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# Finite/fixed-time homogeneous stabilization of infinite dimensional systems

Andrey Polyakov and Yury Orlov

**Abstract**—Finite/fixed-time control design procedure is developed for an infinite dimensional system modeled by abstract evolution equation in a Hilbert space. It is based on solving certain operator equations and inequalities. For a class of PDE models, the corresponding equations/inequalities are shown to be algebraic and solvable in many cases. Theoretical results are supported by examples of controlled PDE models.

## I. INTRODUCTION

### A. State of the art

Homogeneous systems are studied since 1950s for both finite dimensional [1], [2], [3], [4], [5], [6] and infinite dimensional [7], [8], [9] systems. In control theory, the homogeneity is utilized for sliding mode control design [10], [11], finite/fixed-time stability analysis [12], [13], [14], [15], robustness analysis [16], [17], [18] and many other problems [9]. Homogeneous control design for infinite dimensional system governed by an abstract linear evolution equation was first proposed in [19]. The corresponding design procedure was nonconstructive and applicable only for a very restrictive class of generalized homogeneous linear plants. Recently [20], the homogenization method has been developed for controllable MIMO plants modeled by linear ODEs. This method admits a homogeneous (finite/fixed-time) control design for non-homogeneous plants by solving a system of linear algebraic equations/inequalities. The present paper generalizes the homogenization method to infinite dimensional control systems governed by abstract evolution equations.

### B. Motivating Examples

Popular supertwisting and twisting algorithms, which are long recognized for their finite-time stability and robustness features, have recently been extended to PDE setting. The diffusion process

$$\frac{\partial x}{\partial t} = \nu \frac{\partial^2 x}{\partial z^2} + u + \phi(t, x), \quad z \in (a, b), t > 0, \quad (1)$$

$$u(z, t) = -\lambda_1 \frac{x(z, t)}{\|x(\cdot, t)\|_{L^2}^{1/2}} + w(z, t), \quad \frac{\partial w}{\partial t} = -\lambda_2 \frac{x(z, t)}{\|x(\cdot, t)\|_{L^2}} \quad (2)$$

and wave dynamics

$$\frac{\partial^2 x}{\partial t^2} = \mu \frac{\partial^2 x}{\partial z^2} + v + \psi(t, x), \quad z \in (a, b), t > 0, \quad (3)$$

$$v = -l_1 \frac{x(z, t)}{\|x(\cdot, t)\|_{L^2(a, b)}} - l_2 \frac{x_t(z, t)}{\|x_t(\cdot, t)\|_{L^2(a, b)}} \quad (4)$$

driven by the dynamic supertwisting controller (2), and, respectively, twisting controller (4), have been addressed under homogeneous Dirichlet/Neumann boundary conditions.

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In the boundary value problems (1) and (3) subject to either Dirichlet or Neumann boundary conditions the scalar state  $x(z, t)$  is a function of the spatial variable  $z$ , located in the compact domain  $[a, b]$  where  $a < b$ , and of time  $t \geq 0$ ;  $u(\cdot, t), (v, t) \in L^2((a, b), \mathbb{R})$  are distributed control inputs,  $\phi(\cdot, t)$  and  $\psi(\cdot, t)$  are matched external disturbances such that

$$\begin{aligned} \|\phi_t(\cdot, t)\|_{L^2(a, b)} &\leq M_1, \quad \frac{|\phi_t(z, t)|}{M_1} \leq \frac{|x(z, t)|}{\|x(\cdot, t)\|_{L^2(a, b)}} \\ \|\phi_{tz}(\cdot, t)\|_{L^2(a, b)} &\leq M_2, \quad \|\psi(\cdot, t)\|_{L^2(a, b)} \leq M \end{aligned} \quad (5)$$

for all  $t \geq 0$ ,  $z \in [a, b]$  and some *a priori* known constants  $M_1, M_2, M > 0$ . Along with standard magnitude bounds on the temporal and spatial disturbance derivatives, involved into (5) for specifying admissible disturbances, the second inequality of (5) additionally imposes a state-dependent disturbance constraint. In spite of this constraint, an admissible disturbance does not necessarily escape to zero as  $\|x(\cdot, t)\|_{L^2(a, b)} \rightarrow 0$ , similar to that in the finite-dimensional setting. Indeed, the constraint (5) relies on the norm  $\frac{|x(z, t)|}{\|x(\cdot, t)\|_{L^2(a, b)}} \|L^2(a, b)$  which remains unit beyond the origin  $x \equiv 0$  thereby admitting non-vanishing disturbances in the origin.

Under the relations  $\lambda_2 > \max\{2M_1, M_2\}$ ,  $\lambda_1 > \max\{2M, 2M\lambda_2^{-1}\}$  and  $l_2 > M$ ,  $l_1 > l_2 + M$ , imposed on the controller gains, it was shown in [21, Theorems 1&3] that the above diffusion and wave closed-loop systems are globally asymptotically stable regardless of whichever admissible disturbance (5) affects the underlying system.

Open problems had then be raised by the authors of the afore-mentioned work whether the finite-time stability was inherited from their finite-dimensional (super- and twisting) counterparts. Apart from this, the above work did not investigate if the asymptotic stability persists for the infinite horizon along the spatial variable as  $a \rightarrow -\infty$  and  $b \rightarrow \infty$ . These theoretical challenges are addressed in the present work via the homogeneity-based approach to be developed.

Another interesting example treated in the paper is finite/fixed-time stabilization of an interconnected (e.g., in a cascade) system of one-dimensional heat equations.

### C. Contributions of the paper

In this paper a nonlinear homogeneous (finite/fixed-time) control is designed for infinite dimensional systems governed by an abstract evolution equation with an unbounded linear closed densely defined operator and locally Lipschitz nonlinearity/perturbations. The design procedure consists in solving a system of operator equations and inequalities. It is shown that for a class of PDE models, the corresponding operator equations and inequalities become algebraic and can be easily solved in

many practical cases. Finally, to demonstrate the simplicity of the design, a homogeneous finite-time stabilization for an underactuated interconnected system of one-dimensional heat equations is studied. A complimentary contribution is a homogeneity-based finite-time stability analysis of the twisting and supertising systems considered above.

#### D. Notation

$\mathbb{R}$  is the field of reals;  $\mathbb{V}$  is a vector space;  $\mathbb{X}, \mathbb{U}$  are real Banach spaces;  $\|\cdot\|_{\mathbb{V}}$  denotes a norm in  $\mathbb{V}$  (usually, we omit the index  $\mathbb{V}$  if the context is clear);  $\mathbb{H}$  is a real Hilbert space with an inner product  $\langle \cdot, \cdot \rangle$ ;  $\mathbf{0}$  is the zero element of a space;  $\mathcal{D}(G)$  denotes a domain of an operator  $G : \mathcal{D}(G) \subset \mathbb{H} \mapsto \mathbb{H}$ ; let  $\mathcal{D}(G^\infty)$  denote a set of all *entire vectors*<sup>1</sup> of  $G$ ;  $\mathcal{L}(\mathbb{X}, \mathbb{U})$  is the Banach space of linear bounded operators  $\mathbb{X} \mapsto \mathbb{U}$ ;  $I$  is the identity operator  $Ix = x, \forall x$ ;  $\bar{D}(V(x); d) = \limsup_{h \rightarrow 0} \frac{V(x+hd) - V(x)}{h}$  is the upper directional Gateaux derivative at the point  $x \in \mathbb{X}$  of the functional  $V : \mathbb{X} \mapsto \mathbb{R}$  in the direction  $d \in \mathbb{X}$ , whereas  $DV \in \mathcal{L}(\mathbb{X}, \mathbb{R})$  denotes the Frechét derivative of  $V$ ;  $\text{trace}(\Theta)$  is the trace of the matrix  $\Theta \in \mathbb{R}^{n \times n}$ ;  $|z| = \sqrt{z^\top z}$  is the canonical Euclidean norm in  $\mathbb{R}^k$ ;  $|\Gamma| = \sup_{z \neq 0} \frac{|\Gamma z|}{|z|}$  for  $\Gamma \in \mathbb{R}^k$ ;  $C_c^\infty, C^q, L^q$  are well-known function spaces;  $P \succ 0$  means that the square symmetric matrix  $P$  is positive definite;

## II. PROBLEM STATEMENT

Let us consider the controlled non-linear system:

$$\dot{x} = Ax + Bu(x) + f(t, x), \quad t > t_0, \quad x(t_0) = x_0 \in \mathbb{H}, \quad (6)$$

where  $x(t) \in \mathbb{H}$  is the state vector of the system,  $A : \mathcal{D}(A) \subset \mathbb{H} \mapsto \mathbb{H}$ ,  $u(x) \in \mathbb{U}$  is a time-invariant feedback (to be defined),  $B : \mathbb{U} \mapsto \mathbb{H}$ , and  $f(t, x) : \mathbb{R} \times \mathbb{H} \mapsto \mathbb{H}$  is a nonlinear locally Lipschitz function, which models uncertainties/disturbances.

*Assumption 1:* Let a linear closed densely defined operator  $A : \mathcal{D}(A) \subset \mathbb{H} \mapsto \mathbb{H}$  be an infinitesimal generator of a strongly continuous semigroup  $\Phi$  of linear bounded operators on  $\mathbb{H}$  and  $B : \mathbb{U} \mapsto \mathbb{H}$  be a linear bounded operator.

Recall that [24], a continuous function  $x : [t_0, T) \mapsto \mathbb{H}$  is a *mild solution* of the system (6) if  $\forall t \in [t_0, T)$ , one reads

$$x(t) = \Phi(t - t_0)x_0 + \int_{t_0}^t \Phi(t - s)(Bu(x(\tau)) + f(\tau, x(\tau)))d\tau.$$

A mild solution is *strong* if  $x(t) \stackrel{a.e.}{\in} \mathcal{D}(A)$  and (6) is fulfilled almost everywhere on the time interval  $(t_0, T)$ .

