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► **To cite this version:**

Andrey Polyakov, Xubin Ping. Homogeneous SMC Design Using Artificial Neural Network. 17th International Workshop on Variable Structure Systems (VSS2024), Oct 2024, Abu DHABI, United Arab Emirates. hal-04761758

**HAL Id: hal-04761758**

**<https://inria.hal.science/hal-04761758v1>**

Submitted on 31 Oct 2024

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# Homogeneous SMC Design Using Artificial Neural Network

Andrey Polyakov<sup>1</sup> and Xubin Ping<sup>2</sup>

**Abstract**—The paper proposes a renewed design of homogeneous unit sliding mode controller for multi-input system. The so-called homogeneous artificial neural network is utilized for the design. The presented approach simplifies practical implementation of the homogeneous unit control. Theoretical results are supported by numerical simulations.

## I. INTRODUCTION

Systems with sliding modes [1], [2], [3] form an important research domain of control theory, which cover all aspects of a control system design from purely theoretical ideas to various issues of practical implementation. The classical relay [4] and unit [5] sliding mode controllers (SMCs) as well as a more recent super-twisting algorithm [6] are still very popular control solutions [7], [8], [9].

This paper is devoted to issues of practical implementation of the so-called homogeneous unit control proposed recently in [8]. The homogeneous unit control is a modified version of the classical unit control, which can be designed under less conservative restrictions to selected sliding surface. The design principles are based on generalized homogeneity.

Homogeneity is a dilation symmetry [10], [11], which is well-known in the sliding mode control theory as well [12], [13], [14]. The homogeneity simplifies the control systems design and analysis [15], [16], [17], [18], [19], [20]. The key feature of homogeneous system is the equivalence of its local and global behavior due to dilation symmetry of solutions.

The homogeneous unit control is designed by means of the so-called canonical homogeneous norm [21] being an implicit Lyapunov function of the sliding variable. The use of the canonical homogeneous norm essentially simplifies an adjustment of control parameters and allows their tuning based on Linear Matrix Inequalities (LMIs). However, this norm is defined implicitly as a solution of a nonlinear algebraic equation. Therefore, practical implementation of such a control algorithm requires a special numerical algorithm to be developed (see e.g., [20]). In real-time control systems, where the computation time is very limited, an implementation of a computationally hard algorithm is impossible. For such cases, in this paper we develop an alternative approach to implementation of homogeneous unit control based on artificial neural networks.

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From mathematical point of view, Artificial Neural Network (ANN) is a universal approximator of a continuous mapping on a compact (see, e.g., [22], [23]). Theoretically ANN has arbitrary precise approximation under a proper tuning (known also as learning or training) of parameters. Since 2014 the ANNs are realized in microchips [24] allowing their efficient practical realization in real-time systems. ANNs are utilized for SMC design as approximators of uncertain dynamics (see, e.g. [25] and references therein). This paper suggests another way of ANN application in SMC theory.

The generalized homogeneous ANN [26] allows global approximation of homogeneous function to be realized based on local data. The structure of homogeneous ANN is very similar to the structure to the classical ANN, so it does not need any modification/adaptation of learning algorithms or hardware/software realizations of ANNs. The main contribution of this paper is the ANN-based design of the homogeneous unit control. We use the homogeneous ANN for approximation of the homogeneous unit control, which is defined implicitly. In this case, the use of ANN simplifies the practical implementation of the control law.

The paper is organized as follows. First, the problem statement is presented. Next, some preliminaries about homogeneous systems and neural networks are given. After then, the ANN-based homogeneous unit control design is proposed and proven. Finally, a numerical simulation example and concluding remarks are given.

*Notation:*  $\mathbb{R}$  is the field of real numbers;  $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ ;  $\|\cdot\|$  denotes a norm in  $\mathbb{R}^n$  and  $|x| = \sqrt{x^\top x}$  is the canonical Euclidean norm;  $C^n(X, Y)$  is the set of continuously differentiable, at least, up to the order  $n$ ) maps  $X \rightarrow Y$ , where  $X, Y$  are open subsets of finite dimensional spaces; the case  $n = 0$  corresponds to continuous functions;  $\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$ ;  $I_n \in \mathbb{R}^{n \times n}$  is the identity matrix;  $\mathbf{0}$  denotes the zero element, e.g.,  $\mathbf{0} \in \mathbb{R}^n$  is the zero vector but  $\mathbf{0} \in \mathbb{R}^{n \times n}$  is the zero matrix;  $\text{diag}\{\lambda_1, \dots, \lambda_n\}$  - diagonal matrix with elements  $\lambda_i$ ; the order relation  $P \succ 0$  means positive definiteness of the symmetric matrix  $P \in \mathbb{R}^{n \times n}$ ;  $\lambda_{\max}(P)$  and  $\lambda_{\min}(P)$  denote maximal and minimal eigenvalues of the symmetric matrix  $P \in \mathbb{R}^{n \times n}$ ;  $\Re(\lambda)$  denotes the real part of the complex number  $\lambda$ ; the notation  $P^{\frac{1}{2}}$  means that  $P^{\frac{1}{2}} = M$  is such that  $P = M^2$ ; we use  $\mapsto$  for definition of functions, e.g.,  $f : \mathbb{R}^n \mapsto \mathbb{R}^n$  or  $x \mapsto f(x)$ .

