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Modal mass estimation from state-space models and frequency response functions

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Abstract. In operational modal analysis with exogenous inputs, the modal parameters are estimated based on partly measured inputs and outputs. Different algorithms can be used, like combined deterministic stochastic subspace identification (CSI) algorithms or the poly-reference Least Squares Complex Frequency (pLSCF) algorithm. Common for both types of algorithms are the computation of modal mass to normalize the estimated mode shapes according to the inputs. In this work, the modal mass computation for CSI and the pLSCF-algorithm is revisited, where a particular focus is put on the assumption on the intersample behavior that is required for the relation between estimates obtained from the discrete-time system (where the data comes from) and the corresponding continuous-time system (where the modal mass is computed). The discretization methods: zero-order hold, first-order hold, and the impulse invariant discretization are studied and discussed. It is shown that the modal mass estimates are heavily dependent on the discretization approach used. The results are illustrated in the context of Monte Carlo simulation of a six-degrees-of-freedom chain system.

Keywords: Modal mass · subspace identification · pLSCF · FRF · state-space model.

1 Introduction

In experimental modal analysis modal parameters of vibrating systems are obtained from input and output data. While the dynamic behaviour of physical systems is generally represented in the continuous time, data are discrete and are used to characterize a system model in discrete time. The discrete-time model can be formulated in time domain, e.g., with state space modelling, or in frequency domain e.g. with matrix fraction description. When converting from the continuous time to the discrete time, an assumption on the intersample behavior of the continuous time inputs must be made; the common discretization methods are, e.g., zero-order-hold, first-order-hold, and the impulse invariant method [11]. While the estimation of the natural frequencies, damping ratios and mode

shapes, is not influenced by the assumption on the intersample behavior, the modal participation factors, and the underlying modal masses are highly dependant on the discretization approach used.

The aim of this paper is to investigate the influence of the assumed intersample behaviour of the input on the estimation of modal mass. Several discretization methods are used, i.e., zero-order-hold, first-order-hold, and the impulse-invariant method. Two algorithms to obtain quantities for modal mass estimation are considered, i.e., the combined deterministic-stochastic subspace identification approach (CSI) [10,4] and the poly-reference least-squares complex frequency-domain method (pLSCF) [7]. The consistency of modal mass estimation is investigated in a Monte Carlo setting for a toy example of a spring-mass system.

2 Background and modelling

The vibration behavior of a viscously damped, linear time-invariant structural system with m degrees of freedom is described by the differential equation

$$\mathcal{M}\ddot{q}(t) + \mathcal{C}\dot{q}(t) + \mathcal{K}q(t) = u(t) \quad (1)$$

where t denotes continuous time, and $\mathcal{M}, \mathcal{C}, \mathcal{K} \in \mathbb{R}^{m \times m}$ denote mass, damping and stiffness matrices, respectively. Array $q(t) \in \mathbb{R}^m$ contains the continuous-time degrees of freedom, here displacements, and $u(t) \in \mathbb{R}^m$ comprises external loads. Let $\lambda_i \in \mathbb{C}$ and $\psi_i \in \mathbb{R}^m$ denote the i -th ($i = 1 \dots m$) eigenvalue and eigenvector of system (1)

$$(\mathcal{M}\lambda_i^2 + \mathcal{C}\lambda_i + \mathcal{K})\psi_i = 0 .$$

Assume that system (1) is proportionally damped, i.e., matrices \mathcal{M}, \mathcal{C} and \mathcal{K} are diagonalizable by $\psi_i \forall_i$. A set of parameters fully describing the system dynamics comprises the natural frequencies f_i , damping ratios ζ_i , mode shapes ψ_i and modal masses m_i . The modal mass finds a particular interest in engineering applications, e.g., for mode shape scaling, since it describes the contribution of a mode to the system's response, e.g., when the modal mass is relatively large, the mode's contribution to the response is large. The modal mass of the i -th mode writes

$$m_i = \psi_i^T \mathcal{M} \psi_i .$$

While an estimate of m_i can be obtained from a set of input-output response data, it highly depends on the assumed inter-sample behaviour of the input used to discretize the continuous-time system. In what follows, two approaches for modal mass estimation are outlined and their dependence on the input discretization is highlighted.

