysis in [32] so as to have explicit dependence on the polynomial degree for the a priori analysis of the fully-discrete scheme. We consider uniform polynomial degree in the spatial discretization and possibly nonuniform polynomial degree in the time discretization. Static meshes in space are considered.

Our analysis hinges upon deriving stability estimates for the scheme, which are explicit in the polynomial degrees; see Section 2.1. Based on such stability estimates and the properties of an integrated Thomeé operator discussed in Section 2.2, a priori error estimates are proven in Section 2.3. Convergence estimates are obtained in Section 2.4, which are explicit in the spatial mesh size, the time steps, and the polynomial degrees. Optimal convergence for the error measured with respect to L^∞ -type norms in time is shown for sufficiently regular solutions and data for fixed polynomial degrees in time and space.

Second main contribution of the manuscript. For the semi-discrete in time method, we design an error estimator satisfying a posteriori error estimates, with explicit dependence on the polynomial degree distribution in time. The crucial tool in a posteriori error estimates for time-dependent problems is the derivation of a reconstruction operator into smoother spaces. The original idea in the context of parabolic problems traces back to Makridakis and Nochetto [22]; the corresponding p -version analysis is detailed in [27] and later [17]. We design a related operator in the wave equation setting and derive p -optimal approximation estimates in several norms in Section 3.2; such an operator is instrumental for designing an error estimator that is reliable for the error measured in the $L^\infty(L^2)$ norm; see Section 3.3. The upper bound is explicit in the polynomial degree distribution and the time steps, without unknown constants. Since the test and trial test functions have different polynomial degrees in time, the a posteriori error bounds involve extra oscillation terms compared to the parabolic setting.

To the best of our knowledge, we provide for the first time in the literature constant free, reliable a posteriori error estimates for a semi-discrete in time method for the approximation of solutions to the wave equation in second order formulation, which are explicit in the polynomial degree distribution in time and optimal in the time steps. The proposed analysis does not hinge upon any CFL condition, which is greatly advantageous for adaptivity whilst compared to methods based on explicit time stepping; there, for each mesh refinement, one has to check whether the resulting spatial mesh size is sufficiently small compared to the corresponding time step and the polynomial degree [16]. This improvement is even more effective for wave problems involving higher order spatial elliptic operators [24], where the CFL condition poses even stricter constraints on the spatial mesh size.

List of the main results and advances. For the reader's convenience, we list here the main results of the manuscript ([APRI] = a priori analysis; [APOS] = a posteriori analysis):

- [APRI] Theorem 2.1 is concerned with constant free stability estimates for the DG-CG formulation, which are explicit with respect to the polynomial degrees in time and space employed throughout;
- [APRI] Theorem 2.8 discusses a priori estimates that are explicit with respect to the spatial mesh size, the time step distribution, the spatial polynomial degree, and the polynomial degree distribution in time;
- [APOS] we define an error estimator η in (70) for the semi-discrete in time formulation;
- [APOS] corresponding reliability estimates with respect to the $L^\infty(L^2)$ norm of the error are displayed in Proposition 3.6.

Functional setting. Standard notation is used throughout for Sobolev and Bochner spaces. Let D be a Lipschitz domain in \mathbb{R}^d , $d = 1, 2$, and 3 , with boundary ∂D . The space of Lebesgue measurable and square integrable functions over D is $L^2(D)$. The Sobolev space of positive integer order s is $H^s(D)$. We endow $H^s(D)$ with the inner product, seminorm, and norm

$$(\cdot, \cdot)_{s,D}, \quad |\cdot|_{s,D}, \quad \|\cdot\|_{s,D}.$$

Interpolation theory is used to construct Sobolev spaces of positive noninteger order; duality is used to define negative order Sobolev spaces. We shall be particularly using the space $H^{-1}(D)$, which is the dual of $H_0^1(D)$; the duality pairing between the two spaces is $\langle \cdot, \cdot \rangle$.

A trace theorem holds true for the space $H^s(D)$, $1/2 < s < 3/2$. In particular, given s in the above range and g in $H^{s-1/2}(\Gamma_g)$, being Γ_g any subset of ∂D with positive measure in ∂D , we define

$$H_g^s(D, \Gamma_g) := \{v \in H^s(D, \Gamma_D) \mid v|_{\Gamma_D} = g\}.$$

The space of polynomials of nonnegative degree p over D is $\mathbb{P}_p(D)$.

Given \mathcal{X} a real Banach space with norm $\|\cdot\|_{\mathcal{X}}$, an interval I , and t larger than or equal to 1, we define $L^t(I; \mathcal{X})$ as the Bochner space of measurable functions v from I to \mathcal{X} such that the following quantity is finite:

$$\|v\|_{L^t(I; \mathcal{X})} := \begin{cases} \left(\int_I \|v\|_{\mathcal{X}}^t dt \right)^{\frac{1}{t}} & \text{for } 1 \leq t < \infty \\ \text{ess sup}_{t \in I} \|v\|_{\mathcal{X}} & \text{for } t = \infty. \end{cases}$$

For s in \mathbb{N} , the space $H^s(I; \mathcal{X})$ is the space of measurable functions v whose derivatives in time up to order s belong to $L^2(I; \mathcal{X})$. For any real number s larger than or equal to 0, the space $H^s(I; \mathcal{X})$ is constructed using interpolation of integer order Bochner spaces.

To avoid confusion, the seminorm symbol $|\cdot|$ is only used to denote spatial seminorms. Seminorms in time are rather displayed as L^2 norms of a suitable time derivative. The first and second derivative symbols are \cdot' and \cdot'' ; time derivatives of order s larger than 2 are displayed as $\cdot^{(s)}$.

The continuous problem. Let Ω be a polytopic, Lipschitz domain in \mathbb{R}^d , $d = 1, 2, 3$; T a positive final time; $Q_T := (0, T] \times \Omega$ the space-time cylinder; u_0 in $H_0^1(\Omega)$; u_1 in $L^2(\Omega)$; f in $L^2(0, T; H^{-1}(\Omega))$.

Given $\Delta_{\mathbf{x}\cdot}$ and $\nabla_{\mathbf{x}\cdot}$ the spatial Laplace and gradient operator, we consider the following problem: find $u : Q_T \rightarrow \mathbb{R}$ such that

$$\begin{cases} u'' - \Delta_{\mathbf{x}} u = f & \text{in } Q_T \\ u(t, \cdot) = 0 & \text{on } (0, T] \times \partial\Omega \\ u(0, \cdot) = u_0(\cdot) & \text{on } \Omega \\ u'(0, \cdot) = u_1(\cdot) & \text{on } \Omega. \end{cases} \quad (1)$$

Introduce the spaces

$$X := H^2(0, T; H^{-1}(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega)), \quad Y := L^2(0, T; H_0^1(\Omega)),$$

and the bilinear form on $Y \times Y$

$$a(u, v) := (\nabla_{\mathbf{x}} u, \nabla_{\mathbf{x}} v)_{0, \Omega}.$$

We consider the following weak formulation of problem (1):

$$\begin{cases} \text{find } u \in X \text{ such that} \\ \int_0^T [\langle u'', v \rangle + a(u, v)] dt = \int_0^T \langle f, v \rangle dt & \forall v \in Y \\ u(0, \cdot) = u_0(\cdot) \text{ in } H_0^1(\Omega), \quad u'(0, \cdot) = u_1(\cdot) \text{ in } L^2(\Omega). \end{cases} \quad (2)$$

Problem (1) is well posed; see, e.g., [25, Theorem 8.2-2]. To simplify the forthcoming analysis (especially the a posteriori error analysis in Section 3), we assume that f belongs to $L^2(0, T; L^2(\Omega))$.

In [32], homogeneous Dirichlet and inhomogeneous Neumann boundary conditions are considered. This results in further complication in the analysis below. For this reason, we prefer to stick to the setting in (2).

Spatial meshes, time grids, and polynomial degree distributions . We consider either a simplicial or tensor-product conforming mesh \mathcal{T}_h of Ω and a corresponding H^1 -conforming Lagrangian finite element space V_h of uniform order $p_{\mathbf{x}}$. We assume the existence of a constant γ in $(0, 1)$ such that each element of \mathcal{T}_h is star-shaped with respect to a ball of radius larger than or equal to the diameter of that element; moreover, we assume quasi-uniformity of the mesh, i.e., given h_1 and h_2 the diameters of two arbitrary elements with $h_1 \leq h_2$, one has $h_2 \leq \gamma h_1$.¹

We further consider a decomposition $0 = t_0 < t_1 < \dots < t_N = T$ of $[0, T]$ and introduce $\tau_n := t_n - t_{n-1}$ for all $n = 1, \dots, N$. With each time interval $I_n := (t_{n-1}, t_n]$, we associate a local polynomial degree p_n^t ; we collect such polynomial orders (in time) in the vector \mathbf{p}^t in \mathbb{N}_+^N , and set $\mathbf{p}_n^t := p_n^t$. For k in \mathbb{Z} , $\mathbf{p}^t + k$ is the vector of entries $p_n^t + k$.

For all $n = 1, \dots, N$, we set $(v')^+(t_{n-1}, \cdot) := v'_{|I_n}(t_{n-1}, \cdot)$ and $(v')^-(t_{n-1}, \cdot) := v'_{|I_{n-1}}(t_{n-1}, \cdot)$; for all v piecewise continuous in time, we set $v^+(t_{n-1}, \cdot) := v_{|I_n}(t_{n-1}, \cdot)$ and $v^-(t_{n-1}, \cdot) := v_{|I_{n-1}}(t_{n-1}, \cdot)$.

We also define the tensor product space

$$\mathbb{P}_{p_n^t}(I_n; V_h) := \{v_{h, \tau} \in L^2(I_n; V_h) \mid v_{h, \tau} = a(x)b(t), a \in V_h, b \in \mathbb{P}_{p_n^t}(I_n)\}.$$

Throughout we assume that

$$p_n^t \geq 2 \quad \forall n = 2, \dots, N.$$

The fully-discrete DG-CG method. Let $u_{0, h}$ and $u_{1, h}$ be discretizations of u_0 and u_1 in V_h . For instance, we pick $u_{0, h}$ as the elliptic projection of u_0 defined in display (21) below; $u_{1, h}$ as the L^2 projection of u_1 onto V_h . Other variants are possible, but are omitted here.

We define

$$X_{h, \tau} := u_{0, h} + \{u_{h, \tau} \in \mathcal{C}^0(0, T; V_h) \mid u_{h, \tau}|_{I_n} \in \mathbb{P}_{p_n^t}(I_n; V_h) \forall n = 1, \dots, N\}$$

and the upwind jump operator for the time derivative as

$$\llbracket [u'_{h, \tau}(t_{n-1})] \rrbracket (\cdot) = \begin{cases} u'_{h, \tau}|_{I_1}(0, \cdot) - u_{1, h}(\cdot) & \text{if } n = 1 \\ u'_{h, \tau}|_{I_n}(t_{n-1}, \cdot) - u'_{h, \tau}|_{I_{n-1}}(t_{n-1}, \cdot) & \text{if } n = 2, \dots, N. \end{cases}$$

The DG-CG method, see, e.g., [32], reads as follows: find $u_{h, \tau}$ in $X_{h, \tau}$ such that

$$\begin{aligned} \int_{I_n} [(u''_{h, \tau}, v_{h, \tau})_{0, \Omega} + a(u_{h, \tau}, v_{h, \tau})] dt + (\llbracket [u'_{h, \tau}] \rrbracket (t_{n-1}, \cdot), v_{h, \tau}^+(t_{n-1}, \cdot))_{0, \Omega} \\ = (f, v_{h, \tau})_{L^2(I_n; L^2(\Omega))} \quad \forall v_{h, \tau} \in \mathbb{P}_{p_n^t - 1}(I_n; V_h), \quad \forall n = 1, \dots, N. \end{aligned} \quad (3)$$

The initial condition $u_{0, h}$ is imposed strongly in $X_{h, \tau}$; the initial condition $u_{1, h}$ is imposed weakly through the upwind term at the initial time.

¹In the a priori analysis, the only place where quasi-uniformity is used in lieu of local quasi-uniformity is while deriving estimates of the elliptic projector in the L^2 norm; see displays (21) and (22) below

Method (3) is solved time-slab by time-slab as a time marching scheme. Initial conditions on each time slab are assigned taking the values of the solution at the final time of the previous slab and upwinding the first time derivative.

The existence and uniqueness of a solution and the data of method (3) follow, e.g., assuming sufficient smoothness of the solution to problem (2), showing stability estimates as those in Theorem 2.1 below (which imply uniqueness), and using the fact that on each time slab the linear system to solve is square (which entails that existence is equivalent to uniqueness).

The semi-discrete in time DG-CG method. Define the space

$$X_{\tau} := u_0 + \{u_{\tau} \in \mathcal{C}^0(0, T; H_0^1(\Omega)) \mid u_{\tau}|_{I_n} \in \mathbb{P}_{p_n}^t(I_n; H_0^1(\Omega))\}.$$

In Section 3 below, we prove constant free, reliable a posteriori error estimates for the time semi-discrete in time version of (3). More precisely, we look for u_{τ} in X_{τ} such that

$$\int_{I_n} [(u_{\tau}'' , v_{\tau})_{0, \Omega} + a(u_{\tau}, v_{\tau})] dt + (\llbracket u_{\tau}' \rrbracket (t_{n-1}, \cdot), v_{\tau}^{\dagger}(t_{n-1}, \cdot))_{0, \Omega} = (f, v_{\tau})_{L^2(0, T; L^2(\Omega))} \quad (4)$$

$$\forall v_{\tau} \in \mathbb{P}_{p_n}^{t-1}(I_n; H_0^1(\Omega)), \quad \forall n = 1, \dots, N,$$

with u_0 imposed strongly in X_{τ} and u_1 imposed weakly through upwinding.

Let $\Pi_{\mathbf{p}^{t-1}}^0$ denote the piecewise L^2 projector onto $\mathbb{P}_{p_n}^{t-1}(I_n; H_0^1(\Omega))$ for all $n = 1, \dots, N$. We can replace the right-hand side of (4) with the following expression:

$$(f, v_{\tau})_{L^2(I_n; L^2(\Omega))} = (\Pi_{\mathbf{p}^{t-1}}^0 f, v_{\tau})_{L^2(I_n; L^2(\Omega))} \quad \forall n = 1, \dots, N. \quad (5)$$

Structure of the remainder of the paper. We discuss stability and convergence estimates of the fully-discrete method (3) in Section 2, which are explicit in the spatial mesh size, the time steps, and the polynomial degrees. A posteriori error estimates for the semi-discrete in time method (4) are derived in Section 3. We assess the numerical findings with numerical experiments in Section 4, and draw some conclusions in Section 5.

2 A priori error analysis

This section is concerned with proving stability and a priori estimates for method (3): in Section 2.1, we show stability estimates following the analysis in [32] by tracking the explicit dependence on the polynomial degree distributions; in Section 2.2, we introduce and show the approximation properties of an integrated Thomeé-type operator in terms of the spatial mesh size, the time steps, and the polynomial degrees, which are instrumental in deriving abstract error estimates in Section 2.3; standard polynomial approximation results yield convergence estimates in Section 2.4.

2.1 Constant free stability estimates

The stability of method (3) in certain norms is investigated in [32, Theorem 4.5]. The main idea behind the derivation of stability estimates is to take a suitable test function, namely the L^2 projection onto the correct test space of the time derivative of the discrete solution times a weight mimicking an exponential function. The reason for this is that testing only with the time derivative of the discrete solution would yield to stability estimates at the time grid point only, i.e., no global stability estimates would be available; see [32, eq. (4.1)].

The constants in the stability estimates in [32], depend implicitly on the distribution \mathbf{p}^t of polynomial degrees in time; for this reason, we revisit that proof so as to carry out an explicit analysis in terms of the polynomial distribution in time.

To this aim, given a generic element g in the space $X_{h, \tau}$, let $m = m(g)$ so that I_m is the interval where

$$\|g'\|_{L^\infty(I_m; L^2(\Omega))}^2 + |g|_{L^\infty(I_m; H^1(\Omega))}^2 = \max_{n=1}^N \left(\|g'\|_{L^\infty(I_n; L^2(\Omega))}^2 + |g|_{L^\infty(I_n; H^1(\Omega))}^2 \right), \quad (6)$$

The index m depends on the choice of g . We also define

$$\mu_n := \frac{1}{1028(p_n^t)^2(2p_n^t + 1)} \quad \forall n = 1, \dots, N. \quad (7)$$

Theorem 2.1. *Let $u_{h,\tau}$ be the solution to (3) and f be the source term in (1). The following stability estimate holds true:*

$$\begin{aligned} & \mu_m \left(\|u'_{h,\tau}\|_{L^\infty(I_m;L^2(\Omega))}^2 + |u_{h,\tau}|_{L^\infty(I_m;H^1(\Omega))}^2 \right) + \frac{1}{4} \sum_{n=1}^m \|[[u'_{h,\tau}]](t_{n-1}, \cdot)\|_{0,\Omega}^2 \\ & \leq \frac{1}{2} \left(|u_{h,\tau}(0, \cdot)|_{1,\Omega}^2 + \|(u_{h,\tau}^-)'(0, \cdot)\|_{0,\Omega}^2 \right) + \frac{t_m}{\mu_m} \|f\|_{L^2(0,t_m;L^2(\Omega))}^2. \end{aligned} \quad (8)$$

Proof. Define

$$\lambda_n := \frac{1}{4(2p_n^t + 1)\tau_n} \quad \forall n = 1, \dots, N.$$

Let $\Pi_{\mathbf{P}^{t-1}}^0$ denote the piecewise L^2 projector onto $\mathbb{P}_{p_n^t-1}(I_n; L^2(\Omega))$ and consider the following test function already used in [32, Theorem 4.5]

$$v_{h,\tau}|_{I_n} := \Pi_{\mathbf{P}^{t-1}}^0([1 - \lambda_n(t - t_{n-1})]u'_{h,\tau}) \quad \forall n = 1, \dots, N.$$

Picking $v_{h,\tau}$ as above in (3) and proceeding as in the proof of [32, Theorem 4.5], more precisely see [32, eq. (4.2)], yield, for all $n = 1, \dots, N$,

$$\begin{aligned} & (1 - \lambda_n \tau_n) \frac{1}{2} \left(|u_{h,\tau}(t_n, \cdot)|_{1,\Omega}^2 + \|(u_{h,\tau}^-)'(t_n, \cdot)\|_{0,\Omega}^2 \right) + \frac{\lambda_n}{2} \left(\|u'_{h,\tau}\|_{L^2(I_n;L^2(\Omega))}^2 + |u_{h,\tau}|_{L^2(I_n;H^1(\Omega))}^2 \right) \\ & + \frac{1}{2} \|[[u'_{h,\tau}]](t_{n-1}, \cdot)\|_{0,\Omega}^2 \leq \frac{1}{2} \left(|u_{h,\tau}(t_{n-1}, \cdot)|_{1,\Omega}^2 + \|(u_{h,\tau}^-)'(t_{n-1}, \cdot)\|_{0,\Omega}^2 \right) \\ & + ([[u'_{h,\tau}]](t_{n-1}, \cdot), (u_{h,\tau}^+)'(t_{n-1}, \cdot) - v_{h,\tau}^+(t_{n-1}, \cdot))_{0,\Omega} + \int_{I_n} [1 - \lambda_n(\cdot - t_{n-1})](\Pi_{\mathbf{P}^{t-1}}^0 f, u'_{h,\tau})_{0,\Omega} dt. \end{aligned}$$

We estimate the last two terms on the right-hand side separately: one involving the jump of the first derivative at t_{n-1} ; the other involving the source term f .