## II. PROBLEM STATEMENT

Let a control system be modeled by the following ODE:

$$\dot{x} = Ax + B(u + \gamma(t, x)), \quad t > t_0, \quad x(t_0) = x_0, \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the system state,  $u(t) \in \mathbb{R}^m$  is a control input,  $A \in \mathbb{R}^{n \times n}$  is the system matrix,  $B \in \mathbb{R}^{n \times m}$  is the control matrix and  $\gamma : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^m$  is an unknown function, which models uniformly bounded system uncertainties and disturbances.

The traditional goal of SMC design is to stabilize the state of the system to a desired surface  $Cx = \mathbf{0}$  in a finite time. If the unknown function  $\gamma$  is uniformly bounded

$$|\gamma(t, x)| \leq \bar{\gamma}, \quad \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n. \quad (2)$$

and  $p = m$ ,  $\det(CB) \neq 0$  then the unit control [5], [1]

$$u = -(CB)^{-1}(CA)x + (\rho + \|CB\|\bar{\gamma})(CB)^{-1} \frac{Cx}{\|Cx\|}, \quad \rho > 0. \quad (3)$$

solves the considered problem. For  $p \neq m$  this classical technique of SMC design is inapplicable. Under a mild assumption about controllability of the output

$$\sigma = Cx,$$

the so-called *homogeneous unit control* [8]

$$u = \hat{K}x + K\mathbf{d}(-\ln \|Cx\|_{\mathbf{d}})Cx \quad (4)$$

enforces the sliding mode on the surface  $Cx = 0$  even when  $p \neq m$ , where  $\hat{K} \in \mathbb{R}^{m \times n}$ ,  $K \in \mathbb{R}^{m \times p}$  are properly tuned control gains,  $\mathbf{d}(s) = e^{sG_{\mathbf{d}}}$ ,  $s \in \mathbb{R}$  is a linear continuous dilation in  $\mathbb{R}^p$  with a generator  $G_{\mathbf{d}} \in \mathbb{R}^{p \times p}$  and  $\|\cdot\|_{\mathbf{d}}$  in  $\mathbb{R}^p$  is the canonical homogeneous norm defined in a special way (see Section III-A for more details about homogeneity and dilations). Notice that the term  $\mathbf{d}(-\ln \|Cx\|_{\mathbf{d}})Cx$  in (4) is an analog of the unit vector  $\frac{Cx}{\|Cx\|}$  in (3). Moreover, for  $K_0 = -(CB)^{-1}(CA)$ ,  $G_{\mathbf{d}} = I_n$  and  $\|\cdot\|_{\mathbf{d}} = \|\cdot\|$  the control law (4) simply coincides with (3). Therefore, the conventional unit control is a particular case of the feedback (4).

The gains  $K_0$  and  $K$  of the homogeneous unit control can be easily tuned using LMIs (see [8] or Section IV-B for more details).

The key difficulty of the practical implementation of the presented algorithm is an implicit definition of the norm  $\|\cdot\|_{\mathbf{d}}$ . Indeed, to compute  $\|Cx\|_{\mathbf{d}}$  for a given  $Cx$ , we need to solve the nonlinear equation  $\|e^{\bar{s}G_{\mathbf{d}}}Cx\| = 1$  with respect to the variable  $\bar{s} \in \mathbb{R}$  and take, by definition,  $\|Cx\|_{\mathbf{d}} := e^{-\bar{s}}$  (see Section III-A). For  $p \neq m$  it is difficult to solve the mentioned nonlinear equation in a real time since  $G_{\mathbf{d}} \neq I_n$  in this case.

This paper suggests to design SMC algorithm using an approximation of the homogeneous unit control. Notice that existing approximates of the canonical homogeneous norm are not exact and may lead to conservative parameters tuning [21], [27]. In this paper we introduce an arbitrary precise approximation of the homogeneous unit control based on the so-called homogeneous ANN. This allows a non-conservative tuning of control parameters.

### III. PRELIMINARIES

#### A. Generalized Homogeneity

1) *Linear dilations*: Linear dilations in Euclidean spaces and their specific properties are introduced following [20].

A one-parameter family of mappings  $\mathbf{d}(s) : \mathbb{R}^n \mapsto \mathbb{R}^n$ ,  $s \in \mathbb{R}$  is said to be a dilation in  $\mathbb{R}^n$  if it satisfies the **group property**:

$$\mathbf{d}(0)x = x, \quad \mathbf{d}(s+t)x = \mathbf{d}(s) \circ \mathbf{d}(t)x, \quad \forall t, s \in \mathbb{R}, \forall x \in \mathbb{R}^n$$

and the **limit property**:

$$\liminf_{s \rightarrow +\infty} \|\mathbf{d}(s)x\| = +\infty, \quad \limsup_{s \rightarrow -\infty} \|\mathbf{d}(s)x\| = 0, \quad \forall x \neq \mathbf{0}.$$

- A dilation  $\mathbf{d}$  is *continuous* if the function  $s \mapsto \mathbf{d}(s)x$  is continuous for any fixed  $x \in \mathbb{R}^n$ .
- A dilation  $\mathbf{d}$  is *monotone* with respect to a norm  $\|\cdot\|$  in  $\mathbb{R}^n$  if the function  $s \mapsto \|\mathbf{d}(s)x\|$  is strictly increasing.
- A dilation  $\mathbf{d}$  is *strictly monotone* with respect to a norm  $\|\cdot\|$  in  $\mathbb{R}^n$  if there exists  $\beta > 0$  such that  $\|\mathbf{d}(s)x\| \leq e^{\beta s} \|x\|$  for all  $x \in \mathbb{R}^n$  and all  $s < 0$ .