2.1 Modelling and discretization

System (1) and its outputs can be modelled in continuous-time state-space

$$\dot{x}(t) = A_c x(t) + B_c u(t) , \quad (2)$$

$$y(t) = C_c x(t) + D_c u(t) , \quad (3)$$

where $x(t) \in \mathbb{R}^n$ are the states, $u(t) \in \mathbb{R}^u$ are the inputs, $A_c \in \mathbb{R}^{n \times n}$, $B_c \in \mathbb{R}^{n \times u}$, $C_c \in \mathbb{R}^{r \times n}$ and $D_c \in \mathbb{R}^{r \times u}$ are the continuous-time state transition, input, output and feedthrough matrices. Without loss of generality, to simplify the notation assume that all inputs are spatially collocated with outputs. Let a set of eigenvalues and eigenvectors (λ_i, ϕ_i) contain one of each $n = 2m$ eigenvalues and eigenvectors of the state matrix A_c . A mode shape of the system at the observed r degrees of freedom is defined as $\varphi_i^r = C_c \phi_i$ (being a scaled sub-vector of ψ_i) and a mode shape at u input locations φ_i^u is defined as a relevant subset of φ_i^r . A continuous mapping of input to output is given by the transfer function

$$H(s) = C_c (sI_n - A_c)^{-1} B_c + D_c = \sum_{j=1}^m \left(\frac{q_j \varphi_j^r \varphi_j^{uT}}{s - \lambda_j} + \frac{q_j^* \varphi_j^{r*} \varphi_j^{uH}}{s - \lambda_j^*} \right) \in \mathbb{C}^{r \times u} , \quad (4)$$

where I_n is the identity matrix of size $n \times n$, $s = i\omega\tau$, $0 \leq \omega \leq \omega_f$, where ω_f is the Nyquist frequency, $(\cdot)^*$ denotes the complex conjugate, $(\cdot)^H$ denotes the transposed complex conjugate, and q_j are the modal participation factors which are related to the modal masses. Taking $t = k\tau$, where τ is the time step and k is an integer, a solution to (2) is well-known

$$x_{k+1} = e^{A_c \tau} x_k + e^{A_c \tau} \int_{k\tau}^{(k+1)\tau} e^{A_c(k+1)\tau - t} B_c u(t) dt . \quad (5)$$

Assuming some inter-sample behaviour for $u(t)$, (5) and (3) writes

$$x_{k+1} = A x_k + B u_k , \quad (6)$$

$$y_k = C x_k + D u_k , \quad (7)$$

where $x_k \in \mathbb{R}^n$ are the discrete-time states, $u_k \in \mathbb{R}^u$ are the discrete-time inputs, $A = e^{A_c \tau} \in \mathbb{R}^{n \times n}$ with $C \in \mathbb{R}^{r \times n}$ are the discrete-time state transition and output matrices. The discrete-time input $B \in \mathbb{R}^{n \times u}$ and feedthrough $D \in \mathbb{R}^{r \times u}$ matrices depend on the assumed behaviour of $u(t)$ and for now remain undefined. The discrete-time transfer function $H(z)$ is derived from a z -transform of (2)-(3)

$$H(z) = C (zI_n - A)^{-1} B + D = \sum_{j=1}^m \left(\frac{\varphi_j^r \gamma_j^T}{z - e^{\lambda_j \tau}} + \frac{\varphi_j^{r*} \gamma_j^H}{z - e^{\lambda_j^* \tau}} \right) \in \mathbb{C}^{r \times u} , \quad (8)$$

where I_n is the identity matrix of size $n \times n$, $z = e^{i\omega\tau}$ and $\gamma_j \in \mathbb{C}^u$ denotes the j -th column of $(V^{-1}B)^T$ where $V \in \mathbb{C}^{n \times n}$ are the eigenvectors of A .