As for the ‘‘jump’’ term, we invoke [32, Corollary 4.4] and the definition of λ_n , use standard manipulations, and get

$$\begin{aligned} & ([[u'_{h,\tau}]](t_{n-1}, \cdot), (u_{h,\tau}^+)'(t_{n-1}, \cdot) - v_{h,\tau}^+(t_{n-1}, \cdot))_{0,\Omega} \\ & \leq \|[[u'_{h,\tau}]](t_{n-1}, \cdot)\|_{0,\Omega} \left\| (u_{h,\tau}^+)'(t_{n-1}, \cdot) - v_{h,\tau}^+(t_{n-1}, \cdot) \right\|_{0,\Omega} \\ & \leq \|[[u'_{h,\tau}]](t_{n-1}, \cdot)\|_{0,\Omega} \lambda_n \sqrt{(2p_n^t + 1)\tau_n} \|u'_{h,\tau}\|_{L^2(I_n;L^2(\Omega))} \\ & \leq \sqrt{(2p_n^t + 1)\tau_n} \lambda_n \left(\frac{1}{2} \|[[u'_{h,\tau}]](t_{n-1}, \cdot)\|_{0,\Omega}^2 + \frac{\lambda_n}{2} \|u'_{h,\tau}\|_{L^2(I_n;L^2(\Omega))}^2 \right) \\ & = \frac{1}{2} \left(\frac{1}{2} \|[[u'_{h,\tau}]](t_{n-1}, \cdot)\|_{0,\Omega}^2 + \frac{\lambda_n}{2} \|u'_{h,\tau}\|_{L^2(I_n;L^2(\Omega))}^2 \right) \quad \forall n = 1, \dots, N. \end{aligned}$$

As for the ‘‘source’’ term, we write

$$\int_{I_n} [1 - \lambda_n(\cdot - t_{n-1})](\Pi_{\mathbf{P}^{t-1}}^0 f, u'_{h,\tau})_{0,\Omega} dt \leq \|\Pi_{\mathbf{P}^{t-1}}^0 f\|_{L^1(I_n;L^2(\Omega))} \|u'_{h,\tau}\|_{L^\infty(I_n;L^2(\Omega))}.$$

We combine the three displays above: for all $n = 1, \dots, N$,

$$\begin{aligned} & (1 - \lambda_n \tau_n) \frac{1}{2} \left(|u_{h,\tau}(t_n, \cdot)|_{1,\Omega}^2 + \|(u_{h,\tau}^-)'(t_n, \cdot)\|_{0,\Omega}^2 \right) \\ & + \frac{\lambda_n}{4} \left(\|u'_{h,\tau}\|_{L^2(I_n;L^2(\Omega))}^2 + |u_{h,\tau}|_{L^2(I_n;H^1(\Omega))}^2 \right) + \frac{1}{4} \|[[u'_{h,\tau}]](t_{n-1}, \cdot)\|_{0,\Omega}^2 \\ & \leq \frac{1}{2} \left(|u_{h,\tau}(t_{n-1}, \cdot)|_{1,\Omega}^2 + \|(u_{h,\tau}^-)'(t_{n-1}, \cdot)\|_{0,\Omega}^2 \right) + \|\Pi_{\mathbf{P}^{t-1}}^0 f\|_{L^1(I_n;L^2(\Omega))} \|u'_{h,\tau}\|_{L^\infty(I_n;L^2(\Omega))}. \end{aligned}$$

Using that $\lambda_n \tau_n < 1/4$, and $1 - \lambda_n \tau_n \geq 0$, we simplify this inequality:

$$\begin{aligned} & \frac{\lambda_n}{4} \left(\|u'_{h,\tau}\|_{L^2(I_n;L^2(\Omega))}^2 + |u_{h,\tau}|_{L^2(I_n;H^1(\Omega))}^2 \right) + \frac{1}{4} \|[[u'_{h,\tau}]](t_{n-1}, \cdot)\|_{0,\Omega}^2 \\ & \leq \frac{1}{2} \left(|u_{h,\tau}(t_{n-1}, \cdot)|_{1,\Omega}^2 + \|(u_{h,\tau}^-)'(t_{n-1}, \cdot)\|_{0,\Omega}^2 \right) + \|\Pi_{\mathbf{P}^{t-1}}^0 f\|_{L^1(I_n;L^2(\Omega))} \|u'_{h,\tau}\|_{L^\infty(I_n;L^2(\Omega))}. \end{aligned} \quad (9)$$

In [32, eq. (4.1)], a stability estimate is proven taking $u'_{h,\tau}$ as a test function, which gives, for all $n = 1, \dots, N$,

$$\begin{aligned} \frac{1}{2} \left(|u_{h,\tau}(t_{n-1}, \cdot)|_{1,\Omega}^2 + \left\| (u_{h,\tau}^-)'(t_{n-1}, \cdot) \right\|_{0,\Omega}^2 \right) &\leq \frac{1}{2} \left(|u_{h,\tau}(0, \cdot)|_{1,\Omega}^2 + \left\| (u_{h,\tau}^-)'(0, \cdot) \right\|_{0,\Omega}^2 \right) \\ &+ \left\| \Pi_{\mathbf{P}^t-1}^0 f \right\|_{L^1(0,t_{n-1};L^2(\Omega))} \left(\frac{1}{2} \|u'_{h,\tau}\|_{L^\infty(0,t_{n-1};L^2(\Omega))}^2 + \frac{1}{2} |u_{h,\tau}|_{L^\infty(0,t_{n-1};H^1(\Omega))}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Inserting the bound

$$\|u'_{h,\tau}\|_{L^\infty(I_n;L^2(\Omega))} \leq \left(\|u'_{h,\tau}\|_{L^\infty(I_n;L^2(\Omega))}^2 + |u_{h,\tau}|_{L^\infty(I_n;H^1(\Omega))}^2 \right)^{\frac{1}{2}} \quad \forall n = 1, \dots, N$$

in (9), selecting n to be $m = m(u_{h,\tau})$ as in (6), and combining the two displays above give

$$\begin{aligned} \frac{\lambda_m}{4} \left(\|u'_{h,\tau}\|_{L^2(I_m;L^2(\Omega))}^2 + |u_{h,\tau}|_{L^2(I_m;H^1(\Omega))}^2 \right) &+ \frac{1}{4} \sum_{n=1}^m \left\| \llbracket u'_{h,\tau} \rrbracket (t_{n-1}, \cdot) \right\|_{0,\Omega}^2 \\ &\leq \frac{1}{2} \left(|u_{h,\tau}(0, \cdot)|_{1,\Omega}^2 + \left\| (u_{h,\tau}^-)'(0, \cdot) \right\|_{0,\Omega}^2 \right) \\ &+ 2 \left(\sum_{n=1}^m \left\| \Pi_{\mathbf{P}^t-1}^0 f \right\|_{L^1(I_n;L^2(\Omega))} \right) \left(\|u'_{h,\tau}\|_{L^\infty(I_m;L^2(\Omega))}^2 + |u_{h,\tau}|_{L^\infty(I_m;H^1(\Omega))}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

We recall the one dimensional L^∞ to L^2 -norm polynomial inverse inequality in [28, eq. (3.6.4)]:

$$|u_{h,\tau}|_{L^\infty(I_m;H^1(\Omega))}^2 \leq \frac{32(p_m^t)^2}{\tau_n} |u_{h,\tau}|_{L^2(I_m;H^1(\Omega))}^2.$$

We deduce

$$\begin{aligned} \frac{1}{512(p_m^t)^2(2p_m^t+1)} \left(\|u'_{h,\tau}\|_{L^\infty(I_m;L^2(\Omega))}^2 + |u_{h,\tau}|_{L^\infty(I_m;H^1(\Omega))}^2 \right) &+ \frac{1}{4} \sum_{n=1}^m \left\| \llbracket u'_{h,\tau} \rrbracket (t_{n-1}, \cdot) \right\|_{0,\Omega}^2 \\ &\leq \frac{1}{2} \left(|u_{h,\tau}(0, \cdot)|_{1,\Omega}^2 + \left\| (u_{h,\tau}^-)'(0, \cdot) \right\|_{0,\Omega}^2 \right) \\ &+ 2 \left(\sum_{n=1}^m \left\| \Pi_{\mathbf{P}^t-1}^0 f \right\|_{L^1(I_n;L^2(\Omega))} \right) \left(\|u'_{h,\tau}\|_{L^\infty(I_m;L^2(\Omega))}^2 + |u_{h,\tau}|_{L^\infty(I_m;H^1(\Omega))}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Using Young's inequality $2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$, with $\varepsilon = 1028(p_m^t)^2(2p_m^t+1) = \mu_m^{-1}$ for the last term in the above relation and the definition of μ_m in (7), we infer

$$\begin{aligned} \mu_m \left(\|u'_{h,\tau}\|_{L^\infty(I_m;L^2(\Omega))}^2 + |u_{h,\tau}|_{L^\infty(I_m;H^1(\Omega))}^2 \right) &+ \frac{1}{4} \sum_{n=1}^m \left\| \llbracket u'_{h,\tau} \rrbracket (t_{n-1}, \cdot) \right\|_{0,\Omega}^2 \\ &\leq \frac{1}{2} \left(|u_{h,\tau}(0, \cdot)|_{1,\Omega}^2 + \left\| (u_{h,\tau}^-)'(0, \cdot) \right\|_{0,\Omega}^2 \right) + \frac{1}{\mu_m} \left(\sum_{n=1}^m \left\| \Pi_{\mathbf{P}^t-1}^0 f \right\|_{L^1(I_n;L^2(\Omega))} \right)^2. \end{aligned}$$

The stability estimate (8) follows using the Hölder inequality, the stability of the L^2 projector in the $L^2(I_n)$ norm, and the ℓ^2 Cauchy-Schwarz inequality in the last term on the right-hand side above:

$$\sum_{n=1}^m \left\| \Pi_{\mathbf{P}^t-1}^0 f \right\|_{L^1(I_n;L^2(\Omega))} \leq \sum_{n=1}^m \tau_n^{\frac{1}{2}} \|f\|_{L^2(I_n;L^2(\Omega))} \leq t_m^{\frac{1}{2}} \|f\|_{L^2(0,t_m;L^2(\Omega))}. \quad (10)$$

□

Remark 1. The norms appearing on the left-hand side of (8) are of the same sort of those in [32, Theorem 4.5]. Instead, that on the right-hand side involving the source term f differs a bit: here, we employ an $L^2(L^2)$ -type norm weighted with $t_m^{\frac{1}{2}}/\mu_m^{\frac{1}{2}}$, which scales exactly as the $L^1(L^2)$ norm used by Walkington. The reason for this change resides in inequality (10); by proceeding as in [32], we would end up with further suboptimality by two orders in the polynomial degree due to the L^1 stability of the L^2 projector.

2.2 An integrated Thomée-type operator

Given a Banach space \mathcal{X} , we introduce an operator $\mathcal{P}_{\mathbf{p}^t}$ mapping $\mathcal{C}^1(0, T; \mathcal{X})$ into the space of piecewise polynomials with polynomial distribution \mathbf{p}^t over the time grid. This operator is the integrated version of an operator in [30] in the framework of parabolic problems, see also [2], and whose p -approximation properties are derived in [26].

The operator $\mathcal{P}_{\mathbf{p}^t}$ is piecewise defined fixing the following conditions: for all $n = 1, \dots, N$,

$$\begin{cases} (w' - \mathcal{P}_{\mathbf{p}^t}(w)', q_{p_n^t-2})_{L^2(I_n; \mathcal{X})} = 0 & \forall n = 1, \dots, N, \quad \forall q_{p_n^t-2} \in \mathbb{P}_{p_n^t-2}(I_n; \mathcal{X}); \\ \mathcal{P}_{\mathbf{p}^t}(w)'(t_n^-, \cdot) = w'(t_n, \cdot); \quad \mathcal{P}_{\mathbf{p}^t}(w)(t_{n-1}, \cdot) = w(t_{n-1}, \cdot) & \text{in } \mathcal{X}. \end{cases}$$

As shown in [32, Section 5.1], this operator is well defined and the above definition is equivalent to

$$\begin{cases} (w - \mathcal{P}_{\mathbf{p}^t}(w), q_{p_n^t-3})_{L^2(I_n; \mathcal{X})} = 0 & \forall n = 1, \dots, N, \quad \forall q_{p_n^t-3} \in \mathbb{P}_{p_n^t-3}(I_n; \mathcal{X}), \\ \mathcal{P}_{\mathbf{p}^t}(w)'(t_n^-, \cdot) = w'(t_n, \cdot), \quad \mathcal{P}_{\mathbf{p}^t}(w)(t_n, \cdot) = w(t_n, \cdot), \quad \mathcal{P}_{\mathbf{p}^t}(w)(t_{n-1}, \cdot) = w(t_{n-1}, \cdot) & \text{in } \mathcal{X}, \end{cases} \quad (11)$$

where the orthogonality condition in the first line of (11) is not taken into account if $p_n^t = 2$.

The function $\mathcal{P}_{\mathbf{p}^t}(w)$ is globally continuous but not necessarily \mathcal{C}^1 in time and satisfies the following property.

Lemma 2.2. *For all u in $H^2(I_n; \mathcal{X})$ and all $q_{p_n^t-1}$ in $\mathbb{P}_{p_n^t-1}(I_n; \mathcal{X})$, the following identity holds true:*

$$((u - \mathcal{P}_{\mathbf{p}^t}u)'', q_{p_n^t-1})_{L^2(I_n; \mathcal{X})} = ([[\mathcal{P}_{\mathbf{p}^t}u]'] (t_{n-1}, \cdot), q_{p_n^t-1}(t_{n-1}, \cdot))_{\mathcal{X}} \quad \forall n = 1, \dots, N. \quad (12)$$

Proof. We have

$$\begin{aligned} & ((u - \mathcal{P}_{\mathbf{p}^t}u)'', q_{p_n^t-1})_{L^2(I_n; \mathcal{X})} \\ &= -((u - \mathcal{P}_{\mathbf{p}^t}u)', q_{p_n^t-1}')_{L^2(I_n; \mathcal{X})} + ((u - \mathcal{P}_{\mathbf{p}^t}u)'(t_n, \cdot), q_{p_n^t-1}(t_n, \cdot))_{\mathcal{X}} \\ &\quad - ((u - \mathcal{P}_{\mathbf{p}^t}u)'(t_{n-1}, \cdot), q_{p_n^t-1}(t_{n-1}, \cdot))_{\mathcal{X}} \\ &\stackrel{(11)}{=} -((u - \mathcal{P}_{\mathbf{p}^t}u)'(t_{n-1}, \cdot), q_{p_n^t-1}(t_{n-1}, \cdot))_{\mathcal{X}} \stackrel{(11)}{=} ([[\mathcal{P}_{\mathbf{p}^t}u]'] (t_{n-1}, \cdot), q_{p_n^t-1}(t_{n-1}, \cdot))_{\mathcal{X}}, \end{aligned}$$

which is the assertion. \square

The scope of this section is showing the approximation properties of $\mathcal{P}_{\mathbf{p}^t}$. We recall the properties of the original Thomée operator [30]. Introduce $\tilde{\mathcal{P}}_{\mathbf{p}^t}$ mapping $\mathcal{C}^0(0, T; \mathcal{X})$ into the space of piecewise polynomials with degree distribution \mathbf{p}^t over the time grid (in particular, discontinuous functions in time). The operator $\tilde{\mathcal{P}}_{\mathbf{p}^t}$ is defined fixing the following conditions:

$$\begin{cases} (w - \tilde{\mathcal{P}}_{\mathbf{p}^t}(w), q_{p_n^t-1})_{L^2(I_n; \mathcal{X})} = 0 & \forall n = 1, \dots, N, \quad \forall q_{p_n^t-1} \in \mathbb{P}_{p_n^t-1}(I_n; \mathcal{X}); \\ \tilde{\mathcal{P}}_{\mathbf{p}^t}(w)(t_n^-, \cdot) = w(t_n, \cdot) & \text{in } \mathcal{X}. \end{cases} \quad (13)$$

The following approximation results can be found in [26, Lemmas 3.6–3.8, Theorem 3.10] and [5, Lemma 3.3].

Lemma 2.3. *Let $\tilde{\mathcal{P}}_{\mathbf{p}^t}$ be the operator in (13). Then, for all w in $H^{s+1}(I_n; \mathcal{X})$ with $s \geq 0$, the following inequalities hold true for all $n = 1, \dots, N$:*

$$\|w - \tilde{\mathcal{P}}_{\mathbf{p}^t}w\|_{L^2(I_n; \mathcal{X})} \lesssim \|w - \Pi_{\mathbf{p}^t}^0 w\|_{L^2(I_n; \mathcal{X})} + \frac{\tau_n}{p_n^t} \|w'\|_{L^2(I_n; \mathcal{X})}, \quad (14a)$$

$$\begin{aligned} \|w - \tilde{\mathcal{P}}_{\mathbf{p}^t}w\|_{L^2(I_n; \mathcal{X})} &\lesssim \inf_{q_{p_n^t} \in \mathbb{P}_{p_n^t}(I_n; \mathcal{X})} \left(\|w - q_{p_n^t}\|_{L^2(I_n; \mathcal{X})} + \frac{\tau_n}{p_n^t} \|(w - q_{p_n^t})'\|_{L^2(I_n; \mathcal{X})} \right) \\ &\lesssim \left(\frac{\tau_n}{p_n^t} \right)^{\min(s, p_n^t)+1} \|w^{(s+1)}\|_{L^2(I_n; \mathcal{X})}, \end{aligned} \quad (14b)$$

$$\|\tilde{\mathcal{P}}_{\mathbf{p}^t}w\|_{L^2(I_n; \mathcal{X})} \lesssim \|w\|_{L^2(I_n; \mathcal{X})} + \frac{\tau_n}{p_n^t} \|w'\|_{L^2(I_n; \mathcal{X})}, \quad (14c)$$

$$\begin{aligned} \|(w - \tilde{\mathcal{P}}_{\mathbf{p}^t}w)(t_{n-1}, \cdot)\|_{\mathcal{X}} &\lesssim \left(\frac{\tau_n}{p_n^t} \right)^{\frac{1}{2}} \inf_{q_{p_n^t} \in \mathbb{P}_{p_n^t}(I_n; \mathcal{X})} \|(w - q_{p_n^t})'\|_{L^2(I_n; \mathcal{X})} \\ &\lesssim \left(\frac{\tau_n}{p_n^t} \right)^{\min(s, p_n^t)+\frac{1}{2}} \|w^{(s+1)}\|_{L^2(I_n; \mathcal{X})}. \end{aligned} \quad (14d)$$

On the reference interval $\widehat{I} = [-1, 1]^2$, the bounds in Lemma 2.3 are a consequence of the properties of the Legendre polynomials, the identity

$$(w - \widetilde{\mathcal{P}}_{\mathbf{p}^t} w)(t, \cdot) = (w - \Pi_{\mathbf{p}^t}^0 w)(t, \cdot) + (w - \Pi_{\mathbf{p}^t}^0 w)(1, \cdot) L_p(t) \quad \text{in } \mathcal{X}, \quad (15)$$

and sharp estimates for the second term on the right-hand side of (15).

The operators $\mathcal{P}_{\mathbf{p}^t}$ and $\widetilde{\mathcal{P}}_{\mathbf{p}^t}$ in (11) and (13) are related by the following identity: for all t in I_n ,

$$\mathcal{P}_{\mathbf{p}^t}(w)(t, \cdot) = \int_{t_{n-1}}^t (\widetilde{\mathcal{P}}_{\mathbf{p}^{t-1}}(w'))(s, \cdot) ds + w(t_{n-1}, \cdot) \quad \text{in } \mathcal{X} \quad \forall n = 1, \dots, N. \quad (16)$$

Combining (16) with Lemma 2.3, and using [26, Theorem 3.10], we get the following result.

Lemma 2.4. *Let $\mathcal{P}_{\mathbf{p}^t}$ be the operator in (11). Then, for all w in $H^{s+1}(I_n; \mathcal{X})$ with $s \geq 1$, the following inequalities hold true for all $n = 1, \dots, N$:*

$$\|(w - \mathcal{P}_{\mathbf{p}^t} w)'\|_{L^2(I_n; \mathcal{X})} \lesssim \|w' - \Pi_{\mathbf{p}^{t-1}}^0 w'\|_{L^2(I_n; \mathcal{X})} + \frac{\tau_n}{p_n^t} \|w''\|_{L^2(I_n; \mathcal{X})}, \quad (17a)$$

$$\begin{aligned} \|(w - \mathcal{P}_{\mathbf{p}^t} w)'\|_{L^2(I_n; \mathcal{X})} &\lesssim \inf_{q_{p_n^t} \in \mathbb{P}_{p_n^t}(I_n; \mathcal{X})} \left(\|(w - q_{p_n^t})'\|_{L^2(I_n; \mathcal{X})} + \frac{\tau_n}{p_n^t} \|(w - q_{p_n^t})''\|_{L^2(I_n; \mathcal{X})} \right) \\ &\lesssim \left(\frac{\tau_n}{p_n^t} \right)^{\min(s, p_n^t)} \|w^{(s+1)}\|_{L^2(I_n; \mathcal{X})}, \end{aligned} \quad (17b)$$

$$\|(\mathcal{P}_{\mathbf{p}^t} w)'\|_{L^2(I_n; \mathcal{X})} \lesssim \|w'\|_{L^2(I_n; \mathcal{X})} + \frac{\tau_n}{p_n^t} \|w''\|_{L^2(I_n; \mathcal{X})}. \quad (17c)$$

Proof. Inequality (17a) follows from (14a) and (16). Inequality (17c) follows from (14c) and (16). Inequality (17b) follows from (17c), the triangle inequality, and the fact that $\mathcal{P}_{\mathbf{p}^t}$ preserves polynomials of degree p_n^t in time for all $n = 1, \dots, N$. \square

We derive bounds for the operator $\mathcal{P}_{\mathbf{p}^t}$ in L^2 -type norms. To this aim, we introduce the piecewise H^1 projector $\Pi_{\mathbf{p}^{t-1}}^1$ onto polynomials in time as follows:

$$\begin{cases} ((w - \Pi_{\mathbf{p}^{t-1}}^1 w)', q_{p_n^t-1}')_{0, I_n} = 0 & \forall w \in H^1(I_n; \mathcal{X}), \quad q_{p_n^t-1} \in \mathbb{P}_{p_n^t-1}(I_n; \mathcal{X}); \\ (w - \Pi_{\mathbf{p}^{t-1}}^1 w)(t_{n-1}, \cdot) = 0 & \text{in } \mathcal{X} \quad \forall n = 1, \dots, N. \end{cases} \quad (18)$$

On the other hand, the operator $\Pi_{\mathbf{p}^{t-1}}^0$ denotes the piecewise L^2 projector onto polynomials in time.

Some properties of the operator $\mathcal{P}_{\mathbf{p}^t}$ are detailed in the next result.

Proposition 2.5. *Let $\mathcal{P}_{\mathbf{p}^t}$ and $\Pi_{\mathbf{p}^{t-1}}^1$ be the operators in (11) and (18). Then, the following inequality holds true: for all w in $H^{s+1}(I_n; \mathcal{X})$ with $s \geq 1$*

$$\|w - \mathcal{P}_{\mathbf{p}^t} w\|_{L^2(I_n; \mathcal{X})} \lesssim \|w - \Pi_{\mathbf{p}^t}^1 w\|_{L^2(I_n; \mathcal{X})} + \frac{\tau_n^2}{(p_n^t)^2} \|w''\|_{L^2(I_n; \mathcal{X})} \quad \forall n = 1, \dots, N. \quad (19)$$

Since $\mathcal{P}_{\mathbf{p}^t}$ preserves polynomials in time of order p_n^t , we also have

$$\begin{aligned} \|w - \mathcal{P}_{\mathbf{p}^t} w\|_{L^2(I_n; \mathcal{X})} &\lesssim \inf_{q_{p_n^t} \in \mathbb{P}_{p_n^t}(I_n; \mathcal{X})} \left(\frac{\tau_n}{p_n^t} \|(w - q_{p_n^t})'\|_{L^2(I_n; \mathcal{X})} + \frac{\tau_n^2}{(p_n^t)^2} \|(w - q_{p_n^t})''\|_{L^2(I_n; \mathcal{X})} \right) \\ &\lesssim \left(\frac{\tau_n}{p_n^t} \right)^{\min(s, p_n^t)+1} \|w^{(s+1)}\|_{L^2(I_n; \mathcal{X})} \quad \forall n = 1, \dots, N. \end{aligned} \quad (20)$$

Proof. We prove the assertion on the reference interval $\widehat{I} = (-1, 1)$; the general bound follows from a scaling argument.

²With an abuse of notation, we use the same notation for the operators on the reference interval.