In this paper we deal only with a linear continuous dilation, which is defined by the matrix exponential

$$\mathbf{d}(s) = e^{sG_{\mathbf{d}}}, \quad s \in \mathbb{R}$$

where  $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$  is an anti-Hurwitz matrix being the generator of the dilation.

A linear continuous dilation is strictly monotone with respect to the weighted Euclidean norm  $\|\sigma\| = \sqrt{\sigma^{\top} P \sigma}$  with  $0 \prec P = P^{\top} \in \mathbb{R}^{p \times p}$  if and only if [21]

$$PG_{\mathbf{d}} + G_{\mathbf{d}}^{\top} P \succ 0.$$

Below we use this LMI for the homogeneous SMC design.

The standard (or Euler) dilation corresponds to  $G_{\mathbf{d}} = I_n$ . The weighted dilation [10], [28], [15], [17] is the most popular in control systems theory. It is identified by the diagonal generator  $G_{\mathbf{d}} = \text{diag}(r_1, r_2, \dots, r_n) \succ 0$ .

2) *Explicit and implicit homogeneous norms*: Generalized (non-standard) dilation introduce an alternative topology in  $\mathbb{R}^n$  by means of the so-called homogeneous norm. For the weighted dilation  $\mathbf{d}(s) = \text{diag}(e^{sr_1}, \dots, e^{sr_n})$  the homogeneous norm can be introduced, for example, as follows [29], [15], [30]:

$$\|x\|_{\bar{r}} = \left( \sum_{i=1}^n |x_i|^{\frac{\bar{r}}{r_i}} \right)^{\frac{1}{\bar{r}}}, \quad \bar{r} > 0, \quad x = (x_1, \dots, x_n)^{\top} \in \mathbb{R}^n.$$

The considered positive definite function  $x \mapsto \|x\|_{\bar{r}}$  satisfies the identity

$$\|\mathbf{d}(s)x\|_{\bar{r}} = e^s \|x\|_{\bar{r}} \quad (5)$$

being similar to the identity  $\|e^s x\| = e^s \|x\|$  requested for the conventional norm. That is why any positive definite function satisfying (5) was called by *homogeneous norm* [30]. In the case of the standard dilation (i.e., for  $r_1 = \dots = r_n = 1$ ) the homogeneous norm becomes the usual norm if  $\bar{r} \geq 1$ .

A possible design of a homogeneous norm for a linear continuous dilation with a diagonalizable generator is suggested by the following lemma.

*Lemma 1 (Explicit homogeneous norm for diagonalizable dilation):*

Let a linear continuous dilation  $\mathbf{d}$  have a diagonalizable generator, i.e.,  $\exists J \in \mathbb{R}^{n \times n}, \det(J) \neq 0$  such that

$$G_{\mathbf{d}} = J^{-1} \Lambda J, \quad \Lambda = \text{diag}\{r_1, r_2, \dots, r_n\} \succ 0. \quad (6)$$

Then the function  $\|\cdot\|_{\bar{r}} : \mathbb{R}^n \mapsto \mathbb{R}$  defined as follows

$$\|x\|_{\bar{r}} = (\Psi^{\top}(x) Q \Psi(x))^{\frac{1}{2\bar{r}}}, \quad \Psi(x) = \begin{pmatrix} [e_1^{\top} J x]^{\frac{\bar{r}}{r_1}} \\ [e_2^{\top} J x]^{\frac{\bar{r}}{r_2}} \\ \dots \\ [e_n^{\top} J x]^{\frac{\bar{r}}{r_n}} \end{pmatrix}, \quad \bar{r} > 0 \quad (7)$$

is a  $\mathbf{d}$ -homogeneous norm in  $\mathbb{R}^n$  and  $\Psi$  is a homeomorphism on  $\mathbb{R}^n$ , where  $h_i = (0, \dots, 1, \dots, 0)^{\top} \in \mathbb{R}^n$  is the unit vector and  $0 \prec Q = Q^{\top} \in \mathbb{R}^{n \times n}$ .

*Definition 1 (Canonical homogeneous norm):* Let a linear continuous dilation  $\mathbf{d}$  in  $\mathbb{R}^n$  be monotone with respect to a norm  $\|\cdot\|$ . A function  $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \mapsto [0, +\infty)$  defined as follows:  $\|\mathbf{0}\|_{\mathbf{d}} = 0$  and

$$\|x\|_{\mathbf{d}} = e^{s_x}, \quad \text{where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, \quad x \neq \mathbf{0} \quad (8)$$

is said to be a canonical  $\mathbf{d}$ -homogeneous norm in  $\mathbb{R}^n$

*Corollary 1:* [21] Let a linear continuous dilation  $\mathbf{d}$  in  $\mathbb{R}^n$  be strictly monotone with respect to the norm  $\|x\| = \sqrt{x^{\top} P x}$ , where  $0 \prec P = P^{\top} \in \mathbb{R}^{n \times n} : P G_{\mathbf{d}} + G_{\mathbf{d}}^{\top} P \succ 0$ . Then