*The control objective* is to design a feedback law  $u(x)$  such that the closed-loop unperturbed ( $f = \mathbf{0}$ ) system is finite-time (or fixed-time) stable and to characterize a class of perturbations  $f$ , which are rejected by the control.

The class of admissible controllers  $u$  is assumed to be confined to locally bounded mappings  $\mathbb{H} \mapsto \mathbb{U}$  which are locally Lipschitz on  $\mathbb{H} \setminus \{\mathbf{0}\}$  to ensure that strong/mild solution of the closed-loop system (6) is uniquely defined beyond the origin where the feedback controller is possibly non-Lipschitz continuous. The meaning of such a closed-loop

system in the origin is then adopted in the generalized sense of [25, Definition 2.4] as a limiting result obtained through the regularization procedure similar to that proposed for the finite-dimensional systems [26].

Inspired by [19] we develop a design procedure based on the concept of the generalized homogeneity (see below). We expand the approach introduced in [20] to a class of infinite dimensional systems (6). The key idea of this approach is to design a linear feedback law  $K_0x$  such that the operator  $A + BK_0$  becomes homogeneous of a desired degree  $\mu \neq 0$ .

*Assumption 2 (Homogenization Condition):* There exist a linear closed densely defined operator  $G_0 : \mathcal{D}(G_0) \subset \mathbb{H} \mapsto \mathbb{H}$ , a linear closed densely defined operator  $\tilde{G} : \mathcal{D}(\tilde{G}) \subset \mathbb{U} \mapsto \mathbb{U}$  and a linear bounded operator  $K_0 \in \mathcal{L}(\mathbb{H}, \mathbb{U})$  such that

- $G_0$  is an infinitesimal generator of a strongly continuous group  $\mathbf{d}_0$  of linear bounded operators on  $\mathbb{H}$  such that  $\mathbf{d}_0(s)\mathcal{D}(A) \subset \mathcal{D}(A), \forall s \in \mathbb{R}$ ;
- $\tilde{G}$  is an infinitesimal generator of a strongly continuous group  $\tilde{\mathbf{d}}$  of linear bounded operators on  $\mathbb{U}$ ;
- there exist a set  $\mathcal{D}_0 \subset \mathcal{D}(G_0^\infty)$  dense in  $\mathbb{H}$  such that  $G_0^n \mathcal{D}_0 \subset \mathcal{D}(A), \forall n = 0, 1, 2, \dots$  and a set  $\tilde{\mathcal{D}} \subset \mathcal{D}(\tilde{G}^\infty)$  dense in  $\mathbb{U}$  such that  $B\tilde{\mathcal{D}} \subset \mathcal{D}(G_0^\infty)$ ,

$$A_0G_0\sigma - G_0A_0\sigma = A_0\sigma, \quad \forall \sigma \in \mathcal{D}_0, \quad (7)$$

$$G_0Bu = B\tilde{G}u, \quad \forall u \in \tilde{\mathcal{D}}, \quad (8)$$

$$A_0 := A + BK_0. \quad (9)$$

For the finite dimensional system ( $\mathbb{H} = \mathbb{R}^n, \mathbb{U} = \mathbb{R}^m$ ), the fulfillment of this assumption follows from controllability of the pair  $\{A, B\}$  (by taking  $\tilde{G} = \mathbf{0}$ ). Indeed, in the finite-dimensional setting, conditions (7), (8) become the system of linear algebraic equations with respect to  $G_0 \in \mathbb{R}^{n \times n}$ ,  $Y_0 = K_0(G_0 - I_n) \in \mathbb{R}^{n \times m}$ , which has a solution if and only if the pair  $\{A, B\}$  is controllable [20]. To check Assumption 2 in the infinite dimensional case, the system of operator equations (7), (8) must be solved with respect to  $G_0, \tilde{G}$  and  $K_0$ . Below we show that for a class of differential operators  $A$  and some generators  $G_0, \tilde{G}_0$ , the conditions (7), (8) are algebraic equations (see Section V).

Below Assumption 2 is shown to be sufficient to guarantee that the operator  $A + BK_0$  is homogeneous of degree  $\mu \in \mathbb{R}$  with respect to a dilation  $\mathbf{d}$  in  $\mathbb{H}$  generated by the operator  $G_{\mathbf{d}} = I + \mu G_0$ . This is necessary for a homogeneous finite/fixed-time stabilization of the linear plant. We also impose an additional assumption required for the well-posedness of the closed-loop system and provide a sufficient condition for the homogeneous stabilization of the unperturbed ( $f = \mathbf{0}$ ) plant (6), which basically repeats the criterion of linear feedback stabilization (see, e.g., [27]). We study the robustness of exponential, finite-time and fixed-time stability of the perturbed ( $f \neq \mathbf{0}$ ) closed-loop system (6). As an illustration of the results obtained for abstract control system (6) we study a class of differential operators  $A$  which admits the homogeneous control design by solving an algebraic system.

<sup>1</sup>A vector  $x \in \mathbb{H}$  is entire for  $G$  if  $\sum_{i=0}^{\infty} \frac{t^i G^i x}{i!} \in \mathbb{H}$  for all  $t \in \mathbb{R}$ . If  $G$  is a generator of a strongly continuous group of linear bounded operators on  $\mathbb{H}$  then the set of all entire vectors of  $G$  is dense in  $\mathbb{H}$  ([22, page 81], [23]).

### III. PRELIMINARIES

#### A. Linear Dilations in Banach Spaces

Let  $\mathcal{L}(\mathbb{X}, \mathbb{X})$  be the space of linear bounded operators  $\mathbb{X} \mapsto \mathbb{X}$ , where  $\mathbb{X}$  is a real Banach space with a norm  $\|\cdot\|$ . Homogeneity is a dilation symmetry. In this paper, we deal with linear dilation [28], [9, Chapter 6].

*Definition 1:* A mapping  $\mathbf{d} : \mathbb{R} \mapsto \mathcal{L}(\mathbb{X}, \mathbb{X})$  is said to be a linear dilation in  $\mathbb{X}$  if it satisfies

- (Group property)  $\mathbf{d}(0) = I \in \mathcal{L}(\mathbb{X}, \mathbb{X})$  and  $\mathbf{d}(t+s) = \mathbf{d}(t)\mathbf{d}(s)$  for  $t, s \in \mathbb{R}$ ;
- (Limit property)  $\lim_{s \rightarrow -\infty} \|\mathbf{d}(s)u\| = 0$  and  $\lim_{s \rightarrow +\infty} \|\mathbf{d}(s)u\| = \infty$  for all  $u \neq \mathbf{0}$ .

A linear dilation  $\mathbf{d}$  in  $\mathbb{X}$  is

- *strongly continuous* if the mapping  $\mathbf{d}(\cdot)u : \mathbb{R} \mapsto \mathbb{X}$  is continuous for any  $u \in \mathbb{X}$ ;
- *uniformly continuous* if the mapping  $\mathbf{d}(\cdot) : \mathbb{R} \mapsto \mathcal{L}(\mathbb{X}, \mathbb{X})$  is continuous;
- *strictly monotone* if  $\exists \beta > 0 : \|\mathbf{d}(s)\| \leq e^{\beta s}, \forall s \leq 0$ .

A strongly continuous linear dilation always has an infinitesimal generator  $G_{\mathbf{d}} : \mathcal{D}(G_{\mathbf{d}}) \subset \mathbb{X} \mapsto \mathbb{X}$  given by  $G_{\mathbf{d}}u = \lim_{s \rightarrow 0} \frac{\mathbf{d}(s)u - u}{s}$  with the domain  $\mathcal{D}(G_{\mathbf{d}}) = \{u \in \mathbb{X} : \exists \lim_{s \rightarrow 0} \frac{\mathbf{d}(s)u - u}{s}\}$  being the linear subspace dense in  $\mathbb{X}$ . The generator  $G_{\mathbf{d}}$  is a closed linear operator satisfying the property [24, Chapter 1, Theorem 2.4]: for any  $u \in \mathcal{D}(G_{\mathbf{d}})$  and any  $s \in \mathbb{R}$  one holds  $\mathbf{d}(s)u \in \mathcal{D}(G_{\mathbf{d}})$  and the function  $\mathbf{d}(\cdot)u : \mathbb{R} \rightarrow \mathcal{D}(G_{\mathbf{d}})$  is continuously differentiable,  $\frac{d}{ds}\mathbf{d}(s)u = G_{\mathbf{d}}\mathbf{d}(s)u = \mathbf{d}(s)G_{\mathbf{d}}u$ .