Identity (15) implies

$$(w - \mathcal{P}_{\mathbf{p}^t} w)'(t, \cdot) \stackrel{(16)}{=} (w' - \tilde{\mathcal{P}}_{\mathbf{p}^t-1}(w'))(t, \cdot) = (w' - \Pi_{\mathbf{p}^t-1}^0 w')(t, \cdot) + (w' - \Pi_{\mathbf{p}^t-1}^0 w')(1, \cdot) L_{p_n^t-1}(t) \quad \text{in } \mathcal{X}.$$

Since $(w - \mathcal{P}_{\mathbf{p}^t} w)(-1, \cdot) = 0$, an integration by parts gives

$$\begin{aligned} (w - \mathcal{P}_{\mathbf{p}^t} w)(t, \cdot) &= \int_{-1}^t (w - \mathcal{P}_{\mathbf{p}^t} w)'(s, \cdot) ds \\ &= \int_{-1}^t (w' - \Pi_{\mathbf{p}^t-1}^0 w')(s, \cdot) ds + (w' - \Pi_{\mathbf{p}^t-1}^0 w')(1, \cdot) \int_{-1}^t L_{p_n^t-1}(s) ds =: T_1 + T_2. \end{aligned}$$

We estimate the two terms on the right-hand side separately. The properties of the L^2 and H^1 projectors imply $\Pi_{\mathbf{p}^t-1}^0 w' = (\Pi_{\mathbf{p}^t}^1 w)'$. Therefore, recalling from (18) that $w(-1, \cdot) = \Pi_{\mathbf{p}^t}^1 w(-1, \cdot)$, we write

$$T_1 = \int_{-1}^t (w' - (\Pi_{\mathbf{p}^t}^1 w)')(s, \cdot) ds = (w - \Pi_{\mathbf{p}^t}^1 w)(t, \cdot) \quad \text{in } \mathcal{X}.$$

This identity leads to the estimate on the first term on the right-hand side of (19).

As for the term T_2 , we expand w' with respect to Legendre polynomials:

$$w'(t, \cdot) = \sum_{j=0}^{+\infty} w'_j(\cdot) L_j(t) \quad \text{in } \mathcal{X}, \quad w'_j \in \mathcal{X}.$$

Standard properties of the L^2 projector imply

$$T_2 = \left(\sum_{j=p_n^t}^{+\infty} w'_j(\cdot) \right) \int_{-1}^t L_{p_n^t-1}(s) ds \quad \text{in } \mathcal{X}.$$

From [26, eq. (3.5)], we have

$$\left| \sum_{j=p_n^t}^{+\infty} w'_j(\cdot) \right| \lesssim (p_n^t)^{-\frac{1}{2}} \|w''(\cdot)\|_{0, \hat{\Gamma}} \quad \text{in } \mathcal{X}.$$

Using that $\int_{-1}^t L_{p_n^t-1}(s) ds = (2p_n^t - 1)^{-1} (L_{p_n^t} - L_{p_n^t-2})$ and $\|L_{p_n^t}\|_{0, \hat{\Gamma}} \approx (p_n^t)^{-\frac{1}{2}}$ yields

$$\left\| \int_{-1}^t L_{p_n^t-1}(s) ds \right\|_{0, \hat{\Gamma}} \lesssim (p_n^t)^{-1} \left(\|L_{p_n^t}\|_{0, \hat{\Gamma}} + \|L_{p_n^t-2}\|_{0, \hat{\Gamma}} \right) \lesssim (p_n^t)^{-\frac{3}{2}}.$$

Collecting the two displays above implies

$$T_2 \lesssim (p_n^t)^{-2} \|w''(\cdot)\|_{0, \hat{\Gamma}} \quad \text{in } \mathcal{X}.$$

This concludes the proof of (19). As for (20), we add and subtract $q_{p_n^t}$, employ (20), note that $\mathcal{P}_{\mathbf{p}^t}$ piecewise preserves piecewise polynomials of degree p_n^t in time, and use the properties of the projector $\Pi_{\mathbf{p}^t}^1$ as in [28, Corollary 3.15]. \square

2.3 Abstract error analysis

Let $\Pi_h^\mathcal{E} : H_0^1(\Omega) \rightarrow V_h$ denote the elliptic projector defined as

$$a(u - \Pi_h^\mathcal{E} u, v_h) = 0 \quad \forall v_h \in V_h. \quad (21)$$

We have the following stability estimate for the elliptic projector $\Pi_h^\mathcal{E}$ in (21), which is a consequence of the Aubin-Nitsche technique: for any nonnegative s larger than or equal to 1, and possibly equal to ∞ , there exists α in $(0, 1]$ depending on Ω such that

$$\|\Pi_h^\mathcal{E} v\|_{L^s(I_n; L^2(\Omega))} \lesssim \|v\|_{L^s(I_n; L^2(\Omega))} + \frac{h^\alpha}{p_{\mathbf{x}}^\alpha} |v|_{L^s(I_n; H^1(\Omega))} \quad \forall n = 1, \dots, N. \quad (22)$$

Estimate (22) holds true under the assumption of quasi-uniformity of the sequence of spatial meshes. This is the only occasion in the manuscript where local quasi-uniformity cannot be used.

Recall that $\mathcal{P}_{\mathbf{p}^t}$ is given in (11) and let

$$\tilde{u}_{h,\tau} = \mathcal{P}_{\mathbf{p}^t} \Pi_h^\xi u.$$

We analyze a priori estimates of the following error quantity, which we split into two contributions:

$$e_{h,\tau} = u - u_{h,\tau} = (u - \tilde{u}_{h,\tau}) + (\tilde{u}_{h,\tau} - u_{h,\tau}) =: \rho + \xi_{h,\tau}. \quad (23)$$

We show an upper bound on a suitable norm of the two terms on the right-hand side of (23). We begin discussing the term $\xi_{h,\tau}$.

Lemma 2.6. *Consider $m = m(\xi_{h,\tau})$, μ_m , and $\xi_{h,\tau}$ as in (6), (7), and (23). Let u and $u_{h,\tau}$ be the solutions to (2) and (3). Assume that $\Delta_{\mathbf{x}} u$ and u'' belong to $H^2(I_n; L^2(\Omega))$ and $L^2(I_n; H_0^1(\Omega))$ for all $n = 1, \dots, m$. Recall that their initial conditions are u_0 and u_1 , and $u_{0,h}$ and $u_{1,h}$, respectively. Then, the following estimate holds true:*

$$\begin{aligned} & \mu_m \left(\|\xi'_{h,\tau}\|_{L^\infty(I_m; L^2(\Omega))}^2 + |\xi_{h,\tau}|_{L^\infty(I_m; H^1(\Omega))}^2 \right) + \sum_{n=1}^m \left\| \llbracket \xi'_{h,\tau} \rrbracket (t_{n-1}, \cdot) \right\|_{0,\Omega}^2 \\ & \lesssim |(u_0 - u_{0,h})(\cdot)|_{1,\Omega}^2 + \|(u_1 - u_{1,h})(\cdot)\|_{0,\Omega}^2 + \|u_1(\cdot) - (\mathcal{P}_{\mathbf{p}^t} u)'(0, \cdot)\|_{0,\Omega}^2 \\ & \quad + \|(\mathcal{P}_{\mathbf{p}^t} (I - \Pi_h^\xi) u)'(0, \cdot)\|_{0,\Omega}^2 \\ & \quad + \frac{t_m}{\mu_m} \left(\sum_{n=1}^m \|(I - \mathcal{P}_{\mathbf{p}^t}) \Delta_{\mathbf{x}} u\|_{L^2(I_n; L^2(\Omega))}^2 + \sum_{n=1}^m \|(I - \Pi_h^\xi) u''\|_{L^2(I_n; L^2(\Omega))}^2 \right). \end{aligned}$$

The hidden constant is independent of t_n , T , h , $p_{\mathbf{x}}$, and p_n^t for any $n = 1, \dots, m$.

Proof. Let $e_{h,\tau}$ be as in (23). Subtracting (2) and (3), for all $n = 1, \dots, N$, we arrive at

$$\int_{I_n} [(e''_{h,\tau}, v_{h,\tau})_{0,\Omega} + a(e_{h,\tau}, v_{h,\tau})] dt + (\llbracket e'_{h,\tau} \rrbracket (t_{n-1}, \cdot), v_{h,\tau}(t_{n-1}, \cdot))_{0,\Omega} = 0 \quad \forall v_{h,\tau} \in \mathbb{P}_{p_n^t-1}(I_n; V_h).$$

Splitting (23) yields

$$\begin{aligned} & \int_{I_n} [(\xi''_{h,\tau}, v_{h,\tau})_{0,\Omega} + a(\xi_{h,\tau}, v_{h,\tau})] dt + (\llbracket \xi'_{h,\tau} \rrbracket (t_{n-1}, \cdot), v_{h,\tau}(t_{n-1}, \cdot))_{0,\Omega} \\ & = - \int_{I_n} (\rho'', v_{h,\tau})_{0,\Omega} dt - \int_{I_n} a(\rho, v_{h,\tau}) dt - (\llbracket \rho' \rrbracket (t_{n-1}, \cdot), v_{h,\tau}(t_{n-1}, \cdot))_{0,\Omega} \quad \forall v_{h,\tau} \in \mathbb{P}_{p_n^t-1}(I_n; V_h). \end{aligned} \quad (24)$$

We rewrite the right-hand side of (24) as $(f_{\xi_{h,\tau}}, v_{\tau})_{L^2(0,T; L^2(\Omega))}$, i.e., as the right-hand side of method (3), for a suitable $f_{\xi_{h,\tau}}$, which we fix in (27), so as to exploit the stability estimates (8) and deduce the assertion.

We focus on the second term on the right-hand side of (24). Using the fact that $\mathcal{P}_{\mathbf{p}^t}$ and Π_h^ξ commute, definition (21) of Π_h^ξ , the fact that $\mathcal{P}_{\mathbf{p}^t}$ preserves polynomials in time, and an integration by parts, we arrive at

$$\begin{aligned} & \int_{I_n} a(\rho, v_{h,\tau}) dt = \int_{I_n} a((I - \mathcal{P}_{\mathbf{p}^t} \Pi_h^\xi) u, v_{h,\tau}) dt \\ & = \int_{I_n} a((I - \mathcal{P}_{\mathbf{p}^t}) u, v_{h,\tau}) dt = -(\Delta_{\mathbf{x}} (I - \mathcal{P}_{\mathbf{p}^t}) u, v_{h,\tau})_{L^2(I_n; L^2(\Omega))}. \end{aligned} \quad (25)$$

As for the first and third terms on the right-hand side of (24), we use the continuity in time of u' and (12), and get

$$\begin{aligned} & (\rho'', v_{h,\tau})_{L^2(I_n; L^2(\Omega))} + (\llbracket \rho' \rrbracket (t_{n-1}, \cdot), v_{h,\tau}(t_{n-1}, \cdot))_{0,\Omega} \\ & = (u'', v_{h,\tau})_{L^2(I_n; L^2(\Omega))} + (\llbracket u' \rrbracket (t_{n-1}, \cdot), v_{h,\tau}(t_{n-1}, \cdot))_{0,\Omega} \\ & \quad - ((\mathcal{P}_{\mathbf{p}^t} \Pi_h^\xi u)'', v_{h,\tau})_{L^2(I_n; L^2(\Omega))} - \left(\llbracket (\mathcal{P}_{\mathbf{p}^t} \Pi_h^\xi u)' \rrbracket (t_{n-1}, \cdot), v_{h,\tau}(t_{n-1}, \cdot) \right)_{0,\Omega} \\ & = (u'', v_{h,\tau})_{L^2(I_n; L^2(\Omega))} - ((\mathcal{P}_{\mathbf{p}^t} \Pi_h^\xi u)'', v_{h,\tau})_{L^2(I_n; L^2(\Omega))} - \left(\llbracket (\mathcal{P}_{\mathbf{p}^t} \Pi_h^\xi u)' \rrbracket (t_{n-1}, \cdot), v_{h,\tau}(t_{n-1}, \cdot) \right)_{0,\Omega} \\ & \stackrel{(12)}{=} (u'', v_{h,\tau})_{L^2(I_n; L^2(\Omega))} - ((\Pi_h^\xi u)'', v_{h,\tau})_{L^2(I_n; L^2(\Omega))}. \end{aligned}$$

For all $n = 1, \dots, N$, we deduce

$$[(\rho'', v_{h,\tau})_{L^2(I_n; L^2(\Omega))}] + ([[\rho']](t_{n-1}, \cdot), v_{h,\tau}(t_{n-1}, \cdot))_{0,\Omega} = ((I - \Pi_h^\varepsilon)u'', v_{h,\tau})_{L^2(I_n; L^2(\Omega))}. \quad (26)$$

We insert (25) and (26) in (24) and arrive at

$$\begin{aligned} & \int_{I_n} [(\xi''_{h,\tau}, v_{h,\tau})_{0,\Omega} + a(\xi_{h,\tau}, v_{h,\tau})] dt + ([[\xi'_{h,\tau}]](t_{n-1}, \cdot), v_{h,\tau}(t_{n-1}, \cdot))_{0,\Omega} \\ &= ((I - \mathcal{P}_{\mathbf{p}^t})\Delta_{\mathbf{x}}u, v_{h,\tau})_{L^2(I_n; L^2(\Omega))} - ((I - \Pi_h^\varepsilon)u'', v_{h,\tau})_{L^2(I_n; L^2(\Omega))} \\ &= ((I - \mathcal{P}_{\mathbf{p}^t})\Delta_{\mathbf{x}}u - (I - \Pi_h^\varepsilon)u'', v_{h,\tau})_{L^2(I_n; L^2(\Omega))} \quad \forall v_{h,\tau} \in \mathbb{P}_{p_n^t-1}(I_n; V_h), \quad \forall n = 1, \dots, N. \end{aligned}$$

In words, $\xi_{h,\tau}$ is the solution to method (3) with right-hand side given by $(f_{\xi_{h,\tau}}, v_\tau)_{L^2(0,T;L^2(\Omega))}$ with

$$f_{\xi_{h,\tau}} := (I - \mathcal{P}_{\mathbf{p}^t})\Delta_{\mathbf{x}}u - (I - \Pi_h^\varepsilon)u''. \quad (27)$$

We are now in a position to apply the discrete stability estimates (8). Let $m = m(\xi_{h,\tau})$ be defined in (6) and get

$$\begin{aligned} & \mu_m \left(\|\xi'_{h,\tau}\|_{L^\infty(I_m; L^2(\Omega))}^2 + |\xi_{h,\tau}|_{L^\infty(I_m; H^1(\Omega))}^2 \right) + \frac{1}{4} \sum_{n=1}^m \|[[\xi_{h,\tau}]](t_n, \cdot)\|_{0,\Omega}^2 \\ & \leq \frac{1}{2} \left(|\xi_{h,\tau}(0, \cdot)|_{1,\Omega}^2 + \|(\xi_{h,\tau}^-)'(0, \cdot)\|_{0,\Omega}^2 \right) + \frac{t_m}{\mu_m} \|f_{\xi_{h,\tau}}\|_{L^2(0,t_m; L^2(\Omega))}^2 =: T_1 + T_2. \end{aligned}$$

We estimate the two terms on the right-hand side. The triangle inequality implies

$$T_1 \leq \frac{1}{2} \left(|\xi_{h,\tau}(0, \cdot)|_{1,\Omega}^2 + \|(\xi_{h,\tau}^-)'(0, \cdot)\|_{0,\Omega}^2 \right) = \frac{1}{2} [T_{1,1} + T_{1,2}].$$

Using (11) and the stability of the elliptic projection Π_h^ε , we deduce

$$T_{1,1} = |\xi_{h,\tau}(0, \cdot)|_{1,\Omega}^2 = |u_{h,\tau}(0, \cdot) - \Pi_h^\varepsilon u(0, \cdot)|_{1,\Omega}^2 \leq |(u_0 - u_{0,h})(\cdot)|_{1,\Omega}^2.$$

We further have

$$\begin{aligned} T_{1,2} &= \|(\xi_{h,\tau}^-)'(0, \cdot)\|_{0,\Omega}^2 = \|(u_{h,\tau}^-)'(0, \cdot) - (\mathcal{P}_{\mathbf{p}^t} \Pi_h^\varepsilon u)'(0, \cdot)\|_{0,\Omega}^2 = \|u_{1,h}(\cdot) - (\mathcal{P}_{\mathbf{p}^t} \Pi_h^\varepsilon u)'(0, \cdot)\|_{0,\Omega}^2 \\ &\leq 2\|(u_1 - u_{1,h})(\cdot)\|_{0,\Omega}^2 + 2\|u_1(\cdot) - (\mathcal{P}_{\mathbf{p}^t} \Pi_h^\varepsilon u)'(0, \cdot)\|_{0,\Omega}^2 \\ &\leq 2\|(u_1 - u_{1,h})(\cdot)\|_{0,\Omega}^2 + 4\|u_1(\cdot) - (\mathcal{P}_{\mathbf{p}^t} u)'(0, \cdot)\|_{0,\Omega}^2 + 4\|(\mathcal{P}_{\mathbf{p}^t} (I - \Pi_h^\varepsilon)u)'(0, \cdot)\|_{0,\Omega}^2. \end{aligned}$$

Next, we deal with the term T_2 :

$$T_2 = \frac{t_m}{\mu_m} \|f_{\xi_{h,\tau}}\|_{L^2(0,t_m; L^2(\Omega))}^2 =: \frac{t_m}{\mu_m} \sum_{n=1}^m \|(I - \mathcal{P}_{\mathbf{p}^t})\Delta_{\mathbf{x}}u - (I - \Pi_h^\varepsilon)u''\|_{L^2(I_n; L^2(\Omega))}^2 := \frac{t_m}{\mu_m} \sum_{n=1}^m T_{2,n}.$$

We have

$$T_{2,n} \leq 2\|(I - \mathcal{P}_{\mathbf{p}^t})\Delta_{\mathbf{x}}u\|_{L^2(I_n; L^2(\Omega))}^2 + 2\|(I - \Pi_h^\varepsilon)u''\|_{L^2(I_n; L^2(\Omega))}^2.$$

Collecting the bounds on T_1 and T_2 yields the assertion. \square

We deduce an abstract convergence result. Introduce k in $1, \dots, N$ such that

$$\left(\|e'_{h,\tau}\|_{L^\infty(I_k; L^2(\Omega))}^2 + |e_{h,\tau}|_{L^\infty(I_k; H^1(\Omega))}^2 \right) := \max_{n=1}^N \left(\|e'_{h,\tau}\|_{L^\infty(I_n; L^2(\Omega))}^2 + |e_{h,\tau}|_{L^\infty(I_n; H^1(\Omega))}^2 \right). \quad (28)$$

The index k is defined similarly to the index m in (6). However, on the one hand, with k we maximize positive functionals in Bochner spaces and not in space-time finite element spaces; on the other hand, using a different nomenclature of the indices allows us to improve the readability of the estimates in Theorems 2.7 and 2.8 below.

Theorem 2.7. Consider $m = m(\xi_{h,\tau})$, μ_m , k , and $\xi_{h,\tau}$ and ρ as in (6), (7), (28), and (23). Let u and $u_{h,\tau}$ be the solutions to (2) and (3). Assume that $\Delta_{\mathbf{x}}u$ and u'' belong to $H^2(I_n; L^2(\Omega))$ and $L^2(I_n; H_0^1(\Omega))$ for all $n = 1, \dots, m$, respectively. Recall that their initial conditions are u_0 and u_1 , and $u_{0,h}$ and $u_{1,h}$, respectively. Then, the following estimate holds true:

$$\begin{aligned}
& \max_{n=1}^N \left(\|e'_{h,\tau}\|_{L^\infty(I_n; L^2(\Omega))}^2 + |e_{h,\tau}|_{L^\infty(I_n; H^1(\Omega))}^2 \right) \lesssim \frac{1}{\mu_m} \left(\|u_0 - u_{0,h}\|_{1,\Omega}^2 \right. \\
& \quad \left. + \|(u_1 - u_{1,h})(\cdot)\|_{0,\Omega}^2 + \|u_1(\cdot) - (\mathcal{P}_{\mathbf{p}^t} u)'(0, \cdot)\|_{0,\Omega}^2 + \|(\mathcal{P}_{\mathbf{p}^t}(I - \Pi_h^\xi)u)'(0, \cdot)\|_{0,\Omega}^2 \right) \\
& \quad + \frac{t_m}{\mu_m^2} \sum_{n=1}^m \|(I - \mathcal{P}_{\mathbf{p}^t})\Delta_{\mathbf{x}}u\|_{L^2(I_n; L^2(\Omega))}^2 + \frac{t_m}{\mu_m^2} \sum_{n=1}^m \|(I - \Pi_h^\xi)u''\|_{L^2(I_n; L^2(\Omega))}^2 \\
& \quad + \|\rho'\|_{L^\infty(I_k; L^2(\Omega))}^2 + |\rho|_{L^\infty(I_k; H^1(\Omega))}^2 =: \sum_{j=1}^8 \mathfrak{J}_j^2.
\end{aligned} \tag{29}$$

Proof. Using the triangle inequality and the definition of $m = m(\xi_{h,\tau})$ in (6), we deduce

$$\begin{aligned}
& \max_{n=1}^N \left(\|e'_{h,\tau}\|_{L^\infty(I_n; L^2(\Omega))}^2 + |e_{h,\tau}|_{L^\infty(I_n; H^1(\Omega))}^2 \right) \\
& \leq 2 \left(\|\rho'\|_{L^\infty(I_k; L^2(\Omega))}^2 + |\rho|_{L^\infty(I_k; H^1(\Omega))}^2 \right) + 2 \left(\|\xi'_{h,\tau}\|_{L^\infty(I_k; L^2(\Omega))}^2 + |\xi_{h,\tau}|_{L^\infty(I_k; H^1(\Omega))}^2 \right) \\
& \leq 2 \left(\|\rho'\|_{L^\infty(I_k; L^2(\Omega))}^2 + |\rho|_{L^\infty(I_k; H^1(\Omega))}^2 \right) + 2 \left(\|\xi'_{h,\tau}\|_{L^\infty(I_m; L^2(\Omega))}^2 + |\xi_{h,\tau}|_{L^\infty(I_m; H^1(\Omega))}^2 \right).
\end{aligned}$$

The assertion follows using Lemma 2.6. \square

2.4 Convergence estimates

For $m = m(\xi_{h,\tau})$ and μ_m as in (6) and (7), we derive convergence estimates for method (3), which are explicit in the spatial mesh size, the time steps, and the polynomial degrees in space and time, with respect to the norm

$$\max_{n=1}^N \left(\|\cdot\|_{L^\infty(I_n; L^2(\Omega))}^2 + |\cdot|_{L^\infty(I_n; H^1(\Omega))}^2 \right). \tag{30}$$

To this aim, we give explicit bounds on the terms \mathfrak{J}_j , $j = 1, \dots, 8$, appearing on the right-hand side of (29), and collect the resulting estimates in the following result.