- 1)  $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \mapsto \mathbb{R}_+$  is continuous on  $\mathbb{R}^n$ ;
- 2)  $\|\pm \mathbf{d}(s)x\|_{\mathbf{d}} = e^s \|x\|_{\mathbf{d}}, \quad \forall s \in \mathbb{R}, \forall x \in \mathbb{R}^n$ ;
- 3)  $\|x\| = 1 \Leftrightarrow \|x\|_{\mathbf{d}} = 1, \quad \|x\| < 1 \Leftrightarrow \|x\|_{\mathbf{d}} < 1 \quad \text{and}$   
 $\|x\| > 1 \Leftrightarrow \|x\|_{\mathbf{d}} > 1$ ;
- 4) for  $\alpha = \lambda_{\max}(P^{\frac{1}{2}} G_{\mathbf{d}} P^{-\frac{1}{2}} + P^{-\frac{1}{2}} G_{\mathbf{d}}^{\top} P^{\frac{1}{2}})$  and  $\beta = \lambda_{\min}(P^{\frac{1}{2}} G_{\mathbf{d}} P^{-\frac{1}{2}} + P^{-\frac{1}{2}} G_{\mathbf{d}}^{\top} P^{\frac{1}{2}})$  one holds

$$\min\{\|x\|_{\mathbf{d}}^{\alpha}, \|x\|_{\mathbf{d}}^{\beta}\} \leq \|x\| \leq \max\{\|x\|_{\mathbf{d}}^{\alpha}, \|x\|_{\mathbf{d}}^{\beta}\}, \quad \forall x \in \mathbb{R}^n; \quad (9)$$

- 5) the derivative of  $\|\cdot\|_{\mathbf{d}}$  is given by

$$\frac{\partial \|x\|_{\mathbf{d}}}{\partial x} = \|x\|_{\mathbf{d}} \frac{x^{\top} \mathbf{d}^{\top} (-\ln \|x\|_{\mathbf{d}}) P \mathbf{d} (-\ln \|x\|_{\mathbf{d}})}{x^{\top} \mathbf{d}^{\top} (-\ln \|x\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d} (-\ln \|x\|_{\mathbf{d}}) x}, \quad x \neq \mathbf{0}. \quad (10)$$

3) *Homogeneous systems:* Homogeneous mappings and homogeneous systems are studied in [10], [11], [15], [16], [20].

A function  $h : \mathbb{R}^n \mapsto \mathbb{R}^m$  is said to be  $\mathbf{d}$ -homogeneous of degree  $\nu$  if

$$h(\mathbf{d}(s)x) = e^{\nu s} h(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}. \quad (11)$$

The dilation symmetry for vector fields (ODEs) is defined differently. Below we say that  $f : \mathbb{R}^n \mapsto \mathbb{R}^n$  is a vector field only if it defines a right-hand of an ODE

$$\dot{x} = f(x), \quad t > 0. \quad (12)$$

A vector field  $f : \mathbb{R}^n \mapsto \mathbb{R}^n$  is said to be  $\mathbf{d}$ -homogeneous of degree  $\mu$  if

$$f(\mathbf{d}(s)x) = e^{\mu s} \mathbf{d}(s) f(x), \quad \forall x \in \mathbb{R}^n, \quad \forall s \in \mathbb{R}. \quad (13)$$

Homogeneous ODEs have many useful properties for systems analysis. For example, the sign of the homogeneity degree specifies the convergence rate of stable homogeneous systems. The case  $\mu < 0$  corresponds to finite-time stability

typical for SMC systems. The homogeneous unit sliding mode control has been designed in [8] using the linear homogeneity (i.e., homogeneity with respect to linear dilations).

## B. Homogeneous Artificial Neural Networks

Mathematically, Artificial Neural Network (ANN) is a universal approximator of a continuous function on a compact. It has an inspiration from neurobiology and a rather simple mathematical description in a form of a composition of affine/linear mappings with a nonlinear function  $\phi : \mathbb{R} \mapsto \mathbb{R}$  called an *activation function*. Universal approximation theorems [22], [31], [23] characterize functions, which can be approximated by an ANN. Their statements vary dependently of the restriction to the activation function. To the best of authors' knowledge, the weakest restriction to  $\phi$  is given by the following theorem.

*Theorem 1 (Universal approximation theorem, [32]):* Let  $\phi \in C(\mathbb{R}, \mathbb{R})$  be a non-polynomial function and  $\Omega \subset \mathbb{R}^n$  be a compact set. If  $g : \mathbb{R}^n \mapsto \mathbb{R}$  is a continuous function then for any  $\epsilon > 0$  there exists  $N \in \mathbb{N}, \Theta \in \mathbb{R}^{N \times n}, \xi \in \mathbb{R}^N$  and  $W \in \mathbb{R}^{1 \times N}$  such that

$$|g(x) - g_{\epsilon}(x)| \leq \epsilon, \quad \forall x \in \Omega, \quad (14)$$

$$g_{\epsilon}(x) = W \phi(\Theta x + \xi), \quad (15)$$

where the function  $\phi$  in the above formula is applied in the component-wise manner.