*Lemma 1:* Let  $\mathbb{X}$  be a real Banach space. If an operator  $G_0 : \mathcal{D}(G_0) \subset \mathbb{X} \mapsto \mathbb{X}$  is an infinitesimal generator of a strongly continuous group  $\mathbf{d}_0$  of linear bounded operators on  $\mathbb{X}$  then the operator  $G_{\mathbf{d}} = I + \mu G_0$ ,  $\mathcal{D}(G_{\mathbf{d}}) = \mathcal{D}(G_0)$  is a generator of a strongly continuous dilation  $\mathbf{d}$  in  $\mathbb{X}$  for any  $\mu \in \mathbb{R}$  satisfying the inequality

$$1 + \mu\omega > 0, \quad (10)$$

where  $\omega \in \mathbb{R}$  is such that  $\|\mathbf{d}_0(s)\| \leq Me^{\omega s}, \forall s \leq 0$  with  $M \geq 1$ . **Proof.** If  $G_0$  is a generator of a strongly continuous group  $\mathbf{d}_0$  then  $\mu G_0$  with  $\mu \in \mathbb{R}$  is the generator of a strongly continuous group  $\mathbf{d}_\mu$ . Moreover, for any  $\mu \in \mathbb{R}$  the operator  $G_{\mathbf{d}} = I + \mu G_0$  is the generator of the strongly continuous group  $\mathbf{d}$  such that  $\mathbf{d}(s) = e^s \mathbf{d}_\mu(s), \forall s \in \mathbb{R}$ . Since  $\mathbf{d}_0$  is a strongly continuous group then, by [24, Theorem 2.2., p.4], there exist  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that  $\|\mathbf{d}_0(s)\| \leq Me^{\omega s}, \forall s \leq 0$ . Moreover, for any  $\mu \in \mathbb{R}$  we have  $\|\mathbf{d}_\mu(s)\| \leq Me^{\mu\omega s}, \forall s \leq 0$ . Since  $\mathbf{d}_\mu$  is a group then  $\mathbf{d}_\mu(-s)$  is the inverse operator to  $\mathbf{d}_\mu(s)$  and  $1 = \|I\| = \|\mathbf{d}_\mu(s)\mathbf{d}_\mu(-s)\| \leq \|\mathbf{d}_\mu(s)\| \cdot \|\mathbf{d}_\mu(-s)\|$ . Hence, we derive  $\frac{e^{\mu\omega s}}{M} \leq \|\mathbf{d}_\mu(s)\|, \forall s \geq 0$ , and  $\frac{e^{(1+\mu\omega)s}}{M} \leq \|\mathbf{d}(s)\| = \|e^s \mathbf{d}_\mu(s)\|, \forall s \geq 0$ , and  $\|\mathbf{d}(s)\| = \|e^s \mathbf{d}_\mu(s)\| \leq Me^{(1+\mu\omega)s}, \forall s \leq 0$ . Therefore, for any  $\mu \in \mathbb{R}$  such that  $1 + \mu\omega > 0$ , we have  $\|\mathbf{d}(s)\| \rightarrow +\infty$  as  $s \rightarrow +\infty$  and  $\|\mathbf{d}(s)\| \rightarrow 0$  as  $s \rightarrow -\infty$ , i.e., the strongly continuous group  $\mathbf{d}$  is a dilation in  $\mathbb{X}$ . The proof is complete.

A linear dilation  $\mathbf{d}$  in a Hilbert space  $\mathbb{H}$  is strictly monotone [19], [9, page 178] if and only if there exists  $\beta > 0$  such that

$$\langle G_{\mathbf{d}}x, x \rangle \geq \beta \|x\|^2, \quad \forall x \in \mathcal{D}(G_{\mathbf{d}}), \quad (11)$$

where  $\langle \cdot, \cdot \rangle$  is the inner product in  $\mathbb{H}$  and  $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ .

#### B. Canonical homogeneous norm

A strictly monotone dilation  $\mathbf{d}$  in  $\mathbb{X}$  induces an alternative norm topology in  $\mathbb{X}$  by means of the the *canonical homogeneous norm*  $\|\cdot\|_{\mathbf{d}} : \mathbb{X} \mapsto \mathbb{X}$  defined [19] as follows  $\|\mathbf{0}\|_{\mathbf{d}} = 0$ ,

$$\|x\|_{\mathbf{d}} = e^{s_x}, \quad \text{where } s_x \in \mathbb{R} : \|\mathbf{d}(-s_x)x\| = 1, \quad x \neq \mathbf{0}. \quad (12)$$

The function  $\|\cdot\|_{\mathbf{d}}$  is a norm (in the conventional sense) in a real Banach space  $\mathbb{X}_{\mathbf{d}}$  homeomorphic to  $\mathbb{X}$  (see, [9, p.183]). The spaces  $\mathbb{X}_{\mathbf{d}}$  and  $\mathbb{X}$  have the same set of elements but different rules for addition and multiplication of vectors.

*Lemma 2 (Lemmas 7.2 and 7.4, [9]):* Let  $\mathbf{d}$  be a strongly continuous strictly monotone linear dilation in a real Banach space  $\mathbb{X}$ . Then the canonical homogeneous norm  $\|\cdot\|_{\mathbf{d}}$  is continuous on  $\mathbb{X}$  and locally Lipschitz on  $\mathbb{X} \setminus \{\mathbf{0}\}$ . Moreover, if  $\mathbb{X} = \mathbb{H}$  is a real Hilbert space then  $\|\cdot\|_{\mathbf{d}}$  is Fréchet differentiable on  $\mathcal{D}(G_{\mathbf{d}}) \setminus \{\mathbf{0}\}$  and the Fréchet derivative of  $\|\cdot\|_{\mathbf{d}}$  at  $x \in \mathcal{D}(G_{\mathbf{d}}) \setminus \{\mathbf{0}\}$  is given by

$$D\|x\|_{\mathbf{d}}(\cdot) = \frac{\langle \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) \cdot, \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x \rangle}{\langle G_{\mathbf{d}}\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x, \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x \rangle} \|x\|_{\mathbf{d}}. \quad (13)$$

#### C. Homogeneous Operator

A functional  $h : \mathbb{X} \mapsto \mathbb{R}$  is  $\mathbf{d}$ -homogeneous of degree  $\nu \in \mathbb{R}$  if  $h(\mathbf{d}(s)x) = e^{\nu s} h(x), \forall s \in \mathbb{R}, x \in \mathbb{X}$ . Homogeneous operators are defined as follows [4], [9, Chapter 7].

*Definition 2:* An operator  $g : \mathcal{D}(g) \subset \mathbb{X} \mapsto \mathbb{X}$  is said to be  $\mathbf{d}$ -homogeneous of degree  $\mu \in \mathbb{R}$  if  $\mathbf{d}(s)\mathcal{D}(g) \subset \mathcal{D}(g), \forall s \in \mathbb{R}$ ,

$$g(\mathbf{d}(s)x) = e^{\mu s} \mathbf{d}(s)g(x), \quad \forall s \in \mathbb{R}, \forall x \in \mathcal{D}(g), \quad (14)$$

where  $\mathbf{d}$  is a linear dilation in  $\mathbb{X}$ .

For example, the operator  $\frac{\partial^2}{\partial z^2} : H^2(\mathbb{R}, \mathbb{R}) \subset L^2(\mathbb{R}, \mathbb{R}) \mapsto L^2(\mathbb{R}, \mathbb{R})$  is  $\mathbf{d}$ -homogeneous of degree  $2\beta \in \mathbb{R}$  with respect to the linear dilation  $(\mathbf{d}(s)x)(z) = e^{\alpha s} x(e^{\beta s} z), s \in \mathbb{R}, x \in L^2(\mathbb{R}, \mathbb{R}), z \in \mathbb{R}$  and  $\alpha - \beta/2 > 0$ .

*Lemma 3:* If the operators  $A$  and  $B$  satisfy Assumptions 1 and 2 and the dilation  $\mathbf{d}$  is defined by Lemma 1 for  $\mu \in \mathbb{R}$  satisfying (10) then the operator  $A_0 = A + BK_0$  is

- an infinitesimal generator of a strongly continuous semi-group of linear bounded operators on  $\mathbb{H}$ ;
- $\mathbf{d}$ -homogeneous of degree  $\mu$  and

$$A_0 G_{\mathbf{d}} \sigma = (G_{\mathbf{d}} + \mu I) A_0 \sigma, \quad \forall \sigma \in \mathcal{D}_0, \quad (15)$$

where  $G_{\mathbf{d}} = I + \mu G_0$  is an infinitesimal generator of the strongly continuous dilation  $\mathbf{d}$  in  $\mathbb{H}$  and  $\mathcal{D}_0$  is a set dense in  $\mathbb{H}$  defined in Assumption 2.

Moreover, the following identity

$$\mathbf{d}(s)B = e^s B \tilde{\mathbf{d}}(\mu s), \quad (16)$$

holds for all  $s \in \mathbb{R}$ .

**Proof.** Since, by Assumptions 1 and 2,  $B$  and  $K_0$  are bounded operators then the operator  $BK_0$  is bounded as well. Since, by Assumption 1,  $A$  is an infinitesimal generator of a strongly continuous group of linear bounded operators on  $\mathbb{H}$ , then, [24, Theorem 1.1, page 76] the operator  $A_0 = A + BK_0$  is an infinitesimal generator of a strongly continuous group of linear bounded operators on  $\mathbb{H}$ , too. By Assumption 2, we have  $A_0 G_0 \sigma - G_0 A_0 \sigma = A_0 \sigma$ . Multiplying both sides of the latter identity on  $\mu$  we obtain (15). Since  $\mathcal{D}_0 \subset \mathcal{D}(G_0^\infty)$  then

any  $x \in \tilde{\mathcal{D}}_0$  is an entire vector for  $G_{\mathbf{d}} = I + \mu G_0$  too, so  $\mathbf{d}(s)x = \sum_{i=0}^{\infty} \frac{s^n G_{\mathbf{d}}^n x}{n!}$ . Using (15) and  $G_0^n \tilde{\mathcal{D}}_0 \subset \mathcal{D}(A_0), \forall n = 0, 1, 2, \dots$  we derive

$$A_0 \mathbf{d}(s)x = \sum_{i=0}^{\infty} \frac{s^n A_0 G_{\mathbf{d}}^n x}{n!} = \sum_{i=0}^{\infty} \frac{s^n (G_{\mathbf{d}} + \mu I)^n A_0 x}{n!} = e^{\mu s} \mathbf{d}(s) A_0 x$$

for all  $x \in \mathcal{D}_0$ . Taking into account that  $A_0$  is a closed operator and the set  $\mathcal{D}_0$  is dense in  $\mathbb{H}$  we complete the proof of the  $\mathbf{d}$ -homogeneity of  $A_0$ . Similarly, using (8) we derive  $G_{\mathbf{d}} B u = B(I + \mu \tilde{G})u$  for any  $u \in \tilde{\mathcal{D}} \subset \mathcal{D}(\tilde{G}^\infty)$ . Since, by Assumption 2,  $B\tilde{\mathcal{D}} \subset \mathcal{D}(G_0^\infty)$  then

$$\mathbf{d}(s) B u = \sum_{i=0}^{\infty} \frac{s^n G_{\mathbf{d}}^n B u}{n!} = B \sum_{i=0}^{\infty} \frac{s^n (I + \mu \tilde{G})^n u}{n!} = e^s B \tilde{\mathbf{d}}(\mu s).$$

The proof is complete.