Theorem 2.8. Let u and $u_{h,\tau}$ be the solutions to (2) and (3), and $e_{h,\tau}$ be as in (23); u_0 , u_1 , and u be sufficiently smooth; $m = m(\xi_{h,\tau})$, μ_m , and k be as in (6), (7), and (28); α be the elliptic regularity parameter as in (22). Then, the following a priori estimate holds true:

$$\begin{aligned}
& \max_{n=1}^N \left(\|e'_{h,\tau}\|_{L^\infty(I_n; L^2(\Omega))}^2 + |e_{h,\tau}|_{L^\infty(I_n; H^1(\Omega))}^2 \right) \lesssim \frac{1}{\mu_m} \left(\frac{h^{2 \min(p_{\mathbf{x}}+1, s-\frac{1}{2})}}{p_{\mathbf{x}}^{2s-1}} \|u_0(\cdot)\|_{s+\frac{1}{2}, \Omega}^2 \right. \\
& \quad + \frac{h^{2 \min(p_{\mathbf{x}}+1, s-\frac{1}{2})}}{p_{\mathbf{x}}^{2s-1}} \|u_1(\cdot)\|_{s-\frac{1}{2}, \Omega}^2 + \left(\frac{\tau_1}{p_1^t} \right)^{2 \min(p_1^t, s)-1} \|u^{(s+1)}\|_{L^2(I_1, L^2(\Omega))}^2 \\
& \quad + \left(\frac{\tau_1}{p_1^t} \right) \frac{h^{2 \min(p_{\mathbf{x}}+1, s-1)-2(1-\alpha)}}{p_{\mathbf{x}}^{2s-2-2(1-\alpha)}} \|u''\|_{L^2(I_1; H^{s-1}(\Omega))}^2 + \frac{h^{2 \min(p_{\mathbf{x}}+1, s-\frac{1}{2})-2(1-\alpha)}}{p_{\mathbf{x}}^{2s-1-2(1-\alpha)}} \|u_1(\cdot)\|_{s-\frac{1}{2}, \Omega}^2 \Big) \\
& \quad + \frac{t_m}{\mu_m^2} \sum_{n=1}^m \left(\frac{\tau_n}{p_n^t} \right)^{2 \min(s-1, p_n^t+1)} \|\Delta_{\mathbf{x}}u^{(s-1)}\|_{L^2(I_n; L^2(\Omega))}^2 \\
& \quad + \frac{t_m}{\mu_m^2} \sum_{n=1}^m \frac{h^{2 \min(p_{\mathbf{x}}+1, s-1)-2(1-\alpha)}}{p_{\mathbf{x}}^{2s-2-2(1-\alpha)}} \|u''\|_{L^2(I_n; H^{s-1}(\Omega))}^2 \\
& \quad + \left(\frac{h^{2 \min(s, p_k^t)+2} \tau_k}{(p_k^t)^{2s+2}} \left(1 + \frac{h^{-2+2\alpha}}{p_{\mathbf{x}}^{-2+2\alpha}} \right) (\tau_k^{-1} \|u'\|_{L^\infty(I_k; H^{s+1}(\Omega))}^2 \right. \right. \\
& \quad \left. \left. + \frac{\tau_k^{2 \min(s, p_k^t)-1}}{(p_k^t)^{2 \min(s, p_k^t)-2}} \left(\|u^{(s+1)}\|_{L^2(I_k; L^2(\Omega))}^2 + \frac{h^{2\alpha-2}}{p_{\mathbf{x}}^{2\alpha}} h^2 \|u^{(s+1)}\|_{L^2(I_k; H^1(\Omega))}^2 \right) \right) \right) \\
& \quad + \left(\frac{h^{2 \min(p_{\mathbf{x}}, s)} \tau_k^{-1}}{p_{\mathbf{x}}^{2s}} (\tau_k \|u\|_{L^\infty(I_k; H^{s+1}(\Omega))})^2 + \left(\frac{\tau_k}{p_k^t} \right)^{2 \min(s, p_k^t+1)-1} \|u^{(s)}\|_{L^2(I_k, H^1(\Omega))}^2 \right).
\end{aligned}$$

Proof. Estimates on \mathfrak{J}_1 . We recall that $u_{0,h}$ is the elliptic projection of u_0 as in (21) onto V_h . Standard polynomial approximation estimates imply

$$\mu_m \mathfrak{J}_1^2 = \|(u_0 - u_{0,h})(\cdot)\|_{1,\Omega}^2 \lesssim \frac{h^{2 \min(p_{\mathbf{x}}+1, s-\frac{1}{2})}}{p_{\mathbf{x}}^{2s-1}} \|u_0(\cdot)\|_{s+\frac{1}{2},\Omega}^2. \quad (31)$$

Estimates on \mathfrak{J}_2 . We recall that $u_{1,h}$ is the L^2 projection of u_1 onto V_h . Standard polynomial approximation estimates imply

$$\mu_m \mathfrak{J}_2^2 = \|(u_1 - u_{1,h})(\cdot)\|_{0,\Omega}^2 \lesssim \frac{h^{2 \min(p_{\mathbf{x}}+1, s-\frac{1}{2})}}{p_{\mathbf{x}}^{2s-1}} \|u_1(\cdot)\|_{s-\frac{1}{2},\Omega}^2. \quad (32)$$

Estimates on \mathfrak{J}_3 . Using identity (16) and the approximation estimate (14d), we infer

$$\mu_m \mathfrak{J}_3^2 = \|(u - \mathcal{P}_{\mathbf{p}^t} u)'(0, \cdot)\|_{0,\Omega}^2 = \|(u' - \tilde{\mathcal{P}}_{\mathbf{p}^t-1} u')(0, \cdot)\|_{0,\Omega}^2 \lesssim \left(\frac{\tau_1}{p_1^t}\right)^{2 \min(p_1^t, s)-1} \|u^{(s+1)}\|_{L^2(I_1, L^2(\Omega))}^2. \quad (33)$$

Estimates on \mathfrak{J}_4 . Using the triangle inequality, we have

$$\mu_m \mathfrak{J}_4^2 = \|(\mathcal{P}_{\mathbf{p}^t}(I - \Pi_h^\varepsilon)u)'(0, \cdot)\|_{0,\Omega}^2 \leq 2\|((I - \mathcal{P}_{\mathbf{p}^t})(I - \Pi_h^\varepsilon)u)'(0, \cdot)\|_{0,\Omega}^2 + 2\|(I - \Pi_h^\varepsilon)u'(0, \cdot)\|_{0,\Omega}^2 =: \mathfrak{J}_{4,1}^2 + \mathfrak{J}_{4,2}^2.$$

Following the proof of the bound on \mathfrak{J}_3^2 we get the following bound on $\mathfrak{J}_{4,1}^2$:

$$\begin{aligned} \mathfrak{J}_{4,1}^2 &\stackrel{(14d)}{\lesssim} \left(\frac{\tau_1}{p_1^t}\right) \|(I - \Pi_h^\varepsilon)u''\|_{L^2(I_1; L^2(\Omega))}^2 \\ &\lesssim \left(\frac{\tau_1}{p_1^t}\right) \inf_{q_{p_{\mathbf{x}}} \in L^2(I_1; V_h)} \left(\|u'' - q_{p_{\mathbf{x}}}\|_{L^2(I_1; L^2(\Omega))}^2 + \|\Pi_h^\varepsilon(u'' - q_{p_{\mathbf{x}}})\|_{L^2(I_1; L^2(\Omega))}^2 \right) \\ &\stackrel{(22)}{\lesssim} \left(\frac{\tau_1}{p_1^t}\right) \inf_{q_{p_{\mathbf{x}}} \in L^2(I_1; V_h)} \left(\|u'' - q_{p_{\mathbf{x}}}\|_{L^2(I_1; L^2(\Omega))}^2 + \frac{h^{2\alpha}}{p_{\mathbf{x}}^{2\alpha}} \|u'' - q_{p_{\mathbf{x}}}\|_{L^2(I_1; H^1(\Omega))}^2 \right) \\ &\lesssim \left(\frac{\tau_1}{p_1^t}\right) \left(\frac{h^{2 \min(p_{\mathbf{x}}+1, s-1)}}{p_{\mathbf{x}}^{2(s-1)}} \|u''\|_{L^2(I_1; H^{s-1}(\Omega))}^2 + \frac{h^{2\alpha}}{p_{\mathbf{x}}^{2\alpha}} \frac{h^{2 \min(p_{\mathbf{x}}+1, s-1)-2}}{p_{\mathbf{x}}^{2(s-1)-2}} \|u''\|_{L^2(I_1; H^{s-1}(\Omega))}^2 \right) \\ &\lesssim \left(\frac{\tau_1}{p_1^t}\right) \frac{h^{2 \min(p_{\mathbf{x}}+1, s-1)-2(1-\alpha)}}{p_{\mathbf{x}}^{2s-2-2(1-\alpha)}} \|u''\|_{L^2(I_1; H^{s-1}(\Omega))}^2. \end{aligned}$$

As for the term $\mathfrak{J}_{4,2}$, we proceed similarly:

$$\begin{aligned} \mathfrak{J}_{4,2}^2 &= \|(I - \Pi_h^\varepsilon)u_1(\cdot)\|_{0,\Omega}^2 \lesssim \inf_{q_{p_{\mathbf{x}}} \in L^2(I_1; V_h)} \left(\|(u_1 - q_{p_{\mathbf{x}}})(\cdot)\|_{0,\Omega}^2 + \|\Pi_h^\varepsilon(u_1 - q_{p_{\mathbf{x}}})(\cdot)\|_{0,\Omega}^2 \right) \\ &\stackrel{(22)}{\lesssim} \left(\frac{h^{2 \min(p_{\mathbf{x}}+1, s-\frac{1}{2})}}{p_{\mathbf{x}}^{2s-1}} \|u_1(\cdot)\|_{s-\frac{1}{2},\Omega}^2 + \frac{h^{2\alpha}}{p_{\mathbf{x}}^{2\alpha}} \frac{h^{2 \min(p_{\mathbf{x}}+1, s-\frac{1}{2})-2}}{p_{\mathbf{x}}^{2s-3}} \|u_1(\cdot)\|_{s-\frac{1}{2},\Omega}^2 \right) \\ &\lesssim \frac{h^{2 \min(p_{\mathbf{x}}+1, s-\frac{1}{2})-2(1-\alpha)}}{p_{\mathbf{x}}^{2s-1-2(1-\alpha)}} \|u_1(\cdot)\|_{s-\frac{1}{2},\Omega}^2. \end{aligned}$$

Collecting the two displays above gives

$$\mu_m \mathfrak{J}_4^2 \lesssim \left(\frac{\tau_1}{p_1^t}\right) \frac{h^{2 \min(p_{\mathbf{x}}+1, s-1)-2(1-\alpha)}}{p_{\mathbf{x}}^{2s-2-2(1-\alpha)}} \|u''\|_{L^2(I_1; H^{s-1}(\Omega))}^2 + \frac{h^{2 \min(p_{\mathbf{x}}+1, s-\frac{1}{2})-2(1-\alpha)}}{p_{\mathbf{x}}^{2s-1-2(1-\alpha)}} \|u_1(\cdot)\|_{s-\frac{1}{2},\Omega}^2. \quad (34)$$

Estimates on \mathfrak{J}_5 . Using Proposition 2.5, we get

$$\mathfrak{J}_5^2 = \frac{t_m}{\mu_m^2} \sum_{n=1}^m \|(I - \mathcal{P}_{\mathbf{p}^t})\Delta_{\mathbf{x}} u\|_{L^2(I_n; L^2(\Omega))}^2 \lesssim \frac{t_m}{\mu_m^2} \sum_{n=1}^m \left(\frac{\tau_n}{p_n^t}\right)^{2 \min(s-1, p_n^t+1)} \|\Delta_{\mathbf{x}} u^{(s-1)}\|_{L^2(I_n; L^2(\Omega))}^2. \quad (35)$$

Estimates on \mathfrak{J}_6 . Using (22) and the fact that Π_h^ε preserves spatial polynomials of degree $p_{\mathbf{x}}$, we get

$$\begin{aligned}
\mathfrak{J}_6^2 &= \frac{t_m}{\mu_m^2} \sum_{n=1}^m \left\| (I - \Pi_h^\varepsilon) u'' \right\|_{L^2(I_n; L^2(\Omega))}^2 \\
&\lesssim \frac{t_m}{\mu_m^2} \sum_{n=1}^m \inf_{q_{p_{\mathbf{x}}} \in L^2(I_n; V_h)} \left(\|u'' - q_{p_{\mathbf{x}}}\|_{L^2(I_n; L^2(\Omega))}^2 + \left\| \Pi_h^\varepsilon(u'' - q_{p_{\mathbf{x}}}) \right\|_{L^2(I_n; L^2(\Omega))}^2 \right) \\
&\lesssim \frac{t_m}{\mu_m^2} \sum_{n=1}^m \inf_{q_{p_{\mathbf{x}}} \in L^2(I_n; V_h)} \left(\|u'' - q_{p_{\mathbf{x}}}\|_{L^2(I_n; L^2(\Omega))}^2 + \frac{h^{2\alpha}}{p_{\mathbf{x}}^{2\alpha}} \|u'' - q_{p_{\mathbf{x}}}\|_{L^2(I_n; H^1(\Omega))}^2 \right) \\
&\lesssim \frac{t_m}{\mu_m^2} \sum_{n=1}^m \left(\frac{h^{2 \min(p_{\mathbf{x}}+1, s-1)}}{p_{\mathbf{x}}^{2(s-1)}} \|u''\|_{L^2(I_n; H^{s-1}(\Omega))}^2 + \frac{h^{2\alpha}}{p_{\mathbf{x}}^{2\alpha}} \frac{h^{2 \min(p_{\mathbf{x}}+1, s-1)-2}}{p_{\mathbf{x}}^{2(s-1)-2}} \|u''\|_{L^2(I_n; H^{s-1}(\Omega))}^2 \right) \\
&\lesssim \frac{t_m}{\mu_m^2} \sum_{n=1}^m \frac{h^{2 \min(p_{\mathbf{x}}+1, s-1)-2(1-\alpha)}}{p_{\mathbf{x}}^{2s-2-2(1-\alpha)}} \|u''\|_{L^2(I_n; H^{s-1}(\Omega))}^2.
\end{aligned} \tag{36}$$

Estimates on \mathfrak{J}_7 . Using (22) and the definition of ρ in (23) gives

$$\begin{aligned}
\mathfrak{J}_7^2 &= \|\rho'\|_{L^\infty(I_k; L^2(\Omega))}^2 = \|(u - \mathcal{P}_{\mathbf{p}^t} \Pi_h^\varepsilon u)'\|_{L^\infty(I_k; L^2(\Omega))}^2 \\
&\lesssim \left(\|(u - \Pi_h^\varepsilon u)'\|_{L^\infty(I_k; L^2(\Omega))}^2 + \|(u - \mathcal{P}_{\mathbf{p}^t} u)'\|_{L^\infty(I_k; L^2(\Omega))}^2 + \left(\frac{h}{p_{\mathbf{x}}}\right)^{2\alpha} \|(u - \mathcal{P}_{\mathbf{p}^t} u)'\|_{L^\infty(I_k; H^1(\Omega))}^2 \right).
\end{aligned} \tag{37}$$

First, we focus on the first term on the right-hand side: for any $q_{p_{\mathbf{x}}}$ in $W^{1,\infty}(I_k; V_h)$,

$$\begin{aligned}
\|(u - \Pi_h^\varepsilon u)'\|_{L^\infty(I_k; L^2(\Omega))}^2 &\lesssim \|(u - q_{p_{\mathbf{x}}})'\|_{L^\infty(I_k; L^2(\Omega))}^2 + \|\Pi_h^\varepsilon(u - q_{p_{\mathbf{x}}})'\|_{L^\infty(I_k; L^2(\Omega))}^2 \\
&\stackrel{(22)}{\lesssim} \|(u - q_{p_{\mathbf{x}}})'\|_{L^\infty(I_k; L^2(\Omega))}^2 + \frac{h^{2\alpha}}{p_{\mathbf{x}}^{2\alpha}} \|(u - q_{p_{\mathbf{x}}})'\|_{L^\infty(I_k; H^1(\Omega))}^2 \\
&\lesssim \frac{h^{2 \min(s, p_k^t)+2} \tau_k}{(p_k^t)^{2s+2}} (\tau_k^{-1} \|u'\|_{L^\infty(I_k; H^{s+1}(\Omega))}^2) + \frac{h^{2 \min(s, p_k^t)+2\alpha} \tau_k}{(p_k^t)^{2s+2\alpha}} (\tau_k^{-1} \|u'\|_{L^\infty(I_k; H^{s+1}(\Omega))}^2) \\
&= \frac{h^{2 \min(s, p_k^t)+2} \tau_k}{(p_k^t)^{2s+2}} \left(1 + \frac{h^{-2+2\alpha}}{p_{\mathbf{x}}^{-2+2\alpha}} \right) (\tau_k^{-1} \|u'\|_{L^\infty(I_k; H^{s+1}(\Omega))}^2).
\end{aligned}$$

Next, we deal with the second term in the parenthesis on the right-hand side: using a polynomial inverse inequality as in [28, eq. (3.6.4)], for any $q_{p_k^t}$ in $\mathbb{P}_{p_k^t}(I_k; L^2(\Omega))$,

$$\begin{aligned}
\|(u - \mathcal{P}_{\mathbf{p}^t} u)'\|_{L^\infty(I_k; L^2(\Omega))}^2 &\lesssim \|(u - q_{p_k^t})'\|_{L^\infty(I_k; L^2(\Omega))}^2 + \left\| \mathcal{P}'_{\mathbf{p}^t}(u - q_{p_k^t}) \right\|_{L^\infty(I_k; L^2(\Omega))}^2 \\
&\lesssim \|(u - q_{p_k^t})'\|_{L^\infty(I_k; L^2(\Omega))}^2 + \frac{(p_k^t)^2}{\tau_k} \left\| \mathcal{P}'_{\mathbf{p}^t}(u - q_{p_k^t}) \right\|_{L^2(I_k; L^2(\Omega))}^2 \\
&\stackrel{(17c)}{\lesssim} \|(u - q_{p_k^t})'\|_{L^\infty(I_k; L^2(\Omega))}^2 + \frac{(p_k^t)^2}{\tau_k} \left\| (u - q_{p_k^t})' \right\|_{L^2(I_k; L^2(\Omega))}^2 + \tau_k \left\| (u - q_{p_k^t})'' \right\|_{L^2(I_k; L^2(\Omega))}^2.
\end{aligned}$$

Standard polynomial approximation properties in 1D give

$$\begin{aligned}
\left\| (u - q_{p_k^t})' \right\|_{L^\infty(I_k; L^2(\Omega))}^2 &\lesssim \frac{\tau_k^{2 \min(s, p_k^t)-1}}{(p_k^t)^{2 \min(s, p_k^t)-1}} \left\| u^{(s+1)} \right\|_{L^2(I_k; L^2(\Omega))}^2, \\
\frac{(p_k^t)^2}{\tau_k} \left\| (u - q_{p_k^t})' \right\|_{L^2(I_k; L^2(\Omega))}^2 &\lesssim \frac{\tau_k^{2 \min(s, p_k^t)-1}}{(p_k^t)^{2 \min(s, p_k^t)-2}} \left\| u^{(s+1)} \right\|_{L^2(I_k; L^2(\Omega))}^2,
\end{aligned}$$

and

$$\tau_k \left\| (u - q_{p_k^t})'' \right\|_{L^2(I_k; L^2(\Omega))}^2 \lesssim \frac{\tau_k^{2 \min(s, p_k^t)-1}}{(p_k^t)^{2 \min(s, p_k^t)-2}} \left\| u^{(s+1)} \right\|_{L^2(I_k; L^2(\Omega))}^2.$$

We collect the four displays above and get

$$\|(u - \mathcal{P}_{\mathbf{p}^t} u)'\|_{L^\infty(I_k; L^2(\Omega))}^2 \lesssim \frac{\tau_k^{2 \min(s, p_k^t)-1}}{(p_k^t)^{2 \min(s, p_k^t)-2}} \left\| u^{(s+1)} \right\|_{L^2(I_k; L^2(\Omega))}^2.$$

We proceed similarly for the third term on the right-hand side of (37):

$$\begin{aligned} \left(\frac{h}{p_{\mathbf{x}}}\right)^{2\alpha} |(u - \mathcal{P}_{\mathbf{p}^t} u)'|_{L^\infty(I_k; H^1(\Omega))}^2 &\lesssim \left(\frac{h}{p_{\mathbf{x}}}\right)^{2\alpha} \frac{\tau_k^{2 \min(s, p_k^t) - 1}}{(p_k^t)^{2 \min(s, p_k^t) - 2}} \|u^{(s+1)}\|_{L^2(I_k; H^1(\Omega))}^2 \\ &= \frac{h^{2\alpha-2}}{p_{\mathbf{x}}^{2\alpha}} \frac{\tau_k^{2 \min(s, p_k^t) - 1}}{(p_k^t)^{2 \min(s, p_k^t) - 2}} h^2 \|u^{(s+1)}\|_{L^2(I_k; H^1(\Omega))}^2. \end{aligned}$$

Recalling (7), we arrive at

$$\begin{aligned} \mathfrak{J}_7^2 &\lesssim \left[\frac{h^{2 \min(s, p_k^t) + 2} \tau_k}{(p_k^t)^{2s+2}} \left(1 + \frac{h^{-2+2\alpha}}{p_{\mathbf{x}}^{-2+2\alpha}}\right) \tau_k^{-1} \|u\|_{W^{1,\infty}(I_k; H^{s+1}(\Omega))}^2 \right. \\ &\quad \left. + \frac{\tau_k^{2 \min(s, p_k^t) - 1}}{(p_k^t)^{2 \min(s, p_k^t) - 2}} \left(\|u^{(s+1)}\|_{L^2(I_k; L^2(\Omega))}^2 + \frac{h^{2\alpha-2}}{p_{\mathbf{x}}^{2\alpha}} h^2 \|u^{(s+1)}\|_{L^2(I_k; H^1(\Omega))}^2 \right) \right]. \end{aligned} \quad (38)$$