The function  $g_{\epsilon}$  given by (15) is an ANN with one hidden layer. The above theorem shows that the considered ANN is the universal approximator of any continuous mapping on a compact.

In ANN theory, the function  $\phi$  from the above theorem is called *activation function*. The following activation functions are most popular:

- *Sigmoid:*  $\phi(r) = (1 + e^r)^{-1}, r \in \mathbb{R}$ ;
- *Hyperbolic Tangent:*  $\phi(r) = \frac{e^r - e^{-r}}{e^r + e^{-r}}, r \in \mathbb{R}$ ;
- *Rectified Linear Unit:*  $\phi(r) = \max\{0, r\}, r \in \mathbb{R}$ .

All activation functions given above are non-polynomial and continuous, so they satisfy Theorems 1. Below we use the sigmoid being the most classical activation function [22].

In the case of a homogeneous function approximation, the universal approximation theorem can be revisited as follows.

*Theorem 2 (Homogeneous universal approximation theorem, [26]):* Let  $\mathbf{d}$  be a linear continuous dilation in  $\mathbb{R}^n$ . Let  $\|\cdot\|_{\mathbf{d}}$  be an arbitrary (**explicit or implicit**)  $\mathbf{d}$ -homogeneous norm in  $\mathbb{R}^n$ . Let  $\phi \in C(\mathbb{R}, \mathbb{R})$  be a non-polynomial function. If  $h \in C(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbb{R}^m)$  is a  $\mathbf{d}$ -homogeneous function of degree  $\nu \in \mathbb{R}$  then for any  $\epsilon > 0$  there exist  $N \in \mathbb{N}, \Theta \in \mathbb{R}^{N \times n}, \xi \in \mathbb{R}^N$  and  $W \in \mathbb{R}^{m \times N}$  such that

$$|h(x) - h_{\epsilon}(x)| \leq \epsilon \|x\|_{\mathbf{d}}^{\nu}, \quad \forall x \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \quad (16)$$

$$h_{\epsilon}(x) = \|x\|_{\mathbf{d}}^{\nu} W \phi(\Theta \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x + \xi), \quad (17)$$

where the function  $\phi$  in the above formula is applied in the component-wise manner.

The above theorem generalizes the result discovered in [33] for standard dilations. It shows that any homogeneous

function, due to dilation symmetry, can be approximated globally based on the local information. Moreover, for the homogeneous function of the zero degree ( $\nu = 0$ ) the approximation error estimate is uniform on the whole  $\mathbb{R}^n$ . Below we use this feature for the homogeneous SMC design.

#### IV. ANN-BASED HOMOGENEOUS UNIT CONTROL DESIGN

##### A. ANN approximation of the homogeneous projector

The key term in the homogeneous unit control (4) is the vector  $\mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma \in \mathbb{R}^p$  with  $\sigma = Cx \in \mathbb{R}^p$ , which depends of the canonical homogeneous norm  $\|\cdot\|_{\mathbf{d}}$  defined implicitly. By construction, this is a unit vector  $\|\mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma\| = 1$  for any  $\sigma \neq \mathbf{0}$ . The function  $\pi_{\mathbf{d}}(\sigma) = \mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma$  is known as the *homogeneous projector* on the unit sphere [20].

*Corollary 2: Let a diagonalizable linear continuous dilation  $\mathbf{d}$  in  $\mathbb{R}^p$  be strictly monotone with respect to the weighted Euclidean norm  $\|\sigma\| = \sqrt{\sigma^\top P \sigma}$ , where  $0 \prec P = P^\top \in \mathbb{R}^{p \times p}$ . Let  $\|\cdot\|_{\mathbf{d}}$  be the canonical homogeneous norm induced by  $\|\cdot\|$ . Let  $\|\cdot\|_{\tilde{r}}$  be an explicit  $\mathbf{d}$ -homogeneous norm in  $\mathbb{R}^p$  designed by Lemma 1.*

If  $\phi \in C(\mathbb{R}, \mathbb{R})$  is a non-polynomial function then for any  $\epsilon > 0$  there exist  $N \in \mathbb{N}$ ,  $\Theta \in \mathbb{R}^{N \times p}$ ,  $\xi \in \mathbb{R}^N$  and  $W \in \mathbb{R}^{p \times N}$  such that

$$|\mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma - v_\epsilon(\sigma)| \leq \epsilon, \quad \forall \sigma \neq \mathbf{0}, \quad (18)$$

$$v_\epsilon(\sigma) = W\phi(\Theta\mathbf{d}(-\ln \|\sigma\|_{\tilde{r}})\sigma + \xi), \quad (19)$$

where the function  $\phi$  in the above formula is applied in the component-wise manner. Moreover, if

$$\tilde{J}(W, \Theta, \xi) := \sup_{\sigma \in S} |v_\epsilon(\sigma) - \sigma| \leq \epsilon \quad (20)$$

for  $S = \{\sigma \in \mathbb{R}^p : \sigma^\top P \sigma = 1\}$ , then the estimate (18) holds.

In the view of the proven corollary, the training of ANN we can be realized considering the vectors  $\sigma_j$  uniformly distributed on the unit sphere  $S$  and minimizing the functional

$$J(W, \Theta, \xi) = \sum_{j=1}^M (W\phi(\Theta\mathbf{d}(-\ln \|\sigma_j\|_{\tilde{r}})\sigma_j + \xi) - \sigma_j)^2 \rightarrow \min.$$

In practice, the parameters  $\Theta$  and  $\xi$  are frequently selected randomly [34], [35], while the matrix  $W$  is obtained by minimization of the above functional.