#### D. Homogeneous Evolution Equations

A system

$$\dot{x} = Ax + g(x), \quad t > 0, \quad A: \mathcal{D}(A) \subset \mathbb{H} \mapsto \mathbb{H}, \quad g: \mathbb{H} \mapsto \mathbb{H}, \quad (17)$$

is said to be  $\mathbf{d}$ -homogeneous of degree  $\mu$  if the operators  $A$  and  $g$  are  $\mathbf{d}$ -homogeneous of degree  $\mu$ . It is expectable that mathematical objects inherited by homogeneous operators are homogeneous as well [19], [20]. So, both strong and mild solutions of homogeneous system (17) are symmetric with respect to the linear dilation  $\mathbf{d}$  in the following sense

$$x_{\mathbf{d}(s)x_0}(t) = \mathbf{d}(s)x_{x_0}(e^{\mu s}t), \quad (18)$$

where  $x_z$  denotes a solution of (17) with the initial condition  $x(0) = z$  and  $\mu \in \mathbb{R}$  is the homogeneity degree of  $g$ .

A generalization of Zubov-Rosier Theorem (see, [1], [5]) to a class of infinite dimensional systems is given below.

*Theorem 1:* [29] Let  $A$  be a generator of a strongly continuous group of linear bounded operators on  $\mathbb{H}$  and  $g$  be locally Lipschitz continuous on  $\mathbb{H} \setminus \{\mathbf{0}\}$ . Let  $A$  and  $g$  be  $\mathbf{d}$ -homogeneous of degree  $\mu$ . The system (17) is globally uniformly asymptotically stable if and only if there exists a  $\mathbf{d}$ -homogeneous functional  $V: \mathbb{H} \mapsto [0, +\infty)$  of degree 1:

- $\exists k_1, k_2 > 0: k_1 \|x\|_{\mathbf{d}} \leq V(x) \leq k_2 \|x\|_{\mathbf{d}}$ ;
- $V$  is locally Lipschitz continuous on  $\mathbb{H} \setminus \{\mathbf{0}\}$ ;
- $\exists \rho > 0$  such that

$$\bar{D}(V(x); Ax + g(x)) \leq -\rho V^{1+\mu}(x), \quad \forall x \in \mathcal{D}(A) \setminus \{\mathbf{0}\}.$$

The above theorem, in particular, guarantees finite-time and nearly fixed-time stability dependently on the sign of the homogeneity degree  $\mu$ . The definitions of the mentioned stability concepts can be found in [30], [13], [15].

*Example 1:* Let  $\mathbb{H} = L^2(\mathbb{R}, \mathbb{R}^2)$ ,  $a = -\infty, b = \infty$ . The heat system (1), (2) with  $\phi = \mathbf{0}$  is  $\mathbf{d}_1$ -homogeneous of degree  $-1$  for the one-parameter group

$$(\mathbf{d}_1(s)\xi)(z) = \begin{pmatrix} e^{1.75s} x(e^{-0.5s}z) \\ e^{0.75s} w(e^{-0.5s}z) \end{pmatrix}, \quad \xi = (x, w)^\top \in \mathbb{H}, \quad z \in \mathbb{R}$$

being a linear strongly continuous dilation in  $\mathbb{X}$  (please see Lemma 6 for the proof of this fact).

For  $\psi = \mathbf{0}$  the wave system (3), (4) can be rewritten in the form (17) with  $Ax = \begin{pmatrix} x_2 \\ \partial_z^2 x_1 \end{pmatrix}$ ,  $x = (x_1, x_2)^\top \in \mathcal{D}(A)$  and

$$g(x) = \begin{pmatrix} \mathbf{0} \\ -l_1 \frac{x_1}{\|x_1\|_{L^2(\mathbb{R}, \mathbb{R})}} - l_2 \frac{x_2}{\|x_2\|_{L^2(\mathbb{R}, \mathbb{R})}} \end{pmatrix}, \quad x = (x_1, x_2)^\top \in \mathbb{H}.$$

The wave system is  $\mathbf{d}_2$ -homogeneous of degree  $-1$  for

$$(\mathbf{d}_2(s)x)(z) = \begin{pmatrix} e^{1.5s} x_1(e^{-s}z) \\ e^{0.5s} x_2(e^{-s}z) \end{pmatrix}, \quad x = (x_1, x_2)^\top \in \mathbb{H}, \quad z \in \mathbb{R}.$$

By Theorem 1, uniformly asymptotic stability of heat/wave system implies its finite-time stability. The perturbed case  $\phi \neq \mathbf{0}$  (resp.,  $\psi \neq \mathbf{0}$ ) can be studied using homogeneous extension [13], [9, Section 3.7.2] of the system and [9, Theorem 8.6].

#### IV. HOMOGENEOUS CONTROL FOR EVOLUTION SYSTEMS

Let us define the feedback control as follows  $u(\mathbf{0}) = \mathbf{0}$ ,

$$u(x) = K_0 x + \|x\|_{\mathbf{d}}^{1+\mu} \tilde{\mathbf{d}}(\mu \ln \|x\|_{\mathbf{d}}) K \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x, \quad (19)$$

where  $K \in \mathcal{L}(\mathbb{H}, \mathbb{U})$  is a linear bounded operator to be designed later,  $\|\cdot\|_{\mathbf{d}}$  is a canonical homogeneous norm induced by an appropriate norm in  $\mathbb{H}$  (to be defined later),  $K_0 \in \mathcal{L}(\mathbb{H}, \mathbb{U})$  and the strongly continuous group  $\tilde{\mathbf{d}}$  is defined in Assumption 2, whereas  $\mu \in \mathbb{R}$  and  $\mathbf{d}$  are defined by Lemma 1. The structure of the feedback law (19) is inspired by its finite-dimensional analog [20]. If  $\|x\|_{\mathbf{d}} = 1$  then the controller (19) becomes the linear feedback  $u = (K_0 + K)x$ , i.e., the homogeneous controller is obtained by a homogeneous scaling of linear feedback.

*Lemma 4:* If Assumption 2 is fulfilled and  $\mu \in \mathbb{R}$  satisfies (10) then the operator  $g: \mathcal{D}(A) \subset \mathbb{H} \mapsto \mathbb{H}$  given by  $g(\mathbf{0}) = \mathbf{0}$ ,

$$g(x) = (A + BK_0)x + \|x\|_{\mathbf{d}}^{1+\mu} B \tilde{\mathbf{d}}(\mu \ln \|x\|_{\mathbf{d}}) K \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x$$

is  $\mathbf{d}$ -homogeneous of degree  $\mu \in \mathbb{R}$ .

*Proof:* Since the linear operator  $A + BK_0$  is  $\mathbf{d}$ -homogeneous (see, Lemma 3) then we just need to show that the nonlinear operator  $g: \mathbb{H} \mapsto \mathbb{H}$ ,  $g(\mathbf{0}) = \mathbf{0}$  and

$$g(x) = \|x\|_{\mathbf{d}}^\mu B \tilde{\mathbf{d}}(\mu \ln \|x\|_{\mathbf{d}}) K \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x, \quad x \neq \mathbf{0}$$

is  $\mathbf{d}$ -homogeneous of degree  $\mu$ . Indeed, using  $\|\mathbf{d}(s)x\|_{\mathbf{d}} = e^s \|x\|_{\mathbf{d}}$  we derive

$$\begin{aligned} \|\mathbf{d}(s)x\|_{\mathbf{d}}^{1+\mu} B \tilde{\mathbf{d}}(\mu \ln \|\mathbf{d}(s)x\|_{\mathbf{d}}) K \mathbf{d}(-\ln \|\mathbf{d}(s)x\|_{\mathbf{d}}) \mathbf{d}(s)x \\ = e^{(\mu+1)s} \|x\|_{\mathbf{d}}^{1+\mu} B \tilde{\mathbf{d}}(\mu s) \tilde{\mathbf{d}}(\mu \ln \|x\|_{\mathbf{d}}) K \mathbf{d}(-\ln \|x\|_{\mathbf{d}}) x. \end{aligned}$$

Taking into account the identity (16) we obtain  $g(\mathbf{d}(s)x) = e^{\mu s} \mathbf{d}(s)g(x)$ . The proof is complete.  $\blacksquare$

The following assumption is involved to guarantee the well-posedness of the closed-loop system (6), (19).

*Assumption 3:* For any  $R > 0$  there exist  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that the family of linear bounded operators

$$s \mapsto \mathbf{d}(s) B K \mathbf{d}(-s) \in \mathcal{L}(\mathbb{H}, \mathbb{H}), \quad s \in \mathbb{R}$$

satisfies the inequality

$$\|\mathbf{d}(s_1) B K \mathbf{d}(-s_1) - \mathbf{d}(s_2) B K \mathbf{d}(-s_2)\|_{\mathcal{L}(\mathbb{H}, \mathbb{U})} \leq M e^{\omega |s_1 - s_2|},$$

for all  $s_i \in \mathbb{R}: |s_i| < R, i = 1, 2$ , where the dilation  $\mathbf{d}$  is defined by Lemma 1 for  $G_0$  satisfying Assumption 2.

The above assumption is fulfilled, for example, if  $\mathbf{d}$  is a uniformly continuous dilation in  $\mathbb{H}$  or if the generator  $G_{\mathbf{d}}$  of the dilation  $\mathbf{d}$  commutes with  $BK$ .

*Lemma 5:* Under Assumptions 1-3, for any  $x_0 \in \mathbb{H} \setminus \{\mathbf{0}\}$ , (resp.,  $x_0 \in \mathcal{D}(A) \setminus \{\mathbf{0}\}$ ), the closed-loop system (6), (19) has a

unique mild (resp., strong) solution provided that  $f$  is locally Lipschitz continuous. For  $x_0 = \mathbf{0}$ , the system has (possibly non-unique) zero solution provided that  $f(t, \mathbf{0}) = \mathbf{0}$ .