Estimates on \mathfrak{J}_8 . Using the stability of the elliptic projector $\Pi_h^\mathcal{E}$ in $H^1(\Omega)$ and the 1D Sobolev embedding in [19, eq. (1.3)], we write

$$\begin{aligned} \mathfrak{J}_8^2 &= |\rho|_{L^\infty(I_k; H^1(\Omega))}^2 = |u - \mathcal{P}_{\mathbf{p}^t} \Pi_h^\mathcal{E} u|_{L^\infty(I_k; H^1(\Omega))}^2 \\ &\lesssim \left(|u - \Pi_h^\mathcal{E} u|_{L^\infty(I_k; H^1(\Omega))}^2 + |\Pi_h^\mathcal{E} (\text{Id} - \mathcal{P}_{\mathbf{p}^t}) u|_{L^\infty(I_k; H^1(\Omega))}^2 \right) \\ &\leq \left(|u - \Pi_h^\mathcal{E} u|_{L^\infty(I_k; H^1(\Omega))}^2 + |u - \mathcal{P}_{\mathbf{p}^t} u|_{L^\infty(I_k; H^1(\Omega))}^2 \right) \\ &\leq \left(|u - \Pi_h^\mathcal{E} u|_{L^\infty(I_k; H^1(\Omega))}^2 + |u - \mathcal{P}_{\mathbf{p}^t} u|_{L^2(I_k; H^1(\Omega))} |(u - \mathcal{P}_{\mathbf{p}^t} u)'|_{L^2(I_k; H^1(\Omega))} \right). \end{aligned} \quad (39)$$

As for the first term on the right-hand side, polynomial approximation properties (in space) give

$$|u - \Pi_h^\mathcal{E} u|_{L^\infty(I_k; H^1(\Omega))}^2 \leq |u - q_{p_{\mathbf{x}}} u|_{L^\infty(I_k; H^1(\Omega))}^2 \lesssim \frac{h^{2 \min(p_{\mathbf{x}}, s)} \tau_k^{-1}}{p_{\mathbf{x}}^{2s}} (\tau_k \|u\|_{L^\infty(I_k; H^{s+1}(\Omega))})^2.$$

As for the second term on the right-hand side of (39), we use (20) and (17b), and obtain

$$|u - \mathcal{P}_{\mathbf{p}^t} u|_{L^2(I_k; H^1(\Omega))} |(u - \mathcal{P}_{\mathbf{p}^t} u)'|_{L^2(I_k; H^1(\Omega))} \lesssim \left(\frac{\tau_k}{p_k^t}\right)^{2 \min(s, p_k^t + 1) - 1} \|u^{(s)}\|_{L^2(I_k; H^1(\Omega))}^2.$$

Combining the above displays and recalling (7) entail

$$\mathfrak{J}_8^2 \lesssim \left(\frac{h^{2 \min(p_{\mathbf{x}}, s)} \tau_k^{-1}}{p_{\mathbf{x}}^{2s}} (\tau_k \|u\|_{L^\infty(I_k; H^{s+1}(\Omega))})^2 + \left(\frac{\tau_k}{p_k^t}\right)^{2 \min(s, p_k^t + 1) - 1} \|u^{(s)}\|_{L^2(I_k; H^1(\Omega))}^2 \right). \quad (40)$$

Convergence estimates. The assertion follows combining (31), (32), (33), (34), (35), (36), (38), and (40). \square

The a priori estimates in Theorem 2.8 can be simplified under the assumption of elliptic regularity, i.e., assuming the parameter α in (22) to be 1 (this happens for instance if Ω is convex); under more regularity on the solution; requiring the isotropy of the spatial and time meshes; fixing the polynomial degrees in space and time (in what follows, p denotes the polynomial degree of the scheme).

Notably, the next result contains two estimates: the first one holds true for smooth exact solutions and the rate in the time step has optimal order p ; the second one holds true for exact solutions with finite total Sobolev regularity indices s smaller than or equal to p and is optimal in terms of the Sobolev scaling.

Corollary 2.9. *Let u and $u_{h,\tau}$ be the solutions to (2) and (3), and $e_{h,\tau}$ be as in (23). We assume that the initial conditions u_0 and u_1 in (2) and u are smooth in the sense s is larger than or equal to $p_n^t + 2$ on each I_n for all $n = 1, \dots, N$. Assume that the parameter α in (22) is equal*

to 1, i.e., elliptic regularity holds for the domain Ω . Let $\tau_n = \tau$ for all $n = 1, \dots, N$, and choose $h = O(\tau)$. For given p in \mathbb{N} , we further demand that $p_{\mathbf{x}} = p$ and $p_n^t = p$ for all $n = 1, \dots, N$. Recall that $m = m(\xi_{h,\tau})$ is defined in (6). Then, the following a priori error estimate is valid:

$$\max_{n=1}^N \left(\|e'_{h,\tau}\|_{L^\infty(I_n; L^2(\Omega))}^2 + |e_{h,\tau}|_{L^\infty(I_n; H^1(\Omega))}^2 \right)^{\frac{1}{2}} \lesssim \tau^p.$$

On the other hand, if s is smaller than or equal to p , then the following a priori error estimate is valid:

$$\max_{n=1}^N \left(\|e'_{h,\tau}\|_{L^\infty(I_n; L^2(\Omega))}^2 + |e_{h,\tau}|_{L^\infty(I_n; H^1(\Omega))}^2 \right)^{\frac{1}{2}} \lesssim \tau^{s-\frac{1}{2}}. \quad (41)$$

Proof. The proof boils down to using Theorem 2.8 and note that, for the estimates of the terms \mathfrak{J}_3 and \mathfrak{J}_7 in (29), we can use

$$\|u^{(p+1)}\|_{L^2(I_k; L^2(\Omega))} \leq \tau_k^{\frac{1}{2}} \|u^{(p+1)}\|_{L^\infty(I_k; L^2(\Omega))}.$$

All other \mathfrak{J}_j terms, $j = 1, 2, 4, 5, 6, 8$, already gave $\mathcal{O}(\tau^p)$ rates. \square

Remark 2. The error measure in (30) does not involve the jumps at the time nodes of the first time derivative. Some comments about this feature are in order.

- Lemma 2.6 involves an error measure containing the sum of jump terms up to m , $m = m(\xi_{h,\tau})$ as in (6); a modification of the proofs of Theorems 2.7 and 2.8 would lead to estimates for norms of the error containing jumps, but only up to the time node t_m and leading to an optimal convergence rate $\mathcal{O}(\tau^{p-\frac{1}{2}})$, which is however suboptimal for the first term in the error measure (30).
- Walkington's strategy [32] involves the use of a special test function, leading to errors measured in L^∞ -type norms in time. Such norms are different from L^2 -type norms in time, which are typically obtained by testing with "more standard" functions and typically come together with jumps at the time nodes.
- Optimal convergence for a norm involving jumps in our setting may be derived by using stability estimates obtained by using the test function $v_{h,\tau} = u'_{h,\tau}$. This is shown, for instance, in [32, eq. (4.1)] and would give an optimal rate of convergence for the jump terms.
- Even though we did not prove convergence rates for the jumps on the theoretical level, in Section 4 below, we shall investigate their practical behaviour.

3 A posteriori error estimates for the semi-discrete in time method

This section is concerned with introducing an error estimator for the semi-discrete in time method (4), and prove constant free, reliable a posteriori estimates for the error measured in the $L^\infty(L^2)$ norm under extra assumptions discussed in Section 3.1. In Section 3.2, we introduce a novel reconstruction operator and exhibit its approximation properties, which are instrumental in the a posteriori error estimates given in Section 3.3.

3.1 Data assumptions

Throughout, we make the following assumption, which is instrumental in deriving the a posteriori error estimates: for u and u_τ solutions to (2) to (4),

$$\Delta_{\mathbf{x}} u \in L^2(0, T; L^2(\Omega)) \quad (\text{which implies } \Delta_{\mathbf{x}} u_\tau \text{ belongs to } L^2(0, T; L^2(\Omega))). \quad (42)$$

As detailed in Proposition 3.1 below, sufficient conditions such that (42) holds true are:

$$\begin{aligned} & \text{the spatial domain } \Omega \text{ is convex;} \\ & u_0 \text{ and } u_1 \text{ belong to } H^2(\Omega) \cap H_0^1(\Omega) \text{ and } H_0^1(\Omega); \\ & \text{the right-hand side } f \text{ belongs to } H^1(0, T; L^2(\Omega)). \end{aligned} \quad (43)$$

Additional comments on this point are given in Remark 4 below. We now discuss why (43) implies (42).

Proposition 3.1. *Under assumption (43), then, the solution u to (2) is such that*

$$u \in L^\infty(0, T; H^2(\Omega)) \cap W^{1, \infty}(0, T; H_0^1(\Omega)) \cap W^{2, \infty}(0, T; L^2(\Omega)) \cap W^{3, \infty}(0, T; H^{-1}(\Omega)).$$

In particular, property (42) holds true.

Proof. A proof for domains with sufficiently smooth boundary can be found in [8, Theorem 5, Chapter 7.2] and is based on the Faedo-Galerkin technique, based on taking the limits of expansions into eigenfunctions. In turns, the regularity in space only depends on the regularity of the eigenfunctions of $\Delta_{\mathbf{x}}$, which on convex domains (43) is $H^2(\Omega)$; see, e.g., [15]. \square

3.2 A reconstruction operator

In view of deriving a posteriori error estimates in Section 3.3 below, we discuss here the properties of a generalization of the reconstruction operator introduced in [22] for parabolic problems; see also [17, 27] for a proof of p -approximation properties of the operator in [22].

Given a Banach space \mathcal{X} and v_τ in $C^0(0, T; \mathcal{X})$ with $v_\tau|_{I_n}$ in $\mathbb{P}_{p_n}^t(I_n; \mathcal{X})$ and $(v_\tau^-)'(0, \cdot)$ is an element in \mathcal{X} , let $\mathcal{R}_{\mathbf{p}^t+1}$ be piecewise defined for all $n = 1, \dots, N$ as

$$\begin{cases} ((\mathcal{R}_{\mathbf{p}^t+1}v_\tau)'' , q_{p_n^t-1}^t)_{L^2(I_n; \mathcal{X})} = (v_\tau'' , q_{p_n^t-1}^t)_{L^2(I_n; \mathcal{X})} + (\llbracket v_\tau' \rrbracket (t_{n-1}, \cdot), q_{p_n^t-1}^t(t_{n-1}, \cdot))_{\mathcal{X}} \\ \mathcal{R}_{\mathbf{p}^t+1}v_\tau(t_{n-1}, \cdot) = v_\tau(t_{n-1}, \cdot), \quad (\mathcal{R}_{\mathbf{p}^t+1}v_\tau)'(t_{n-1}, \cdot) = (v_\tau^-)'(t_{n-1}, \cdot) \text{ in } \mathcal{X} \quad \forall q_{p_n^t-1}^t \in \mathbb{P}_{p_n^t-1}^t(I_n; \mathcal{X}). \end{cases} \quad (44)$$

The corresponding operator for parabolic problems also allows for spatial mesh changes, see, e.g., [13], a topic that is still open for the wave equation in second order formulation and that we shall investigate in the future.

We begin by proving the following property of $\mathcal{R}_{\mathbf{p}^t+1}u_\tau$.

Proposition 3.2. *The function $\mathcal{R}_{\mathbf{p}^t+1}v_\tau$ is a C^1 piecewise polynomial in time reconstruction of a C^0 piecewise polynomial in time.*

Proof. The assertion follows using the last condition in (44) and proving that

$$(\mathcal{R}_{\mathbf{p}^t+1}v_\tau)'(t_n, \cdot) = (v_\tau^-)'(t_n, \cdot), \quad \mathcal{R}_{\mathbf{p}^t+1}v_\tau(t_n, \cdot) = v_\tau(t_n, \cdot) \quad \text{in } \mathcal{X}. \quad (45)$$

Proving the first identity in (45). Taking $q_{p_n^t-1}^t = c_X$ to be fixed in \mathcal{X} below and independent of time in the first line of (44) for all $n = 1, \dots, N$, and integrating by parts on both sides lead to

$$\begin{aligned} & ((\mathcal{R}_{\mathbf{p}^t+1}v_\tau)'(t_n, \cdot) - (\mathcal{R}_{\mathbf{p}^t+1}v_\tau)'(t_{n-1}, \cdot), c_X)_{\mathcal{X}} \\ & = ((v_\tau^-)'(t_n, \cdot) - \underbrace{(v_\tau^+)'(t_{n-1}, \cdot) + (v_\tau^+)'(t_{n-1}, \cdot) - (v_\tau^-)'(t_{n-1}, \cdot)}_{=0}, c_X)_{\mathcal{X}}. \end{aligned}$$

Using that $(\mathcal{R}_{\mathbf{p}^t+1}v_\tau)'(t_{n-1}, \cdot) = (v_\tau^-)'(t_{n-1}, \cdot)$ in \mathcal{X} and taking c_X equal to $(\mathcal{R}_{\mathbf{p}^t+1}v_\tau)'(t_n, \cdot) - (v_\tau^-)'(t_n, \cdot)$, we deduce the first identity in (45).

Proving the second identity in (45). We take $q_{p_n^t-1}^t = (t - t_{n-1})\tilde{c}_X$ with \tilde{c}_X to be fixed in \mathcal{X} below and independent of time in the first condition of (44) for all $n = 1, \dots, N$, integrate by parts, and get

$$((\mathcal{R}_{\mathbf{p}^t+1}v_\tau)'(t_n, \cdot), \tau_n \tilde{c}_X)_{\mathcal{X}} - ((\mathcal{R}_{\mathbf{p}^t+1}v_\tau)' , \tilde{c}_X)_{L^2(I_n; \mathcal{X})} = ((v_\tau^-)'(t_n, \cdot), \tau_n \tilde{c}_X)_{\mathcal{X}} - (v_\tau', \tilde{c}_X)_{L^2(I_n; \mathcal{X})}.$$

Using $(\mathcal{R}_{\mathbf{p}^t+1}v_\tau)'(t_n, \cdot) = (v_\tau^-)'(t_n, \cdot)$ entails

$$((\mathcal{R}_{\mathbf{p}^t+1}v_\tau)' , \tilde{c}_X)_{L^2(I_n; \mathcal{X})} = (v_\tau', \tilde{c}_X)_{L^2(I_n; \mathcal{X})}.$$

Integrating by parts again gives

$$(\mathcal{R}_{\mathbf{p}^t+1}v_\tau(t_n, \cdot) - \mathcal{R}_{\mathbf{p}^t+1}v_\tau(t_{n-1}, \cdot), \tilde{c}_X)_\mathcal{X} = (v_\tau(t_n, \cdot) - v_\tau(t_{n-1}, \cdot), \tilde{c}_X)_\mathcal{X}.$$

Using that $\mathcal{R}_{\mathbf{p}^t+1}v_\tau(t_{n-1}, \cdot) = v_\tau(t_{n-1}, \cdot)$ in \mathcal{X} , which is the first initial condition in (44), and choosing \tilde{c}_X equal to $\mathcal{R}_{\mathbf{p}^t+1}v_\tau(t_n, \cdot) - v_\tau(t_n, \cdot)$ yield the second identity in (45). \square

We have additional properties on the operator $\mathcal{R}_{\mathbf{p}^t+1}$.

Lemma 3.3. *Consider a Banach space \mathcal{X} and v_τ in $\mathcal{C}^0(0, T; \mathcal{X})$ with $v_\tau|_{I_n}$ in $\mathbb{P}_{p_n^t}(I_n; \mathcal{X})$. For all $n = 1, \dots, N$, the following identities hold true:*

$$\|(v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau)'\|_{L^2(I_n; \mathcal{X})}^2 = \tau_n c_1(p_n^t)^2 \|[[v'_\tau]](t_{n-1}, \cdot)\|_{\mathcal{X}}^2, \quad (46a)$$

$$\|(v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau)'\|_{L^\infty(I_n; \mathcal{X})}^2 = \|[[v'_\tau]](t_{n-1}, \cdot)\|_{\mathcal{X}}^2, \quad (46b)$$

$$\|v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau\|_{L^2(I_n; \mathcal{X})}^2 \leq \tau_n^3 c_2(p_n^t)^2 \|[[v'_\tau]](t_{n-1}, \cdot)\|_{\mathcal{X}}^2, \quad (46c)$$

where

$$c_1(p_n^t)^2 := \frac{p_n^t}{(2p_n^t - 1)(2p_n^t + 1)}, \quad c_1(p_n^t) \approx (p_n^t)^{-\frac{1}{2}}, \quad (47a)$$

$$c_2(p_n^t)^2 := \begin{cases} \frac{1}{4} \frac{p_n^t}{(p_n^t - 2)(p_n^t - 1)(2p_n^t - 1)(2p_n^t + 1)} & \text{if } p_n^t \geq 3, \\ \frac{2}{15\pi^2} & \text{if } p_n^t = 2, \end{cases} \quad c_2(p_n^t) \approx (p_n^t)^{-\frac{3}{2}}. \quad (47b)$$

Proof. Identity (46a) is proven in [27, Theorem 2]. Identity (46b) is essentially proven in [17, Lemma 1]. As for inequality (46c), we first observe that definition (44), an integration by parts, the first identity in (45), and the smoothness of $\mathcal{R}_{\mathbf{p}^t+1}v_\tau$ imply, for all $n = 1, \dots, n$,

$$\begin{aligned} & - ([[v'_\tau]](t_{n-1}, \cdot), q_{p_n^t-1}(t_{n-1}, \cdot))_\mathcal{X} \stackrel{(44)}{=} ((v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau)'', q_{p_n^t-1})_{L^2(I_n; \mathcal{X})} \\ & \stackrel{(\text{IBP})}{=} -((v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau)', q'_{p_n^t-1})_{L^2(I_n; \mathcal{X})} \\ & \quad + ((v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau)'(t_n, \cdot), q_{p_n^t-1}(t_n, \cdot))_\mathcal{X} - ((v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau)'(t_{n-1}, \cdot), q_{p_n^t-1}(t_{n-1}, \cdot))_\mathcal{X} \\ & \stackrel{(45)}{=} -((v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau)', q'_{p_n^t-1})_{L^2(I_n; \mathcal{X})} - ([[v'_\tau]](t_{n-1}, \cdot), q_{p_n^t-1}(t_{n-1}, \cdot))_\mathcal{X} \quad \forall q_{p_n^t-1} \in \mathbb{P}_{p_n^t-1}(I_n; \mathcal{X}). \end{aligned}$$

Proving (46c) for p_n^t larger than 2. A further integration by parts and the smoothness of v_τ and $\mathcal{R}_{\mathbf{p}^t+1}v_\tau$ at the endpoints of each I_n entail

$$(v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau, q''_{p_n^t-1})_{L^2(I_n; \mathcal{X})} = 0 \quad \forall q_{p_n^t-1} \in \mathbb{P}_{p_n^t-1}(I_n; \mathcal{X}). \quad (48)$$

For all t in I_n , we pick

$$q_{p_n^t-1}(t, \cdot) := \int_{t_{n-1}}^t \int_{t_{n-1}}^s \Pi_{\mathbf{p}^t-3}^0(v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau)(r, \cdot) dr ds \quad \text{in } \mathcal{X}$$

and deduce

$$(v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau, \Pi_{\mathbf{p}^t-3}^0(v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau))_{L^2(I_n; \mathcal{X})} = 0. \quad (49)$$

Using [28, Theorem 3.11] entails

$$\|(I - \Pi_{\mathbf{p}^t-3}^0)(v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau)\|_{L^2(I_n; \mathcal{X})} \leq \frac{1}{2} \frac{\tau_n}{\sqrt{(p_n^t - 2)(p_n^t - 1)}} \|(v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau)'\|_{L^2(I_n; \mathcal{X})}. \quad (50)$$

Approximation properties as in [28, Theorem 3.11] imply

$$\begin{aligned} & \|v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau\|_{L^2(I_n; \mathcal{X})}^2 = (v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau, v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau)_{L^2(I_n; \mathcal{X})} \\ & \stackrel{(49)}{=} (v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau, (I - \Pi_{\mathbf{p}^t-3}^0)(v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau))_{L^2(I_n; \mathcal{X})} \\ & \stackrel{(50)}{\leq} \frac{1}{2} \frac{\tau_n}{\sqrt{(p_n^t - 2)(p_n^t - 1)}} \|v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau\|_{L^2(I_n; \mathcal{X})} \|(v_\tau - \mathcal{R}_{\mathbf{p}^t+1}v_\tau)'\|_{L^2(I_n; \mathcal{X})}. \end{aligned}$$

The first line in inequality (46c) follows recalling identity (46a).

Proving (46c) for p_n^t equal to 2. The last display above modifies as follows: the 1D Poincaré inequality in time implies

$$\|v_\tau - \mathcal{R}_{\mathbf{p}^{t+1}} v_\tau\|_{L^2(I_n; \mathcal{X})}^2 \leq \frac{\tau_n^2}{\pi^2} \|(v_\tau - \mathcal{R}_{\mathbf{p}^{t+1}} v_\tau)'\|_{L^2(I_n; \mathcal{X})}^2.$$

The second line in inequality (46c) follows recalling identity (46a) with $p_n^t = 2$. □

3.3 An error estimator for the semi-discrete in time method

We construct a computable error estimator and prove constant free, reliable a posteriori error estimates. We introduce

$$v(t, \cdot) = \int_t^\xi (u - \mathcal{R}_{\mathbf{p}^{t+1}} u_\tau)(s, \cdot) ds \quad \text{in } \mathcal{X}, \quad (51)$$

with ξ in I_m , $1 \leq m \leq N$. The function v belongs to $\mathcal{C}^1(0, T; L^2(\Omega))$. In definition (44), we set

$$(\mathcal{R}_{\mathbf{p}^{t+1}} u_\tau)'(0, \cdot) = u_1(\cdot) \quad \text{in } L^2(\Omega). \quad (52)$$

We pick ξ in (51) such that

$$\|(u - \mathcal{R}_{\mathbf{p}^{t+1}} u_\tau)(\xi, \cdot)\|_{0, \Omega} = \|u - \mathcal{R}_{\mathbf{p}^{t+1}} u_\tau\|_{L^\infty(0, T; L^2(\Omega))} \quad \text{with } \xi \in I_m. \quad (53)$$

We state an auxiliary technical result.