An estimate of the modal mass is typically obtained from an estimate of $H(z)$ and an assumption that $H(s) = H(z)$, see e.g., [2], whereafter a least-square problem to obtain the modal participation factors and consequently the

modal masses follow. The aforementioned assumption that $H(s) = H(z)$ does not hold for every inter-sample discretization of $u(t)$; in what follows we convert the analog transfer function $H(s)$ to the digital transfer function $H(z)$ in such a way that the underlying impulse response functions are the same at the discrete sampling instants. This yields the following discrete-time system matrices

$$B = A_c B_c \tau, \quad D = C_c B_c \tau,$$

which are used to obtain the modal masses in the remainder of this paper.

2.2 Modelling and discretization in frequency domain

A linear time-invariant system can be modelled in frequency domain using e.g., a left or right matrix fraction description, or a common denominator description [8]. In the following, a description of the right matrix fraction description is given.

Let $\beta(s) \in \mathbb{R}^{r \times u}$ and $\alpha(s) \in \mathbb{R}^{u \times u}$ denote the numerator and denominator polynomial, then a right matrix fraction description of the Frequency Response Functions (FRFs) can be modelled as

$$H(s) = \beta(s)\alpha^{-1}(s), \quad \beta(s) = \sum_{j=0}^n \beta_j s^j, \quad \alpha(s) = \sum_{j=0}^n \alpha_j s^j. \quad (9)$$

By taking the z-transform the same expression is obtained with z instead of s and the system can be modelled in the discrete-time domain. This expression is the basis for the identification of the polynomial coefficients from the FRF estimates, and subsequently the eigenvalues λ_i and participation vectors L_i , using pLSCF.

In a second step, $H(z) = H(s)$ is assumed, and the mode shapes can be retrieved in a least-squares sense by equating

$$H(s) = \sum_{j=1}^m \left(\frac{\varphi_j^r L_j^T}{s - \lambda_j} + \frac{\varphi_j^{r*} L_j^H}{s - \lambda_j^*} \right) \quad (10)$$

to the discrete-time FRF estimates (up to some lower and upper residual terms in case there are modes outside the frequency band of interest), in what is called the least-squares frequency-domain (LSFD) method [5]. It should be noted that this step implicitly assumes that the discrete-time and continuous-time FRFs are equal. Such a property strongly depends on the actual intersample behavior (in time) of the inputs, and is not given in general. A good approximation is given by the impulse invariant discretization of the inputs, whereas ZOH or FOH inputs yield stronger discrepancies between discrete-time and continuous-time FRFs.

3 Modal mass estimation

In this section we outline two approaches that are typically used to obtain modal mass from a set of input-output data.

3.1 Modal mass estimation with subspace identification

The modal mass of the i -th mode can be defined from the properties of state-space model (2)-(3) and (6)-(7) and its consistent estimate can be obtained by plugging consistent estimates obtained from the chosen subspace identification approach e.g. [6,4]. The resultant modal mass writes

$$m_i = s_i o_i \frac{(\varphi_i^u)^T \gamma_i^*}{\gamma_i^T \gamma_i^*}, \quad (11)$$

where s_i is a scaling factor which depends on the input discretization and o_i is a scaling factor which depends on the type of output, e.g., $o_i = \lambda_i^2$ for accelerations. It can be shown that for an impulse invariant scaling s_i writes

$$s_i = \frac{e^{\lambda_i \tau}}{(\lambda_i - \lambda_i^*)} \tau.$$

Note that when computing the modal mass with scaled mode shapes, like a scaling with the k -th mode shape component [3,9], the vector γ_i has to be adapted accordingly. Then, the computations of this section can be carried out equivalently using $\varphi_i/(\varphi_i)_k$ and $(\varphi_i)_k \gamma_i$, instead of φ_i and γ_i .

Also, remark that the expression for the modal mass (11) is hereby developed for a system adhering to an impulse invariant assumption on the intersample behaviour of the input. In case input discretization is different, (11) should be reformulated, which is beyond the scope of this paper.