*Proof:* Since the canonical homogeneous norm  $\|\cdot\|_{\mathbf{d}}$  is locally Lipschitz on  $\mathbb{H} \setminus \{\mathbf{0}\}$  then, due to Assumption 3, the mapping  $x \mapsto Bu(x)$  is locally Lipschitz on  $\mathbb{H} \setminus \{\mathbf{0}\}$ . In the view of [29, Lemma 2], the closed-loop system has a unique mild solution for any  $x_0 \in \mathbb{H} \setminus \{\mathbf{0}\}$ . For  $x_0 \in \mathcal{D}(A) \setminus \{\mathbf{0}\}$ , the corresponding solution turns out to be strong due to [24, Corollary 2.11, Chap. 4]. If  $A\mathbf{0} + Bu(\mathbf{0}) + f(t, \mathbf{0}) = \mathbf{0}$  then, for  $x_0 = \mathbf{0}$ , the system always possesses the zero solution, which may be non-unique. ■

The parameters  $\mu$  and group  $\tilde{\mathbf{d}}$  specify the properties of the feedback law at zero. Indeed, since  $\pi(x) = \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x$  is uniformly bounded by virtue of  $\|\pi(x)\| = 1$  then the condition

$$e^{(1+\mu)s} \|B\tilde{\mathbf{d}}(\mu s)K\|_{\mathcal{L}(\mathbb{H}, \mathbb{H})} \rightarrow \mathbf{0} \text{ as } s \rightarrow -\infty \quad (20)$$

guarantees  $Bu(x) \rightarrow 0$  as  $x \rightarrow \mathbf{0}$ . In the finite-dimensional case [20], the group  $\tilde{\mathbf{d}}$  is ignored (i.e.,  $\tilde{\mathbf{d}}(s) = I, \forall s \in \mathbb{R}$ ) and continuity of the control depends only on the homogeneity degree  $\mu$ . In particular,  $\mu = -1$  corresponds to the case of the sliding mode control, but  $\mu < -1$  means that the control is unbounded at zero. The following theorem generalizes [19].

*Theorem 2:* Let Assumptions 1-3 be fulfilled,  $f$  be locally Lipschitz,  $f(t, \mathbf{0}) = \mathbf{0}$  and the parameter  $\mu \in \mathbb{R}$  satisfy (10). If there exist numbers  $\beta, \rho \in \mathbb{R}_+$ , an operator  $K \in \mathcal{L}(\mathbb{H}, \mathbb{U})$  and a self-adjoint positive operator  $P \in \mathcal{L}(\mathbb{H}, \mathbb{H})$  such that

$$\langle P(A_0 + BK + \rho G_{\mathbf{d}})x, x \rangle \leq 0, \quad \forall x \in \mathcal{D}_0, \quad (21)$$

$$\langle PG_{\mathbf{d}}x, x \rangle \geq \beta \langle x, x \rangle, \quad \forall x \in \mathcal{D}_0. \quad (22)$$

then the closed-loop system (6), (19) with  $f$  satisfying

$$\exists \kappa \in (0, \rho) : \frac{\langle P\mathbf{d}(-\ln \|x\|_{\mathbf{d}})f(t, x), \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x \rangle}{\langle PG_{\mathbf{d}}\mathbf{d}(-\ln \|x\|_{\mathbf{d}})x, \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x \rangle} \leq \kappa \|x\|_{\mathbf{d}}^{\mu}, \quad (23)$$

is **i)** *finite-time stable* for  $\mu < 0$ :  $x(t) = \mathbf{0}, \forall t \geq t_0 + \frac{\|x_0\|_{\mathbf{d}}^{-\mu}}{(-\mu)(\rho-\kappa)}$ , **ii)** *exponentially stable* for  $\mu = 0$ :  $\|x(t)\|_{\mathbf{d}} \leq e^{-(\rho-\kappa)(t-t_0)} \|x_0\|_{\mathbf{d}}, \forall t \geq t_0$ ; **iii)** *nearly fixed-time stable* for  $\mu > 0$ :  $\|x(t)\|_{\mathbf{d}} \leq r, \forall t \geq \frac{1}{(\rho-\kappa)r^{\mu}}, \forall r > 0, \forall x_0 \in \mathbb{H}$ , where the canonical homogeneous norm  $\|x\|_{\mathbf{d}}$  is induced by the norm  $\|x\| = \sqrt{\langle Px, x \rangle}$  and  $\mathbf{d}$  is generated by  $G_{\mathbf{d}}$ .

*Proof:* First, let us consider the case  $f = \mathbf{0}$  and denote  $g(x) = Ax + Bu(x)$ . On one hand, by taking into account  $\mathcal{D}_0 \subset \mathcal{D}(G_{\mathbf{d}}) \cap \mathcal{D}(A)$  and using (13) we derive

$$\bar{D}(\|x\|_{\mathbf{d}}; g(x)) = \|x\|_{\mathbf{d}} \frac{\langle P\mathbf{d}(-\ln \|x\|_{\mathbf{d}})g(x), \pi(x) \rangle}{\langle PG_{\mathbf{d}}\pi(x), \pi(x) \rangle}, \quad x \in \mathcal{D}_0 \setminus \{\mathbf{0}\},$$

where  $\pi(x) = \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x$  is the homogeneous projector on the unit sphere (i.e.,  $\|\pi(x)\| = 1$ ). On the other hand, using Lemma 3 we conclude that  $g$  is  $\mathbf{d}$ -homogeneous operator of degree  $\mu$ , thereby arriving at  $g(\pi(x)) = \|x\|_{\mathbf{d}}^{-\mu} \mathbf{d}(-\ln \|x\|_{\mathbf{d}})g(x)$  and

$$\bar{D}(\|x\|_{\mathbf{d}}; g(x)) = \|x\|_{\mathbf{d}}^{1+\mu} \frac{\langle P(A_0 + BK)\pi(x), \pi(x) \rangle}{\langle PG_{\mathbf{d}}\pi(x), \pi(x) \rangle} \leq -\rho \|x\|_{\mathbf{d}}^{1+\mu},$$

where inequalities (21), (22) are utilized on the last step. By Lemma 5 the closed-loop system has a strong solution  $x_{x_0}(t)$  for any  $x_0 \in \mathcal{D}(A) \setminus \{\mathbf{0}\}$ , which is locally Lipschitz and satisfies (6) almost everywhere (as long as  $x_{x_0}(t) \neq \mathbf{0}$ ). Since  $x \mapsto \|x\|_{\mathbf{d}}$  is locally Lipschitz on  $\mathbb{H} \setminus \{\mathbf{0}\}$  then the function

$t \mapsto \|x_{x_0}(t)\|_{\mathbf{d}}$  is absolutely continuous. Taking into account that  $\mathcal{D}_0$  is dense in  $\mathbb{H}$  we derive

$$\frac{d\|x_{x_0}(t)\|_{\mathbf{d}}}{dt} \stackrel{a.e.}{=} \bar{D}(\|y\|_{\mathbf{d}}, g(y)) \Big|_{y=x_{x_0}(t)} \leq -\rho \|x_{x_0}(t)\|_{\mathbf{d}}^{1+\mu}$$

as long as  $x_{x_0}(t) \neq \mathbf{0}$ . Hence, dependently of  $\mu$ , we derive the finite-time, exponential or nearly fixed-time convergence of the strong solutions to zero. Similar to, e.g., [19] the same conclusions are established to hold for the closed-loop mild solutions as well. For  $f \neq \mathbf{0}$  we derive

$$\bar{D}(\|x\|_{\mathbf{d}}; g(x) + f) \leq -\rho \|x\|_{\mathbf{d}}^{1+\mu} + \frac{\|x\|_{\mathbf{d}} \langle P\mathbf{d}(-\ln \|x\|_{\mathbf{d}})f, \pi(x) \rangle}{\langle PG_{\mathbf{d}}\pi(x), \pi(x) \rangle},$$

and using (23) we complete the proof. ■

Therefore, to design the homogeneous stabilizer (19) we need to solve, first, the system of operator equations (7), (8) with respect to  $G_0, K_0, G$  and, next, the system of operator inequalities (21), (22) with respect to  $K, P$ . The latter step is similar to a linear feedback design [27].

Notice that the global fixed-time stabilizer for the system (6) can be designed applying Theorem 2 for  $\mu = \mu_1 < 0$  and  $\mu = \mu_2 > 0$  (see, [9, Theorem 9.2] for more details).

## V. HOMOGENEOUS CONTROL FOR PDES

In this section we show how the above abstract results can be applied to a particular class of modes of control systems.

### A. Linear dilations in function spaces

Let  $\mathbb{V}$  be a vector space of functions  $x : \mathbb{R}^n \mapsto \mathbb{R}^k$ . If  $\Theta \in \mathbb{R}^{n \times n}$  and  $\Gamma \in \mathbb{R}^{k \times k}$  then the family of linear operators

$$(\mathbf{d}(s)x)(z) = e^{\Gamma s} x(e^{s\Theta} z), \quad z \in \mathbb{R}^n, \quad x \in \mathbb{V}, \quad s \in \mathbb{R} \quad (24)$$

is a group. To be a linear dilation in a normed vector space  $\mathbb{V}$ , the operator  $\mathbf{d}(s)$  must be bounded for any  $s \in \mathbb{R}$  and the group  $\mathbf{d}$  must satisfy the limit property  $\|\mathbf{d}(s)x\| \rightarrow 0$  as  $s \rightarrow -\infty$  and  $\|\mathbf{d}(s)x\| \rightarrow +\infty$  as  $s \rightarrow +\infty$ . The group (24) may be a dilation in a function space  $\mathbb{V}$  provided that the matrices  $\Gamma, \Theta$  as well as a norm in  $\mathbb{V}$  are properly defined.