Therefore, the following algorithm suggests a possible design of an ANN approximation of the unit control.

*Algorithm 1 (Approximation of  $\mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma$ ):*

**Initialization.** Select  $\epsilon > 0$  and a natural  $N \in \mathbb{N}$

**Step 1.** Generate uniformly distributed random numbers  $\xi_i \in [0, 1]$  and uniformly distributed random row-vectors  $\theta_i \in [0, 1]^p$  with  $i = 1, 2, \dots, N$ .

**Step 2.** Generate uniformly distributed random points  $\sigma_j \in \mathbb{R}^n$  on the unit sphere  $\sigma^\top P \sigma = 1$  with  $j = 1, 2, \dots, M$ .

**Step 3.** Compute  $M \times N$ -matrix  $G$  with the elements

$$G_{j,i} = \frac{1}{1 + e^{\xi_i + \theta_i \mathbf{d}(-\ln \|\sigma_j\|_{\tilde{r}})\sigma_j}} \quad (21)$$

**Step 4.** Compute  $w_k \in \mathbb{R}^{1 \times N}$ ,  $k = 1, \dots, p$  by the formula

$$w_k = G^+ \begin{pmatrix} e_k^\top \sigma_1 \\ e_k^\top \sigma_2 \\ \vdots \\ e_k^\top \sigma_M \end{pmatrix} \quad (22)$$

where  $e_k = (0, \dots, 0, 1, 0, \dots, 0)^\top \in \mathbb{R}^p$ ,  $k = 1, 2, \dots, p$  is the identity vector and  $G^+$  denotes the Moore pseudo inverse matrix to  $G$ .

**Step 5.** Define the homogeneous ANN by the formula (19) with the parameters

$$W = \begin{pmatrix} w_1 \\ \vdots \\ w_p \end{pmatrix}, \quad \Theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_N \end{pmatrix}, \quad \xi = (\xi_1, \dots, \xi_N)^\top.$$

**Step 5.** If  $\tilde{J}(W, \Theta, \xi) > \epsilon$  then  $N \leftarrow N+1$  and go to **Step 1**.

For a sufficiently dense grid of points  $\sigma_j$  on the unit sphere and a sufficiently large number of neurons  $N$  with uniformly randomly selected parameters  $\Theta$  and  $\xi$ , the above algorithm guarantees an approximation of the homogeneous projector  $\mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma$  by ANN with the given approximation precision  $\epsilon > 0$ . This can be proven in the usual way [34], [35]. In this paper we claim this as a conjecture without proof.

##### B. SMC control design

Inspired by the above derivations and the homogeneous unit control design developed in [8], we present the following algorithm of SMC control design.

*Algorithm 2 (ANN-based homogeneous unit control):*

**Step 1.** Given  $C \in \mathbb{R}^{p \times m}$ , solve the linear algebraic equation

$$C(A + B\tilde{K})(I_n - C^\top(CC^\top)^{-1}C) = \mathbf{0} \quad (23)$$

and compute

$$A_\sigma = C(A + B\tilde{K})C^\top(CC^\top)^{-1}, \quad B_\sigma = CB. \quad (24)$$

If the equation (23) has no solution with respect to  $\tilde{K}$  or the pair  $\{A_\sigma, B_\sigma\}$  is not controllable, then the developed homogeneous unit SMC design scheme is not applicable.

**Step 2.** Solve the linear algebraic equation

$$A_\sigma G_0 - G_0 A_\sigma + B_\sigma Y_0 = A_\sigma, \quad G_0 B_\sigma = \mathbf{0} \quad (25)$$

with respect to  $Y_0 \in \mathbb{R}^{m \times p}$ ,  $G_0 \in \mathbb{R}^{p \times p}$  and define

$$G_{\mathbf{d}} = I_n - G_0, \quad \mathbf{d}(s) = e^{sG_{\mathbf{d}}}, \quad s \in \mathbb{R}, \quad (26)$$

$$A_0 = A_\sigma - B_\sigma Y_0 G_{\mathbf{d}}^{-1}. \quad (27)$$

If the pair  $\{A_\sigma, B_\sigma\}$  is controllable then a solution  $(G_0, Y_0)$  exists, the matrix  $G_{\mathbf{d}}$  is anti-Hurwitz and diagonalizable, but the matrix  $A_0$  is nilpotent [8].

**Step 3.** For  $\epsilon > 0$  and  $\rho > 0$ , solve the linear algebraic system

$$\begin{aligned} (A_0 + \rho G_{\mathbf{d}})X + X(A_0 + \rho G_{\mathbf{d}})^\top + B_\sigma Y + Y^\top B_\sigma^\top + \hat{\gamma}X &= -\hat{\gamma}B_\sigma B_\sigma^\top, \\ G_{\mathbf{d}}X + XG_{\mathbf{d}}^\top &\succ 0, \quad X = X^\top \succ 0 \end{aligned} \quad (28)$$

with respect to  $X \in \mathbb{R}^{n \times n}$ ,  $Y \in \mathbb{R}^{m \times n}$ , where  $\hat{\gamma} = \bar{\gamma} + \epsilon$ .