3.2 Modal mass estimation with pLSCF

For the modal parameter estimation the pLSCF and LSFD algorithms are used, where the last step assumes equality of the discrete-time and continuous-time FRFs, yielding estimates of the continuous-time poles λ_j , participation vectors L_j and mode shapes φ_j satisfying Eq. (10). From this equation, the modal mass is estimated, as the transfer function $H(s)$ writes

$$H(s) = \sum_{j=1}^m \left(\frac{\varphi_j^r L_j^T}{s - \lambda_j} + \frac{\varphi_j^{r*} L_j^H}{s - \lambda_j^*} \right) = \sum_{j=1}^m \left(\frac{q_j \varphi_j^r \varphi_j^{uT}}{s - \lambda_j} + \frac{q_j^* \varphi_j^{r*} \varphi_j^{uH}}{s - \lambda_j^*} \right). \quad (12)$$

The modal scaling factor q_j is determined at measurement degrees of freedom with both input and output using $o_i^{-1} \varphi_j^u L_j^{uT} = q_j \varphi_j^u \varphi_j^{uT}$ identified from Eq. (12), where o_i is the scaling factor as defined in Section 3.1. The modal mass can then be calculated as in [2]

$$q_j = \frac{L_j^{uT} \varphi_j^{u*}}{o_i \varphi_j^{uT} \varphi_j^{u*}}, \quad m_j = \frac{1}{q_j (\lambda_j - \lambda_j^*)}. \quad (13)$$

Likewise for the modal mass computation with subspace identification, the modal mass computation with pLSCF can be carried out using scaled mode shapes as long as the participation vectors L_j is adapted accordingly.

4 Numerical application

In this section we conduct an empirical analysis of the influence of the intersample behaviour on the modal mass estimation. We suppose the impulse invariant discretization for the modal mass computation (as shown in the previous section), whereas the actual intersample behavior of the inputs may be different.

For this purpose consider a spring-mass system modelled with each mass weighing 0.05 kg, spring constants $k_i = [100 \ 200 \ 100 \ 200 \ 100 \ 200]$ N/m, and with 2 % proportional damping at each mode; see Figure 1 for a sketch of the system.

The system is excited with white noise inputs measured at DOF 2 and 3, and acceleration output is measured at DOF 2, 3, and 6. The sampling frequency is 40 Hz. Firstly, we analyze the effects of the intersample behaviour on the modal mass estimates based on a single input-output data realization. Secondly, the analysis is performed in a Monte Carlo setting with 1000 data realizations, where an equal amount of noise is added to the response signals. The amount of noise added corresponds to a signal-to-noise ratio (SNR) of 20 for the first output at location 2.

The discrete input-output time signals are converted to power spectral densities to compute the FRFs using the H1-estimator. This estimator is unbiased and consistent when inputs are noise free [1]. The frequency span considered is 0-20 Hz and is divided into 1200 lines. The computed FRFs are used as inputs to the pLSCF algorithm from [7]. The discrete-time inputs and outputs are used with the CSI algorithm from [4]. For both methods a model order of 12 is considered.

The modal mass is computed for mode shapes normalized with respect to the component corresponding to output at DOF 2, and compared with the exact modal mass $m_j = [0.81 \ 0.17 \ 0.33 \ 1.26 \ 0.19 \ 0.26]$. Figure 2 shows the normalized error for the modal mass computation when the actual intersample behavior of the inputs satisfies three different discretization methods, i.e., zero-order-hold (zoh), first-order-hold (foh), and the impulse invariance method (ii). It can be seen that in general the normalized error is relatively large when using zoh or foh, compared to when using the impulse invariance method. This applies for both the CSI and pLSCF algorithms.

Figure 3 shows histograms for the modal mass of the second mode obtained using CSI (top) and pLSCF (bottom) for zoh (left), foh (middle), and ii (right). It is seen that all histograms appear normal and when using CSI with ii assumption, the estimates seem unbiased. There is a small bias for pLSCF using ii, while the bias is considerable when using zoh and foh for both CSI and pLSCF.