*Lemma 6:* The family of operators (24) is

- a strongly continuous group of linear bounded operators on  $L^p(\mathbb{R}^n, \mathbb{R}^k)$ ,  $p \geq 1$ ;
- a strongly continuous linear dilation in  $L^p(\mathbb{R}^n, \mathbb{R}^k)$  provided that the matrix  $\tilde{\Gamma} := \Gamma - \frac{\text{trace}(\Theta)}{p} I_k$  is anti-Hurwitz<sup>2</sup>

*Proof:* For  $\mathbf{d}(s)$  given by (24), using the change of the integration variable in the Lebesgue integral, we derive

$$\|\mathbf{d}(s)x\|_{L^p}^p = \int_{\mathbb{R}^n} |e^{s\Gamma} x(e^{s\Theta} z)|^p dz = \int_{\mathbb{R}^n} \det(e^{-s\Theta}) |e^{\Gamma s} x(\tilde{z})|^p d\tilde{z}.$$

Since  $\det(e^{-s\Theta}) = e^{-s \text{trace}(\Theta)}$  then

$$\|\mathbf{d}(s)x\|_{L^p}^p = \int_{\mathbb{R}^n} |e^{s(\Gamma - \frac{\text{trace}(\Theta)}{p} I_k)} x(\tilde{z})|^p d\tilde{z} \leq |e^{s\tilde{\Gamma}}|^p \|x\|_{L^p}^p.$$

Therefore,  $\mathbf{d}(s)$  is a group of bounded linear operators on  $L^p(\mathbb{R}^n, \mathbb{R}^k)$  and  $\|\mathbf{d}(s)\| = \sup_{x \neq \mathbf{0}} \frac{\|\mathbf{d}(s)x\|_{L^p}}{\|x\|_{L^p}} \leq |e^{s\tilde{\Gamma}}|, \forall s \in \mathbb{R}$ . Since, by assumption,  $\tilde{\Gamma} = \Gamma - \frac{\text{trace}(\Theta)}{p} I_k$  is anti-Hurwitz matrix then  $e^{s\tilde{\Gamma}} r \rightarrow \mathbf{0}$  as  $s \rightarrow -\infty$  and  $e^{s\tilde{\Gamma}} r \rightarrow \infty$  as  $s \rightarrow +\infty$  for any  $r \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ . Hence, we conclude that the limit property is fulfilled and  $\mathbf{d}$  is a linear dilation in  $L^p(\mathbb{R}^n, \mathbb{R}^k)$ .

<sup>2</sup>A matrix  $G \in \mathbb{R}^{n \times n}$  is anti-Hurwitz if  $-G$  is Hurwitz.

Let the vector  $x \in L^p(\mathbb{R}^n, \mathbb{R}^k)$  be chosen arbitrarily. Let us show that for any  $\varepsilon > 0$  there exists  $\bar{s} > 0$  such that  $\|\mathbf{d}(s)x - x\|_{L^p} \leq \varepsilon$  for all  $s \in [-\bar{s}, \bar{s}]$ . Let  $M_1 = \sup_{s \in [-1, 1]} \|\mathbf{d}(s)\| \leq \sup_{s \in [-1, 1]} |e^{s\Gamma}| < +\infty$ . Since the space of infinitely smooth compactly supported functions is densely embedded in  $L^p$  then there exists  $x_\infty \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^k)$  such that  $\|x - x_\infty\|_{L^p} \leq \frac{\varepsilon}{3(M_1+1)}$ . This implies that

$$\|\mathbf{d}(s)x - x\|_{L^p} \leq \|\mathbf{d}(s)x_\infty - x_\infty\|_{L^p} + \varepsilon/3$$

for all  $s \in [-1, 1]$ . Since  $x_\infty$  is compactly supported then there exists  $r > 0$  such that  $x_\infty(z) = \mathbf{0}$  for all  $z : |z| > r$ . If  $M_2 = \sup_{s \in [-1, 1]} |e^{\Theta s}| \geq 1$  then  $x_\infty(e^{\Theta s}z) - x_\infty(z) = \mathbf{0}$  for all  $z : |z| > rM_2$  and  $s \in [-1, 1]$ . In this case, we derive

$$\|\mathbf{d}(s)x_\infty - x_\infty\|_{L^p} = \left( \int_{\mathbb{R}^n} |e^{s\Gamma} x_\infty(e^{\Theta s}z) - x_\infty(z)|^p dz \right)^{\frac{1}{p}} \leq |e^{s\Gamma} - I_m| \cdot \|x_\infty\|_{L^p} + \left( \int_{|z| \leq rM_2} M_3^p |(e^{\Theta s} - I_n)z|^p dz \right)^{\frac{1}{p}},$$

where  $M_3 = \sup_{|z| \leq M_2 r} |\nabla x_\infty| < +\infty$  (due to  $x_\infty \in C_c^\infty$ ) and the identity

$$x_\infty(z+h) - x_\infty(z) = \left( \int_0^1 \nabla x_\infty(z + \lambda h) d\lambda \right) h, \quad \forall h \in \mathbb{R}^n$$

is utilized on the last step. Since  $M_4 = \sup_{s \in [-1, 1]} \frac{|e^{s\Gamma} - I_k|}{|s|} < +\infty$ ,

$$M_5 = \sup_{s \in [-1, 1]} \frac{|e^{s\Theta} - I_n|}{|s|} < +\infty \text{ and } M_6 = \int_{|z| \leq rM_2} dz < +\infty,$$

then, for  $|s| \leq \bar{s} = \min \left\{ 1, \frac{\varepsilon}{3M_4\|x_\infty\|}, \frac{\varepsilon}{3rM_2M_3M_5M_6^{1/p}} \right\}$ ,

$$\|\mathbf{d}(s)x_\infty - x_\infty\|_{L^p} \leq |s|M_4\|x_\infty\|_{L^p} + |s|rM_2M_3M_5M_6^{\frac{1}{p}} \leq \frac{2\varepsilon}{3}.$$

Therefore,  $\forall \varepsilon > 0, \exists \bar{s} > 0 : \|\mathbf{d}(s)x - x\|_{L^p} \leq \varepsilon$  for all  $s \in [-\bar{s}, \bar{s}]$ , i.e., the group  $\mathbf{d}$  is strongly continuous at  $s = 0$ . Using the group property  $\mathbf{d}(s+t) = \mathbf{d}(s)\mathbf{d}(t)$ ,  $\forall s, t \in \mathbb{R}$  we prove the strong continuity for all  $s \in \mathbb{R}$ . ■

The matrix  $\Gamma$  may be Hurwitz and determine a dilation in  $L^p$ .

*Example 2:* Let  $\mathbb{V} = L^2(\mathbb{R}^n, \mathbb{R}^k)$  and  $\mathbf{d}$  is given by (24) with  $\Theta = -I_n$  and  $\Gamma = -I_k$ . Using the change of the integration variable in the Lebesgue integral, we derive

$$\|\mathbf{d}(s)x\|_{\mathbb{V}}^2 = \int_{\mathbb{R}^n} |e^{-s}x(e^{-s}z)|^2 dz = e^{s(-2+n)}\|x\|_{\mathbb{V}}^2.$$

Therefore,  $\mathbf{d}$  is a dilation in  $L^2(\mathbb{R}^n, \mathbb{R}^k)$  if  $n \geq 3$ , but it is not a dilation in  $L^2(\mathbb{R}^n, \mathbb{R}^k)$  for  $n = 1$  and  $n = 2$ .

*Corollary 1:* If the matrix  $Q = Q^\top \in \mathbb{R}^{k \times k}$  satisfies

$$\left( \Gamma - \frac{\text{trace}(\Theta)}{p} I_k \right)^\top Q + Q \left( \Gamma - \frac{\text{trace}(\Theta)}{p} I_k \right) \succ 0, \quad Q \succ 0 \quad (25)$$

then the dilation (24) is strictly monotone with respect to

$$\|x\|_{L_*^p} = \left( \int_{\mathbb{R}^n} (x^\top(z)Qx(z))^{\frac{p}{2}} dz \right)^{\frac{1}{p}}. \quad (26)$$

*Proof:* The inequality (25) implies that  $\frac{d}{ds}\|\mathbf{d}(s)x\|_*^p \geq \frac{p\lambda}{2}\|\mathbf{d}(s)x\|_*^p$  for any  $x \neq \mathbf{0}$  and any  $s \in \mathbb{R}$ , where  $\lambda > 0$  is a minimal eigenvalue of  $Q^{-1/2}\Gamma^\top Q^{1/2} + Q^{1/2}\Gamma Q^{-1/2}$ . ■

A generator of the dilation (24) in  $L^p$  is characterized below.

*Lemma 7:* The infinitesimal generator of the dilation (24) in  $L^p(\mathbb{R}^n, \mathbb{R}^k)$  is the linear closed densely defined operator  $G_{\mathbf{d}} : \mathcal{D}(G_{\mathbf{d}}) \subset L^p(\mathbb{R}^n, \mathbb{R}^k) \mapsto L^p(\mathbb{R}^n, \mathbb{R}^k)$  given by

$$G_{\mathbf{d}}x = \Gamma x + ((\Theta z) \cdot \nabla)x, \quad z \in \mathbb{R}^n \quad (27)$$

whose domain  $\mathcal{D}(G_{\mathbf{d}})$  is a completion of the space  $C_c^\infty(\mathbb{R}^n, \mathbb{R}^k)$  with respect to the norm  $\|x\|_{G_{\mathbf{d}}} = \|x\|_{L^p} + \|G_{\mathbf{d}}x\|_{L^p}$ , where  $((\Theta z) \cdot \nabla) = \sum_{i=1}^n (\Theta z)_i \frac{\partial}{\partial z_i}$ .