Lemma 3.4. *Given v and ξ as in (51) and (53), the following estimates hold true:*

$$\inf_{q_{p_n^t} \in \mathbb{P}_{p_n^t}(I_n; L^2(\Omega))} \|v - q_{p_n^t}\|_{L^\infty(I_n; L^2(\Omega))} \leq \frac{\tau_n}{\pi} c_3(p_n^t) \|v'\|_{L^\infty(I_n; L^2(\Omega))}, \quad (54a)$$

$$\|v\|_{L^\infty(t_{m-1}, \xi; L^2(\Omega))} \leq \tau_m \|v'\|_{L^\infty(t_{m-1}, \xi; L^2(\Omega))}, \quad (54b)$$

where we have set, also for future convenience,

$$c_3(p_n^t) := \begin{cases} \sqrt{\pi} & \text{if } p_n^t = 0, 1, 2 \\ \frac{1}{p_n^t - 2} & \text{if } p_n^t \geq 3, \end{cases} \quad c_3(p_n^t) \approx (p_n^t)^{-1}. \quad (55)$$

Proof. Inequality (54a) in the case p_n^t larger than 2 is the Bochner version of [31, Theorem 7.2] up to a scaling argument. Inequality (54a) in the case p_n^t equal to 0, 1, and 2 follows from [19, eq. (1.3)] and the 1D Poincaré inequality in time. Inequality (54b) follows from the Cauchy-Schwarz inequality, the definition of v in (51), and the fact that $v(\xi, \cdot) = 0$ and $|\xi - t_{m-1}| \leq \tau_m$. □

Proposition 3.5. *Let assumption (43) hold true, which implies (42). Consider u and u_τ the solutions of (2) and (4), and $\mathcal{R}_{\mathbf{p}^{t+1}}$ the operator in (11). Given v , ξ and m , $c_2(p_n^t)$, and $c_3(p_n^t)$ as in (51), (53), (47b), and (55), we have*

$$\begin{aligned} & \|u - \mathcal{R}_{\mathbf{p}^{t+1}} u_\tau\|_{L^\infty(0, T; L^2(\Omega))} \\ & \leq 2 \left(\left(\sum_{n=1}^{m-1} \frac{\tau_n}{\pi} c_3(p_n^t - 1) \|f - \Pi_{\mathbf{p}^{t-1}}^0 f\|_{L^1(I_n; L^2(\Omega))} + \tau_m \|f - \Pi_{\mathbf{p}^{t-1}}^0 f\|_{L^1(I_m; L^2(\Omega))} \right) \right. \\ & \quad \left. + \left(\sum_{n=1}^{m-1} \frac{\tau_n}{\pi} c_3(p_n^t - 1) \|\Delta_{\mathbf{x}}(u_\tau - \Pi_{\mathbf{p}^{t-1}}^0 u_\tau)\|_{L^1(I_n; L^2(\Omega))} + \tau_m \|\Delta_{\mathbf{x}}(u_\tau - \Pi_{\mathbf{p}^{t-1}}^0 u_\tau)\|_{L^1(I_m; L^2(\Omega))} \right) \right) \\ & \quad \left. + \left(\sum_{n=1}^{m-1} \frac{\tau_n^3}{\pi} c_2(p_n^t) c_4(p_n^t - 3) \|[\Delta_{\mathbf{x}} u'_\tau](t_{n-1}, \cdot)\|_{L^2(\Omega)} + \tau_m^3 c_2(p_m^t) \|[\Delta_{\mathbf{x}} u'_\tau](t_{m-1}, \cdot)\|_{L^2(\Omega)} \right), \end{aligned} \quad (56)$$

with $c_4(p_n^t - 3)$ defined as

$$c_4(p_n^t - 3) := \begin{cases} \pi \frac{|t_m - t_{n-1}|}{\tau_n} & \text{if } p_n^t = 2 \\ c_3(p_n^t - 3) & \text{if } p_n^t \geq 3. \end{cases} \quad (57)$$

Proof. Let $\mathcal{R}_{\mathbf{p}^t+1}$ be the reconstruction operator in (44). Using (4), we have the identity

$$((\mathcal{R}_{\mathbf{p}^t+1}u_\tau)'' , v_\tau)_{L^2(I_n; L^2(\Omega))} - \langle \Delta_{\mathbf{x}}u_\tau, v_\tau \rangle = \langle f, v_\tau \rangle \quad \forall v_\tau \in \mathbb{P}_{p_n^t-1}^t(I_n; H_0^1(\Omega)) \quad \forall n = 1, \dots, N.$$

Recall the time semi-discrete right-hand side in (5). Due to (42), for all $n = 1, \dots, N$, we have the following identity in $L^2(I_n; L^2(\Omega))$ inside each time interval:

$$(\mathcal{R}_{\mathbf{p}^t+1}u_\tau)'' - \Pi_{\mathbf{p}^t-1}^0 \Delta_{\mathbf{x}}u_\tau = \Pi_{\mathbf{p}^t-1}^0 f. \quad (58)$$

Using $\Delta_{\mathbf{x}}\Pi_{\mathbf{p}^t-1}^0 u_\tau = \Pi_{\mathbf{p}^t-1}^0 \Delta_{\mathbf{x}}u_\tau$, and subtracting (58) to (1), we get the following identity in $L^2(I_n; L^2(\Omega))$:

$$(u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)'' - \Delta_{\mathbf{x}}(u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau) = (f - \Pi_{\mathbf{p}^t-1}^0 f) + \Delta_{\mathbf{x}}(\mathcal{R}_{\mathbf{p}^t+1}u_\tau - \Pi_{\mathbf{p}^t-1}^0 u_\tau). \quad (59)$$

Next, we derive an error equation testing (59) with a particular function mimicking that proposed in [3, eq. (3.7)], i.e., the function in (51). It is immediate to check

$$-v'(t, \cdot) = (u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)(t, \cdot) \quad (60)$$

and

$$v(\xi, \cdot) = 0. \quad (61)$$

Multiplying (59) with the function in (51), integrating in space, and integrating by parts, we arrive at the following identity in $L^2(0, T)$:

$$\begin{aligned} & ((u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)'' , v)_{0,\Omega} + a(u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau, v) \\ &= (f - \Pi_{\mathbf{p}^t-1}^0 f, v)_{0,\Omega} + (\Delta_{\mathbf{x}}(\mathcal{R}_{\mathbf{p}^t+1}u_\tau - \Pi_{\mathbf{p}^t-1}^0 u_\tau), v)_{0,\Omega}. \end{aligned}$$

Using (60), we readily deduce the following identity in $L^2(0, T)$:

$$((u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)'' , v)_{0,\Omega} - a(v', v) = (f - \Pi_{\mathbf{p}^t-1}^0 f, v)_{0,\Omega} + (\Delta_{\mathbf{x}}(\mathcal{R}_{\mathbf{p}^t+1}u_\tau - \Pi_{\mathbf{p}^t-1}^0 u_\tau), v)_{0,\Omega}.$$

The following trivial but crucial identity holds true: given sufficiently smooth functions \aleph and \beth , we have

$$(\aleph'' , \beth)_{0,\Omega} = (\aleph', \beth)'_{0,\Omega} - (\aleph', \beth)'_{0,\Omega} \quad \text{in } L^2(0, T).$$

This identity and (60) imply

$$\begin{aligned} & ((u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)' , v)'_{0,\Omega} + ((u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)', u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)_{0,\Omega} - a(v', v) \\ &= (f - \Pi_{\mathbf{p}^t-1}^0 f, v)_{0,\Omega} + (\Delta_{\mathbf{x}}(\mathcal{R}_{\mathbf{p}^t+1}u_\tau - \Pi_{\mathbf{p}^t-1}^0 u_\tau), v)_{0,\Omega} \quad \text{in } L^2(0, T). \end{aligned}$$

Equivalently, we write

$$\begin{aligned} & ((u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)' , v)'_{0,\Omega} + \frac{1}{2}(\|u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau\|_{0,\Omega}^2)' - \frac{1}{2}a(v, v)' \\ &= (f - \Pi_{\mathbf{p}^t-1}^0 f, v)_{0,\Omega} + (\Delta_{\mathbf{x}}(\mathcal{R}_{\mathbf{p}^t+1}u_\tau - \Pi_{\mathbf{p}^t-1}^0 u_\tau), v)_{0,\Omega} \quad \text{in } L^2(0, T). \end{aligned}$$

We integrate in time over $(0, \xi)$, ξ as in the choice of the test function in (51). We arrive at

$$\begin{aligned} & \frac{1}{2}\|(u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)(\xi, \cdot)\|_{0,\Omega}^2 - \frac{1}{2}a(v(\xi, \cdot), v(\xi, \cdot)) = \frac{1}{2}\|(u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)(0, \cdot)\|_{0,\Omega}^2 - \frac{1}{2}a(v(0, \cdot), v(0, \cdot)) \\ & + \int_0^\xi ((f - \Pi_{\mathbf{p}^t-1}^0 f, v)_{0,\Omega} + (\Delta_{\mathbf{x}}(\mathcal{R}_{\mathbf{p}^t+1}u_\tau - \Pi_{\mathbf{p}^t-1}^0 u_\tau), v)_{0,\Omega} - ((u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)', v)'_{0,\Omega}) dt. \end{aligned}$$

Using (44) and (52), we have $(u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)'(0, \cdot) = 0$, whence we write

$$\begin{aligned} & - \int_0^\xi ((u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)', v)'_{0,\Omega} dt \\ &= -((u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)'(\xi, \cdot), v(\xi, \cdot))_{0,\Omega} + ((u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)'(0, \cdot), v(0, \cdot))_{0,\Omega} \\ &\stackrel{(61)}{=} ((u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)'(0, \cdot), v(0, \cdot))_{0,\Omega} = 0. \end{aligned}$$

Combining the two above displays again with (61), adding and subtracting u_τ in the last term on the right-hand side, and using that $(u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)(0, \cdot) = 0$ yield

$$\begin{aligned} & \frac{1}{2} \|(u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)(\xi, \cdot)\|_{0,\Omega}^2 + \frac{1}{2} a(v(0, \cdot), v(0, \cdot)) \\ &= \int_0^\xi ((f - \Pi_{\mathbf{p}^t-1}^0 f, v)_{0,\Omega} + (\Delta_{\mathbf{x}}(u_\tau - \Pi_{\mathbf{p}^t-1}^0 u_\tau), v)_{0,\Omega} + (\Delta_{\mathbf{x}}(\mathcal{R}_{\mathbf{p}^t+1}u_\tau - u_\tau), v)_{0,\Omega}) dt. \end{aligned}$$

Let $q_{p_n^t-1}$ and $q_{p_n^t-3}$ realize (54a) of degree $p_n^t - 1$ and $p_n^t - 3$ (with $q_{p_n^t-3} = 0$ for $p_n^t = 2$), respectively. Using the properties of $\Pi_{\mathbf{p}^t-1}^0$, the orthogonal property (48), and the choice of ξ in (53) (including the fact that ξ lies in the time interval I_m), we end up with

$$\begin{aligned} & \frac{1}{2} \|u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau\|_{L^\infty(0,T;L^2(\Omega))}^2 + \frac{1}{2} a(v(0, \cdot), v(0, \cdot)) \\ & \leq \sum_{n=1}^{m-1} \int_{I_n} (f - \Pi_{\mathbf{p}^t-1}^0 f, v - q_{p_n^t-1})_{0,\Omega} dt + \int_{t_{m-1}}^\xi (f - \Pi_{\mathbf{p}^t-1}^0 f, v)_{0,\Omega} dt \\ & + \sum_{n=1}^{m-1} \int_{I_n} (\Delta_{\mathbf{x}}(u_\tau - \Pi_{\mathbf{p}^t-1}^0 u_\tau), v - q_{p_n^t-1})_{0,\Omega} dt + \int_{t_{m-1}}^\xi (\Delta_{\mathbf{x}}(u_\tau - \Pi_{\mathbf{p}^t-1}^0 u_\tau), v)_{0,\Omega} dt \\ & + \sum_{n=1}^{m-1} \int_{I_n} (\Delta_{\mathbf{x}}(\mathcal{R}_{\mathbf{p}^t+1}u_\tau - u_\tau), v - q_{p_n^t-3})_{0,\Omega} dt + \int_{t_{m-1}}^\xi (\Delta_{\mathbf{x}}(\mathcal{R}_{\mathbf{p}^t+1}u_\tau - u_\tau), v)_{0,\Omega} dt = \sum_{i=1}^6 T_i. \end{aligned} \tag{62}$$

As for the the term T_1 , we use Hölder's inequality, inequality (54a) (with constant $c_3(p_n^t - 1)$), and the fact that $\|v'\|_{L^\infty(I_n;L^2(\Omega))}$ is smaller than $\|v'\|_{L^\infty(0,T;L^2(\Omega))}$, and end up with

$$\begin{aligned} T_1 & \leq \sum_{n=1}^{m-1} \|f - \Pi_{\mathbf{p}^t-1}^0 f\|_{L^1(I_n;L^2(\Omega))} \|v - q_{p_n^t-1}\|_{L^\infty(I_n;L^2(\Omega))} \\ & \leq \sum_{n=1}^{m-1} \frac{\tau_n}{\pi} c_3(p_n^t - 1) \|f - \Pi_{\mathbf{p}^t-1}^0 f\|_{L^1(I_n;L^2(\Omega))} \|v'\|_{L^\infty(I_n;L^2(\Omega))} \\ & \leq \left(\sum_{n=1}^{m-1} \frac{\tau_n}{\pi} c_3(p_n^t - 1) \|f - \Pi_{\mathbf{p}^t-1}^0 f\|_{L^1(I_n;L^2(\Omega))} \right) \|v'\|_{L^\infty(0,T;L^2(\Omega))}. \end{aligned} \tag{63}$$

Similarly, using (54b) and the fact that $\|v'\|_{L^\infty(t_{m-1},\xi;L^2(\Omega))}$ is smaller than $\|v'\|_{L^\infty(0,T;L^2(\Omega))}$, we also infer the following bound on the term T_2 :

$$\begin{aligned} T_2 & \leq \|f - \Pi_{\mathbf{p}^t-1}^0 f\|_{L^1(t_{m-1},\xi;L^2(\Omega))} \|v\|_{L^\infty(t_{m-1},\xi;L^2(\Omega))} \\ & \leq \tau_m \|f - \Pi_{\mathbf{p}^t-1}^0 f\|_{L^1(I_n;L^2(\Omega))} \|v'\|_{L^\infty(t_{m-1},\xi;L^2(\Omega))} \\ & \leq \tau_m \|f - \Pi_{\mathbf{p}^t-1}^0 f\|_{L^1(I_m;L^2(\Omega))} \|v'\|_{L^\infty(0,T;L^2(\Omega))}. \end{aligned} \tag{64}$$

Next, we bound the term T_3 using again (54a) with constant $c_3(p_n^t - 1)$:

$$\begin{aligned} T_3 & \leq \sum_{n=1}^{m-1} \|\Delta_{\mathbf{x}}(u_\tau - \Pi_{\mathbf{p}^t-1}^0 u_\tau)\|_{L^1(I_n;L^2(\Omega))} \|v - q_{p_n^t-1}\|_{L^\infty(I_n;L^2(\Omega))} \\ & \leq \sum_{n=1}^{m-1} \frac{\tau_n}{\pi} c_3(p_n^t - 1) \|\Delta_{\mathbf{x}}(u_\tau - \Pi_{\mathbf{p}^t-1}^0 u_\tau)\|_{L^1(I_n;L^2(\Omega))} \|v'\|_{L^\infty(I_n;L^2(\Omega))} \\ & \leq \left(\sum_{n=1}^{m-1} \frac{\tau_n}{\pi} c_3(p_n^t - 1) \|\Delta_{\mathbf{x}}(u_\tau - \Pi_{\mathbf{p}^t-1}^0 u_\tau)\|_{L^1(I_n;L^2(\Omega))} \right) \|v'\|_{L^\infty(0,T;L^2(\Omega))}. \end{aligned} \tag{65}$$

We bound the term T_4 based on (54b) and proceeding as in the bound of T_3 :

$$\begin{aligned} T_4 & \leq \|\Delta_{\mathbf{x}}(u_\tau - \Pi_{\mathbf{p}^t-1}^0 u_\tau)\|_{L^1(t_{m-1},\xi;L^2(\Omega))} \|v\|_{L^\infty(t_{m-1},\xi;L^2(\Omega))} \\ & \leq \tau_m \|\Delta_{\mathbf{x}}(u_\tau - \Pi_{\mathbf{p}^t-1}^0 u_\tau)\|_{L^1(t_{m-1},\xi;L^2(\Omega))} \|v'\|_{L^\infty(t_{m-1},\xi;L^2(\Omega))} \\ & = \tau_m \|\Delta_{\mathbf{x}}(u_\tau - \Pi_{\mathbf{p}^t-1}^0 u_\tau)\|_{L^1(I_m;L^2(\Omega))} \|v'\|_{L^\infty(0,T;L^2(\Omega))}. \end{aligned} \tag{66}$$

Using estimates (54a) with constant $c_3(p_n^t - 3)$ and (46c) with constant $c_2(p_n^t)$, we show an upper bound on the term T_5 for $p_n^t \geq 3$:

$$\begin{aligned}
T_5 &\leq \sum_{n=1}^{m-1} \tau_n^{\frac{1}{2}} \|\Delta_{\mathbf{x}}(\mathcal{R}_{\mathbf{p}^t+1} u_{\boldsymbol{\tau}} - u_{\boldsymbol{\tau}})\|_{L^2(I_n; L^2(\Omega))} \|v - q_{p_n^t-3}\|_{L^\infty(I_n; L^2(\Omega))} \\
&\leq \sum_{n=1}^{m-1} \frac{\tau_n^{\frac{3}{2}}}{\pi} c_3(p_n^t - 3) \|\Delta_{\mathbf{x}}(\mathcal{R}_{\mathbf{p}^t+1} u_{\boldsymbol{\tau}} - u_{\boldsymbol{\tau}})\|_{L^2(I_n; L^2(\Omega))} \|v'\|_{L^\infty(I_n; L^2(\Omega))} \\
&\leq \left(\sum_{n=1}^{m-1} \frac{\tau_n^3}{\pi} c_2(p_n^t) c_3(p_n^t - 3) \|\llbracket \Delta_{\mathbf{x}} u'_{\boldsymbol{\tau}} \rrbracket(t_{n-1}, \cdot)\|_{L^2(\Omega)} \right) \|v'\|_{L^\infty(0, T; L^2(\Omega))}.
\end{aligned} \tag{67}$$

For $p_n^t = 2$, using the definition of v and ξ in (51) and (53), and the fact that $|\xi - t_{n-1}|$ is larger than $|\xi - t_n|$, we have

$$\|v\|_{L^\infty(I_n; L^2(\Omega))} \leq |\xi - t_{n-1}| \|v'\|_{L^\infty(I_n; L^2(\Omega))} \leq |t_m - t_{n-1}| \|v'\|_{L^\infty(I_n; L^2(\Omega))}.$$

Using the definition of $c_4(p_n^t - 3)$ in (57), we have the following bound on the term T_5 for all p_n^t larger than or equal to 2:

$$\begin{aligned}
T_5 &\leq \sum_{n=1}^{m-1} \tau_n^{\frac{1}{2}} \|\Delta_{\mathbf{x}}(\mathcal{R}_{\mathbf{p}^t+1} u_{\boldsymbol{\tau}} - u_{\boldsymbol{\tau}})\|_{L^2(I_n; L^2(\Omega))} \|v\|_{L^\infty(I_n; L^2(\Omega))} \\
&\leq \sum_{n=1}^{m-1} |t_m - t_{n-1}| \tau_n^{\frac{1}{2}} \|\Delta_{\mathbf{x}}(\mathcal{R}_{\mathbf{p}^t+1} u_{\boldsymbol{\tau}} - u_{\boldsymbol{\tau}})\|_{L^2(I_n; L^2(\Omega))} \|v'\|_{L^\infty(I_n; L^2(\Omega))} \\
&\leq \left(\sum_{n=1}^{m-1} |t_m - t_{n-1}| \tau_n^2 c_2(p_n^t) \|\llbracket \Delta_{\mathbf{x}} u'_{\boldsymbol{\tau}} \rrbracket(t_{n-1}, \cdot)\|_{L^2(\Omega)} \right) \|v'\|_{L^\infty(0, T; L^2(\Omega))}.
\end{aligned}$$

We bound the term T_6 using estimates (46c) and (54b):

$$\begin{aligned}
T_6 &\leq \tau_m^{\frac{1}{2}} \|\Delta_{\mathbf{x}}(\mathcal{R}_{\mathbf{p}^t+1} - u_{\boldsymbol{\tau}})\|_{L^2(t_{m-1}, \xi; L^2(\Omega))} \|v\|_{L^\infty(t_{m-1}, \xi; L^2(\Omega))} \\
&\leq \tau_m^{\frac{3}{2}} \|\Delta_{\mathbf{x}}(\mathcal{R}_{\mathbf{p}^t+1} - u_{\boldsymbol{\tau}})\|_{L^2(I_m; L^2(\Omega))} \|v'\|_{L^\infty(0, T; L^2(\Omega))} \\
&\leq \tau_m^3 c_2(p_m^t) \|\llbracket \Delta_{\mathbf{x}} u'_{\boldsymbol{\tau}} \rrbracket(t_{m-1}, \cdot)\|_{L^2(\Omega)} \|v'\|_{L^\infty(0, T; L^2(\Omega))}.
\end{aligned} \tag{68}$$

Finally, using the fact that $\|v'\|_{L^\infty(0, T; L^2(\Omega))}$ is equal to $\|u - \mathcal{R}_{\mathbf{p}^t+1} u_{\boldsymbol{\tau}}\|_{L^\infty(0, T; L^2(\Omega))}$, the assertion follows inserting (63), (64), (65), (66), (67), and (68) in (62). \square

The error measure on the left-hand side of (56) contains the reconstruction operator $\mathcal{R}_{\mathbf{p}^t+1}$, which we do not want to compute in practice. For this reason, we elaborate further.