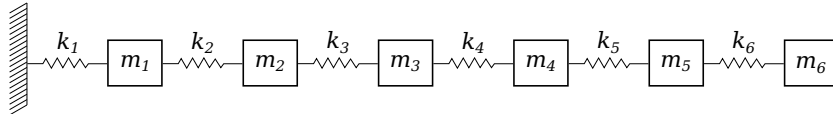


Fig. 1. Sketch of spring-mass system.

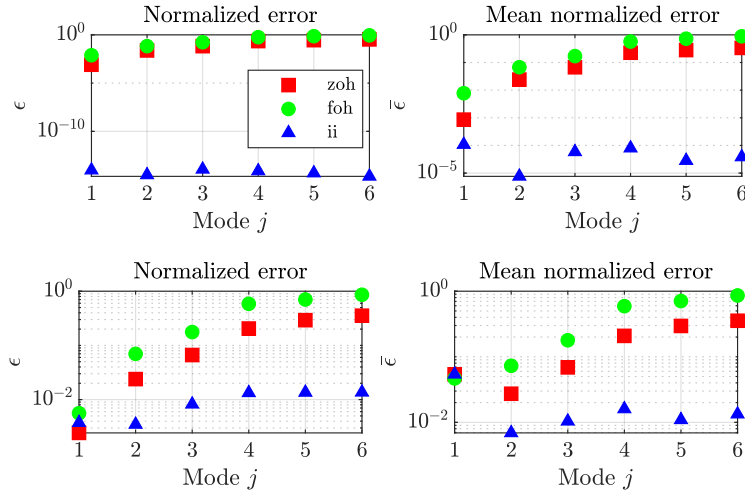


Fig. 2. The normalized error for the modal mass computation using the three different discretization methods; zoh, foh, and ii for both the CSI (top) and pLSCF algorithm (bottom). The error is shown for one realization without noise (left) and 1000 realizations with noise on the outputs (right).

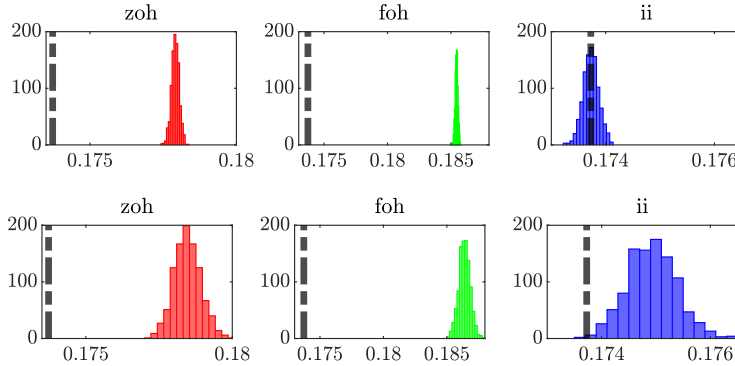


Fig. 3. Histograms for the 2nd estimated modal mass with noise on outputs using CSI (top) and pLSCF algorithm (bottom) for the three discretization methods: zoh (left), foh (middle), and ii (right). The dashed line is the exact value for the modal mass.

5 Conclusion

In this work, three different discretization methods (zero-order-hold, first-order-hold, and the impulse invariance method) for the intersample behavior of inputs have been used for discrete-time domain models and their impact on the modal mass computation has been showcased on a simulated six degrees of free-

dom mass-spring system. Two modelling approaches have been used; state-space modelling in time domain and a right matrix fraction description in frequency domain, whereas the CSI- and pLSCF algorithms have been used to estimate the modal masses in time and frequency domain, respectively. While the assumption on the intersample behaviour is explicit in the CSI approach through the input and feedthrough matrices B and D , the pLSCF assumes equality of the discrete-time and continuous-time FRFs $H(z) = H(s)$ where the appropriate discretization method appears to be the impulse invariant method.

It was shown that the intersample behaviour has a large impact on the modal mass computation, and relatively large errors appear in the computations if the dataset has been generated under a different intersample behavior than the one assumed in the modal mass computation. Besides, it was illustrated that for the modal mass computation with noise on the output, all histograms appeared normal.

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