*Proof:* Taking  $x_\infty \in C_c^\infty$  we derive

$$\lim_{s \rightarrow 0} \frac{\mathbf{d}(s)x_\infty - x_\infty}{s} = \Gamma x_\infty + ((\Theta z) \cdot \nabla)x_\infty.$$

Making completion of  $C_c^\infty$  in the norm  $\|x\|_{G_{\mathbf{d}}}$  we obtain  $\mathcal{D}(G_{\mathbf{d}})$ . Since  $\mathbf{d}$  is a strongly continuous group in  $L^p$  then the operator  $G_{\mathbf{d}} : \mathcal{D}(G_{\mathbf{d}}) \subset L^p \mapsto L^p$  is closed by Hille–Yosida Theorem [24]. ■

### B. Homogeneity of linear differential operators

Let us consider the differential operator  $A : \mathcal{D}(A) \subset L^2(\mathbb{R}^n, \mathbb{R}^k) \mapsto L^2(\mathbb{R}^n, \mathbb{R}^k)$  defined as follows

$$(Ax)(z) = \sum_{|\alpha| \leq q} \Xi_\alpha(z) \frac{\partial^{\alpha_1} \dots \partial^{\alpha_n} x(z)}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}}, \quad (28)$$

where  $q \in \mathbb{N}$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)^\top \in \mathbb{R}^n$ ,  $z = (z_1, \dots, z_n)^\top \in \mathbb{R}^n$ , and  $\Xi_\alpha \in C^\infty(\mathbb{R}^n, \mathbb{R}^{k \times k})$ .

Let a dilation  $\mathbf{d}$  in  $L^2(\mathbb{R}^n, \mathbb{R}^k)$  be defined by the formula (24). As a straightforward corollary of Lemma 3 we conclude that the operator  $A$  is  $\mathbf{d}$ -homogeneous of the degree  $\mu \in \mathbb{R}$  if

$$AG_{\mathbf{d}}x_\infty = (\mu I + G_{\mathbf{d}})Ax_\infty, \quad \forall x_\infty \in C^\infty(\mathbb{R}^n, \mathbb{R}^k),$$

where  $G_{\mathbf{d}}$  is given by (27), or equivalently,

$$\mathbf{0} = \sum_{|\alpha| \leq q} (\Xi_\alpha(z)\Gamma - (\Gamma + \mu I_k)\Xi_\alpha(z)) \frac{\partial^{\alpha_1} \dots \partial^{\alpha_n}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}} + \left( \Xi_\alpha(z) \frac{\partial^{\alpha_1} \dots \partial^{\alpha_n}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}} \right) ((\Theta z) \cdot \nabla) - ((\Theta z) \cdot \nabla) \left( \Xi_\alpha(z) \frac{\partial^{\alpha_1} \dots \partial^{\alpha_n}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}} \right).$$

This identity is *algebraic* and can be checked in many cases.

### C. On homogeneous control for PDEs

In this subsection, we consider the system (6) with  $\mathbb{H} = L^2(\mathbb{R}^n, \mathbb{R}^k)$ ,  $A$  given by (28) and  $B \in \mathcal{L}(\mathbb{U}, \mathbb{H})$  given by

$$(Bu)(z) = \Omega(z)u(z), \quad z \in \mathbb{R}^n, \quad \Omega \in C^\infty(\mathbb{R}^n, \mathbb{R}^{k \times m}), \quad (29)$$

where  $\mathbb{U}$  is a Banach space of functions  $\mathbb{R}^n \mapsto \mathbb{R}^m$  such that  $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$  contains a subset dense in  $\mathbb{U}$ . Let the operator  $K_0 \in \mathcal{L}(\mathbb{H}, \mathbb{U})$  be defined in the form:

$$(K_0x)(z) = \Upsilon_0(z)x(z), \quad z \in \mathbb{R}^n, \quad \Upsilon_0 \in C^\infty(\mathbb{R}^n, \mathbb{R}^{m \times k}). \quad (30)$$

In this case, the operator  $A_0 = A + BK$  has the same structure as the operator  $A$ , but the term  $\Xi_0$  in (28) is replaced with

$$\tilde{\Xi}_0(z) = \Xi_0(z) + \Omega(z)\Upsilon_0(z), \quad z \in \mathbb{R}^n. \quad (31)$$

Selecting the generator of the group  $G_0$  in the form

$$G_0x = \Gamma_0x + ((\Theta z) \cdot \nabla)x, \quad z \in \mathbb{R}^n, \quad (32)$$

where  $\Xi_0 \in \mathbb{R}^{n \times n}$  and  $\Gamma_0 \in \mathbb{R}^{k \times k}$ , the condition (7) becomes the algebraic system

$$\mathbf{0} = \left( \tilde{\Xi}_0(z)\Gamma_0 - (\Gamma_0 + I_k)\tilde{\Xi}_0(z) \right) - \sum_{i=1}^n (\Theta z)_i \frac{\partial}{\partial z_i} \tilde{\Xi}_0(z), \quad (33)$$

$$\begin{aligned} \mathbf{0} &= \sum_{1 \leq \alpha \leq q} (\Xi_\alpha(z)\Gamma_0 - (\Gamma_0 + I_k)\Xi_\alpha(z)) \frac{\partial^{\alpha_1} \dots \partial^{\alpha_n}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}} + \\ &\Xi_\alpha(z) \frac{\partial^{\alpha_1} \dots \partial^{\alpha_n}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}} ((\Theta z) \cdot \nabla) - ((\Theta z) \cdot \nabla) \left( \Xi_\alpha(z) \frac{\partial^{\alpha_1} \dots \partial^{\alpha_n}}{\partial z_1^{\alpha_1} \dots \partial z_n^{\alpha_n}} \right). \end{aligned} \quad (34)$$

Since  $\frac{\partial}{\partial z_i} (\Omega(z)u(z)) = \left( \frac{\partial}{\partial z_i} \Omega(z) \right) u(z) + \Omega(z) \frac{\partial}{\partial z_i} u(z)$  then the condition (8) holds if

$$\Gamma_0 \Omega(z) + \sum_{i=1}^n (\Theta z)_i \frac{\partial}{\partial z_i} \Omega(z) = \Omega(z) \tilde{\Gamma}, \quad (35)$$

$$\tilde{G}u = \tilde{\Gamma}u + ((\Theta z) \cdot \nabla)u, \quad \tilde{\Gamma} \in \mathbb{R}^{m \times m}. \quad (36)$$

Therefore, to define a homogeneous control (19) for the system (6) with the differential operator  $A$  given by (28) and the linear bounded operator  $B$  given by (29), we just need to solve the system of algebraic equations (33), (34), (35). Moreover, the operator inequalities (21), (22) may also be reduced to linear matrix inequalities (LMIs) in various particular cases.

*Remark 1:* If  $\Xi_0(z) = \Xi_0 \in \mathbb{R}^{k \times k}$  and  $\Omega(z) = \Omega \in \mathbb{R}^{k \times m}$  are constant matrices then conditions (33) and (35) become

$$(\Xi_0 + \Omega \Upsilon_0)\Gamma_0 = (\Gamma_0 + I_k)(\Xi_0 + \Omega \Upsilon_0), \quad \Gamma_0 \Omega = \Omega \tilde{\Gamma} \quad (37)$$

with  $\Upsilon_0 \in \mathbb{R}^{m \times k}$ ,  $\Gamma_0 \in \mathbb{R}^{k \times k}$ ,  $\tilde{\Gamma} \in \mathbb{R}^{m \times m}$ .

*Remark 2:* If  $\Omega(z) = \Omega \in \mathbb{R}^{k \times m}$  is a constant matrix, the conditions (33), (34), (35) are fulfilled for  $\tilde{\Gamma} = \mathbf{0}$  and the operator  $K \in \mathcal{L}(\mathbb{H}, \mathbb{U})$  is defined as

$$Kx = \Upsilon x, \quad \Upsilon \in \mathbb{R}^{k \times m},$$

then Assumption 3 holds and the control (19) has the form

$$u = \Upsilon_0 x + \|x\|_{\mathbf{d}}^{1+\mu} \Upsilon e^{-\Gamma \ln \|x\|_{\mathbf{d}}} x. \quad (38)$$

Indeed, since  $(\mathbf{d}(s_1)x)(z) = e^{s_1 \Gamma} x(e^{s_1 \mu \Theta} z)$ ,  $x \in \mathbb{H}$ ,  $s_1 \in \mathbb{R}$ ,  $(\tilde{\mathbf{d}}(s_2)u)(z) = e^{s_2 \tilde{\Gamma}} u(e^{s_2 \Theta} z)$ ,  $u \in \mathbb{U}$ ,  $z \in \mathbb{R}^n$ ,  $s_2 \in \mathbb{R}$  then

$$(\tilde{\mathbf{d}}(s_2)\Upsilon \mathbf{d}(s_1)x)(z) = e^{s_2 \tilde{\Gamma}} \Upsilon e^{\Gamma s_1} x(e^{\Theta(\mu s_1 + s_2)} z).$$

Hence, for  $\tilde{\Gamma} = \mathbf{0}$ ,  $s_1 = -\ln \|x\|_{\mathbf{d}}$ ,  $s_2 = \mu \ln \|x\|_{\mathbf{d}}$ , we have  $\tilde{\mathbf{d}}(\mu \ln \|x\|_{\mathbf{d}})\Upsilon \mathbf{d}(-\ln \|x\|_{\mathbf{d}})x = \Upsilon e^{-\Gamma \ln \|x\|_{\mathbf{d}}} x$ , i.e., the formula (38) corresponds to (19). Moreover, since  $\mathbf{d}(s)BK\mathbf{d}(-s) = e^{s\Gamma} \Omega \Upsilon e^{-s\Gamma}$  then the mapping  $s \mapsto \mathbf{d}(s)BK\mathbf{d}(-s)$  is uniformly continuous.

*Remark 3:* If  $n = 1$  and the linear matrix inequality (LMI)

$$\left( \Gamma - \frac{\mu \Theta}{2} I_k \right) X + X \left( \Gamma - \frac{\mu \Theta}{2} I_k \right)^\top \succ 0, \quad X \succ 0, \quad (39)$$

then the operator inequality (22) holds for  $Px = X^{-1}x$ .