Recall that m is as in (53). Consider the time steps and time polynomial degree distributions $\boldsymbol{\tau}$ and \mathbf{p}^t as discussed in Section 1; $c_1(p_n^t)$, $c_2(p_n^t)$, $c_3(p_n^t)$, and $c_4(p_n^t)$ as defined in (47a), (47b), (55), and (57). Define

$$\eta_1 := \max_{n=1, \dots, N} \tau_n (c_1(p_n^t) c_2(p_n^t))^{\frac{1}{2}} \|\llbracket u'_{\boldsymbol{\tau}} \rrbracket(t_{n-1}, \cdot)\|_{0, \Omega}.$$

Given m as in (53), for all $n = 1, \dots, m-1$, we also define

$$\eta_{2, n} := \begin{cases} \frac{2}{\pi} \left(\tau_n c_3(p_n^t - 1) \|\Delta_{\mathbf{x}}(u_{\boldsymbol{\tau}} - \Pi_{\mathbf{p}^t-1}^0 u_{\boldsymbol{\tau}})\|_{L^1(I_n; L^2(\Omega))} \right. \\ \quad \left. + \tau_n^3 c_2(p_n^t) c_4(p_n^t - 3) \|\llbracket \Delta_{\mathbf{x}} u'_{\boldsymbol{\tau}} \rrbracket(t_{n-1}, \cdot)\|_{L^2(\Omega)} \right) & \text{if } n = 1, \dots, m-1 \\ 2 \left(\tau_m \|\Delta_{\mathbf{x}}(u_{\boldsymbol{\tau}} - \Pi_{\mathbf{p}^t-1}^0 u_{\boldsymbol{\tau}})\|_{L^1(I_n; L^2(\Omega))} + c_2(p_n^t) \tau_m^3 \|\llbracket \Delta_{\mathbf{x}} u'_{\boldsymbol{\tau}} \rrbracket(t_{n-1}, \cdot)\|_{L^2(\Omega)} \right) & \text{if } n = m, \end{cases}$$

and the data oscillation terms

$$\text{osc}_n(f) := \begin{cases} \frac{2\tau_n}{\pi} c_3(p_n^t - 1) \|f - \Pi_{\mathbf{p}^t-1}^0 f\|_{L^1(I_n; L^2(\Omega))} & \text{if } n = 1, \dots, m-1 \\ 2\tau_m \|f - \Pi_{\mathbf{p}^t-1}^0 f\|_{L^1(I_n; L^2(\Omega))} & \text{if } n = m. \end{cases}$$

With this at hand, for m as in (53), we introduce

$$\eta_2 := \sum_{n=1}^m \eta_{2,n}, \quad \text{osc}(f) := \sum_{n=1}^m \text{osc}_n(f). \quad (69)$$

We are now in a position to derive a constant free, reliable a posteriori upper bound for the $L^\infty(L^2)$ error with respect to the error estimator

$$\eta := \eta_1 + \eta_2. \quad (70)$$

Proposition 3.6. *Let u and u_τ be the solutions to (2) and (4); η as in (70), and $\text{osc}(f)$ as in (69). Then, the following bound holds true:*

$$\|u - u_\tau\|_{L^\infty(0,T;L^2(\Omega))} \leq \eta + \text{osc}(f). \quad (71)$$

Proof. The triangle inequality implies

$$\|u - u_\tau\|_{L^\infty(0,T;L^2(\Omega))} \leq \|u - \mathcal{R}_{\mathbf{p}^t+1}u_\tau\|_{L^\infty(0,T;L^2(\Omega))} + \|u_\tau - \mathcal{R}_{\mathbf{p}^t+1}u_\tau\|_{L^\infty(0,T;L^2(\Omega))}.$$

Next, we pick $\tilde{m} = 1, \dots, N$ and $\tilde{\xi}$ such that

$$\left\| (u_\tau - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)(\tilde{\xi}) \Big|_{0,\Omega} \right\| = \|u_\tau - \mathcal{R}_{\mathbf{p}^t+1}u_\tau\|_{L^\infty(0,T;L^2(\Omega))} \quad \text{with } \tilde{\xi} \in I_{\tilde{m}}.$$

An upper bound for the first term in the maximum is a consequence of (56). As for the second term, we use the (1D in time) Sobolev embedding $H^1 \hookrightarrow L^\infty$, see, e.g., [19, eq. (1.3)], and observe

$$\begin{aligned} \|u_\tau - \mathcal{R}_{\mathbf{p}^t+1}u_\tau\|_{L^\infty(0,T;L^2(\Omega))}^2 &\leq \|u_\tau - \mathcal{R}_{\mathbf{p}^t+1}u_\tau\|_{L^2(I_{\tilde{m}};L^2(\Omega))} \|(u_\tau - \mathcal{R}_{\mathbf{p}^t+1}u_\tau)'\|_{L^2(I_{\tilde{m}};L^2(\Omega))} \\ &\stackrel{(46a),(46c)}{\leq} \tau_{\tilde{m}}^2 c_1(p_{\tilde{m}}^t) c_2(p_{\tilde{m}}^t) \|\llbracket u'_\tau \rrbracket(t_{\tilde{m}-1}, \cdot)\|_{0,\Omega}^2. \end{aligned}$$

As \tilde{m} and $\tilde{\xi}$ cannot be determined a priori, we further elaborate the above bound as follows:

$$\|u_\tau - \mathcal{R}_{\mathbf{p}^t+1}u_\tau\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq \max_{n=1,\dots,N} \tau_n^2 c_1(p_n^t) c_2(p_n^t) \|\llbracket u'_\tau \rrbracket(t_{n-1}, \cdot)\|_{0,\Omega}^2.$$

The assertion follows combining the above displays. \square

Remark 3. For the case $p_n^t = 2$, the parameter $c_4(p_n^t - 3)$ in (57) scales as τ_n^{-1} , which reduces by one order the convergence rate of the term T_6 compared to the case $p_n^t \geq 3$. On the other hand, the optimal convergence order for the term T_6 for $p_n^t \geq 3$ is $\mathcal{O}(\tau^{p_n^t+2})$, which is one order higher compared to the error $\|u - u_\tau\|_{L^\infty(0,T;L^2(\Omega))}$. Therefore, for $p_n^t = 2$, the term T_6 converges with the same rate as that of the error $\|u - u_\tau\|_{L^\infty(0,T;L^2(\Omega))}$.

Remark 4. Property (42) is crucial in deriving bounds (71). In principle, we may also assume the weaker requirement that $\Delta_{\mathbf{x}}u_\tau$ belongs to $L^2(0,T;H^{-1}(\Omega))$ but end up with an error estimator involving a negative norm, which is harder to realize in practice than the $L^2(0,T;L^2(\Omega))$ norm.

4 Numerical experiments

We assess numerically the a priori and a posteriori error estimates proven in Theorem 2.8 and Corollary 2.9, and Proposition 3.6.

Spatial and time discretization. In what follows, we always consider the spatial domain $\Omega = (-1, 1)^2$ partitioned into uniform tensor-product meshes; we use tensor Lagrangian (equidistributed) nodal basis functions. As for the time discretization, we fix $T = 1$ and take the Lagrangian basis functions in time for simplicity.

Error measures. We are interested in investigating different error measures for the a priori and a posteriori error estimates.

Given u and $u_{h,\tau}$ the solutions to (2) and (3), and $e_{h,\tau} := u - u_{h,\tau}$, we consider the two error measures appearing on the left-hand side of (30) separately:

$$\max_{n=1}^N \|e'_{h,\tau}\|_{L^\infty(I_n;L^2(\Omega))}, \quad \max_{n=1}^N |e_{h,\tau}|_{L^\infty(I_n;H^1(\Omega))}. \quad (72)$$

We also consider the errors

$$|e_{h,\tau}|_{L^2(0,T;H^1(\Omega))}, \quad \|e'_{h,\tau}\|_{L^2(0,T;L^2(\Omega))}, \quad \|e_{h,\tau}\|_{L^\infty(0,T;L^2(\Omega))}, \quad (73)$$

and the jump error

$$\left(\sum_{n=1}^N \|[[e'_{h,\tau}]](t_{n-1}, \cdot)\|_{0,\Omega}^2 \right)^{\frac{1}{2}}. \quad (74)$$

The L^2 -type norms in time are computed with Gauß-Legendre quadrature formulas of order $2p_n^t + 3$; the L^∞ -type norms in time are evaluated at $2p_n^t + 3$ equally distributed nodes in time for each time interval; the spatial norms are computed by using tensor product Gauß-Legendre quadrature formulas of order $2p_x + 3$.

Imposing Dirichlet boundary conditions. We consider homogeneous Dirichlet boundary conditions, which are imposed strongly at the boundary degrees of freedom. The case of inhomogeneous boundary conditions is discussed in [32, Sections 3 and 5.3].

4.1 Convergence under uniform refinements

We investigate the rate of convergence of the fully-discrete method (3) under uniform time step and polynomial degree in time refinements in various norms for three tests.

The initial and boundary conditions, and the right-hand side of (1) are computed accordingly to the explicit formula of the different solutions we pick below.

4.1.1 Uniform refinements: test case 1

We consider the analytic solution

$$u(x, y, t) := (1 - x^2)(1 - y^2) \cos(4t). \quad (75)$$

As for the spatial discretization, we fix $p_x = 2$ and $h = 0.4$. Since the exact solution is a quadratic polynomial in space and we use a quadratic nodal tensor product basis, up to machine precision, the spatial error is zero.

Convergence under uniform time step refinements. For p_n^t in $\{2, 3\}$, we pick τ in $\{2 \times 10^{-1}, 10^{-1}, 5 \times 10^{-2}, 2.5 \times 10^{-2}, 1.25 \times 10^{-2}\}$; for p_n^t in $\{4, 5\}$, we pick τ in $\{2 \times 10^{-1}, 1.25 \times 10^{-1}, 9.09 \times 10^{-2}, 7.15 \times 10^{-2}, 5.88 \times 10^{-2}\}$. We display the errors in (72), (73), and (74) in Figure 1.

The expected optimal convergence rates as discussed in Corollary 2.9 are observed for the errors in (72); the same convergence rate is achieved by the other error measures, which is not covered by the theoretical results from Section 2. In particular, the error in (74) has order $\mathcal{O}(\tau^{p_n^t - \frac{1}{2}})$; see also Remark 2 for further comments on this point.

Convergence under uniform polynomial degree in time refinements. We pick p_n^t in $\{2, 3, 4, 5, 6\}$ and τ in $\{2 \times 10^{-1}, 10^{-1}\}$. We display the errors in (72) (73), and (74) in Figure 2.

We observe exponential convergence for all the errors. Even though this is not covered by the results in Section 2, we can expect this convergence from the smoothness of the function in (75) and standard p -FEM techniques [28].

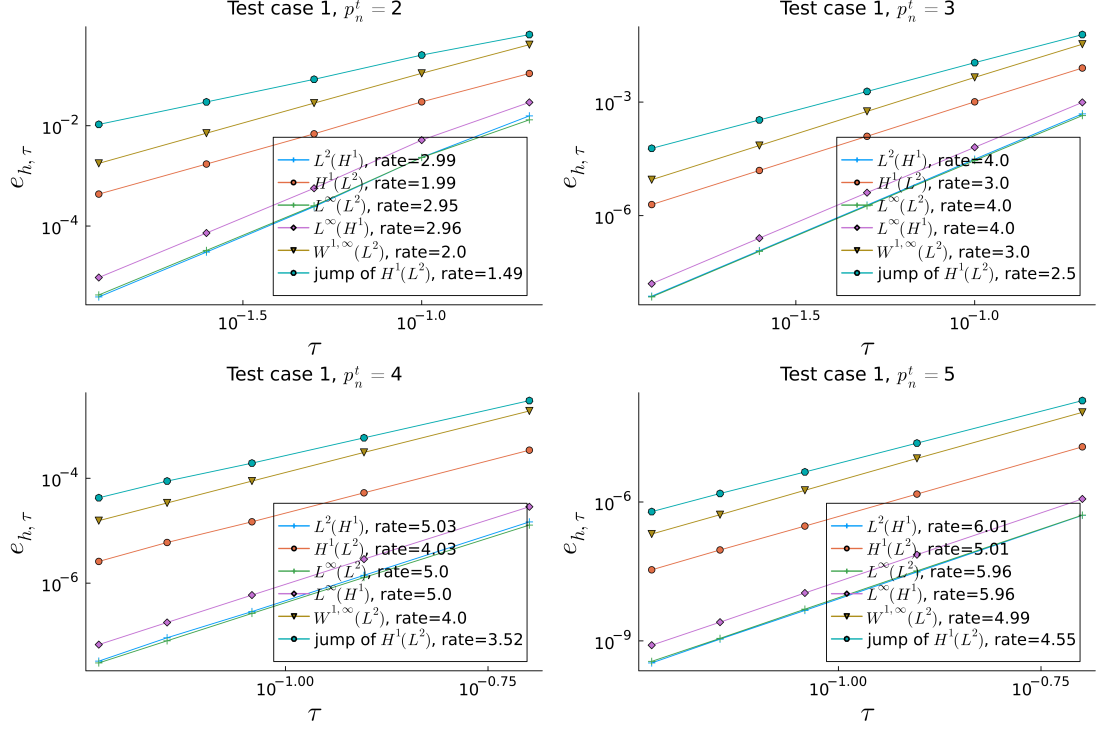


Figure 1: Exact solution as in (75), uniform τ -refinement.

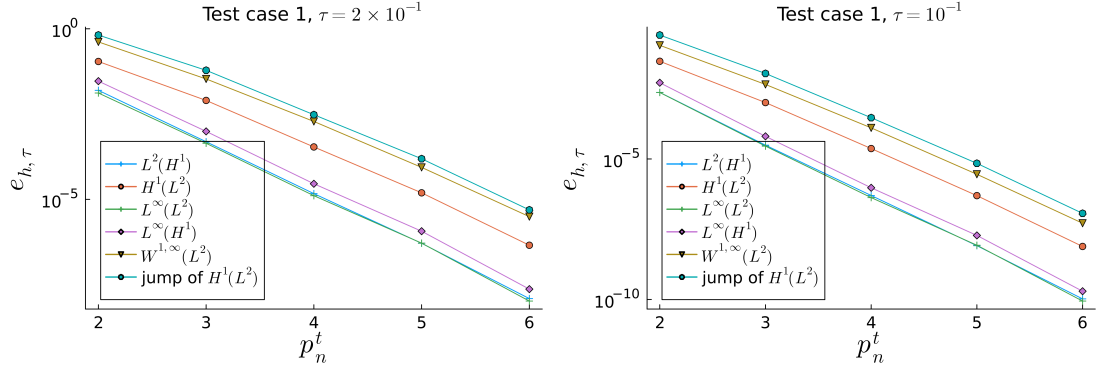


Figure 2: Exact solution as in (75), uniform p_n^t -refinement.

4.1.2 Uniform refinements: test case 2

We consider analytic in space solutions

$$u(x, y, t) := (1 - x^2)(1 - y^2)t^\alpha, \quad \alpha \in \{1.75, 2.5\}, \quad (76)$$

which belong to $H^{2.25-\epsilon}(0, T; \mathcal{C}^\infty(\Omega))$ and $H^{3-\epsilon}(0, T; \mathcal{C}^\infty(\Omega))$ for $\alpha = 1.75$ and 2.25 , respectively, for arbitrarily small, positive ϵ . We fix $p_x = 2$ and $h = 0.4$.

Convergence under uniform time step refinements. We pick $p_n^t = 2$ and τ in $\{2 \times 10^{-1}, 10^{-1}, 5 \times 10^{-2}, 2.5 \times 10^{-2}, 1.25 \times 10^{-2}, 6.13 \times 10^{-3}, 3.06 \times 10^{-3}, 1.53 \times 10^{-3}\}$ for $\alpha = 1.75$ and τ in $\{2 \times 10^{-1}, 10^{-1}, 5 \times 10^{-2}, 2.5 \times 10^{-2}, 1.25 \times 10^{-2}\}$ for $\alpha = 2.5$. We do not consider higher polynomial degrees in time, since they deliver the same convergence rates. The results are exhibited in Figure 3.

We observe optimal convergence for the errors in (72) as dictated by Corollary 2.9; similar rates are achieved by the error measures in (73). The error measured in the $W^{1,\infty}(L^2)$ -seminorm confirms estimate (41); that seminorm converges with the same rate of the error in (74).

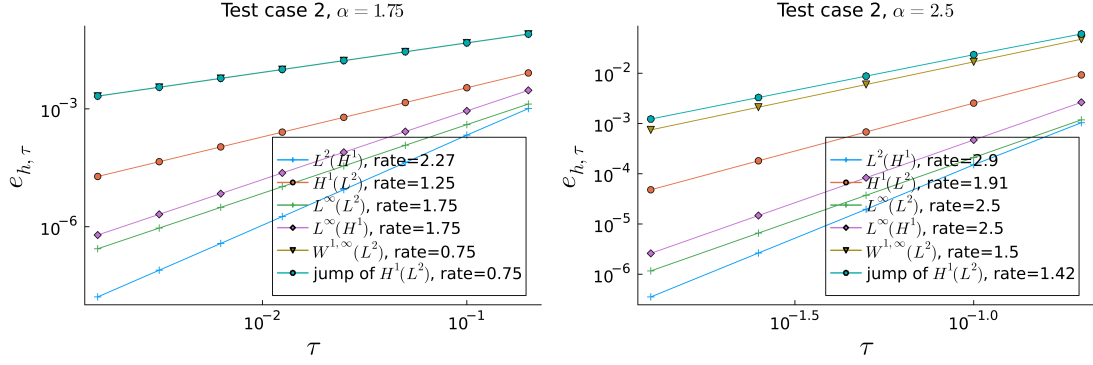


Figure 3: Exact solution as in (76), uniform τ -refinement.

Convergence under uniform polynomial degree in time refinements. We pick p_n^t in $\{2, \dots, 10\}$ and $\tau = 0.2$. The results are displayed in Figure 4.

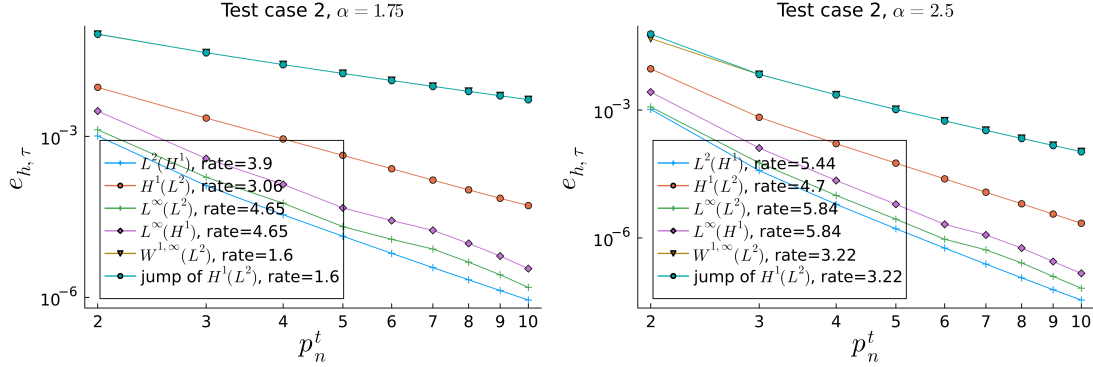


Figure 4: Exact solution as in (76), p_n^t -refinement.

We observe doubling order convergence in p_n^t , which is standard in p -FEM while approximating functions with growth of t^α type [28, Section 3.3.5], for the jump of the $H^1(L^2)$ -seminorm, whereas other quantities display a super-convergence phenomenon.

4.1.3 Simultaneous space–time uniform refinements: test case 3

We consider the analytic solution

$$u(x, y, t) := \sin(\pi x) \sin(\pi y) \cos(\sqrt{2}\pi t), \quad (77)$$

and pick $p_x = p_n^t + 1$ and $h = \tau$, i.e., we are interested in simultaneous space and time refinements. On the other hand, for $p_n^t \in \{2, 3\}$, we choose τ in $\{2 \times 10^{-1}, 10^{-1}, 5 \times 10^{-2}, 2.5 \times 10^{-2}, 1.25 \times 10^{-2}\}$; for $p_n^t = 4$, we choose τ in $\{2 \times 10^{-1}, 10^{-1}, 6.67 \times 10^{-2}, 5 \times 10^{-2}, 4 \times 10^{-2}\}$; for $p_n^t = 5$, we choose τ in $\{5 \times 10^{-1}, 3.33 \times 10^{-1}, 2.5 \times 10^{-1}, 2 \times 10^{-1}, 1.67 \times 10^{-1}\}$. The results are displayed in Figure 5.

We observe optimal convergence rates for the errors (72) and (73).

4.2 Efficiency of the error estimator for the semi-discrete in time scheme

Henceforth, the semi-discrete formulation (4) is considered. We check the efficiency of the error estimator η in (70) (with $m = N$ and $\xi = t_N$ in (69)), compared to the error measured in the $L^\infty(0, T; L^2(\Omega))$ norm. In particular, we investigate the behaviour of the effectivity index

$$\kappa := \frac{\eta}{\|e_{h,\tau}\|_{L^\infty(0,T;L^2(\Omega))}}. \quad (78)$$

We focus on the test cases with exact solutions in (75) and (76), and consider here uniform time steps and polynomial in time refinements.