**Step 4.** Using Lemma 1 define an explicit homogeneous norm  $\|\cdot\|_{\tilde{r}}$  in the form (7).

**Step 5.** For

$$\epsilon \in \left(0, \frac{\epsilon}{\sqrt{\lambda_{\max}(X^{-1}Y^\top Y X^{-1})}}\right) \quad \text{and} \quad P = X^{-1} \quad (29)$$

apply Algorithm 1 and find the parameters  $W, \Theta, \xi$  of the ANN approximation

$$v_\epsilon(\sigma) = W\phi(\Theta \mathbf{d}(-\ln \|\sigma\|_{\bar{r}})\sigma + \xi), \quad \phi(r) = \frac{1}{1+e^r}, \quad r \in \mathbb{R} \quad (30)$$

of the homogeneous projector  $\mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma$ .

**Step 5.** Define the feedback law

$$u = \tilde{K}x + K_0Cx + Kv_\epsilon(Cx), \quad K = YX^{-1}. \quad (31)$$

Due to identity (23) the dynamics of the sliding variable  $\sigma = Cx$  for the closed-loop system (1), (31) is given by [8]

$$\dot{\sigma} = A_0\sigma + B_\sigma(Kv_\epsilon + \gamma(t, x)). \quad (32)$$

This dynamics can be rewritten as follows

$$\dot{\sigma} = A_0\sigma + B_\sigma(K\mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma + \tilde{\gamma}), \quad (33)$$

where

$$\tilde{\gamma} = \gamma + K(v_\epsilon(\sigma) - \mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma).$$

Due to selection of parameter  $\epsilon$  (see Algorithm 2) we have

$$|\tilde{\gamma}| \leq |\gamma| + |K(v_\epsilon(\sigma) - \mathbf{d}(-\ln \|\sigma\|_{\mathbf{d}})\sigma)| \leq \bar{\gamma} + \epsilon \sqrt{\lambda_{\max}(K^\top K)} = \bar{\gamma} + \epsilon = \hat{\gamma}.$$

Applying Theorem 3 from [8] we complete the proof of the following theorem.

*Theorem 3:* Let the feedback law (31) be defined by Algorithm 2, the estimate (20) hold and the uncertain function  $\gamma$  satisfies the condition (2). Then the system (32) is globally uniformly finite-time stable and

$$\sigma(t) = \mathbf{0}, \quad \forall t \geq \|\sigma(0)\|_{\mathbf{d}}/\rho.$$

The above theorem show that the ANN-based homogeneous unit control (31) has the same properties as the original (implicit) homogeneous unit controller. Namely, it insures the sliding mode on the surface  $Cx = \mathbf{0}$  and rejects the bounded matched perturbation of the magnitude  $\bar{\gamma}$ .

## V. NUMERICAL EXAMPLE

Let us consider the example from [8]:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using (23) we derive  $\tilde{K} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$ , so

$$A_\sigma = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & 3 \end{pmatrix}, \quad B_\sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The pair  $\{A_\sigma, B_\sigma\}$  is controllable, so we can design a SMC. A solution of (25) is given by

$$G_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0.5 & 0 & 0 \end{pmatrix}, \quad Y_0 = \begin{pmatrix} -0.5 & -2.5 & -0.5 \\ -1.5 & 1.5 & -3.5 \end{pmatrix},$$

so the generator of the dilation  $\mathbf{d}$  is has the form

$$G_{\mathbf{d}} = \begin{pmatrix} 2 & 0 & 0 \\ 0.5 & 1 & 0 \\ -0.5 & 0 & 1 \end{pmatrix}, \quad \text{and } K_0 = \begin{pmatrix} 0 & 0 & 0.5000 & -2.5000 & -0.5000 \\ 0 & 0 & -1.5000 & 1.5000 & -3.5000 \end{pmatrix}.$$

Solving the linear system (28) for  $\bar{\gamma} = 0.4, \epsilon = 0.1$  and  $\rho = 1$  we derive the control parameters

$$K = \begin{pmatrix} 5.8379 & -2.5103 & 1.1938 \\ -5.8379 & 1.1938 & -2.5103 \end{pmatrix}, \quad P = \begin{pmatrix} 5.3112 & -1.3265 & 1.3265 \\ -1.3265 & 0.5413 & -0.2751 \\ 1.3265 & -0.2751 & 0.5413 \end{pmatrix}.$$

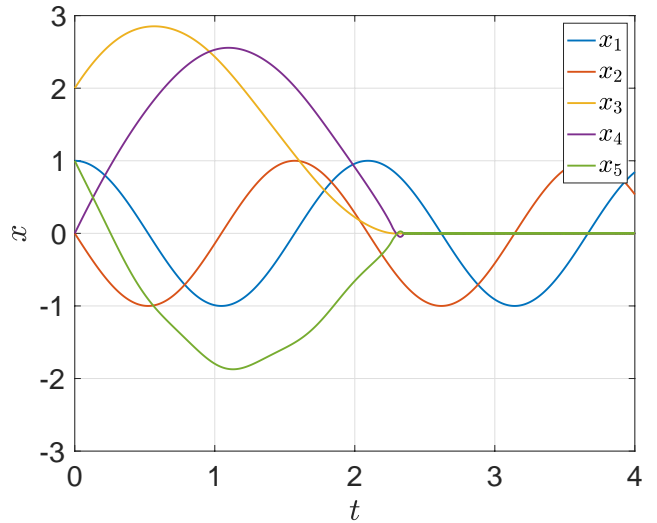


Fig. 1. The states of the system with the ANN-based homogeneous unit control (31)