Indeed, using the integration by parts we derive  $\langle z X^{-1} \frac{\partial}{\partial z} x, x \rangle = -\frac{1}{2} \langle X^{-1} x, x \rangle$  for all  $x \in C_c^\infty$ . Hence,  $\langle X^{-1} G_{\mathbf{d}} x, x \rangle = \langle X^{-1} (\Gamma - \frac{\mu \Theta}{2} I_k) x, x \rangle$ , so that the LMI (39) is equivalent to (22) in the considered case.

*Remark 4:* Let conditions of Remarks 1, 2, 3 be fulfilled and  $q = 2$ ,  $\tilde{\Xi}_0 = \Xi_0 + \Omega \Upsilon_0$ ,  $\Xi_1(z) = \gamma z I_k$  with  $\gamma \geq 0$ ,  $\Xi_2(z) = \tilde{\Xi}_2 \in \mathbb{R}^{k \times k}$ ,  $z \in \mathbb{R}$ . If the system of LMIs (39) and

$$\tilde{\Xi}_0 X + \tilde{\Xi}_0^\top X + \Omega \Upsilon + \Upsilon^\top \Omega^\top - \gamma X \prec 0, \quad \tilde{\Xi}_2 X + X \tilde{\Xi}_2^\top \succ 0 \quad (40)$$

is feasible with respect to  $X \in \mathbb{R}^{k \times k}$  and  $Y \in \mathbb{R}^{m \times k}$  then the inequality (21) is fulfilled for  $Px = X^{-1}x$  and  $Kx = YX^{-1}x$  for some  $\rho > 0$ .

Indeed, using integration by parts we derive

$$\langle z \gamma X^{-1} \frac{\partial x}{\partial z}, x \rangle = \frac{-\gamma \langle X^{-1} x, x \rangle}{2}, \quad \langle X^{-1} \tilde{\Xi}_2 \frac{\partial^2 x}{\partial z^2}, x \rangle = -\langle Q \frac{\partial x}{\partial z}, \frac{\partial x}{\partial z} \rangle,$$

where  $2Q = X^{-1} \tilde{\Xi}_2 + \tilde{\Xi}_2^\top X^{-1}$ . Since, by assumption,  $A_0 x = \tilde{\Xi}_0 x + z \gamma \frac{\partial x}{\partial z} + \tilde{\Xi}_2 \frac{\partial^2 x}{\partial z^2}$  then  $2 \langle P A_0 x, x \rangle = \langle (X^{-1} \tilde{\Xi}_0 + \tilde{\Xi}_0^\top X^{-1} - \gamma X^{-1}) x, x \rangle - 2 \langle Q \frac{\partial x}{\partial z}, \frac{\partial x}{\partial z} \rangle$ . Since  $\langle X^{-1} G_{\mathbf{d}} x, x \rangle = \langle X^{-1} (\Gamma - \frac{\mu \Theta}{2} I_k) x, x \rangle$  then the inequality (21) holds if

$$\begin{aligned} \langle (X^{-1} \tilde{\Xi}_0 + \tilde{\Xi}_0^\top X^{-1} + \Omega \Upsilon + \Upsilon^\top \Omega^\top - \gamma X^{-1}) x, x \rangle \leq \\ -2 \rho \langle X^{-1} (\Gamma - \frac{\mu \Theta}{2} I_k) x, x \rangle. \end{aligned}$$

It holds for a sufficiently small  $\rho > 0$  provided that (40) is fulfilled.

If the pair  $\{\tilde{\Xi}_0, \Omega\}$  is controllable and  $\tilde{\Xi}_2 = \nu I_n$  with some  $\nu > 0$  then the LMIs (39), (40) are always feasible [20] and the inequalities (21), (22) are feasible for any  $\rho > 0$  and  $\mu \in \mathbb{R}$ . This implies null controllability of the system (6).

## VI. EXAMPLE: INTERCONNECTED HEAT SYSTEM

Let us consider the system

$$\frac{\partial x(z,t)}{\partial t} = \nu \frac{\partial^2 x(z,t)}{\partial z^2} + \Xi_0 x(z,t) + \Omega u(z,t) + f(t, x(z,t)), \quad (41)$$

where  $\nu > 0$  is a diffusion coefficient,  $z \in \mathbb{R}$  is the space variable,  $t \geq 0$  is the time variable,  $x(\cdot, t) \in L^2(\mathbb{R}, \mathbb{R}^k)$  is the distributed state of the system,  $u(z, t) \in L^2(\mathbb{R}, \mathbb{R}^m)$  is a distributed control input be defined in the state feedback form,  $\Xi_0 \in \mathbb{R}^{k \times k}$ ,  $\Omega \in \mathbb{R}^{n \times m}$  are constant matrices and  $f: \mathbb{R} \times L^2(\mathbb{R}, \mathbb{R}^k) \mapsto L^2(\mathbb{R}, \mathbb{R}^k)$  is a locally Lipschitz nonlinearity. The system (41) written in a component-wise manner can be interpreted as a system of interconnected heat equations with the interconnection weights, given by the matrix  $\Xi_0$ , and with the matrix  $\Omega$ , specifying the control inputs of the system. For system (41) we design the homogeneous stabilizer (38).

In the view of Remark 1, we define the operator  $K_0 \in \mathcal{L}(L^2(\mathbb{R}, \mathbb{R}^m), L^2(\mathbb{R}, \mathbb{R}^k))$  by solving the system of equations

$$\Xi_0 \Gamma_0 - \Gamma_0 \Xi_0 + \Omega Y_0 = \Xi_0, \quad \Gamma_0 \Omega = \mathbf{0}$$

with respect to  $\Gamma_0 \in \mathbb{R}^{k \times k}$ ,  $Y_0 \in \mathbb{R}^{m \times k}$ . This system of equations is linear and it always has a solution if the pair  $\{\Xi_0, \Omega\}$  is controllable (see, [20] for more details). Moreover, the matrix  $\Gamma_0 - I_k$  is invertible and  $K_0$  is given by

$$K_0 x = Y_0 (\Gamma_0 - I_n)^{-1} x.$$

Hence, to fulfill Assumption 2 we need to find  $\Theta \in \mathbb{R}$  such that the algebraic condition (34) holds. In our case, the required condition is

$$0 = -\frac{\partial^2}{\partial z^2} + 2\Theta \frac{\partial^2}{\partial z^2}$$

i.e.,  $\Theta = 0.5$ . Using Lemma 1 we define the dilation in  $L^2(\mathbb{R}, \mathbb{R}^k)$  as follows

$$\mathbf{d}(s) = e^{s\Gamma} x(e^{0.5\mu s} z), \quad \Gamma = I_k + \mu \Gamma_0$$

with  $\mu < 2/3$ . The group  $\tilde{\mathbf{d}}$  is defined as follows

$$(\tilde{\mathbf{d}}(s)u)(z) = u(e^{0.5s} z), \quad z \in \mathbb{R}$$



Let us denote  $\tilde{\Xi}_0 = \Xi_0 + \Omega Y_0 (\Gamma_0 - I_n)^{-1}$ . In the view of Remark 4, to design the homogeneous stabilizer (38) it is sufficient to solve the system of LMIs (39), (40) with  $\gamma = 0$ ,  $\Xi_2 = \nu I_k$ . The corresponding system of matrix inequalities is always feasible provided that the pair of matrices  $\{\Xi_0, \Omega\}$  is controllable [20]. So, the feedback gain in (38) is defined as  $\Upsilon = YX^{-1}$  and the canonical homogeneous norm  $\|x\|_d$  is induced by the norm  $\|x\| = \sqrt{\langle X^{-1}x, x \rangle}$ . By Theorem 2, the closed-loop system with  $f = \mathbf{0}$  is finite-time stable for  $\mu < 0$ , exponentially stable for  $\mu = 0$  and nearly-fixed-time stable for  $\mu > 0$ . Moreover, if the nonlinearity  $f$  satisfies the condition (23), then the asymptotic stability and the convergence type of the closed-loop system is preserved. By [29, Corollary 2], the closed-loop system is Input-to-State Stable [31], [32] with respect to measurement noises.

Notice that the homogeneous norm  $\|\cdot\|_d$  is defined implicitly by the formula (12) for any  $x = (x_1, \dots, x_k) \in L^2(\mathbb{R}, \mathbb{R}^k)$ . The equation (12) has the unique solution due to monotonicity of the dilation. In our case, the corresponding equation can be rewritten as an algebraic equation  $Q(\|x\|_d, \alpha_{12}, \dots, \alpha_{nn}) = 1$  with  $\alpha_{ij} = \langle x_i, x_j \rangle_{L^2}$ ,  $i, j = 1, \dots, k$ , where  $Q: \mathbb{R}^{k^2+1} \mapsto \mathbb{R}_+$  is a strictly monotone function of the first argument for any  $x \neq \mathbf{0}$ , so that the homogeneous norm  $\|x\|_d$  can always be computed numerically using, for example, bisection method [9, Section 9.3.2]. For  $k = 2$  it may admit an explicit solution as, for example, in [9, Example 7.2]. Development of numerical method for consistent discretization of the closed-loop system is a difficult interesting problem, which goes out of the scope of this paper.

## VII. CONCLUSIONS AND DISCUSSIONS

In this paper, homogeneous finite/fixed time stabilizers are designed for abstract evolution equations in Hilbert spaces using homogenization method (Theorem 2). Several useful auxiliary results, characterizing dilations in abstract and function spaces, are proven as well. Various PDE models are considered to exemplify theoretical results. The design procedure in these cases consists in solving the system of rather simple algebraic equations/inequalities similar to the finite-dimensional setting [20]. The key restriction of the studied PDE models is the unboundedness of the domain for the space variable (namely,  $z \in \mathbb{R}^n$ ). A relaxation of this restriction is an important problem for the future research.

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