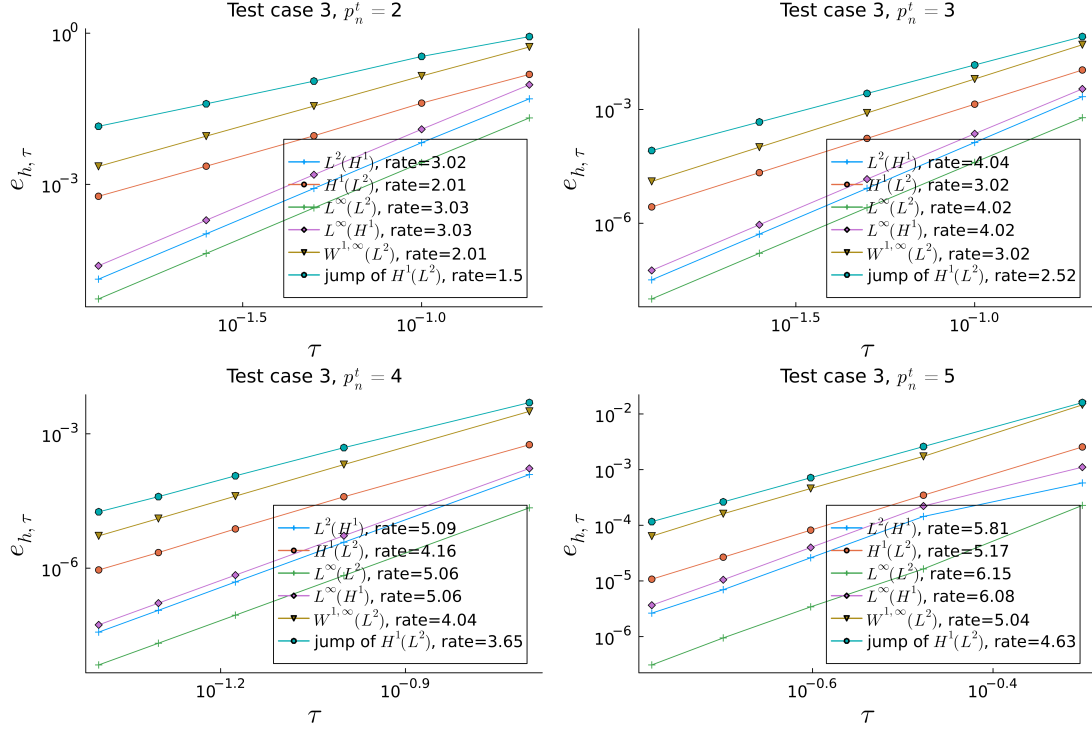


Figure 5: Exact solution as in (77), τ -refinement.

In Figure 6, we present the results we obtained under uniform time steps refinements. For the test case with exact solution as in (75), we pick p_n^t in $\{2, 3, 4\}$; for the test case with exact solution as in (76), we pick $p_n^t = 2$, $\alpha = 1.75$.

The estimator has the optimal convergence rate as the error measured in the $L^\infty(0, T; L^2(\Omega))$ norm. Notably, the effectivity index in (78) is stable with respect to τ , i.e., is uniformly bounded by a constant with respect to τ .

Then, in Figure 7, the same tests are investigated with $N = 5$, i.e., for a fixed $\tau = 0.2$ under p -refinements in time; for the test case with exact solution in (76) we only consider $\alpha = 1.75$.

Also in this case, the estimator has the same convergence rate as that of the $L^\infty(0, T; L^2(\Omega))$ norm of the error. For the test case with exact solution in (75), the effectivity index κ is uniformly bounded in terms of p_n^t ; for the test case with exact solution in (76), κ increases with rate $1/2$ in terms of p_n^t .

4.3 Convergence under adaptive refinements in time

Since the data oscillation terms in the a posteriori error estimates (71) are not dominant, we omit them for simplicity. We consider here an adaptive algorithm with the usual structure

$$\text{SOLVE} \quad \Rightarrow \quad \text{ESTIMATE} \quad \Rightarrow \quad \text{MARK} \quad \Rightarrow \quad \text{REFINE.}$$

The **ESTIMATE** step is driven by using the error estimator associated with η ; to this aim, we propose an algorithm for the localization of ξ , which is the point of $(0, T]$ where the error estimator η_1 attains the maximum. As for the **MARK** step, we use Dörfler's marking with a given threshold θ in $(0, 1]$. The **REFINE** step is realized by the bisection of the marked time intervals.

In what follows, we denote the Kronecker delta function by $\delta_{i,j}$.

Adaptive algorithm for the localization of each ESTIMATE step.

1. Find the interval index m where η_1 attains the maximum and set $\xi = t_{m-1}$.

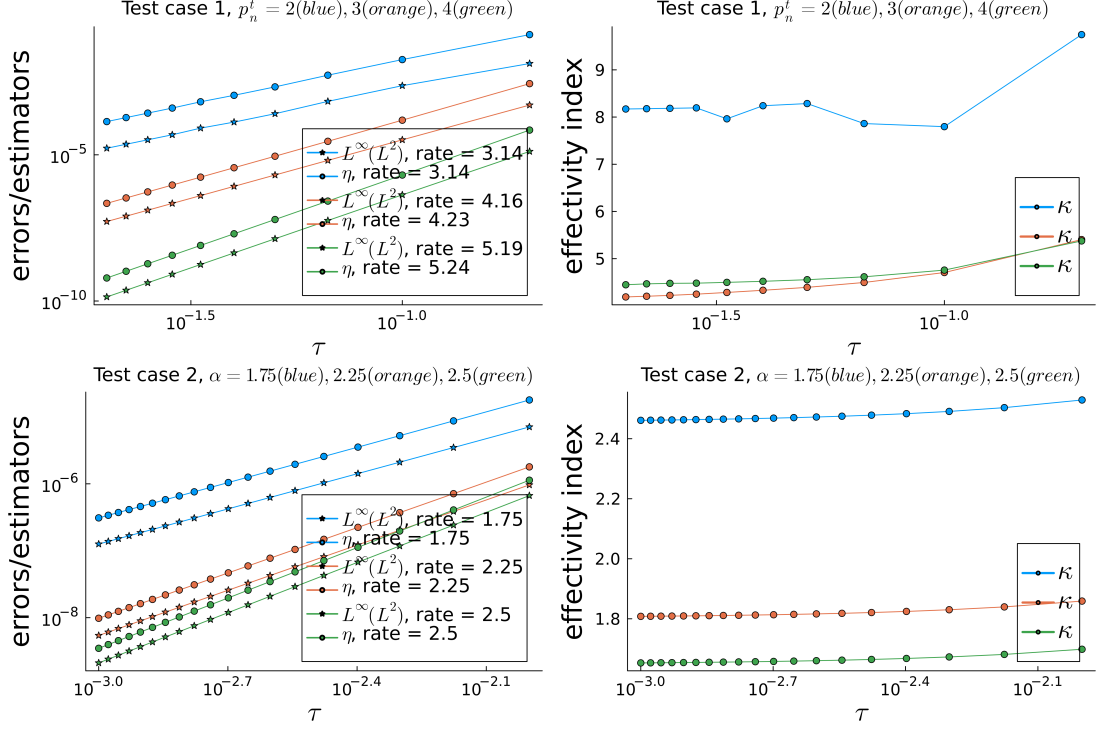


Figure 6: Exact solution as in (76), uniform τ -refinement.

2. Compute $\eta_{2,n}$ for $n = 1, \dots, m$ and η_2 by (69) with the above given m .
3. Set $\eta_{2,n} = 0$ for $n = m + 1, \dots, N$.
4. Denote $\eta_n = \delta_{m,n}\eta_1 + \eta_{2,n}$ the local error estimator on the time interval I_n for $n = 1, \dots, N$.

4.3.1 Numerical results: the adaptive algorithm

We consider the test case with exact solution as in (76) and $\alpha = 1.75$; p_n^t in $\{2, 3, 4\}$; $p_x = 2$; 5 nodes in each space direction; Dörfler's marking parameter $\theta = 0.5$. We define $DoFs = N \times p_n^t \times \text{card}(V_h)$. In Figure 8, we display the $L^\infty(L^2)$ uniform and adaptive errors and estimators η in the left panels; the effectivity indices κ are presented in the right panels.

Some remarks are in order:

- adaptive algorithm delivers optimal convergence in terms of the number $DoFs$ of the adaptive algorithm;
- the effectivity index is uniformly bounded for fixed p_n^t , and the adaptive algorithm asymptotically returns smaller effectivity indices.

In Figure 9, we illustrate the final time meshes produced with $p_n^t = 2$ and 4 from the adaptive and uniform refinement algorithms.

The adaptive algorithm generates a time mesh with a strong grading towards the initial time. Such temporal meshes are more graded in the case $p_n^t = 4$ compared to the case $p_n^t = 2$.

5 Conclusions

For a DG-CG discretization of the wave equation in second order formulation

- we derived a priori estimates for the fully discrete method, which are explicit in the spatial mesh size, the time steps, and the polynomial degree distributions in space and time (the errors were measured in L^∞ -type norms in time);

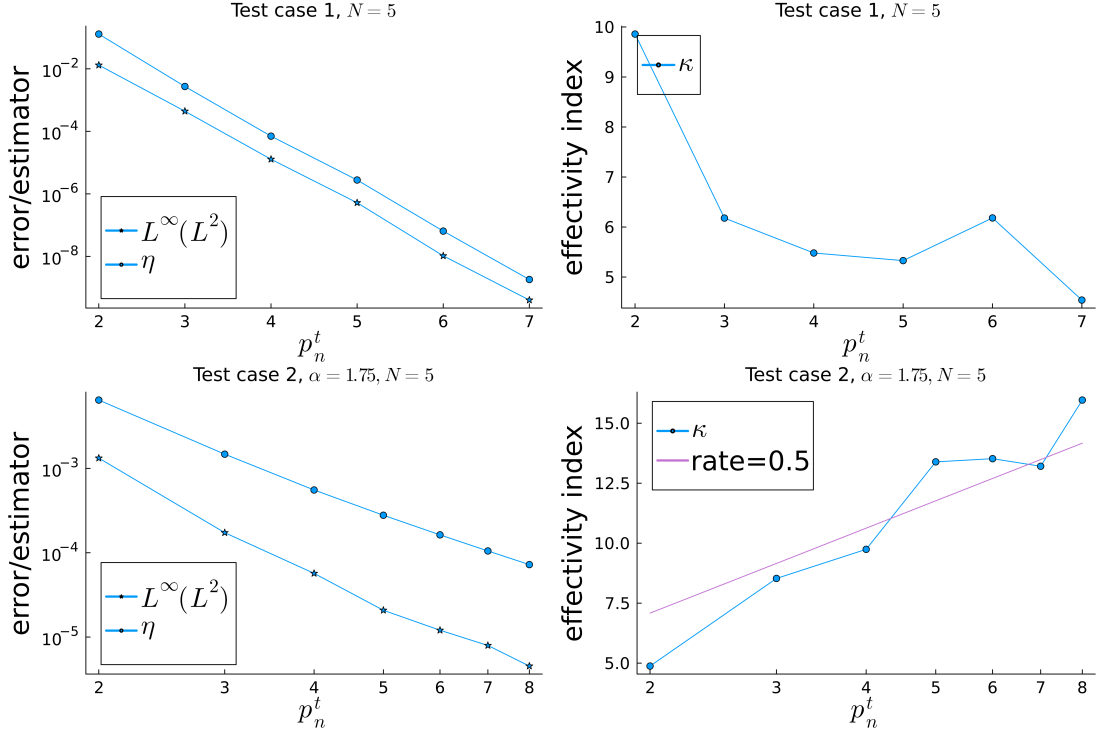


Figure 7: Exact solution as in (76), p_n^t -refinement.

- we derived constant free, reliable a posteriori estimates for the $L^\infty(L^2)$ error in terms of a novel error estimator, which involves jumps of the time derivatives and the spatial Laplacian at the time nodes.

A campaign of numerical experiments revealed that

- the convergence estimates under uniform refinements are optimal also for L^2 -type errors in time;
- the proposed error estimator is efficient under time step refinements and may be inefficient under p -refinements in time.

References

- [1] S. Adjerid. A posteriori finite element error estimation for second-order hyperbolic problems. *Comput. Methods Appl. Mech. Engrg.*, 191(41-42):4699–4719, 2002.
- [2] A. K. Aziz and P. Monk. Continuous finite elements in space and time for the heat equation. *Math. Comp.*, 52(186):255–274, 1989.
- [3] G. A. Baker. Error estimates for finite element methods for second order hyperbolic equations. *SIAM J. Numer. Anal.*, 13(4):564–576, 1976.
- [4] Ch. Bernardi and E. Süli. Time and space adaptivity for the second-order wave equation. *Math. Models Methods Appl. Sci.*, 15(02):199–225, 2005.
- [5] P. Castillo, B. Cockburn, D. Schötzau, and Ch. Schwab. Optimal a priori error estimates for the hp -version of the local discontinuous Galerkin method for convection–diffusion problems. *Math. Comp.*, 71(238):455–478, 2002.
- [6] Th. Chaumont-Frelet. Asymptotically constant-free and polynomial-degree-robust a posteriori estimates for space discretizations of the wave equation. *SIAM J. Sci. Comput.*, 45(4):A1591–A1620, 2023.
- [7] Th. Chaumont-Frelet and A. Ern. Damped energy-norm a posteriori error estimates for fully discrete approximations of the wave equation using C^2 -reconstructions. <https://arxiv.org/abs/2403.12954>, 2024.
- [8] L. C. Evans. *Partial Differential Equations*, volume 19. American Mathematical Society, 2022.
- [9] D. A. French. A space-time finite element method for the wave equation. *Comput. Methods Appl. Mech. Engrg.*, 107(1-2):145–157, 1993.

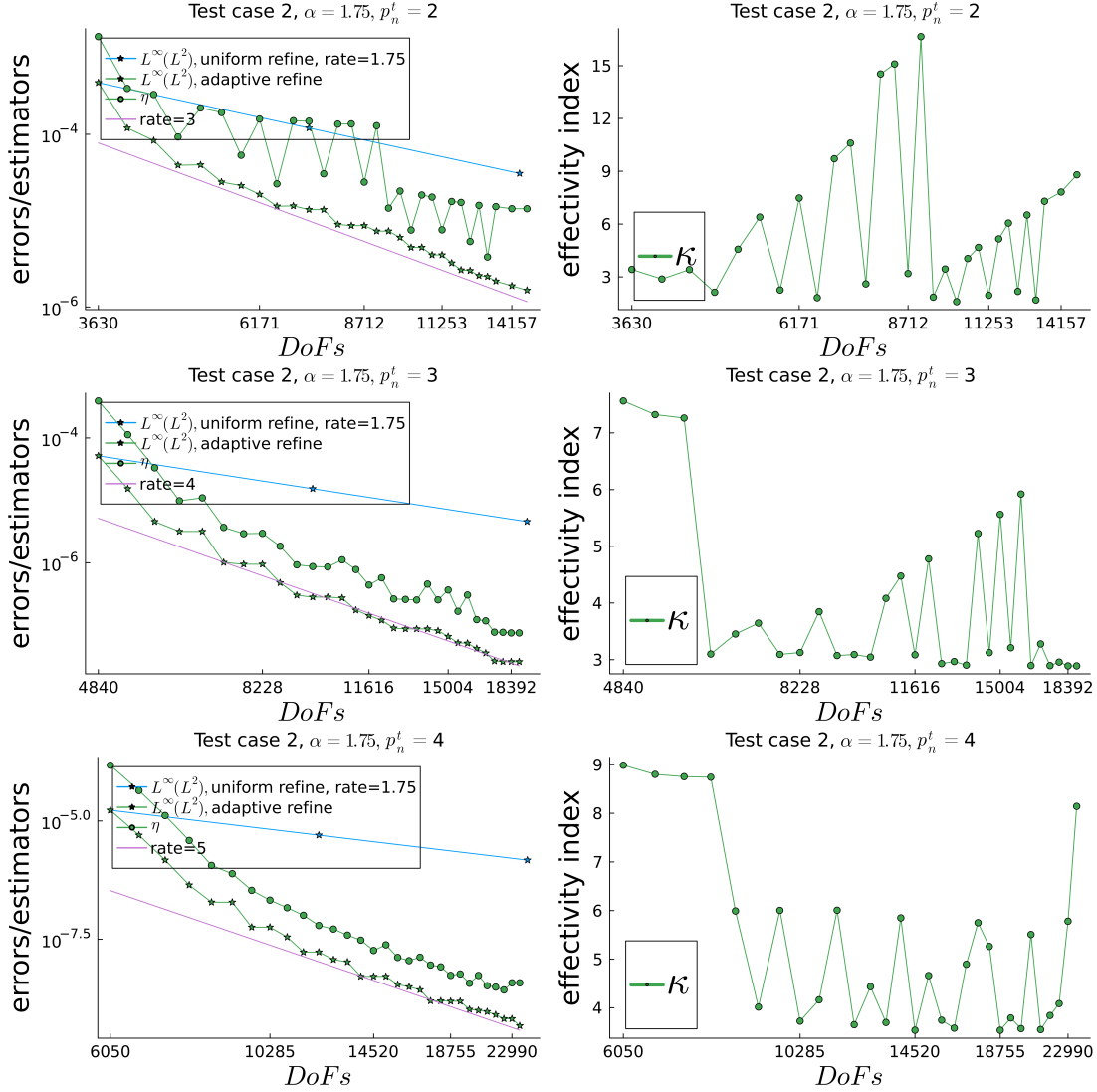


Figure 8: Exact solution as in (76).

- [10] D. A. French and T. E. Peterson. A continuous space-time finite element method for the wave equation. *Math. Comp.*, 65(214):491–506, 1996.
- [11] E. H. Georgoulis, O. Lakkis, and Ch. Makridakis. A posteriori $L^\infty(L^2)$ -error bounds for finite element approximations to the wave equation. *IMA J. Numer. Anal.*, 33(4):1245–1264, 2013.
- [12] E. H. Georgoulis, O. Lakkis, Ch. Makridakis, and J. M. Virtanen. A posteriori error estimates for leap-frog and cosine methods for second order evolution problems. *SIAM J. Numer. Anal.*, 54(1):120–136, 2016.
- [13] E. H. Georgoulis, O. Lakkis, and Th. P. Wihler. A posteriori error bounds for fully-discrete hp -discontinuous Galerkin timestepping methods for parabolic problems. *Numer. Math.*, 148:363–386, 2021.
- [14] O. Gorynina, A. Lozinski, and M. Picasso. Time and space adaptivity of the wave equation discretized in time by a second-order scheme. *IMA J. Numer. Anal.*, 39(4):1672–1705, 2019.
- [15] P. Grisvard. *Elliptic Problems in Nonsmooth Domains*, volume 69 of *Classics in Applied Mathematics*. SIAM, 2011.
- [16] M. Grote, S. Michel, and S. Sauter. Stabilized leapfrog based local time-stepping method for the wave equation. *Math. Comp.*, 90(332):2603–2643, 2021.
- [17] B. Holm and Th. P. Wihler. Continuous and discontinuous Galerkin time stepping methods for nonlinear initial value problems with application to finite time blow-up. *Numer. Math.*, 138(3):767–799, 2018.
- [18] G. M. Hulbert and Th. J. R. Hughes. Space-time finite element methods for second-order hyperbolic equations. *Comput. Methods Appl. Mech. Engrg.*, 84(3):327–348, 1990.
- [19] A. Ilyin, A. Laptev, M. Loss, and S. Zelik. One-dimensional interpolation inequalities, Carlson–Landau inequalities, and magnetic Schrödinger operators. *Int. Math. Res. Not. IMRN*, 2016(4):1190–1222, 2016.

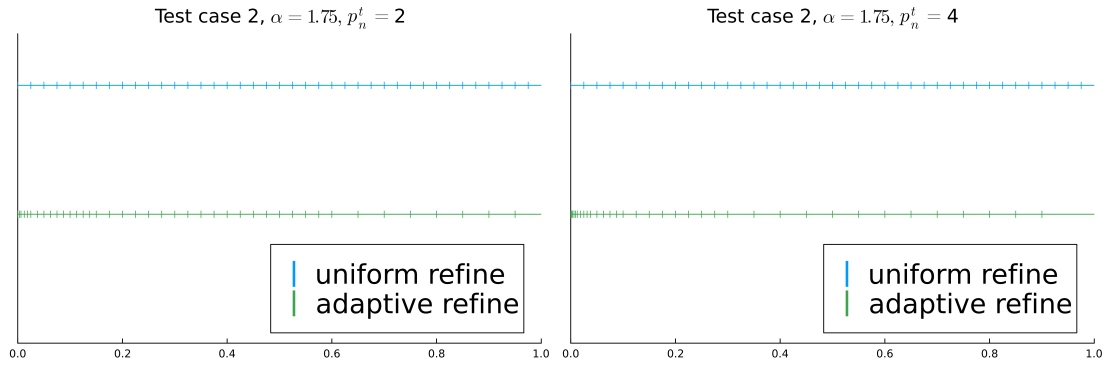


Figure 9: Exact solution as in (76), time mesh visualization.

- [20] C. Johnson. Discontinuous Galerkin finite element methods for second order hyperbolic problems. *Comput. Methods Appl. Mech. Engrg.*, 107(1-2):117–129, 1993.
- [21] O. Karakashian and Ch. Makridakis. Convergence of a continuous Galerkin method with mesh modification for nonlinear wave equations. *Math. Comp.*, 74(249):85–102, 2005.
- [22] Ch. Makridakis and R. H. Nochetto. Elliptic reconstruction and a posteriori error estimates for parabolic problems. *SIAM J. Numer. Anal.*, 41(4):1585–1594, 2003.
- [23] Ch. Makridakis and R. H. Nochetto. A posteriori error analysis for higher order dissipative methods for evolution problems. *Numer. Math.*, 104(4):489–514, 2006.
- [24] N. Nataraj, R. Ruiz-Baier, and A. Yousuf. Semi and fully-discrete analysis of lowest-order nonstandard finite element methods for the biharmonic wave problem. <https://arxiv.org/abs/2407.03777>, 2024.
- [25] P.-A. Raviart and J.-M. Thomas. *Introduction à l'analyse numérique des équations aux dérivées partielles*. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983.
- [26] D. Schötzau and Ch. Schwab. Time discretization of parabolic problems by the hp -version of the discontinuous Galerkin finite element method. *SIAM J. Numer. Anal.*, 38(3):837–875, 2000.
- [27] D. Schötzau and Th. P. Wihler. A posteriori error estimation for hp -version time-stepping methods for parabolic partial differential equations. *Numer. Math.*, 115(3):475–509, 2010.
- [28] Ch. Schwab. *p - and hp -Finite Element Methods: Theory and Applications in Solid and Fluid Mechanics*. The Clarendon Press, Oxford University Press, New York, 1998.
- [29] E. Süli. A posteriori error analysis and adaptivity for finite element approximations of hyperbolic problems. In *An Introduction to Recent Developments in Theory and Numerics for Conservation Laws: Proceedings of the International School on Theory and Numerics for Conservation Laws, Freiburg/Littenweiler, October 20–24, 1997*, pages 123–194. Springer, 1999.
- [30] V. Thomée. *Galerkin Finite Element Methods for Parabolic Problems*, volume 25. Springer Science & Business Media, 2007.
- [31] L. N. Trefethen. *Approximation Theory and Approximation Practice, Extended Edition*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2019.
- [32] N. J. Walkington. Combined DG–CG time stepping for wave equations. *SIAM J. Numer. Anal.*, 52(3):1398–1417, 2014.