We define the explicit homogeneous norm in the form (7) with  $\bar{r} = 1, \Lambda = \text{diag}(1, 1, 2)$  and

$$J = \begin{pmatrix} 0.5000 & 0 & 1.0000 \\ -0.5000 & 1.0000 & 0 \\ 1.2247 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.3777 & -0.1602 & 0.3701 \\ -0.1602 & 0.3777 & -0.3701 \\ 0.3701 & -0.3701 & 1.3224 \end{pmatrix}.$$

Finally, we approximate the homogeneous projector by ANN using Algorithm 1 with  $N = 10$  and  $M = 218$ . The identified parameters of the homogeneous ANN are

$$W^\top = \begin{pmatrix} 13.5596 & -6.8008 & -5.7874 \\ -13.6193 & 26.8979 & -24.0310 \\ 13.0903 & -23.1810 & 11.5070 \\ -41.5002 & -5.2537 & 25.6086 \\ 0.2463 & 3.6766 & -6.3507 \\ 10.6307 & 21.3408 & -13.0933 \\ -7.6098 & 10.5104 & -13.1711 \\ 10.6975 & 7.0618 & 0.3810 \\ 26.5101 & -27.4186 & 13.7912 \\ 0.8133 & -10.5075 & 7.6427 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 0.2700 & 0.3608 & -0.0579 \\ 0.2469 & -0.1935 & 0.2956 \\ -0.1847 & -0.1939 & 0.0514 \\ 0.2194 & 0.0286 & 0.0576 \\ -0.4158 & -0.3793 & -0.3712 \\ -0.2647 & -0.0624 & 0.1480 \\ 0.3531 & 0.1680 & 0.4585 \\ 0.3081 & -0.2913 & -0.2017 \\ 0.3476 & 0.0574 & 0.3879 \\ -0.2990 & -0.3997 & 0.1517 \end{pmatrix}, \quad \xi = \begin{pmatrix} -0.1190 \\ -0.0282 \\ 0.4260 \\ -0.4978 \\ -0.4933 \\ -0.1830 \\ 0.4405 \\ 0.2207 \\ 0.1702 \\ -0.3840 \end{pmatrix}.$$

The simulation results of the closed-loop system (1) with the designed SMC controller (31) and the matched perturbation  $\gamma = 0.4 \begin{pmatrix} \sin(2t) \\ \cos(10t) \end{pmatrix}$  are depicted on Figures 1 and 2.

Homogeneous Control Systems Toolbox for MATLAB<sup>1</sup> has been utilized for simulations and comparison reasons. The system has been discretized by means of zero-order-hold (sample-and-hold) method with step size  $h = 0.001$ . The trajectories of the system (1), (31) are very similar (see [8]) to ones obtained for the implicitly defined homogeneous unit control (4), but the time of numerical simulation has been reduced in 10 times! This indirectly confirms that the use of ANN may reduce the computational load and essentially simplify a practical implementation of the homogeneous unit control.

<sup>1</sup> <http://researchers.lille.inria.fr/~polyakov/hcs/>

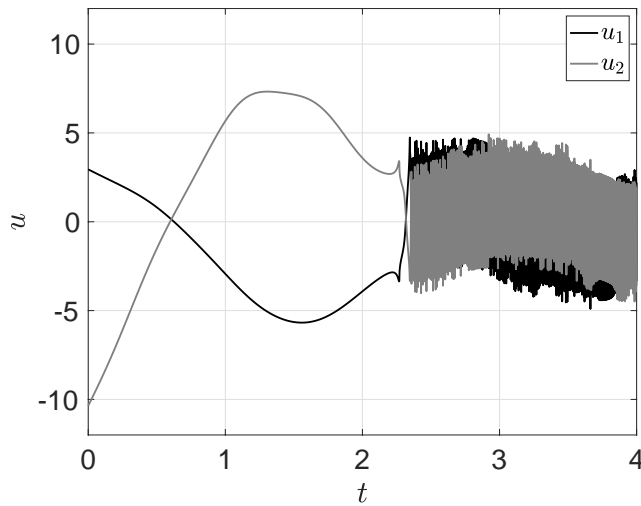


Fig. 2. The control signal generated by the ANN-based homogeneous unit control (31)

## VI. DISCUSSIONS AND CONCLUSIONS

In this paper, an ANN-based design of a homogeneous unit SMC is developed. The ANN is utilized for the approximation of the homogeneous projector defined implicitly. The homogeneous projector is an analog of the standard projector  $\frac{\sigma}{\|\sigma\|}$  on the unit sphere. In the case of homogeneous homogeneous unit control, this projector is defined implicitly. The use of ANN simplifies the practical realization of the control in real-time systems, since ANNs can be implemented in microchips. Moreover, the numerical simulation show that the computation time of the ANN-based homogeneous unit control can be reduced in 10 times comparing with its original (implicit) form. Further development of ANN-based application of homogeneous algorithms is the interesting direction of future research.

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